AN ELEMENTARY INTRODUCTION TO STABLE HOMOTOPY THEORY

HOWARD BECK

Abstract. This document serves as an elementary introduction to the field of stable homotopy theory. We start with a review of the concepts we will use from a standard undergraduate course in topology, followed by a brief detour into algebraic topology and category theory on a need-to-know basis, which will contextualize the object of interest in homotopy theory. Then, we introduce stable homotopy theory, seeing how constructions we have unstably become well-behaved in the stable setting.

1. Introduction

Broadly, homotopy theory concerns looking at topological spaces up to a natural equivalence relation called homotopy equivalence. Homotopy theorists make frequent use of tools from algebra, such as groups and rings to simplify their study of spaces. It becomes clear quite quickly that spaces by themselves can be poorly behaved, so many homotopy theorists prefer to work with topological spectra, which can be thought of as a stable version of the concept of spaces. This field of investigation is called stable homotopy theory.

At face value, homotopy theory is just about topology. However, as is frequent all throughout mathematics, there is cross-pollination with other areas. Algebra is the most obvious example. The most elementary proof that all subgroups of a free group are also free is a topological argument using fundamental groups. Algebra has also benefited from the introduction of cohomology - an invariant originally developed to study topological spaces. In fact, topological spaces themselves appear here through classifying spaces of groups. There are further applications of topology in algebraic geometry, which is a growing area of research called motivic homotopy theory. There is also a school of thought concerning "higher algebra" [Lur17], which posits that certain constructions inspired by stable homotopy theory are the correct generalization of algebra.

In general, it is clearer than ever now that homotopy theory is a powerful tool, even when applied to other settings. This document is not a thorough introduction to stable homotopy theory. Instead, the focus is on quickly building up from a standard introduction to topology course. Proofs will often be emitted in favor of explanation and motivation of definitions. The few proofs that are present have a more pedagogical purpose, as a kind of exercise in using all the definitions and notations introduced. There are great sources to learn the material more rigorously, such as [May98].

1.1. **Elementary Topology.** We start with an informal review of topology. All of this material can be found in any standard topology textbook, such as [Mun00]. The point of this review is to help establish topological notations used throughout the rest of this paper, along with introducing perspectives on certain constructions that may be overlooked or not

Date: November 15th, 2024.

given proper care in an introductory topology course.

A topological space, or just a space, is a set X with a collection of subsets which we declare to be the open subsets of X, called its topology. The topology must be closed under arbitrary union and under finite intersection, and further, it must contain the empty set $\emptyset$ and the entirety of the set X. We will refer X itself as being a topological space, even though we formally should also include the data of its topology. For two topological spaces X and Y, we define a function $f : X \to Y$ on the underlying sets to be continuous if the preimage of any open set U in Y is an open set $f^{-1}(U)$ in X. We really only care about continuous functions, so from here on out, a function or map $f : X \to Y$ will be a continuous function from X to Y. Some examples of topological spaces include the real numbers $\mathbb{R}$, n-dimensional Euclidean spaces $\mathbb{R}^n$, the interval $I = [0, 1]$, n-dimensional (hollow) unit spheres $S^n \subset \mathbb{R}^{n+1}$, n-dimensional (filled in) unit disks $D^n \subset \mathbb{R}^n$, and a singular point $\star$.

Topological spaces also interact with each other in useful ways. For two spaces X and Y, we may define a product space $X \times Y$. As a set, it has all pairs of elements (x, y) with $x \in X$ and $y \in Y$. There are continuous projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ that just return one of the coordinates. The topology on the product space is given by taking any subset $U \subset X \times Y$ to be open if both images $\pi_X(U)$ and $\pi_Y(U)$ are open in the respective spaces X and Y. For a subset $A \subseteq X$, we may also take the subspace A. Open sets in A are precisely those sets contained inside of open sets in X. The inclusion function $i : A \hookrightarrow X$ is then a continuous map. For an equivalence relation $\sim_X$ between elements of a space X, we may form the quotient space $X/\sim$, which has a continuous projection map $\pi : X \to X/\sim_X$ by sending a point x to its equivalence class $[x]_\sim$ under the relation. Open sets in $X/\sim_X$ are precisely those that are images under π of open sets in X. We may also write X/A for the quotient space given by the equivalence relation $x \sim y$ if and only if $x, y \in A$ - in other words, we collapse all of A into a point. Two spaces can also be disjointly added together - this may either be called the disjoint union or the coproduct of spaces X and Y and is denoted either $X \sqcup Y$ or $X \coprod Y$. As a set, it just has disjoint copies of X and Y. An open set in the coproduct is just a disjoint union of an open set in X and an open set in Y. We may also take arbitrary products and coproducts, although there is some care we need to take in giving an infinite product space a suitable topology.

Continuous functions give rise to our first notion of an equivalence of topological spaces. Namely, we say that spaces X and Y are homeomorphic, written as $X \cong Y$, if there exist (continuous) functions $f : X \to Y$ and $g : Y \to X$ which are bijections on sets and are each other's inverses. The notion of homeomorphism by construction captures all the data about spaces. While this is nice, it is not very flexible. We will often study spaces via maps in and out of them, since they are frequently more accessible to work with than the spaces themselves. Such maps have a natural notion of equivalence, which is what we will care about. We say that two functions $f, g : X \to Y$ are homotopic, written as $f \simeq g$, if there exists a function h, called a homotopy, $h : X \times I \to Y$ agreeing with f, g on $X \times \partial I$. That is, we may choose f to correspond to the point $0 \in I$ and g to the point $1 \in I$. Then, h is an extrapolation between the two - they can be deformed into each other through moving along I. Therefore, we will often think of a homotopy as a deformation through time of functions, where the "time" variable moves in the interval. We call a function f that is homotopic to a constant map nullhomotopic or contractible. Homotopy is an equivalence relation on the set of functions between any two topological spaces, so any map lives in a homotopy class of maps. We then get a set $[X, Y]$ of homotopy classes of maps from X to Y.

A weaker but more useful notion of topological spaces being similar, compared to homeomorphism, is that when all the homotopy classes of maps into or out of spaces are the same. That is, we will equate X to Y if for all other spaces Z, $[X, Z] \simeq [Y, Z]$ and $[Z, X] \simeq [Z, Y]$. This happens precisely when there is a function $f : X \to Y$ and a homotopy inverse $g : Y \to X$, such that they compose to the identities *up to homotopy* only. That is, we have $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Here, we define id_X and id_Y as the identity maps on X and Y respectively, that leave any point unchanged. In this case, we say that X and Y are homotopy equivalent, written as $X \simeq Y$. We call a space X contractible if it is homotopy equivalent to a point: that is, $X \simeq \star$.

We may also look at pointed topological spaces, which are topological spaces with the extra information of some preferred basepoint $\star \in X$. When working over pointed spaces, we require that all the maps be pointed maps: namely, for basepoints $\star_X$ and $\star_Y$, we require that $f(\star_X) = \star_Y$ for a function $f : X \to Y$. All the definitions from earlier apply, only with minor modifications. A pointed homeomorphism is a homeomorphism fixing the basepoint. Products $X \times Y$ have a basepoint $(\star_X, \star_Y)$. Subspaces $A \subset X$ that have the basepoint $\star_X$ may inherit the basepoint $\star_X \in A$. Quotient spaces $X/\sim$ get the basepoint given by the image of $\star_X$ into the equivalence class $[\star_X]_\sim$. A pointed homotopy between functions is a map $f : X \times I \to Y$ that fixes sends the basepoint $\star_X$ to $\star_Y$ for all times $t \in I$. Then, we get a pointed homotopy equivalence precisely in the same way, requiring that X and Y have maps between each other that are pointed homotopy inverses. We may then form the pointed homotopy classes of maps between X and Y, denoted as $[X, Y]_\star$. Any unpointed space can always be turned into a pointed space via an operation $(-)_+$ that adds a disjoint basepoint to X. Pointed spaces also admit some new constructions, that are pointed versions of products and coproducts. The wedge sum between two spaces X and Y, denoted $X \vee Y$, is the pointed space $(X \coprod Y)_{\star_X \sim \star_Y}$. In other words, we take X and Y and glue them together only at the basepoint, giving a preferred basepoint where they meet. There is also a smash product, denoted $X \wedge Y$, which is a quotient of $X \times Y$ by $X \vee Y$. In other words, we may embed the wedge sum into the product, and then just identify all the points in the wedge sum into the same basepoint in the smash product. For example, a pointed homotopy between pointed spaces X and Y can also be thought of as a pointed map $X \wedge I_+ \to Y$. Also, in all these definitions, note that we may synonymously say based instead of pointed, to refer to the basepoint $\star_X \in X$.

As mentioned earlier, algebra provides a good lens through which we can analyze topology. For any pointed space X, we have the fundamental group $\pi_1(X)$, also called the first homotopy group, given by homotopy classes of maps out of a circle:

$$\pi_1(X) = [S^1, X]_\star$$

We will sometimes call elements of $\pi_1(X)$ loops, as they are ways to insert the circle S^1 into a space X. Implicitly, we should pick a basepoint of the circle for this to work - we may just view S^1 as being a subset of $\mathbb{C}$ and just pick the basepoint $1 \in \mathbb{C}$. A priori, the fundamental group is just a set, but in fact it will be an honest group. To compose two maps $f, g : S^1 \to X$, we may think of them as maps $[0, 1] \to X$ that send 0 and 1 to the basepoint $\star_X$. Then, we may form a combined map by compressing f into $[0, 1/2] \to X$ and g into $[1/2, 1] \to X$ and gluing them together at $1/2$. Of course, we should check that this composition is associative, admits an inverse, and is homotopy invariant - we will not do that here. While a useful description, here we will prefer another viewpoint of the group structure. Specifying two elements of $[S^1, X]_\star$ is the same as giving an element of $[S^1 \vee S^1, X]_\star$ - f acts on the first circle, g acts on the other one, and they agree at the common basepoint. Then, we may use the pinch map $S^1 \to S^1 \vee S^1$: taking $S^1 \subset \mathbb{C}$, we may pinch two points on an equator together. The picture to have in mind is that of pinching the Earth's equator into the center, giving two spherical blobs glued together at a point. However, for now we only work over the 1-sphere, so we will take the points $\pm i$ that define an equator for S^1 and identify them together by pinching. The claim is that the quotient space $S^1/\{\pm 1\}$ is (pointedly) homeomorphic to the wedge sum $S^1 \vee S^1$. The quotient map $S^1 \to S^1/\sim = S^1 \vee S^1$ then lets us get group composition:

Definition 1.1.1. The group operation on the fundamental group $\pi_1(X)$ is given as follows. For two classes $[f], \in \pi_1(X)$, we may pick representatives $f, g : S^1 \to X$. This naturally corresponds to a map $f \vee g : S^1 \vee S^1 \to X$. We get a "multiplication" $f \cdot g$ by precomposing the wedge of loops with the pinch map:

$$f \cdot g : S^1 \xrightarrow{\text{pinch}} S^1 \vee S^1 \xrightarrow{f \wedge g} X$$

In fact, the homotopy class of $f \cdot g$ does not depend on the homotopy classes of f or g, so we get an honest group operation:

$$[f] \cdot = [f \cdot g]$$

This confirms that in fact, $\pi_1(X)$ is a group. Further, $\pi_1(X)$ is very well-behaved, in the following way:

Proposition 1.1.2. *Maps $f : X \to Y$ give induced group homomorphisms on $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$. Further, the induced map does not depend on the homotopy class of f. A map with a homotopy inverse gives an isomorphism of fundamental groups - that is, when X and Y are homotopy equivalent, their fundamental groups are isomorphic.*

Proof sketch. We get that functions $f : X \to Y$ induce group homomorphisms $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$, where we take a homotopy class of maps $S^1 \to X$ and postcompose with f:

$$\pi_1(f) : \pi_1(X) \to \pi_1(Y)$$

$$[S^1 \xrightarrow{\gamma} X] \mapsto [S^1 \xrightarrow{\gamma} X \xrightarrow{f} Y]$$

We should check that these construction does not depend on the homotopy class of γ, and in fact this turns out to be true. In fact, it does not even depend on the homotopy class of f. For homotopy equivalent spaces $X \simeq Y$, the homotopy classes of maps $[S^1, X]_\star$ and $[S^1, Y]_\star$ are in bijection - in general, these sets are in bijection for any input space, by our definition of homotopy equivalence. Therefore, as sets, $\pi_1(X)$ and $\pi_1(Y)$ are in bijection. But in fact, the functions $f : X \rightleftarrows Y : g$ specifying this homotopy equivalence must induce bijective group homomorphisms $\pi_1(f) : \pi_1(X) \rightleftarrows \pi_1(Y) : \pi_1(g)$. This tells us that $\pi_1(f)$ and $\pi_1(g)$ are in fact group isomorphisms. □

This is the key insight - $\pi_1(-)$ preserves homotopies, both on the level of spaces and on the level of functions. This will be our first example of a homotopy invariant.

1.2. **Basics of Algebraic Topology.** We are thus inspired to see what other kinds of homotopy invariants we can find in the algebra world. Here, we will discuss three more types of homotopy invariants to motivate the discussion of stable homotopy theory. All of these constructions can be found in algebraic topology textbooks, such as [Hat02] or [Mil21].

We will first generalize the fundamental group as an example, as there is no reason we should just stick to maps out of a one dimensional circle. We may actually form higher homotopy groups:

Definition 1.2.1. For a point space X and $n \geq 1$, we define the n^{th} homotopy group $\pi_n(X)$ to be

$$\pi_n(X) = [S^n, X]_\star$$

As before, we get a group structure via pinch maps for $n \geq 1$. We take some embedding of a lower dimensional sphere $S^{n-1} \subset S^n$, and then collapse it to a point. This gives a map:

$$S^n \xrightarrow{\text{pinch}} S^n/S^{n-1} \cong S^n \vee S^n$$

and a group structure via, for two homotopy classes of maps $[f], \in \pi_n(X)$,

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{\gamma} X$$

In fact, something remarkable happens here, in that all the higher homotopy groups $\pi_n(X)$ for $n \geq 2$ are all abelian, although we won't show this.

$\pi_0(X)$ also exists - it consists of pointed homotopy classes of maps $\{\pm 1\} \to X$. The basepoint $1 \in S^0$ must get sent to the basepoint $\star_X \in X$. Therefore, we just look at unpointed homotopy classes of maps $-1 \cong \star \to X$. But a homotopy of maps between a point and X is just a path $[0, 1] \to X$. Therefore, $\pi_0(X)$ is just a set and consists of the path-connected components of X. There is no lower dimensional sphere for which we can use the pinch trick to get a group structure, and in fact there is no natural group structure on $\pi_0(X)$ in general.

While the homotopy groups have clear geometric descriptions, as based loops in a space, they are terribly difficult to compute. Another natural invariant that is easier to study is called homology. We will not explicitly define homology here, but instead we will just say it comes from maps $\Delta^n \to X$, where Δ^n is the (topological) n-simplex - the subspace of $\mathbb{R}^{n+1}$ given by points:

$$\Delta^n = \{(t_1, t_2, \ldots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_i \geq 0, t_1 + \ldots + t_{n+1} \leq 1\} \subset \mathbb{R}^{n+1}$$

These spaces are the geometric generalizations of triangles or tetrahedra into higher dimensions. As topological spaces, they are all contractible. Therefore, all maps $\sigma : \Delta^n \to X$ are nullhomotopic, so at first it seems we gain nothing useful out of them. However, there are $n+1$ ways to geometrically embed the $(n-1)$-simplex into the n-simplex. For example, the 2-simplex, a triangle, has 3 edges that are each lines, which are 1-simplices. Going up a dimension, tetrahedra are 3-simplices which have 4 triangular faces, which are 2-simplices.

Homology at degree n, denoted by $H_n(X)$ for a space X, measures, in some informal sense, which maps $\sigma : \Delta^n \to X$ do not come from restrictions of higher maps $\Delta^{n+1} \to X$. At each n, homology has an abelian group structure - it is sometimes thought of as the abelian version of homotopy. In fact, we may take homology with coefficients $H_n(X; G)$ in some *abelian* group G, and then there will be an action of G on the group $H_n(X; G)$. When not specified, $H_n(X)$ is homology with a coefficient group $\mathbb{Z}$. We may even take homology with coefficients in a ring R, or in an an R-module M for some ring R. Then, $H_n(X;R)$ and $H_n(X;M)$ are both R-modules. Homology will also enjoy induced homomorphisms $H_n(f) : H_n(X;G) \to H_n(Y;G)$ coming from continuous maps $f : X \to Y$. Informally, these come from the composition:

$$\Delta^n \to X \xrightarrow{f} Y$$

Homology will also be a homotopy invariant:

Proposition 1.2.2. *The homology groups $H_n(X;G)$ for any coefficient group (or ring, or module) G are a homotopy invariant. Namely, the induced maps $H_n(f) : H_n(X;G) \to H_n(Y)$ do not depend on the homotopy class of f. As such, $H_n(X;G)$ itself does not depend on the homotopy equivalence class of X.*

For a vector space V over a field $\mathbb{F}$, we may form its dual space:

$$V^* = \mathrm{Hom}_{\mathrm{Vect}(\mathbb{F})}(V, \mathbb{F})$$

where by $\mathrm{Hom}_{\mathrm{Vect}(\mathbb{F})}(A, B)$ for two $\mathbb{F}$-vector spaces A, B, we mean the vector space of linear functions $f : A \to B$. Something interesting happens for dual constructions: maps of $\mathbb{F}$-vector spaces $f : V \to W$ induce maps going *backwards* on the dual spaces. Namely, we have a (co)induced map $f^* : W^* \to V^*$ given by precomposition now:

$$f^* : W^* = \mathrm{Hom}_{\mathrm{Vect}(\mathbb{F})}(W, \mathbb{F}) \to \mathrm{Hom}_{\mathrm{Vect}(\mathbb{F})}(V, \mathbb{F}) = V^*$$

$$\{\lambda : W \to \mathbb{F}\} \mapsto \{V \xrightarrow{f} W \xrightarrow{\lambda} \mathbb{F}\}$$

Again, without going into the weeds of the rigorous definition involving simplices, there is a dual notion to homology, called cohomology, denoted in degree n as $H^n(X;G)$ for a space X and coefficient group G. Cohomology also has homotopy-invariant induced group homomorphisms, although as with dual vector spaces, these maps go the other way. Namely, we get, for a map $f : X \to Y$, a homomorphism $H^n(f) : H^n(Y;G) \to H^n(X;G)$.

Proposition 1.2.3. *The cohomology groups $H^n(X;G)$ for any coefficient group (or ring, or module) G are a homotopy invariant. Namely, the induced maps $H^n(f) : H^n(Y;G) \to H^n(X;G)$ do not depend on the homotopy class of f. As such, $H^n(X;G)$ itself does not depend on the homotopy equivalence class of X.*

Homology and cohomology groups are not arbitrary. In fact, they are uniquely determined, by certain axioms, up to isomorphism. In general, we may actually define relative homology $H_n(X, A; G)$ and cohomology $X^n(X, A; G)$ for topological spaces $A \subseteq X$, where there is additional data considered based on maps $\Delta^n \to A$. We will again not give a rigorous definition, but instead, all we will say is that we may construct standard homology from relative homology by picking a basepoint $\star \in X$:

$$H_n(X;G) \simeq H_n(X, \star; G) \oplus H_n(\star; G) \qquad H^n(X;G) \simeq H^n(X, \star; G) \oplus H^n(\star; G)$$

Then, the following axioms uniquely characterize homology:

Axioms 1.2.4 ([ES15] and [Mil62], as formulated in [Mil21] and informally stated here)**.** The Eilenberg-Steenrod axioms give that a homology theory is an assignment $H_n(X;G)$ of abelian groups to spaces X for each coefficient group and $n \in \mathbb{Z}$ (we take them to be the trivial groups for negative degree), along with maps $\delta_n : H_n(X, A) \to H_{n-1}(A)$, satisfying:

ES1 (Homotopy invariance) There are induced maps on homology from maps $f : X \to Y$ as we saw that we do not depend on the homotopy class of f. Further, there are induced maps on relative homology too.

ES2 (Excision) For a subset A whose closure $\bar{U}$ lies in the interior of A, the inclusion of the pair $(X-U, A-U)$ into (X, A) induces an isomorphism $H_n(X-A, A-u; G) \simeq H_n(X, A; G)$.

ES3 (Long exact sequence) For the inclusion map $i : A \hookrightarrow X$ and inclusion of the pair $j : (X, \emptyset) \hookrightarrow (X, A)$, the sequence

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \to \cdots$$

is long exact - meaning the kernel of one map is the same as the image of the previous map.

ES4 (Dimension) All non-zero degrees have zero homology groups: $H_n(X; G) = 0$. Further, $H_0(X; G) \simeq G$.

ES5 (Milnor) Homology turns coproducts into direct sums. That is,

$$H_n\left(\coprod_\alpha X_\alpha; G\right) \simeq \bigoplus_\alpha H_n(X_\alpha; G)$$

We won't concern ourselves with excision or long exact sequences. At best, we will use a version of the Milnor axiom later on. We will also generally care more about cohomology theories, as they are better behaves in some ways. Cohomology theories have their own axiomatization, which is remarkably similar to the Eilenberg-Steenrod axioms. We will care most about the dimension axiom here, since we get a beautiful theory of cohomology when we get rid of it and allow for cohomology to not be concrentrated in degree 0.

One reason cohomology is nicer is that it is representable. This means there is a space, in fact, a CW complex, $K(G, n)$ such that:

$$H^n(X; G) = [X, K(G, n)]$$

These spaces $K(G, n)$ are called Eilenberg-Maclane spaces. As CW complexes, they are determined uniquely up to homotopy by the requirement that they have the following homotopy groups:

$$\pi_k(K(G, n)) \simeq \begin{cases} 0 & k \neq 0 \\ G & k = n \end{cases}$$

In other words, their homotopy is concentrated in degree n. We will look at these spaces in more detail later, but for now, we will be satisfied with just saying that somehow, they have enough structure to make $H^n(X; G)$ a group, even though a priori it is just a set.

Before we move on, we should establish another notation common in algebraic topology. We frequently work with various gradings, such as the degrees n on homology, cohomology, and homotopy groups. Instead of explicitly writing n, we will often instaed use $\bullet$. We may also use the notation $f_\bullet$ or $f^\bullet$ for induced maps. For example, we will use this notation to discuss induced maps on cohomology in terms of the Eilenberg-Maclane spaces. A map

$f : X \to Y$ induces a map

$$f^\bullet : H^\bullet(Y) \to H^\bullet(X)$$
$$[\lambda : Y \to K(G,n)] \mapsto [X \xrightarrow{f} Y \xrightarrow{\lambda} K(G,n)]$$

We should also say something about homotopy equivalence. While it is a useful invariant, there is a weaker one that is actually much more useful.

Definition 1.2.5. For two pointed spaces X, Y, we say they are weakly equivalent if there is a map $f : X \to Y$ that induces isomorphisms on all homotopy groups:

$$\pi_n(f) : \pi_n(X) \xrightarrow{\sim} \pi_n(Y)$$

This definition is remarkable, since it is in general not an equivalence relation. In fact, one of the strongest statements we can say is that, for CW complexes, weak equivalences admit homotopy inverses. That is, the notions of weak equivalence and homotopy equivalence line up for CW spaces. However, this is not true in general. From here on out, when we say two spaces are equivalent, we mean *weakly* equivalent.

1.3. **Categorical Language.** So far, we have been using very loose language to discuss the kinds of objects we work with, and how they transform into other objects. We will introduce some terminology from category theory that homotopy theorists use freely, in order to have a better language to talk about these notions. Sets will be too restrictive of an environment to work in, however. For example, it is a famous result that there is no "set of sets." However, we want to be able to talk about all topological spaces simultaneously, for example. This causes a problem, as we can give any set a topology by declaring all sets to be open - this is called the discrete topology. Therefore, we will instead work with classes, which may be much larger than sets. We will not give an axiomatic treatment of classes, besides saying we form classes for most reasonable things. More information about category theory can be found in a textbook such as [Lei14].

Definition 1.3.1. A category $\mathcal{C}$ is the data of a class of objects $\mathrm{ob}(\mathcal{C})$, along with a class of morphisms $\mathrm{mor}(\mathcal{C})$. Morphisms can be thought of as maps, or arrows, between objects. Namely, there are functions:

$$\text{source} : \mathrm{mor}(C) \to \mathrm{ob}(C)$$
$$\text{target} : \mathrm{mor}(C) \to \mathrm{ob}(C)$$

that give us the source and target of any morphism. We also can compose morphisms. If a morphism $f \in \mathrm{mor}(\mathcal{C})$ has source x and target y, and another morphism $g \in \mathrm{mor}(\mathcal{C})$ has source y and target z, we may form the composite morphism $g \circ f$. We also require there to be an identity morphism for each object that just trivially points back at it. In other words, there is a function

$$\mathrm{id} : \mathrm{ob}(C) \to \mathrm{mor}(C)$$

such that the output of $\mathrm{id}(x)$ has source x and target x.

At this point, we should take a step back. For the purposes of this paper, we do not need to fully process this information, but can instead get an intuition from some examples.

Example 1.3.2. There is a category of topological spaces, called Top. The objects in Top are topological spaces. The morphisms in Top are continuous maps between the spaces. Similarly, pointed topological spaces form a category $\mathrm{Top}_\star$, with the morphisms being basepoint-preserving continuous functions.

Example 1.3.3. We may similarly form the naive homotopy category $h\,\mathrm{Top}$. The objects in $h\,\mathrm{Top}$ are topological spaces, and the morphisms are *homotopy classes of maps* between spaces.

Example 1.3.4. Using weak equivalences, we may form the (genuine) homotopy category Ho(Top). We get from $h\,\mathrm{Top}$ to Ho(Top) by adding morphisms $f^{-1} : Y \to X$ for any weak equivalence $f : X \to Y$, therefore making weak equivalences invertible.

Example 1.3.5. Grp is the category of groups. The objects are groups, and morphisms are homomorphisms between the groups. There is a *full subcategory* Ab $\subset$ Grp of abelian groups.

Example 1.3.6. Set is the category of sets. The objects are sets, and the morphisms are just functions between sets. Notably, Set is not a set, but it is a category.

Morphisms also allow us to extend the notion of isomorphism:

Definition 1.3.7. We say two objects $x, y \in \mathrm{ob}(\mathcal{C})$ are isomorphic if there exist morphisms f, g:

$$f : x \rightleftarrows y : g$$

such that $f \circ g = \mathrm{id}(g)$ and $g \circ f = \mathrm{id}(f)$. We call either one of f and g an isomorphism.

For example, in the categories from earlier, we have:

- Isomorphisms in Top are homeomorphisms, and on $\mathrm{Top}_\star$ are pointed homeomorphisms.
- Isomorphisms in $h\,\mathrm{Top}$ are homotopy equivalences.
- We recover the definition of group isomorphisms in Grp.

We may also transform between categories:

Definition 1.3.8. A covariant functor F between categories $\mathcal{C}$ and $\mathcal{D}$ is the data of assignments $F(x) \in \mathrm{ob}(\mathcal{D})$ for each $x \in \mathrm{ob}(\mathcal{C})$, and assignments $F(m) \in \mathrm{mor}(\mathcal{D})$ for each morphism $m \in \mathrm{mor}(\mathcal{C})$, that converts objects in $\mathcal{C}$ to objects in $\mathcal{D}$ in a way that commutes with morphisms.

We can summarize this definition with a commutative diagram, where going from any object to another through the arrows does not depend on the path taken:

$$\begin{tikzcd} \mathrm{ob}(\mathcal{C}) \ni x \arrow[r, "F"] \arrow[d, "\mathrm{mor}(\mathcal{C}) \ni m"'] & F(x) \in \mathrm{ob}(\mathcal{D}) \arrow[d, "F(m) \in \mathrm{mor}(\mathcal{D})"] \\ \mathrm{ob}(\mathcal{C}) \ni y \arrow[r, "F"'] & F(y) \in \mathrm{ob}(\mathcal{D}) \end{tikzcd}$$

We get our first examples of functors from algebraic topology:

Example 1.3.9. $\pi_n(-) : \mathrm{Ho}(\mathrm{Top}_\star) \to \mathrm{Grp}$ is a covariant functor. It transforms spaces into groups, and continuous maps into group homomorphisms via induced maps.

Similarly, $H_n(-) : \mathrm{Ho}(\mathrm{Top}) \to \mathrm{Ab}$ is also a covariant functor, for the same reason.

In fact, we have already seen functors strictly in topology:

Example 1.3.10. $(-)_+ : \mathrm{Top} \to \mathrm{Top}_\star$ is the covariant functor that adds a basepoint to a space.

Cohomology is a different kind of functor - it is contravariant. Instead of defining contravariant functors directly, we will do what most authors do, and instead define opposite categories:

Definition 1.3.11. For a category $\mathcal{C}$, its opposite category $\mathcal{C}^{\mathrm{op}}$ is the category with the same objects, but the morphism source and targets are reversed. That is, the target and source maps on $\mathcal{C}^{\mathrm{op}}$ are swapped from those on $\mathcal{C}$. We may then define a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ as a covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Now we have a way to talk about the coinduced maps on cohomology:

Example 1.3.12. Cohomology is a contravariant functor $\mathrm{Ho}(\mathrm{Top}^{\mathrm{op}}) \to \mathrm{Ab}$.

We will frequently study categories by looking at their morphisms. For categories that are not too small, we can package all morphisms between two objects into not just a class, but a set:

Definition 1.3.13. We call a category $\mathcal{C}$ locally small if, for every two objects $x, y \in \mathrm{ob}(\mathcal{C})$, the subclass of morphisms between them is actually a set. In that case, we define the Hom-set between x and y:

$$\mathrm{Hom}_{\mathcal{C}}(x, y) = \{m \in \mathrm{mor}(\mathcal{C}) \mid \mathrm{source}(m) = x, \mathrm{target}(m) = y\}$$

In fact, we may think of this as a functor:

$$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$$

where the first term is the contravariant term, and the second term is the covariant term. It is a useful exercise to think about how $\mathrm{Hom}_{\mathcal{C}}(-, -)$ is a functor and why the first term is contravariant - it follows from arguments we have used already.

Example 1.3.14. In $h\,\mathrm{Top}$, $\mathrm{Hom}_{h\,\mathrm{Top}}(X, Y)$ is the set $[X, Y]_\star$

Example 1.3.15. For a group G and an automorphism group $\mathrm{GL}(V)$ of a vector space V, we have that $\mathrm{Hom}_{\mathrm{Grp}}(G, \mathrm{Aut}(V))$ is the set of group representations of G over the vector space V.

Something special happens in Top once we restrict our notion of what a topological space. There is a broad class of topological spaces such that we can give $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ a natural topology, for any restricted enough X and Y. From here on out, we will actually take Top to be the category of compactly generated topological spaces, as described in, for example, [Mil21], Chapter 40. In this special setting, we have internal Homs, where the Hom-sets may be viewed as elements of Top.

Frequently, when transforming between categories through functors, we may want a way to go backwards. We may not always have an inverse functor, but in special cases, we get the weaker notion of adjunction:

Definition 1.3.16. An adjunction is a pair of functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ such that, for all $x \in \mathrm{ob}(\mathcal{C})$ and $y \in \mathrm{ob}(\mathcal{D})$, we get:

$$\mathrm{Hom}_{\mathcal{D}}(F(x), y) \simeq \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

We call F the left adjoint, G the right adjoint, and the pair (F, G) an adjoint pair.

In other words, we may not define inverses on the level of objects, but we almost can on the level of morphisms.

Example 1.3.17. In representation theory, Frobenius reciprocity is a special case of the adjunction:

$$\mathrm{Ind}_H^G : \mathrm{R}_{\mathbb{F}}(H) \rightleftharpoons \mathrm{R}_{\mathbb{F}}(G) : \mathrm{Res}_H^G$$

where we take the category of representations $\mathrm{R}_{\mathbb{F}}(G)$ of a group G over some field $\mathbb{F}$. $\mathrm{Ind}_H^G(-)$ is the induced representation functor that creates G representations out of H-representations for a subgroup $H \subseteq G$, and $\mathrm{Res}_H^G(-)$ is the restriction representation functor that converts a G-representation into its underlying H-representation.

Example 1.3.18. We have a forgetful functor $f : \mathrm{Grp} \to \mathrm{Set}$ that just returns the underlying group structure. It has a left-adjoint given by the free group functor $* : \mathrm{Set} \to \mathrm{Grp}$, that sends a set to the group freely generated on that set.

From here on out, we will not be careful about distinguishing between categories $\mathcal{C}$ and the objects of the category $\mathcal{C}$. Namely, we will freely use notations such as $X \in \mathrm{Top}_\star$, and take it to mean that $x \in \mathrm{ob}(\mathrm{Top}_\star)$.

2. Stable Homotopy Theory

We will see that homotopy theory can be quite badly behaved. In fact, if we know all the homotopy groups of a space, it is likely since the space can be constructed as a product of Eilenberg-Maclane spaces. The simplest example is given by the circle:

$$\pi_n(S^1) \simeq \begin{cases} 0 & n \neq 1 \\ \mathbb{Z} & n = 1 \end{cases}$$

where by 0 we mean the trivial group. This will turn out to be the only sphere whose homotopy groups we know. Calculating $\pi_n(S^m)$ for all n, m is still an open problem. However, there is a lot we do know. Informally, the groups $\pi_n(S^{n+m})$ stabilize for large m. Theis kind of phenonemon is not unique to the circle, and stable homotopy theory concerns the study of such stable behaviors. A good reference for much of the modern formulation of stable homotopy theory can be found in [May98].

2.1. Stable Homotopy Groups.

We will expore these kinds of stable behaviors. The reason the spheres stabilize the way they do is because we may form spheres through smashing other spheres. Namely, $S^n \wedge S^m \simeq S^{n+m}$. In general, the best we can say about how spaces behave stably is about how $\Sigma^n \vee X$ behave:

Definition 2.1.1. For a topological space X, we define its suspension ΣX as:

$$\Sigma X = S^1 \wedge X$$

and its loop space ΩX as:

$$\Omega X = \mathrm{Map}_\star(S^1, X)$$

Here, we take the pointed mapping space $\mathrm{Map}_\star(A, B)$ to be the set of pointed (continuous) functions $A \to B$, which will carry with it an appropriate topology.

The name of the topology on ΩX is called the compact-open topology, although we will not define it. It only exists when the spaces A and B are nice enough - soon we will restrict our attention to a broad class of spaces that are quite reasonable.

Remark 2.1.2. Note that in fact, Σ and Ω are functors:

$$\Sigma : \mathrm{Top}_\star \to \mathrm{Top}_\star \qquad\qquad \Omega : \mathrm{Top}_\star \to \mathrm{Top}_\star$$

In fact, they are adjoint functors:

$$\Sigma : \mathrm{Top}_\star \rightleftharpoons \mathrm{Top}_\star : \Omega \tag{2.1.1}$$

To see the adjunction, we will need rthat specifying a map $\Sigma X \to Y$ is the same as specifying a map $X \to \Omega Y$. This can be thought of as a currying argument. For sets X, Y, denote by X^Y to be the set of functions (of sets) $X \to Y$. Then, for sets A, B, C we have that a map $A \times B \to C$ is the same as a map $A \to C^B$ - we simply remove one parameter from the left and put it on the right. That is, we send:

$$\{f : A \times B \to C\} \mapsto \{g(a) = \{b \mapsto f(a, b)\}\}$$

and in fact, this map is bijective. This is also rephrased as saying $\left(C^B\right)^A \simeq C^{A\times B}$.

This kind of argument is quite general. In pointed topological spaces, multiplication is given by the smash product, and exponentiation is given by pointed mapping spaces. Therefore, we see that indeed, a map $\Sigma X = X \wedge S^1 \to Y$ is the same data as a map $X \to \mathrm{Map}_\star(S^1, X)$. The currying argument also gives us an explicit formula for n-fold suspensions and loopings:

Proposition 2.1.3. *The iterated suspensions and loopings are given by:*

$$\Sigma^n X \cong S^n \wedge X \qquad\qquad \Omega^n \cong \mathrm{Map}_\star(S^n, X)$$

Proof. This follows inductively by noting that $S^n \wedge S^m \cong S^{n+m}$:

$$\begin{aligned} \Sigma^n X &= S^1 \wedge (\Sigma^{n-1} X) \\ &= S^1 \wedge S^{n-1} \wedge X \\ &= S^n \wedge X \end{aligned} \qquad \begin{aligned} \Omega^n X &= \mathrm{Map}_\star(S^1, \Omega^{n-1} X) \\ &= \mathrm{Map}_\star(S^1, \mathrm{Map}_\star(S^{n-1} X)) \\ &= \mathrm{Map}_\star(S^1 \wedge S^{n-1}, X) \\ &= \mathrm{Map}_\star(S^n, X) \end{aligned}$$

We get from maps $S^1 \to \mathrm{Map}_\star(S^{n-1}, X)$ to maps $S^1 \wedge S^{n-1} \to X$ through currying. □

This means that, using the adjunction:

$$\pi_n(\Omega X) = [S^n, \Omega X]_\star \simeq [\Sigma S^n, X]_\star \simeq [S^{n+1}, X]_\star = \pi_{n+1}(X)$$

With this language, we are now able to state what it means for spheres to have stable homotopy groups. More generally, for some space X, we may first suspend it, and then loop it. This gives a map $i : X \to \Omega(\Sigma X)$. The adjunction means we get an induced map:

$$\pi_n(i) : \pi_n(X) \to \pi_n(\Omega(\Sigma X)) \simeq \pi_{n+1}(\Sigma X) \tag{2.1.2}$$

Theorem 2.1.4 (Freudenthal Suspension Theorem, [Fre38])**.** *Upon successive application, Equation (2.1.2) is eventually an isomorphism. That is, for large enough k, the groups $\pi_{n+k}(\Sigma^k X)$ conveREGE.*

Therefore, we are inclined to ask about the stable values of homotopy groups.

Definition 2.1.5. For a pointed space X, the stable homotopy groups $\pi_n^S(X)$ are given by:

$$\pi_n^S(X) = \lim_{\to k} \pi_{n+k}(\Sigma^k X)$$

2.2. **Generalized Cohomology and Brown Representability.** Something interesting happens on the stable homotopy groups $\pi_n^S(X)$ of a space. One can verify that $\pi_n^S(X)$ satisfies all the requirements for a homology theory, from the Eilenberg-Steenrod axioms, apart from the dimension axiom. Somehow, this gives us a generalized homology theory. We saw that standard cohomology comes from Eilenberg-Maclane spaces, and now we are seeing a homology theory come from spheres. In fact, this is part of a broader pattern.

Definition 2.2.1. A generalized cohomology theory E is a contravariant functor $E : \mathrm{Ho}(\mathrm{Top}_\star)^{\mathrm{op}} \to \mathrm{Ab}$ that satisfies all the parallels of the Eilenberg-Steenrod axioms for cohomology theories, except for the dimension axiom.

Here, we have chosen to work over pointed spaces - this amounts to providing a relative cohomology theory $E^n(X, \star)$, which we can bring back up to the level of unbased spaces via the usual construction:

$$E^n(X) \simeq E^n(X, \star) \oplus E^n(\star)$$

In general, is a theorem of Brown that certain well-behaved contravariant functors $\mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ are always representable, in the same way that standard cohomology is.

Corollary 2.2.2 (of the Brown Representability Theorem, [Bro62])**.** *Every cohomology theory on* Ho(Top) *is representable by maps into a pointed CW complex:*

$$E^n(X) \simeq [X, E(n)]_\star$$

for some sequence of spaces $E(n)$.

However, the spaces $E(n)$ are not arbitrary. For example, observe what happens when we suspend X:

$$E_n(\Sigma X) \simeq [\Sigma X, E(n)]_\star \simeq [X, \Omega\, E(n)]$$

Without the specifics, it is hard to see what happens next. We will brush this to the side, and just state that one of the Eilenberg-Steenrod axioms for cohomology theories required such a thing called a Mayer-Vietoris sequence. The Mayer-Vietoris sequence readily calculates that:

$$E^n(\Sigma X) \simeq E^{n-1}(X)$$

Then, we get that:

$$E^n(\Sigma X) \simeq [X, \Omega\, E(n)]_\star \simeq [X, E(n-1)]_\star \simeq E^{n-1}(X)$$

for all spaces X. In fact, this will imply that:

$$E(n) \simeq \Omega\, E(n+1)$$

Note that all we need is really that they are weakly equivalent, but we said that $E(n)$ are CW spaces, so that they must be homotopy equivalent anyways.

2.3. **Stable Homotopy Category and Spectra.** What we saw in the case of representing spaces for cohomology theories is that to go down in cohomology, you must loop a representing space, up to weak equivalence. This captures the adjoint notion of suspending spaces shifting cohomological degree by -1.

In this sense, the collection $\{E(n)\}$ is in some sense a "stable" version of a space. We will actually want to go further. We mentioned earlier that $\pi_n^S(-)$ is a homology theory - we will want to ask where this comes from. The general theory will require looking at these spaces

$E(n)$ again, but putting some stronger restrictions on how they interact with each other. By the end, we will be able to take cohomologies, homologies, and homotopy groups of not only standard spaces, but also of these "stable spaces".

The original definition that people worked with was the following:

Definition 2.3.1. A prespectrum is a sequence of pointed spaces $E(n) \in \mathrm{Top}_\star$ with structure maps $\tilde{\sigma} : E(n) \to \Omega E(n+1)$

Prespectra are not quite what we found represent cohomology theories - earlier, we said that the structure maps needed to be weak equivalences (or alternative, homotopy equivalences between CW spaces). It became clearer that a better definition was due to May, and it took a while to catch on due to its complexity:

Definition 2.3.2. A spectrum is a sequence of pointed spaces $E(n) in \, \mathrm{Top}_\star$ with structure maps σ_n that are homeomorphisms:

$$\tilde{\sigma}_n : E(n) \xrightarrow{\simeq} \Omega\, E(n+1)$$

Of course, these spectra are much more complicated. Every space $E(n)$ is a loop space of $E(n+1)$, a double loop space of $E(n+2)$, and so on. We call these infinite loop spaces, and they are horrible to compute with.

However, they are related. We may form a category of prespectra, PreSpectra, with objects that are prespectra, and morphisms between them are maps from the underlying spaces that commute with the structure maps:

$$\begin{array}{ccc} E(n) & \xrightarrow{\tilde{\sigma}_n^E} & \Omega\, E(n+1) \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \Omega\, f_{n+1}} \\ F(n) & \xrightarrow[\tilde{\sigma}_n^F]{} & \Omega\, F(n+1) \end{array} \tag{2.3.1}$$

We similarly define the category of spectra, Spectra. These definitions of spectra are related through an adjunction:

Proposition 2.3.3 ([LMS86], Appendix§1, Theorem 1.1)**.** *There is a forgetful functor ℓ : Spectra $\to$ PreSpectra that takes a spectrum and forgets that the structure maps are homeomorphisms. ℓ admits a left-adjoint spectrification functor L:*

$$L : \mathrm{PreSpectra} \rightleftharpoons \mathrm{Spectra} : \ell$$

Further, $L\ell$ is an isomorphism in Spectra*.*

Namely, the procedure of taking a spectrum, forgetting that the structure maps are homeomorphisms, and then forcing the structure maps to be homeomorphisms again, gives us back the original data. The reason we care about the adjunction is the following. Prespectra are easy to work with and construct, while genuine spectra are horrible to work with. We may create spectra by spectrifying some constructions on prespectra. For example, we may form a spectrum by suspending a space many times:

Definition 2.3.4. For an unbased space X, we may suspending it into a prespectrum through the suspension prespectrum functor Π^∞:

$$\begin{aligned}\Pi^\infty X : \mathrm{Top} &\to \mathrm{PreSpectra}\\ X &\mapsto \{E(n) = \Sigma^n X_+\}\end{aligned}$$

The structure maps are given by noting that

$$\Sigma E(n) = \Sigma\Sigma^n X_+ \cong \Sigma^{n+1} X_+ = E(n+1)$$

We may apply the adjunction from Equation (2.1.1) to get the structure map:

$$E(n) \to \Omega\, E(n+1)$$

By spectrifying, we get an honest suspension spectrum functor:

$$\begin{aligned}\Sigma^\infty : \mathrm{Top} &\to \mathrm{Spectra}\\ X &\mapsto L\Pi^\infty X\end{aligned}$$

Definition 2.3.5. We also form a "shifted" version for all k, called the shifted desuspension prespectrum functor Π_k^∞ which has n-th space given by:

$$\begin{aligned}\Pi_k^\infty : \mathrm{Top} &\to \mathrm{PreSpectra}\\ X &\mapsto \left\{ E(n) = \begin{cases}\Sigma^{n-k} X_+ & k \leq n\\ \star & \text{otherwise}\end{cases}\right\}\end{aligned}$$

We have the same structure maps as with the suspension presctrum functor, but shifted to the left by k. From this, we may define the proper shift desuspension functor Σ_k^∞, again by spectrification:

$$\begin{aligned}\Sigma_k^\infty : \mathrm{Top} &\to \mathrm{Spectra}\\ X &\mapsto L\Pi_k^\infty X\end{aligned}$$

This terminology may seem confusing at first, but it comes from the fact that Σ_k^∞ "desuspends" the usual Σ^∞ functor through shifting backwards by k.

Our next goal is to define homotopies between spectra. Once we are there, we will define the stable version of the spheres S^n, and this will give us a notion of homotopy groups of spectra. We keep doing the same as we do unstably: we will then create a homotopy category of spectra by adjoining formal inverses to all weak equivalences.

Unstably, we defined homotopies of functions $f, g : X \to Y$ through smashing X with I_+ and finding a map on the smash product that agreed with f, g on ∂I. We will do the same thing here, by defining the smash product of a spectrum with a space:

Definition 2.3.6. For a prespectrum E and a pointed space X, we may form the prespectrum $E \wedge X$:

$$(E \wedge X)(n) = E(n) \wedge X$$

with structure maps $\sigma_n^{E\wedge X} = \sigma_n^E \wedge \mathrm{id}_X$. We do the usual trick for E a spectrum to form the spectrum $E \wedge X$ by forgetting the proper spectrum structure, working in prespectra, and passing back to spectra:

$$E \wedge X = L\left((\ell E) \wedge X\right)$$

In the case that $E = S^n$ is a sphere, we get a suspension functor Σ^n on Spectra:

$$\begin{aligned} \Sigma^n : \text{Spectra} &\to \text{Spectra} \\ E &\mapsto E \wedge S^n \end{aligned}$$

There will also be a way to take mapping space spectra:

Definition 2.3.7. For a prespectrum E and a pointed space X, we may also form the prespectrum $\mathrm{Map}_\star(X, E)$, which is defined space-wise as:

$$\mathrm{Map}_\star(X, E)(n) = \mathrm{Map}_\star(X, E(n))$$

The structure maps are given by

$$\begin{aligned} \mathrm{Map}_\star(X, E(n)) &\to \mathrm{Map}_\star(X, \Omega\, E(n+1)) \\ &\cong \mathrm{Map}_\star(\Sigma X, E(n+1)) \\ &\cong \mathrm{Map}_\star(S^1 \wedge X, E(n+1)) \\ &\cong \Omega\, \mathrm{Map}_\star(X, E(n+1)) \end{aligned}$$

The first map is given by postcomposition:

$$X \to E(n) \xrightarrow{\sigma_n} \Omega\, E(n)$$

The second map is given by the adjunction, by taking a map $X \to \Omega\, E(n+1)$ and giving a map $\Sigma X \to E(n+1)$. The third map is given by the definition of Σ, and the fourth map is given by currying.

We use the usual trick to give a definition of the spectrum $\mathrm{Map}_\star(X, E)$:

$$\mathrm{Map}_\star(X, E) = L\, \mathrm{Map}_\star(X, \ell(E))$$

As before, when $E = S^n$, we get a looping functor Ω^n:

$$\begin{aligned} \Omega^n : \text{Spectra} &\to \text{Spectra} \\ E &\mapsto \mathrm{Map}_\star(S^n, E) \end{aligned}$$

Using suspensions of spectra by spaces, we may form sphere spectra:

Definition 2.3.8. For an integer n, we may form the sphere spectrum $\mathbb{S}^n$ as :

$$\mathbb{S}^n_G = \begin{cases} \Sigma^\infty S^n & n \geq 0 \\ \Sigma^\infty_{|n|} S^0 & n < 0 \end{cases}$$

When we do not specify a superscript, we mean by $\mathbb{S}$ to be the spectrum $\mathbb{S}^0$.

We are in a good now position to define homotopies of maps between spectra, along with homotopy groups of spectra:

Definition 2.3.9. We define a homotopy in Spectra between functions $f, g : X \to Y$ to be a map of spectra $X \wedge I_+ \to Y$ agreeing with f, g on $X \wedge (\partial I)_+$. We can then form the homotopy classes of maps $[X, Y]$ as usual, by taking the set of homotpy equivalence classes of maps $X \to Y$ between spectra. The homotopy groups of spectra $\pi_n(X)$, as with of spaces, are given by homotopy classes of maps out of the sphere spectrum:

$$\pi_n(X) = [\mathbb{S}^n, X]$$

One should check that these are in fact groups. That does turn out to be correct - we may think of it as an extension of the argument by pinching maps.

We will now identify together spectra that are weakly equivalent. There turn out to be two identical definitions of weak equivalence on spectra:

Definition 2.3.10 (Weak equivalence of spectra, 1)**.** A weak equivalence in Spectra between X, Y is a map $f : X \to Y$ that induces isomorphisms on all the homotopy groups at all H for all $n \in \mathbb{Z}$:

$$\pi_n(f) : \pi_n(X) \xrightarrow{\sim} \pi_n(Y)$$

Alternatively, we may take weak equivalences to be defined space-wise:

Definition 2.3.11 (Weak equivalence of spectra, 1)**.** We may also say that a map of spectra $f : X \to Y$ is a weak equivalence if the maps on spaces are weak equivalences:

$$\pi_n(f_i) : \pi_n(X(i)) \xrightarrow{\sim} \pi_n(Y(i))$$

In either case, we actually get the same notion of weak equivalence, although it is still not an equivalence relation, as with the unstable case. We form a homotopy category in the same way as before:

Definition 2.3.12. We define the stable homotopy category Sp to be Spectra with inverse morphisms $f^{-1} : Y \to X$ formally adjointed for any weak equivalence $f : X \to Y$.

In some sense, Sp contains more data than $\text{Top}_\star$. You can make any space into a spectrum via the suspension functor Σ^∞, but not all spectra come from suspensions. In practice, frequently we take Sp to be our "spectra," instead of Spectra or PreSpectra, especially as identifying weak equivalences together tends to make computations easier.

One way to think about the importance of spectra is that they let us invert Σ and Ω as each other, whereas beforehand, in spaces, they were only adjoint functors. By Σ and Ω here, we mean the functors $- \wedge S^1$ and $\text{Map}_\star(S^1, -)$. For example, if we suspend and then loop, we get the following homotopy groups:

$$\pi_n(\Omega\,\Sigma X) = [\mathbb{S}^n, \Omega\,\Sigma X] \simeq [\Sigma\mathbb{S}^n, \Sigma X] = [\mathbb{S}^{n+1}, \Sigma X]$$

On the level of prespectra, we are looking at maps $f_n : S^{n+1} \to \Sigma X(n)$ that commute for each n. This will be no different than maps $g_n : S^n \to X(n)$ that commute for all n, so that we recover:

$$\pi_n(\Omega\,\Sigma X) \simeq \pi_n(X)$$

Therefore, taking $X \mapsto \Omega\,\Sigma X$ is a weak equivalence, so that we say that Σ has a left-inverse of Ω in the homotopy category. Other inverses follow by similar arguments.

2.4. **Homology and Cohomology Calculations and Eilenberg-Maclane Spectra.**

Now, we can state what the stable homotopy groups $\pi_n^S(X)$ mean. We require these to exist in the limit, which is the same as requiring that we find functions $S^n \to X$ that commute with all suspensions. This is in fact just a map of prespectra $\mathbb{S}^n \to \Sigma^\infty X$. Precisely in this sense, we get a homology theory: That is, we get that:

$$\pi_n^S(X) = \pi_n(\Sigma^\infty X) = [\mathbb{S}^n, \Sigma^\infty X]$$

where the right happens in spectra. We already saw a generalization for other spectra, where we computed homotopy groups of a spectrum.

We want a way to think about cohomology in this framework. We know from Brown representability that we have Eilenberg-Maclane spaces $K(G,n)$ for any abelian group. We also saw that they fit into a prespectrum, via homotopy equivalence:

$$K(G,n) \simeq \Omega K(G,n+1)$$

By spectrification, we may then get the Eilenberg-Maclane spectra HG. In fact, we may define standard cohomology in terms of spectra, but we will first need a way to "multiply" two cohomology theories, via a smash product.

Definition 2.4.1. On the level of prespectra, we may form a smash product $E \wedge F$ on spaces:

$$(E \wedge F)(n) = \begin{cases} E(k) \wedge F(k) & n = 2k \\ E(k+1) \wedge F(k) & n = 2k+1 \end{cases}$$

In other words, we alternate which spaces we smash, going up each time. The construction will not actually depend on how we smash the spaces, as long as we hit $E(n) \wedge F(m)$ for arbitarily large n, m. We may of course get back to spectra:

Definition 2.4.2. On spectra, we define a smash product $E \wedge F$ as usual:

$$E \wedge F = L(\ell(E) \wedge \ell(F))$$

Then, for any spectrum E, we may take homology and cohomology with respect to other spectra:

Definition 2.4.3. For spectra E and X, we define the E-valued homology of X as:

$$E_n(X) = [\mathbb{S}^n, E \wedge X] = \pi_n(E \wedge X)$$

and the E-valued cohomology of X as:

$$E^n(X) = [\mathbb{S}^{-n} \wedge X, E]$$

Note that $E_n(X)$ is equal to $X_n(E)$, and that X is in the covariant part of the Hom, so that homology is covariant. On the other hand, X is in the contravariant part of the Hom in $E^n(X)$, so that cohomology is contravariant.

We conclude by stating, without proof, that in fact these constructions give back ordinary homology and cohomology of spaces, in the same way as we formed stable homotopy groups:

Proposition 2.4.4. *For a topological space X, taking homology and cohomology, valued at the Eilenberg-Maclane spectrum HG representing homology, of suspensions gives back the usual notions of homology and cohomology:*

$$HG_n(\Sigma^\infty X) \simeq H_n(X; G)$$
$$HG^n(\Sigma^\infty X) \simeq H^n(X; G)$$

3. Closing Remarks

Now that we have a basic understanding of the modern formulation of stable homotopy theory, we may ask what is next. Much of the current research deals with how spectra interact with each other. There is a spectrum MU called complex cobordism that classifies complex-oriented cohomology theories, which have a beautiful connection to algebra through formal group laws - this area of study is called chromatic homotopy theory. There are also spectra called Morava $K(n)$-theories which are able to detect certain periodic behaviors. Recently, the Telescope Conjecture of Ravenel was solved in [BHLS23], which concerns how the telescope spectra $T(n)$ interact with the Morava K-theories. As mentioned earlier, we can think of spectra as being related to generalizations of algebra. In fact, there was a recent generalization of Hilbert's Nullstellensatz, involving spectra, called the chromatic nullstellensatz [BSY22].

We may also build upon homotopy theory and add more constraints. For example, equivariant homotopy theory concerns spaces that have group actions. This field of study was used to prove the famous long-standing Kervaire invariant one problem in [HHR16], which at face value seems very geometric and has nothing to do with groups. We may also extend the tools of homotopy theory to other settings - there has been recent work concerning motivic homotopy theory, which studies objects from algebraic geometry, and connecting it back to homotopy groups of spheres.

While we did not cover many of the proofs and left out details, hopefully this should contextualize much of the modern work in stable homotopy theory and give an appreciation for the subject and its nuances.

References

[BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M Schlank. K-theoretic counterexamples to Ravenel's telescope conjecture. *arXiv preprint arXiv:2310.17459*, 2023.

[Bro62] Edgar H Brown. Cohomology theories. *Annals of Mathematics*, 75(3):467–484, 1962.

[BSY22] Robert Burklund, Tomer M Schlank, and Allen Yuan. The chromatic nullstellensatz. *arXiv preprint arXiv:2207.09929*, 2022.

[ES15] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*, volume 2193. Princeton University Press, 2015.

[Fre38] Hans Freudenthal. Über die klassen der sphärenabbildungen i. große dimensionen. *Compositio Mathematica*, 5:299–314, 1938.

[Hat02] A. Hatcher. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Lei14] Tom Leinster. *Basic category theory*, volume 143. Cambridge University Press, 2014.

[LMS86] L.G. Lewis, J.P. May, and M. Steinberger. *Equivariant Stable Homotopy Theory*. Lecture notes in mathematics. Springer-Verlag, 1986.

[Lur17] Jacob Lurie. Higher algebra. Unpublished. Available online at `https://www.math.ias.edu/~lurie/`, 09 2017.

[May98] JP May. Stable algebraic topology and stable topological algebra. *Bulletin of the London Mathematical Society*, 30(3):225–234, 1998.

[Mil62] J Milnor. On axiomatic homology theory. *Pacific J. Math.*, 12(4):337–341, 1962.

[Mil21] Haynes R Miller. *Lectures on algebraic topology*. World Scientific, 2021.

[Mun00] J.R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

Email address: `hbeck@mit.edu`

URL: `https://www.mit.edu/~hbeck`