NOTES ON CHROMATIC BLUESHIFT

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ABSTRACT. This document serves as a place for me to catalog results, notations, definitions, and perhaps most importantly, sources, that are background knowledge and useful information I was missing to understand the Chromatic Blueshift Conjecture of Burkland, Schlank, and Yuan [BSY22] (Conjecture 9.9) in its full generality. Therefore, it serves as reference for me to come back to and brush up on what I may be missing, and have easy access to further information instead of having to comb through papers and websites continuously finding a good source.

CONTENTS

A note on the references: this is not meant to be a survey on work, nor is it meant to be a source meant to learn the concepts in detail. It is meant to be a unified recounting of stories told in many different places, in a hopefully more organized fashion. The citations are chosen to launch you (and me) in the right direction to learn more. I prioritize citing references that are more useful for learning more, as opposed to citing original work. Also, computers and typesetting programs have of course revolutionized math publishing, and as such modern work is easier on the eyes to read. Therefore, I often complement older and harder-to-read works with newer sources such as summaries or lecture notes. Much of the "story-telling" parts of the material come from a combination of Jacob Lurie's lecture notes [Lur10], the inter-connected highway of pages in nLab's beautiful documentation, and of course, the Chromatic Nullstellensatz paper itself [BSY22].

There are likely minor notational inconsistencies or strange choices made in notation. However, I hope there are no outright lies.

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1. Localization of Spectra

1.1. **Notation and Reminders.** Throughout this paper, I will use the following conventions:

Spectra (taken to be consistent with [BSY22])

- Take some model category of spectra, such as PreSpectra or Spectra, and let Sp be its homotopy category (the choice does not matter) - from here on out, ob(Sp) are our "spectra."
- We may create a prespectrum by forming a sequence $\{X(n)\}$ of pointed CW complexes with cellular maps $\Sigma X(n) \to X(n+1)$, and then back down to Sp.
- For a topological space $Y \in \mathrm{Top}_\star$, let $\Sigma^\infty Y$ be its suspension spectrum.
- I use $\otimes$ for the smash product on spectra despite using $\wedge$ for the smash product on spaces. This makes the symmetric monoidal structure on Sp more obvious to keep in mind, but frankly it is only because [BSY22] use it and I should be consistent with them here.
- $\mathbb{E}_\infty$ spectra may also be called $\mathbb{E}_\infty$-rings (note that all $\mathbb{E}_\infty$ spectra are in fact ring spectra).
- For a subcategory $\mathcal{C}$ of spectra, let $\mathrm{CAlg}(\mathcal{C})$ be the full subcategory of $\mathcal{C}$ containing $\mathbb{E}_\infty$ spectra. Namely, CAlg(Sp) are all the $\mathbb{E}_\infty$ spectra.
- We will let $\mathbb{S}^n$ be the sphere spectrum $\Sigma^\infty S^n$. When n is not specified, we take $\mathbb{S} = \mathbb{S}^0$.

Algebra

- Let $\mathbb{Z}_{(p)}$ be the p-localized integers, where all primes but p are invertible.
- Let $\mathbb{F}_p$ be the finite field with p elements.
- For a group G, we denote by e the identity element.

Homology theories

Fix spectra $E, X, Y \in \mathrm{Sp}$.

- The homotopy groups $\pi_\bullet(X)$ are given by:
$$\pi_n(X) = [\mathbb{S}^n, X]$$
- $\pi_\bullet(R)$ forms a ring when R is a ring spectrum.
- We define the E-homology of X:
$$E_n(X) = \pi_n(E \otimes X) = [\mathbb{S}^n, E \otimes X]$$
- A map $f : X \to Y$ induces maps on E-homology via:
$$E_n(f) : E_n(X) \to E_n(Y)$$
$$[\mathbb{S}^n \xrightarrow{\phi} E \otimes X] \mapsto [\mathbb{S}^n \xrightarrow{\phi} E \otimes X \xrightarrow{\mathrm{id}_E \otimes f} E \otimes Y] \tag{1.1.1}$$

1.2. *E*-**local and** *E*-**acyclic Spectra, Bousfield Localization.** Following [Lur10], Lecture 20, we define E-local and -acyclic spectra:

Definition 1.2.1. Let $E, X \in \mathrm{Sp}$.

- X is E-acyclic if the smash product is homotopic (isomorphic, since we work in the homotopy category) to the 0 spectrum: $E \otimes X \simeq 0$
- X is E-local if every map $Y \to X$ for an E-acyclic spectrum Y is nullhomotopic

Notation 1.2.2. For a spectrum $E \in \mathrm{Sp}$, we let Sp_E be the full subcategory of E-local spectra.

An important example comes from p-local and p-complete spectra:

Definition 1.2.3. For an abelian group G, the Moore spectrum $SG \in \mathrm{Sp}$ is characterized by:

$$\pi_{<0}(SG) \simeq 0$$
$$\pi_0(SG) \simeq G$$
$$H_{>0}(SG; \mathbb{Z}) \simeq 0$$

Definition 1.2.4. We say a spectrum R is p-local for a prime p if it is $S\mathbb{Z}_{(p)}$-local, or p-complete if it is $S\mathbb{F}_p$-local.

For some spectra X, E, we may want to force X to be E-local. As we will soon see, in fact we may always do so in such a way that retains the same information, from the viewpoint of E:

Definition 1.2.5. An E-equivalence of spectra is a map $f : X \to Y$ such that $\mathrm{id}_E \otimes f : E \otimes X \to E \otimes Y$ is a homotopy equivalence

In fact, in this sense, any spectrum X is E-equivalent to an E-local spectrum $L_E X$:

Theorem 1.2.6 (Bousfield Localization - [Bou79])**.** *For spectra $E, X \in \mathrm{Sp}$, we have the E-localization X denoted $L_E X$, such that $L_E X$ is E-local and there is an E-equivalence $X \to L_E X$. Further, $L_E(-)$ is a functor:*

$$L_E : \mathrm{Sp} \to \mathrm{Sp}_E$$

Because there is an E-equivalence $f : X \to L_E X$, we have that the induced map on E-homology $E_*(f)$ is an isomorphism. This follows immediately by noting $E \otimes X \to E \otimes L_E X$ is an isomorphism in Equation (1.1.1). In fact, we may localize with respect to multiple spectra, but will not do so here. There is also a related notion of localization of a topological space, but we will only work with spectra.

Definition 1.2.7. A functor $L_E : \mathrm{Sp} \to \mathrm{Sp}_E$ is an E-equivalence between the inputs and corresponding outputs is called a localization functor.

In general, these localization functors can be complicated. However, we get lucky in the case of p-localizations, where we just end up needing to smash the spectrum with $S\mathbb{Z}_{(p)}$. This will be a special case of a broader class of localization functors:

Definition 1.2.8. A localization functor L is smashing if there is a spectrum K where

$$L(X) \simeq K \otimes X$$

The p-localization functors will be smashing, along with all localizations at Moore spectra at localized integers:

Proposition 1.2.9 ([Bou79], Proposition 2.4)**.** *For a set of primes J, let $G = \mathbb{Z}_{(J)}$. The localization functor L_{SG} is smashing:*

$$L_{SG}(X) \simeq SG \otimes X$$

Further,

$$\pi_\bullet(L_{SG} X) \simeq G \otimes \pi_\bullet(X)$$

and X is SG-local if and only if p is invertible in all the $\pi_\bullet(X)$ for $p \notin J$.

1.3. **Morava K-theories and Periodicity.** We define the Morava K-theories, which are spectra closely related to heights of formal group laws. One construction of them is given by the iomorphic summands of $\bmod p$ complex K-theory. We will not explicitly make these spectra, but instead just state some properties:

Definition 1.3.1 (partial definition, see [Rav92a], Proposition 1.5.2 for more details)**.** For a fixed p, the Morava K-theories $K(n)$ are spectra with coefficient rings:

$$\pi_\bullet(K(n)) \simeq \begin{cases} \mathbb{Q} & n = 0 \\ \mathbb{F}_p[v_n^{\pm 1}] & n \geq 1 \end{cases} \tag{1.3.1}$$

where $|v_n| = 2(p^n - 1)$. The value of p is fixed and not noted, but is rather implied.

Remark 1.3.2. The strange exception at $n = 0$ happens since we take $K(0) = H\mathbb{Q}$. Notably, all the $K(0)$ are the same at all primes. On the other extreme, we set $K(\infty) = H\mathbb{F}_p$. In this sense, Morava K-theories can be thought of as extrapolating between these two Eilenberg-Maclane spectra.

A more extensive treatment of the Morava K-theories can be found in Proposition 1.5.2. in [Rav92a] or in [Lur10], Lecture 22.

The Morava K-theories are important in chromatic homotopy theory for many reasons. Of central interest to us, they are well-suited to detect periodic behavior. We already see some trace of this in the homotopy groups, Equation (1.1.1), where there is an element v_n for the theories at $n \geq 1$. These elements give $\pi_\bullet(K(n))$ a periodic structure through shifts up and down in degree by v_n or v_n^{-1}.

Another reason for the importance of the Morava K-theories is that they are, in a very meaningful sense, the homotopy theoretic version of prime fields. We will not state what this means in full detail, but one of the implications is a way to calculate the $K(n)$-homologies of other spectra:

Proposition 1.3.3. *For another spectrum X, the smash product $K(n) \otimes X$ decomposes as some direct sum:*

$$K(n) \otimes X \simeq \bigoplus_{\alpha} \Sigma^{k_\alpha} K(n)$$

for some powers k_α. In particular, we may calculate the $K(n)$-homology:

$$K(n)_\bullet(X) = \pi_\bullet(K(n) \otimes X) \simeq \bigoplus_{\alpha} \pi_\bullet\left(\Sigma^{k_\alpha} K(n)\right) \simeq \bigoplus_{\alpha} \pi_{\bullet - k_\alpha}(K(n))$$

In particular, it is a free $\pi_\bullet(K(n))$-module.

We turn back to the periodic behavior of Morava K-theories. They interact particularly well with finite spectra:

Definition 1.3.4. A finite spectrum is one given by a prespectrum such that the spaces X_n are finite CW complexes, and the structure maps $\sigma : \Sigma X_n \to X_{n+1}$ are subcomplex inclusions.

Lemma 1.3.5 ([Lur10], Lecture 26, Lemma 4)**.** *For a finite, p-local, spectrum X, suppose $K(n)_\bullet(X) = 0$. Then, $K(n-1)_\bullet(X) = 0$.*

This lets us define the type of a finite p-local spectrum, which is where the chain ends:

Definition 1.3.6. The type $n \geq 0$ of a finite p-local spectrum X is the greatest n such that $K(m)_\bullet(X) = 0$ for $m < n$. Hence, $K(n)_\bullet(X) \neq 0$. Note that any non-zero X will have $n \geq 0$.

Now, we define self-maps of spectra:

Definition 1.3.7 ([HS98], Definition 1)**.** A self map is a map $\Sigma^k X \to X$ for $X \in \mathrm{Sp}$.

Note that we can iterate self-maps as follows:

$$f^n : \Sigma^{kn} X = \Sigma^{k(n-1)}\left(\Sigma^k X\right) \xrightarrow{\Sigma^{k(n-1)} f} \Sigma^{k(n-1)}(X) = \Sigma^{k(n-2)}\left(\Sigma^k X\right) \xrightarrow{\Sigma^{k(n-2)} f} \cdots \to \Sigma^k X \xrightarrow{f} X$$

We specialize to a particularly nice class of self-maps on finite p-local spectra:

Definition 1.3.8 ([HS98], Definition 8)**.** For a finite, p-local, spectrum X, a v_n-self map is a self map $f : \Sigma^k X \to X$ such that

- $K(0)_\bullet(f)$ is multiplication by a rational number $q \in \mathbb{Q}^\times$.
- $K(n)_\bullet(f)$ is an isomorphism.
- $K(m)_\bullet(f)$ is nilpotent, for $m \neq n$.

Here, we think of $K(n)_\bullet(f)$ as a map from $K(n)_\bullet(E) \to K(n)_{\bullet-k}(E)$. By Proposition 1.3.3, we know that both of these are the same vector field over the field $\pi_\bullet(K(n))$. These homotopy groups have the same periodic behavior from earlier through shifting up and down by v_n, which lets us identify the $-k$-shifted grading with the original grading. In this sense, we think of $K(n)_\bullet(f)$ from an algebraic object to itself, and we can then define multiplication, isomorphisms, and nilpotent maps from there.

It is natural to ask when a finite p-local spectrum admits a v_n-self map. Hopkins and Smith provide a necessary and sufficient condition:

Theorem 1.3.9 (Periodicity Theorem - [HS98], Theorem 9)**.** *Any finite spectrum X admits a v_n-self map if and only if X is p-local and type $\geq n$. Further, the v_n-self map is a $K(n)$-equivalence.*

Further, while these maps may not be unique, their iterates are eventually the same:

Lemma 1.3.10 ([HS98], Lemma 3.6, or [Lur10], Lecture 27, Lemma 9)**.** *For two v_n-self maps $f, g : \Sigma^k X \to X$, there exist $a, b > 0$ such that:*

$$f^a \simeq g^b$$

The great insight is that these spectra exist at any prescribed type n:

Theorem 1.3.11 ([Mit85], Theorem B)**.** *For every $n \geq 0$, there is a finite, p-local, spectrum X of type n.*

Note that Theorem 1.3.9 means such an X admits a v_n-self map.

1.4. **Telescopic Localization and Bousfield Classes.** Following [Rav92b], we define telescopes:

Definition 1.4.1. For a finite, p-local, spectrum X of type n with v_n-self map $f : \Sigma^k X \to X$, we define the telescope $\hat{X}$ as the homotopy direct limit of:

$$X \xrightarrow{\Sigma^{-k} f} \Sigma^{-k} X \xrightarrow{\Sigma^{-2k} f} \Sigma^{-2k} X \xrightarrow{\Sigma^{-3k} f} \cdots$$

By Proposition 1.3.11, this does not depend on the choice of f. This colimit is sometimes also denoted as $X[f^{-1}]$. Because f is a $K(n)$-equivalence, such as by Theorem 1.3.9, we have that the map $i : X \to \hat{X}$ is a $K(n)$-equivalence too. Further,X and all of its inverse suspensions are still p-local, giving us that the telescope $\hat{X}$ is also p-local.

We may ask how different these spectra $\hat{X}$ are. To answer this, we need the notion of Bousfield equivalence:

Definition 1.4.2 ([HS98], Definition 13)**.** Two spectra $E, F \in \mathrm{Sp}$ are Bousfield equivalent if for all other spectra X,

$$E \otimes X \simeq 0 \iff F \otimes X \simeq 0$$

We note the (Bousfield-)equivalence class of X as the Bousfield class $\langle X \rangle$.

This precisely means the notions of E-acyclicity and F-acyclicity are the same. Further, if $\langle E \rangle = \langle F \rangle$, then for a fixed nullhomotopic map $Y \to X$, Y is E-acyclic if and only if it is F-acyclic by definition. Therefore, the notions of E-local and F-local also are the same. It is not hard to see then that E-localization and F-localization are then also the same.

An equivalent formulation is that $E_\bullet(X) \simeq 0$ if and only if $F_\bullet(X) \simeq 0$. This follows by noting $E_\bullet(X) = \pi_\bullet(E \otimes X)$, and the smash product $E \otimes X$ has homotopy groups 0 precisely when it is (weakly) contractible, and thus 0 in the homotopy category Sp.

The claim is that the choice of X doesn't matter. Therefore, the functors $L_E \hat{X}$ align, and as do the notions of $\hat{X}$-acyclic and $\hat{X}$-local. To show this, we introduce a technical definition from the Hopkins-Smith paper, which connects Bousfield classes of finite spectra to $K(n)$-localization:

Definition 1.4.3 ([HS98], Theorem 14)**.** For X a finite spectrum, we define:

$$\mathrm{Cl}(X) = \{(n,p) \mid K(n)_\bullet(X) \not\simeq 0 \text{ at prime } p\}$$

In other words, this is the set of (n,p) where X is not $K(n)_\bullet$-acyclic. The insight is that these sets determine the Bousfield classes:

Theorem 1.4.4 ([HS98], Theorem 14)**.** *For finite spectra X, Y, $\langle X \rangle = \langle Y \rangle$ if and only if $\mathrm{Cl}(X) = \mathrm{Cl}(Y)$.*

Claim 1.4.5. *For all finite, p-local, spectra X of type n, the telescopes $\hat{X}$ are in the same Bousfield class.*

Proof. **WARNING: THIS PROOF IS WRONG** The goal is to show $\mathrm{Cl}(\hat{X}) = \{m \mid m \geq n\} \times \{p\} \cup \{0\} \times \{p \text{ prime}\}$. At p, there is a $K(m)$-equivalence $X \to \hat{X}$, so that $K(m)_\bullet(\hat{X}) \neq 0$ precisely when $m \geq n$, which proves the statement at p. At some other prime $q \neq p$, denote by $K^q(m)$ the Morava K-theories at q. For these, we use the fact $\hat{X}$ is p-local and use Proposition 1.2.9, so that:

$$K^q(m) \otimes X \simeq K^q(m) \otimes L_{S\mathbb{Z}_{(p)}} \hat{X} \simeq K^q(m) \otimes S\mathbb{Z}_{(p)} \otimes \hat{X} \simeq \big(S\mathbb{Z}_{(p)} \otimes K^q(m)\big) \otimes \hat{X}$$

The case $n = 0$ is not interesting, as X is always type ≥ 0 so that $K(0)_\bullet(\hat{X}) = 0$. This follows by noting that all the $K(0)$ at all primes are the same, and they are just $H\mathbb{Q}$.

For other values of n, we take the homotopy groups, using Proposition 1.2.9 again:

$$\pi_\bullet(S\mathbb{Z}_{(p)} \otimes K^q(M)) \simeq \mathbb{Z}_{(p)} \otimes \mathbb{F}_q[v_n^{\pm 1}] \simeq 0$$

Therefore, p-localized Morava K-theory at q is (weakly equivalent to) 0, so that:

$$K^q(m) \otimes \hat{X} \simeq 0 \otimes \hat{X} \simeq 0 \implies K^q(m)_\bullet(\hat{X}) = 0$$

and $\hat{X}$ is $K(m)$-acyclic at any $p, m \geq 1$. Therefore, $\mathrm{Cl}(\hat{X})$ does not depend on the choice of X. Applying Theorem 1.4.4 gives the desired result. $\square$

Using this result, we define "the" telescope spectrum. We have done nothing different than just constructing the telescope $\hat{X}$, but this notation makes it seem like there is a dependence on X, which there is not up to Bousfield equivalence.

Definition 1.4.6. Fix some finite, p-local, spectrum X of type n. We will denote as the telescope spectrum $T(n)= \hat{X}$.

Note that we make a choice when constructing $T(n)$, but the notions of $T(n)$-localization, $T(n)$-locality, and $T(n)$-acyclicity do not depend on the choice. As with the Morava K-theories, this definition also depends on p, but we leave it implied.

1.5. **Telescope Conjecture.** This section is about the Telescope Conjecture, which is not relevant to the Chromatic Blueshift Conjecture that we are building up to. However, it was a huge unresolved conjecture in stable homotopy theory, and it was recently settled in full generality. It would be irresponsible to not cover it now that we have all the required terminology.

It is known that all $K(n)$-local spectra are also $T(n)$ local, that is:

$$\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$$

It was conjectured that in fact, this inclusion is an equality. This is called the telescope conjecture, and we now know the answer:

Theorem 1.5.1 (Telescope Conjecture, posed by Douglas Ravenal)**.** *The Telescope Conjecture asserts that* $\mathrm{Sp}_{K(n)} = \mathrm{Sp}_{T(n)}$. *It can also be rephrased as the assertion that for finite p-local spectra X of type n, the telescopes$\hat{X}$ are just given by $K(n)$-localization:*

$$\hat{X} \simeq L_{K(n)}X$$

It is true at $n = 0, 1$ for all primes, but false at all primes for $n \geq 2$.

Proof. At $n = 0$, a finite p-local spectrum X of type 0 has the identity as a v_0-self map. Therefore, its telescope $\hat{X}$ it just itself. The identity is a $K(0)$-equivalence, so that $L_{K(n)}X = X = \hat{X}$.

At $n = 1$ and $p = 2$, it is true by [Mah81], Theorem 6.3. The case of $n = 2$ and $p > 2$ is also true by [Mil81], Theorem 4.11.

The case of $n \geq 2$ and p arbitrary was disproved recently in [BHLS23], Theorem A. □

2. Equivariant Homotopy Theory

The last missing puzzle piece comes from the world of equivariant homotopy theory. This will concern studying topological spaces up to homotopy, but where we have a group action on the spaces. We will build up to a stable equivariant homotopy theory, which gives us equivariant versions of standard spectra we are used to.

To avoid point problems, we will restrict our attention to compact Lie groups G. However, for our purposes, we will really only need to work over finite abelian groups. What is important is that G be a topological group so that the group action is continuous, and that we may form orbit spaces G/H. Of course, dicrete groups, such as finite groups, are given the discrete topology. We will also require that all subgroups $H \subseteq G$ be closed as topological subspaces, so that we do not run into point-set issues. Although some of the results here will not strictly require working with closed subgroups, we will not lose anything by only looking at them.

2.1. **Equivariant Spaces.** The foundations of equivariant homotopy theory start with the same story we're used to. Roughly, we define a category of equivariant topological spaces where we can form products and mapping spaces. From there, we may define the usual notions of homotopies and homeomorphisms.

We will work in an appropriate category of spaces with a left G-action:

Definition 2.1.1. For a fixed topological group G, we define G-Top as the category of spaces $X \in \mathrm{Top}$ with a continuous left G-action $G \times X \to X$. We call these objects G-spaces, and the morphisms are equivariant continuous maps $f : X \to Y$, so that they satisfy $f(g \cdot x) = g \cdot f(x)$.

It's worth taking some time to look at some constructions within G-Top:

Construction 2.1.2. Some important examples of spaces in G-Top are:

- a space G, with the action given by left group multiplication G. Note that we required G carry a topology, so that this definition makes sense.
- orbit spaces (also called coset spaces) G/H, whose points are cosets gH. These carry the quotient topology from G, by identifying together $g_1 \sim g_2$ such that $g_1H = g_2H$. It inherits a G-action from G, that factors through the quotient.
- products $X \times Y$, where G acts diagonally (as in, $g \cdot (x, y) = (g \cdot x, g \cdot y)$).
- mapping spaces $\mathrm{Map}^G(X, Y)$ of G-equivariant functions $X \to Y$, with the compact-open topology. G acts by setting $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$.

We similarly define a pointed version of the story, which behaves nicely up to homotopy in the same way that we will often prefer to work over $\mathrm{Top}_\star$.

Definition 2.1.3. We similarly define, G-$\mathrm{Top}_\star$ as the category of pointed spaces $X \in \mathrm{Top}_*$ with left G-actions as before. We will similarly call objects of this category pointed G-spaces. We take that the point $\star \in X$ is fixed by G, and require that all morphisms are equivariant and basepoint-preserving continuous maps.

Of course, we have similar versions of the constructions in G-Top:

Construction 2.1.4. Some important examples of spaces in G-$\mathrm{Top}_\star$ are:

- G again, where we pick the basepoint $e \in G$. This means the action map $G \times X \to X$ for a pointed G-space X is an equivariant pointed map.
- G/H again, where we pick the basepoint as the coset eH.
- smash products of pointed spaces $X \wedge Y$, by forming the usual quotient of $X \times Y$. The G-action here descends from the G-action on the product.
- mapping spaces $\mathrm{Map}_\star^G(X, Y) \subseteq \mathrm{Map}^G(X, Y)$ of based G-equivariant functions that send the basepoint $\star_X$ to the basepoint $\star_Y$. These inherit a topology and group action from $\mathrm{Map}^G(X, Y)$. We can treat these as pointed spaces by choosing the basepoint $\{x \mapsto \star_Y\} \in \mathrm{Map}_\star^G(X, Y)$.

We will also set the convention when referencing some topological space X in an equivariant context, we will give it the trivial action. In particular, we will often reference the interval I, the disks D^n, the spheres S^n, and the Euclidean spaces $\mathbb{R}^n$. We will understand G acts trivially on all of these spaces.

With that in mind, we also get extensions of the usual notions around homotopy:

Definition 2.1.5. Take G-spaces $X, Y \in G$-Top. By taking the interval I, a G-homotopy between functions $f, g : X \to Y$ is a G-equivariant map $h : X \times I \to Y$ that agrees with f, g on $X \times \partial I$. This lets us define G-homotopy equivalent as usual, and gives us a notion of G-contractability when a space is G-homotopic to a point. We also can form G-homotopic classes of maps, which we denote $[X, Y]^G$. If there exist (equivariant, continuous) maps $f : X \rightleftarrows Y : g$, we say that X and Y are G-homeomorphic.

Note that these are all strengthenings of the usual notions from standard topology. Also, we have similar obvious notions for pointed spaces, which we won't state.

In the non-equivariant world, CW complexes are of central importance due to CW approximation. As a review, we form a space X by taking a colimit over n-skeleta X_n formed by pushout squares:

$$\begin{tikzcd}
\coprod_{i \in \mathcal{I}} S^n \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d] \\
\coprod_{i \in \mathcal{I}} D^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_n
\end{tikzcd}$$

We want to understand how to make an equivariant parallel. To give a CW complex a G-action, it is tempting to define an action on each cell D^n that restricts to an action on the image of the boundary ∂D^n in the $(n-1)$-skeleton. However, this definition is much too restrictive and unwieldy to be computationally useful. The more natural thing to do is to encode the G-action by permuting the indices $\mathcal{I}$ instead. The G-orbits always take the form of coset spaces G/H, which means the thing to do is really to make G-cells look like $G/H \times D^n$ for some subgroup $H \subseteq G$, with the trivial G-action on D^n. It turns out that this construction works remarkably well and lets us extend the usual results about CW complexes to the equivariant setting.

The takeaway is that the equivariant parallel of disks and spheres are often products of the same with G/H. Interestingly, this means many constructions depend on a choice of H, which we will say more about later. For now, we have the following example of this general phenomenon. Namely, there is a sense in which the space G with the left G-action by multiplication acts like a point:

Proposition 2.1.6. *As spaces in* Top, $X \cong \mathrm{Map}^G(G, X)$.

Proof. A G-equivariant map $f : G \to X$ is fully determined by the choice of $f(e)$ via $f(g) = f(g \cdot e) = g \cdot f(e)$. This gives a map

$$\phi : X \to \mathrm{Map}^G(G, X)$$
$$x \mapsto \{g \mapsto g \cdot x\}$$

that is clearly invertible, and both directions are continuous - thus giving a homeomorphism in Top. ☐

It is important to consider these spaces as being homeomorphic in Top, and not in G-Top. X and $\mathrm{Map}^G(G, X)$ have incompatible G-actions, so that ϕ is just a continuous map and not an equivariant continuous map. To see why, note that:

$$\phi(h \cdot x) = \{g \mapsto (gh) \cdot x\}$$

while

$$h \cdot \phi(x) = \{g \mapsto (ghh^{-1}) \cdot x\} = \phi(x)$$

This is quite remarkable - $\mathrm{Map}^G(G, X)$ preserves topological information about X but forgets the G-action. Of course, there is nothing special about G as opposed to any orbit space G/H, and it fits into a general pattern:

Proposition 2.1.7 (Case of [GM95], Lemma 1.1)**.** *The mapping space* $\mathrm{Map}^G(G/H, X)$ *is homeomorphic (in* Top*) to the H-fixed points of the G-action on X.*

Proof. A map from $G/H \to X$ extends to an equivariant map $G \to X$ that is H-invariant. Thus, G-equivariant maps $G/H \to X$ are those maps $g \mapsto g \cdot x$ for a constant $x \in X$ where $h \cdot x = x$ for each $h \in H$. ☐

In this sense, the orbit spaces G/H are generalizations of points that only encode H-actions.

Of course, for different subgroups $H_1, H_2 \subseteq G$, the H_i-fixed points are not unrelated. When $H_1 \subseteq H_2$, all H_2-fixed points are also H_1-fixed points. A similar story happens with conjugation: if H_1 and H_2 are conjugate subgroups, then their fixed points are homeomorphic in Top. To see this, take some $h \in H_1$ and $ghg^{-1} \in H_2$ when $H_2 = gHg^{-1}$. Then, for a fixed point x of h, gx is a fixed point of ghg^{-1}.

This relation between subgroups is called subconjugacy, and as we saw above, it is a very natural relation to think about when considering fixed points and how they vary over subgroups. Thus motivated, we will look at fixed points over subgroups related by subconjugacy.

Definition 2.1.8 ([May96], Definition 4.5)**.** A family $\mathcal{F}$ is a collection of closed subgroups $H \subseteq G$ which is closed under subconjugation. That is, if $H \in \mathcal{F}$ and $g^{-1}Kg \subset H$, then $K \in \mathcal{F}$.

Note that when G is a finite abelian group, a family is just a collection of subgroups closed under subgroups.

We will often form constructions for subgroups $H \subset G$ that will extend naturally to families. For example, we have fixed point functors, which we informally defined earlier. These will turn out to be quite useful:

Definition 2.1.9. For a subgroup $H \subseteq G$, we define the H-fixed point functor:

$$\begin{aligned}(-)^H : G\text{-Top} &\to \text{Top}\\ X &\mapsto \{x \in X \mid h \cdot x = x, \forall h \in H\}\end{aligned}$$

For a family $\mathcal{F}$, we may also take the $\mathcal{F}$-fixed point functor:

$$\begin{aligned}(-)^{\mathcal{F}} : G\text{-Top} &\to \text{Top}\\ X &\mapsto \{x \in X \mid h \cdot x = x, \forall h \in H, \forall H \in \mathcal{F}\}\end{aligned}$$

For now, we will only restrict our attention to H-fixed points. Strictly speaking, X^H still retains a G-action. However, it can be best not to think of it as a functor back into G-Top, as ignoring the G-action on X^H lets us say that:

Proposition 2.1.10. *$(-)^H$ is a corepresentable functor. Specifically,*

$$X^H \cong \text{Map}^G(G/H, X)$$

Again, we should be careful about which category we are working over. Remember how cohomology is representable via:

$$H^n(X; \pi) \simeq [X, K(\pi, n)]$$

However, H^n is a functor Top $\to$ Ab, and as written, all we get is a functor Top $\to$ Set. We get lucky in that the specific sets that get outputted have a canonical abelian group structure coming from the fact that $K(\pi, n) \simeq \Omega K(\pi, n+1)$. Namely, by adjunction we get a map $S^1 \wedge K(\pi, n) = \Sigma K(\pi, n) \to K(\pi, n+1)$, and we can get the group action by precomposition with the pinch map $S^1 \to S^1 \wedge S^1$. In the same way, X^H is corepresentable, so we get the opposite. Now, $\text{Map}^G(G/H, X)$ has too much structure, as we can think of it as a G-space. But recall that it has the trivial G-action, so we really lose nothing by just looking at the underlying topological space.

Fixed point functors are our parallel to invariant subsets of G-actions on sets. We similarly will be interested in looking at stabilizer subgroups:

Definition 2.1.11. The isotropy group G_x is the subgroup that fixes some $x \in G$:

$$G_x = \{g \in G \mid g \cdot x = x\}$$

So far, we have defined fixed points of a space. There is also a similarly defined notion of the G-orbits of a space, which we will not do here. However, there are homotopy-theoretic issues with the naive definitions. Recall that $\text{Map}^G(G, X)$ gave us the G-fixed points of X. We will define homotopy fixed points similarly, but replacing G with its universal space:

Definition 2.1.12. For a nice enough topological group, such as a compact Lie group, we have a universal space EG, a space in G-Top that is contractible in Top and has a free G-action.

In fact, these constructions always exist, and are unique in Top up to weak equivalence. There is a beautiful theory behind them, which relies on simplicial sets and geometric realization functors. This description can be found in any introductory book on topology book, such as [Mil21], Chapter 57. For our purposes, we will just assume their existence. In fact, we may form universal spaces for a family $\mathcal{F}$ too, which we will do later.

The upshot is that the correct way, according to homotopy theory, to take fixed points and orbits are given as follows, using the universal space EG:

Definition 2.1.13. For $X \in G$-Top and EG the universal space of G, we define the homotopy fixed points:

$$X^{hG} = \text{Map}^G(EG, X)$$

and the homotopy orbit space:

$$X_{hG} = \frac{EG \times X}{G} = \frac{EG \times X}{(e, x) \sim (ge, gx)}$$

We will not really concern ourselves with either of these definitions, but they are worth mentioning in their own right. We will also have a similar notion of fixed points of spectra, and this sort of construction will come up again when making a homotopically better version.

2.2. **The Equivariant Homotopy Category.** From here, the usual story is that we would define homotopy groups $\pi_n(X)$, and weak equivalences would be maps that induce isomorphisms on them. We get a better understanding of what weak equivalences are by passing to CW complexes via CW approximation where they are just honest homotopy equivalences. We would then form a homotopy category by adjoining formal inverses to weak equivalences. However, as we saw in the last section, the correct notion of cells in the equivariant setting depends on a choice of H. As such, the correct way to define homotopy groups involves the orbit spaces G/H:

Definition 2.2.1 ([GM95], Chapter 1)**.** We define the n-th homotopy group $\pi_n^H(X)$ of $X \in G\text{-Top}_\star$ at a subgroup $H \subseteq G$ as:

$$x_n^H(X) = [G/H \wedge S^n, X]_\star^G$$

where as before, we take G/H with the left multiplication action by G and S^n with the trivial action.

We take a smash product instead of a standard product since we work with based spaces. Implicitly, we are choosing a basepoint in S^n, and already have one in G/H from before. These groups look quite complicated at first, but we can greatly simplify the calculations using the H-fixed point functor:

Definition 2.2.2. (Alternative definition) We can equivalently define $\pi_n^H(X)$ as:

$$\begin{aligned}
& & \pi_n^H(X) &= [G/H \wedge S^n, X]_\star^G \\
&\text{(Currying)} & &= [S^n, \mathrm{Map}^G(G/H, X)]_\star^G \\
&\text{(Trivial } G\text{-actions)} & &= [S^n, \mathrm{Map}^G(G/H, X)]_\star \\
&\text{(Corepresentability of } (-)^H\text{)} & &= [S^n, X^H]_\star \\
& & &= \pi_n(X^H)
\end{aligned}$$

In other words, we can calculate the G homotopy groups non-equivariantly by passing to the H-fixed points.

Back to homotopy groups: remember that classically, we use induced maps on π_n to define weak equivalences, and then get a homotopy category by identifying weakly equivalent spaces together. In the equivariant setting, we have some choice to make about which H we care about. However, the π_n^H do not vary arbitrarily with H, and we will again encode this information using families.

We can now define the equivariant version of weak equivalences as per usual, by requiring maps induce isomorphisms on homotopy groups at each H in a given family:

Definition 2.2.3. A $\mathcal{F}$-weak equivalence with respect to a family $\mathcal{F}$ is an equivariant pointed map $f : X \to Y$ between G-spaces that induces isomorphisms on all homotopy groups at $H \in \mathcal{F}$: that is, $\pi_n^H(f) : \pi_n^H(X) \xrightarrow{\sim} \pi_n^H(Y)$. When the family is not specified, we take $\mathcal{F}$ to be all closed subgroups and call such a map f a G-weak equivalence.

We may then form the homotopy category of G-spaces by formally adjoining G-weak equivalences. This presentation is not too abstract, but it is difficult to work with. It would be irresponsible of me to not introduce the more commonly used one, that is more computationally tractable. It will first require the machinery of the orbit category:

Definition 2.2.4 ([May96], Chapter I.4)**.** The orbit spaces G/H for closed subgroups $H \subseteq G$ form a (full) subcategory of G-Top, which we will call $\mathcal{O}_G$ - the orbit category of G.

This motivates the strange definition of a family from Definition 2.2.7. Morphisms $G/K \to G/H$ exist if and only if K is subconjugate to H, so that $\mathcal{F}$ are the objects of some subcategory of $\mathcal{O}_G$. We notice that, for a given space $X \in G$-Top, we have a canonical map from $\mathcal{O}_G \to \mathrm{Top}$ that sends G/H to the H-fixed points X^H. This gives a functor Φ from G-Top to $\mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})$, presheaves on the orbit category. This will be a convenient way to think about G-spaces, but we will need to give it a definition of homotopy to formalize this notion:

Definition 2.2.5. For an object $T \in \mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})$ (sometimes called an $\mathcal{O}_G$-space) and a space $X \in \mathrm{Top}$, we may define $T \times X$ via the composition:

$$\mathcal{O}_G \xrightarrow{T} \mathrm{Top} \xrightarrow{\times X} \mathrm{Top}$$

Then, we define a homotopy in $\mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})$ of functions between $\mathcal{O}_G$-spaces $f, g : T \to U$ as a map $h : T \times I \to U$ that agrees with f, g on $T \times \partial I$.

Now that we have a notion of homotopy in both G-Top and $\mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})$, we find out that the map Φ from earlier actually becomes a homotopy equivalence:

Theorem 2.2.6 (Elmendorf's Theorem, [Elm83], Theorem 1)**.** *The functor*

$$\Phi : G\text{-Top} \to \mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})$$
$$X \mapsto (G/H \mapsto X^H)$$

admits a homotopy inverse Ψ, that together define a homotopy equivalence between the categories G-Top and $\mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})$. In fact, restricting to G-CW complexes on the left and taking homotopy categories on both sides gives us that Ψ is a right-adjoint for Φ:

$$\Phi : \mathrm{Ho}(G\text{-CW}) \rightleftharpoons \mathrm{Ho}(\mathrm{Fun}(\mathcal{O}_G, \mathrm{Top})) : \Psi$$

Of course, the homotopy equivalence requires a backwards map between the categories. Elmendorf's Theorem at first looks like some weird technical result, but the beauty comes from the map Ψ. The existence of the backwards map tells us that knowing the H-fixed points for all H gives us a lot of information about X - enough to reconstruct it up to weak equivalence.

The choice of family $\mathcal{F}$ gives rise to a different notion of G-equivariant homotopy theory. We may require that the subgroups H appearing in G-CW complexes lie in a family $\mathcal{F}$. There are also times when we only care about $\mathcal{F}$-weak equivalences. In fact, there is a $\mathcal{F}$-Elmendorf's Theorem, where we localize at $\mathcal{F}$-weak equivalences on the left and only have spaces G/H for $H \in \mathcal{F}$ in the orbit category on the right. The extreme ends of all the choices for $\mathcal{F}$ come up often and as such have names:

Definition 2.2.7. We call the case $\mathcal{F} = \{\{e\}\}$ Borel-equivariance. When $\mathcal{F} = \{\text{all closed subgroups}\}$, this is called genuine equivariance.

Note that genuine equivariant is precisely when $G \in \mathcal{F}$, as every subgroup is subconjugate to the parent group.

2.3. **Representations and Representation Spheres.** In this section, we will say a bit about representation theory and introduce constructions in equivariant homotopy theory coming from representations. These concepts will let us encode more data about the group action when stabilizing the equivariant homotopy category.

As a reminder, a group representation is a homomorphism of a group into the automorphism group of a vector space. In other words, a representation is a map $\rho : G \to \mathrm{GL}(V)$ for a $\mathbb{F}$-vector space V. Here, we will focus on real representations of compact Lie groups. Therefore, all representations can be made orthogonal, and we will require that the maps $\rho : G \to O(V)$ also be continuous. In representation theory, we usually think of a "representation" as the map ρ. However, in equivariant homotopy theory, it will be more useful for us to think of a representation as being the data of the G-space space V with the G-action coming from the representation. Namely, for $g \in G$ and $v \in V$:

$$g \cdot (v) = \rho(g)(v)$$

Given a representation, we may form a "representation sphere" - a space homeomorphic (in Top) to a sphere that has a G-action encoding the representation:

Definition 2.3.1 ([May96], Chapter IX.1)**.** For a representation V, we form the representation sphere S^V as the one-point compactification of V. G acts on V through the representation action, and it fixes ∞. We may view S^V as a pointed space in G-$\mathrm{Top}_\star$ with the basepoint given by $0 \in V$.

Representation spheres will be useful as representations encode a lot of data about group G, and further, it is difficult to give a sphere a G-action. However, it is worth noting that not all spheres with group actions come from representation spheres. Representations always have exactly two fixed points - namely, $0 \in V$ and the point ∞. As a simple counterexample, S^n with the antipodal C_2-action then cannot be made into a representation sphere, as it has no fixed points. Note that the standard n-sphere S^n (with the trivial G-action) is the representation sphere corresponding to the n-dimensional trivial action of G on $\mathbb{R}$. This will be part of a general pattern, where we may replace non-equivariant topological degrees n with the trivial n-dimensional representation $\mathbb{R}^n$.

We go from unstable homotopy theory to the stable setting by studying suspensions and loops. Non-equivariantly, we have, for a based space X,

$$\Sigma X = S^1 \wedge X \qquad \Omega X = \mathrm{Map}_\star(S^1, X)$$

The iterated suspension and loop functors follow inductively by noting that the smash product is commutative, and that $S^n \wedge S^m \cong S^{n+m}$:

$$\begin{aligned}\Sigma^n X &= S^1 \wedge (\Sigma^{n-1} X)\\ &= S^1 \wedge S^{n-1} \wedge X\\ &= S^n \wedge X\end{aligned} \qquad \begin{aligned}\Omega^n X &= \mathrm{Map}_\star(S^1, \Omega^{n-1} X)\\ &= \mathrm{Map}_\star(S^1, \mathrm{Map}_\star(S^{n-1} X))\\ &= \mathrm{Map}_\star(S^1 \wedge S^{n-1}, X)\\ &= \mathrm{Map}_\star(S^n, X)\end{aligned}$$

We get from maps $S^1 \to \mathrm{Map}_\star(S^{n-1}, X)$ to maps $S^1 \wedge S^{n-1} \to X$ through a standard argument by currying. Of course, there is nothing stopping us from substituting in other spheres in the equivariant setting, so we get loop and suspension functors for each representation:

Definition 2.3.2 ([May96], Chapter IX.1)**.** For each G-representation V, we get a suspension functor Σ^V and a loop functor Ω^V:

$$\begin{aligned}\Sigma^V : G\text{-Top} &\to G\text{-Top}\\ X &\mapsto \Sigma^V \wedge X\end{aligned} \qquad \begin{aligned}\Omega^V : G\text{-Top} &\to G\text{-Top}\\ X &\mapsto \mathrm{Map}_\star^G(S^V, X)\end{aligned}$$

By the usual arguments, these are adjoint functors, and further, they compose well. Namely, they respect direct sums of representations: $\Sigma^{V\oplus W} X \cong \Sigma^V \Sigma^W X$ and $\Omega^{V\oplus W} X \cong \Omega^V \Omega^W X$.

It will turn out that, in the equivariant setting, cohomology theories will be graded by representations instead of the integers $\mathbb{Z}$. However, representations differ from integers in that they don't admit subtraction. We will want some way to go "down" in degree, so we introduce formal differences to our grading. We start with the category $\mathrm{R}(G)$ of isomorphism classes of (real, orthogonal, continuous) G-representations. The direct product $V \oplus W$ on representations turns $\mathrm{R}(G)$ into a symmetric monoidal abelian category, meaning the following definition makes sense:

Definition 2.3.3 ([May96], Chapter IV.5)**.** Let $\mathrm{RO}(G)$ be the Grothendieck group of $\mathrm{R}(G)$. That is, we formally adjoin elements of the form $V \ominus W$. The tensor product $V \otimes W$ turns $\mathrm{RO}(G)$ into a ring. Objects in $\mathrm{RO}(G)$ will be the degrees of our genuine cohomology theories. As it encodes the real representations in such a way that has a ring structure, we call $\mathrm{RO}(G)$ the real representation ring of G. We call elements of the form $V \ominus W$ virtual representations and call any element $V \ominus V$ (which are all equal), the 0 representation.

Note that $\mathrm{RO}(\{e\}) \simeq \mathbb{Z}$ for the trivial group by taking direct sums of trivial representations. So in fact, we recover the usual grading in the non-equivariant case.

2.4. **Stable Equivariant Homotopy Theory and Spectra.** gNow that we have spheres that encode data about G and have an appropriate grading, we are in a good position to investigate spectra.

Previously, spectra were graded on $\mathbb{N}$. The equivariant versions of spectra are graded similarly to $\mathrm{R}(G)$. Note that standard cohomology is $\mathbb{Z}$-graded, which is the Grothendieck group of $\mathbb{N}$. Similarly, we said that equivariant cohomology would be $\mathrm{RO}(G)$-graded, which is the Grothendieck group of $\mathrm{R}(G)$. This is not a coincidence.

We will grade spectra only over nice enough representations, which we combine as spaces into the data of a "G-universe."

Definition 2.4.1 ([GM95], Chapter 2)**.** We call a countably infinite dimensional real vector space U a G-universe if it:

- has an inner product.
- has an isometric G-action with respect to the norm from its inner product.

- decomposes as the direct sum of representations, with each one that appears having countably infinitely many copies.
- contains a copy of the trivial representation. By the last point, it must appear infinitely many many times, so that there is a copy of $\mathbb{R}^\infty$.

It is complete if it contains all irreducible representations.

The equivariant version of spectra will be indexed over nice enough subspaces of the universe:

Definition 2.4.2 ([GM95], Chapter 2)**.** An indexing space $V \subset U$ is a finite-dimensional inner product subspace closed under the G-action on U.

We can think of indexing spaces as G-representations, with the G-action determining the automorphism group assignments as before.

Over this special grading, we will define G-spectra, which will be very close to being the objects of our equivariant stable homotopy category.

Definition 2.4.3 ([GM95], Chapter 2)**.** A G-spectrum E indexed over a G-universe U has based G-spaces $E(V) \in G\text{-Top}_\star$ for all indexing spaces $V \subset U$. For each pair of indexing spaces $V \subset W \subset U$, there are structure maps:

$$\tilde{\sigma}_{VW} : E(V) \xrightarrow{\simeq} \Omega^{W-V} E(W)$$

that are based homeomorphisms. Furthermore, the structure maps must satisfy transitivity, making the following diagram commute, for all indexing spaces $V \subset W \subset T$:

$$\begin{tikzcd} E(V) \arrow[r, "\tilde{\sigma}_{VW}"] \arrow[rr, bend right, "\tilde{\sigma}_{VT}"'] & \Omega^{W-V} E(W) \arrow[r, "\Omega^{W-V} \tilde{\sigma}_{WT}"] & \Omega^{T-V} E(T) \end{tikzcd}$$

We call E a genuine spectrum if U is a complete G-universe. The category of G-spectra indexed over U is called $G\text{-Spectra}_U$, and morphisms $f : E \to F$ are maps $f_V : E(V) \to F(V)$ between the underlying spaces that commute with the structure maps. By a G-action on a spectrum, we will mean the data of all the G-actions on all the underlying spaces $E(V)$.

We will drop the universe U for much of the next discussion as we will only work with one G-universe for now.

Our next goal is to define a homotopy category of G-spectra by localizing at weak equivalences as usual. This means we need to define homotopy groups of G-spectra. As in the non-equivariant case, we will want functors that suspend a space into a spectrum along with a notion of a sphere spectrum. However, we run into issues with these constructions as G-spectra are hard to work with directly, given that each space $E(V)$ is an infinite loop space. We may instead form equivariant prespectra, which are easier to construct, and then pass back to honest spectra:

Definition 2.4.4 ([GM95], Chapter 2)**.** A G-prespectrum indexed over a G-universe U is the same data as a G-spectrum, but where the structure maps $\tilde{\sigma}_{VW} : E(V) \to \Omega^{W-V} E(W)$ may be arbitrary instead of being required to be homeomorphisms. The category of G-prespectra indexed over U is called $G\text{-PreSpectra}_U$.

Again, we will drop the U for now, but it will come back later.

Of course, these definitions of G-spectra are adjointly related, which we will use to form G-spectra out of the - easier to define - G-prespectra:

Proposition 2.4.5 ([LMS86], Appendix§1, Theorem 1.1)**.** *We take the forgetful functor $\ell : G\text{-Spectra} \to G\text{-PreSpectra}$. ℓ admits a left-adjoint spectrification functor L:*

$$L : G\text{-PreSpectra} \rightleftharpoons G\text{-Spectra} : \ell$$

Further, $L\ell$ is an isomorphism in $G\text{-Spectra}$.

Namely, the procedure of taking a G-spectrum, forgetting that the structure maps are homeomorphisms, and then forcing the structure maps to be homeomorphisms again, gives us back the original data.

We will want a way to form equivariant sphere spectra, which we do non-equivariantly using suspension functors and shift desuspension functors that produce spectra out of spaces. Our cohomology theories will be graded by $\mathrm{RO}(G)$, so we need to be able to produce sphere spectra at each element of the real representation ring - both proper and virtual representations. The equivariant version of the classical suspension functor between spectra will be useful for constructing sphere spectra at virtual representations, and along the way, we get a notion of homotopy of maps between G-spectra by developing a way to smash a spectrum with a G-space. The next few definitions will simply be constructing all of these functors from spaces to spectra and between spectra, following almost unchanged from the usual story.

Definition 2.4.6 ([GM95], Example 2.1)**.** For a G-space X, we may form the suspension prespectrum $\Pi^\infty X$ through suspending the space many times:

$$(\Pi^\infty X)(V) = \Sigma^V X_+$$

We may then form the suspension functor Σ^∞:

$$\begin{aligned}\Sigma^\infty : G\text{-Top} &\to G\text{-Spectra}\\ X &\mapsto L\Pi^\infty X\end{aligned}$$

Definition 2.4.7 ([GM95], Example 2.1)**.** We also form a "shifted" version of the suspended prespectrum, giving us $\Pi^\infty_Z X$ which has Z-th space given by:

$$(\Pi^\infty_Z X)(V) = \begin{cases}\Sigma^{Z-V} X_+ & V \subseteq Z\\ \star & \text{otherwise}\end{cases}$$

From this, we may define the proper shift desuspension functor Σ^∞_Z:

$$\begin{aligned}\Sigma^\infty_Z : G\text{-Top} &\to G\text{-Spectra}\\ X &\mapsto L\Pi^\infty_Z X\end{aligned}$$

This terminology may seem confusing at first, but it comes from the fact that Σ^∞_V "desuspends" the usual Σ^∞ functor through shifting backwards by V. Now we find a way to smash a spectrum with a space, and as a case of this construction, we get a suspension functor between spectra:

Definition 2.4.8 ([GM95], Chapter 2 and [May98])**.** For a G-prespectrum E and a pointed G-space X, we may form the G-prespectrum $E \wedge X$:

$$(E \wedge X)(V) = E(V) \wedge X$$

with structure maps $\sigma^{E\wedge X}_{VW} = \sigma^E_{VW} \wedge \mathrm{id}_X$. We do the usual trick for E a G-spectrum to form the G-spectrum $E \wedge X$ by forgetting the proper spectrum structure, working in prespectra, and passing back to spectra:

$$E \wedge X = L\left((\ell E) \wedge X\right)$$

In the case that $E = S^V$ is a representation sphere, we get a suspension functor Σ^V on G-Spectra:

$$\begin{aligned}\Sigma^V : G\text{-Spectra} &\to G\text{-Spectra}\\ E &\mapsto E \wedge S^V\end{aligned}$$

Using all of these tools, we may form G-sphere spectra:

Definition 2.4.9 ([May96], Chapter 2)**.** For an element T of the real representation ring $\mathrm{RO}(G)$, we may form the G-sphere spectrum $\mathbb{S}^T_G$ as:

$$\mathbb{S}^T_G = \begin{cases}\Sigma^\infty S^T & T \in \mathrm{R}(G)\\ \Sigma^\infty_{-T} S^0 & -T \in \mathrm{R}(G)\\ \Sigma^V \Sigma^\infty_W S^0 & T = V \ominus W\end{cases}$$

As a special case, we get G-sphere spectra $\mathbb{S}^n_G$ by viewing n as the n-dimensional trivial representation, as usual:

$$\mathbb{S}^n_G = \begin{cases} \Sigma^\infty S^n & n \geq 0 \\ \Sigma^\infty_{|n|} S^0 & n < 0 \end{cases}$$

where we take all S^n with the trivial G-action. Similarly, we will take $\mathbb{S}_G$ without an exponent to indicate the spectrum $\mathbb{S}^0_G$ for the virtual 0 representation.

The subscript G will be dropped from now on, since we will not have to distinguish between equivariant and non-equivariant sphere spectra.

We are in a good now position to define homotopies of maps between spectra and homotopy groups of spectra:

Definition 2.4.10 ([May96], Chapter 2)**.** We define a G-homotopy in G-Spectra between functions $f, g : X \to Y$ to be a map of spectra $X \wedge I_+ \to Y$ agreeing with f, g on $X \wedge (\partial I)_+$, with I and I_+ endowed with the trivial G-action. We can then form the homotopy classes of maps $[X, Y]^G$ as usual. The homotopy groups of spectra $\pi^H_V(X)$, as with of G-spaces, are given at subgroups $H \subseteq G$ and indexed by the representation ring as $V \in \mathrm{RO}(G)$:

$$\pi^H_V(X) = [\mathbb{S}^V \wedge G/H_+, X]^G$$

where we again abbreviate $\pi^H_n(X) = \pi^H_{\mathbb{R}^n}(X)$ to be indexed at a trivial n-dimensional representation.

We now define weak equivalences of G-spectra in order to form the stable equivariant homotopy category. However, something highly nontrivial happens here, in that weak equivalences will only care about the homotopy groups at n. We can alternatively define weak equivalences between non-equivariant spectra $E \to F$ by requiring that the maps between spaces $E(n) \to F(n)$ are weak equivalences. The claim is that if we define weak equivalences in this way for G-spectra - where the maps $E(V) \to F(V)$ are all weak equivalences for all indexing spaces V, then it is still equivalent to looking at the homotopy groups at trivial representations. In other words, the trivial representations encode enough information about G-spectra to tell us when they are weakly equivalent in the sense of underlying spaces. The interested reader may want to look at [GM95], Theorem 2.3 for additional discussion or [LMS86], I§7, Theorem 7.12 for a proof. For our purposes, we will stick to the definition involving homotopy groups:

Definition 2.4.11. A weak equivalence in G-Spectra between X, Y is a map $f : X \to Y$ that induces isomorphisms on all the homotopy groups at all H for all $n \in \mathbb{Z}$:

$$\pi^H_n(f) : \pi^H_n(X) \xrightarrow{\sim} \pi^H_n(Y)$$

From here on out, our definition of "G-spectra" will change from G-Spectra to instead be the homotopy category of G-Spectra, localized at weak equivalences:

Definition 2.4.12. We define the stable equivariant homotopy category G-Sp_U to be G-$\mathrm{Spectra}_U$ indexed over a G-universe U, localized at weak equivalences by adjoining formal inverses to all weak equivalences.

Of course, there are ways to define smash products $X \otimes Y$ in any of G-PreSpectra, G-Spectra, and G-Sp. We will not get into the explicit construction here, but details can be found in [May96], Chapter XII.1-XII.5.

2.5. Geometric Fixed Points and the Tate Construction.

So far, we have been ignoring the G-universe U we work over. As in the case of choosing families, the choice of universe gives us different notions of spectra that are still related. We will have names for the extremes:

Definition 2.5.1. When U is a complete universe, we call a G-spectrum indexed over U a genuine G-spectrum. On the other extreme, when U has the trivial G-action (it is a sum of countably infinitely many trivial representations), we say that a spectrum indexed over U is a naive G-spectrum. We have similar notions for G-prespectra.

There is a critical insight here that we may develop the theory of non-equivariant spectra precisely the same way as we do for G-spectra. We just need to drop all requirements about G-actions - alternatively, we can work over a G-universe U that is fixed by G. That is, it only has trivial representations, or as a G-space, it is its own G-fixed points: $U^G = U$. In this case, we also do not care about G-actions on the underlying spaces and not require maps be equivariant. In fact, working over a universe can make certain arguments cleaner, such as the existence of smash products in Spectra. [May96], Chapter XII.1-XII.4 gives a good overview of how this development plays out. Therefore, we are motivated to investigate what the G-fixed point functor on universes does to G-spectra.

From here on out, we will assume U is a complete G-universe so that we work over genuine G-spectra. We will also go back to working on the level of G-Spectra, although the constructions will descend back down to G-Sp.

Definition 2.5.2. The inclusion map $i\colon U^G \to U$ induces a forgetful functor i^* from G-spectra indexed over U to G-spectra indexed over the fixed universe U^G:

$$\begin{aligned} i^* : G\text{-Spectra}_U &\to G\text{-Spectra}_{U^G} \\ \{E(V)\}_{V\subset U} &\mapsto \{E(i(V))\}_{V\subset U^G} \end{aligned}$$

In other words, we collapse all subspaces with non-trivial actions and retain only the trivial representations.

Importantly, the spaces underlying the spectrum $i^*(E)$ still have G-actions. In this way, we have forgotten the G-action on the entire spectrum. However, by forgetting these actions, we get standard spectra:

Definition 2.5.3. For a genuine G-spectrum E, the underlying non-equivariant spectrum of E is the object in Spectra with the same spaces, indexed over the non-equivariant universe U^G, as $i^*(E)$, but forgetting their G-action.

In general, we can view any naive G-spectrum as a non-equivariant spectrum by just forgetting the G-action on the underlying spaces.

We will also get a left-adjoint to i^* that includes non-trivial representations into a naive G-spectrum to turn it into a genuine G-spectrum. This construction feels quite mysterious, so it deserves at least a proof idea:

Proposition 2.5.4 ([LMS86], II§1, Definition 1.1)**.** *i^* admits a left-adjoint i_*:*

$$i_* : G\text{-Spectra}_{U^G} \rightleftharpoons G\text{-Spectra}_U : i^*$$

Proof (outline). We start on the level of G-PreSpectra. For $E \in G\text{-PreSpectra}_{U^G}$ and an indexing space $V \oplus \mathbb{R}^n \subset U$ for a non-trivial representation V, we define:

$$(i_*E)(V \oplus \mathbb{R}^n) = E(\mathbb{R}^n) \wedge S^V$$

Of course, we can elevate this to the level of G-Spectra the usual way:

$$\begin{aligned} i_* : G\text{-Spectra}_{U^G} &\to G\text{-Spectra}_U \\ E &\mapsto L(i_*(\ell E)) \end{aligned}$$

□

Now, we are able to produce "fixed points of spectra." The meaning is quite obvious on naive spectra, but we will want a way to bump the definition up to genuine spectra using the adjunction we just defined:

Definition 2.5.5. For a family $\mathcal{F}$ of subgroups of G, we define the (naive) $\mathcal{F}$-fixed point functor on naive G-spectra:

$$\begin{aligned} (-)^{\mathcal{F}} : G\text{-Spectra}_{U^G} &\to G\text{-Spectra}_{U^G} \\ \{E(V)\}_{V\subset U^G} &\mapsto \{E(V)^{\mathcal{F}}\}_{V\subset U^G} \end{aligned}$$

where we take H-fixed points at the level of spaces, for all subgroups $H \in \mathcal{F}$. We make use of the previous adjunction to extend the definition to genuine G-spectra. The (genuine) $\mathcal{F}$-fixed point functor on genuine G-spectra is given by:

$$\begin{aligned} (-)^{\mathcal{F}} : G\text{-Spectra}_U &\to G\text{-Spectra}_U \\ E &\mapsto i_*((i^*E)^{\mathcal{F}}) \end{aligned}$$

by first passing to naive spectra, taking fixed points, and then passing back up to genuine spectra. Of course, these definitions make sense for H-fixed points too.

While a convenient definition, it suffers from some disturbing problems. Taking the G-fixed points of free G-spectrum does not give us a G-spectrum with a trivial action (on the underlying spaces). Another example of the failure of this fixed point spectrum is that we would like fixed points to commute with suspension. However, in general,

$$(\Sigma^\infty X)^G \not\simeq \Sigma^\infty(X^G)$$

The rest of this section will be dedicated to finding the "correct" fixed point functor that succeeds where $(-)^{\mathcal{F}}$ fails.

In the same way we can form the space EG, we may also form the space $E\mathcal{F}$ for a family $\mathcal{F}$. To do so, we must first define what an $\mathcal{F}$-space is:

Definition 2.5.6. An $\mathcal{F}$-space X is a G-space where all isotropy groups are in the family: $G_x \in \mathcal{F}$ for all x.

Definition 2.5.7. We say a G-space X is $\mathcal{F}$-contractible if it is H-contractible for each $H \in \mathcal{F}$. That is, viewed as an H-space, there is an H-homotopy $X \simeq \star$ for each $H \in \mathcal{F}$.

We can take any G-space X and form a $\mathcal{F}$-contractible space $E\mathcal{F}$.

Definition 2.5.8. The universal space $E\mathcal{F}$ is the space of homotopy type of a G-CW complex. It carries, for every $\mathcal{F}$-space X, a a unique homotopy class of maps $X \to E\mathcal{F}$. In fact, it is stronger than being $\mathcal{F}$-contractible -it is *only* $\mathcal{F}$-contractible:

$$(E\mathcal{F})^H \simeq \star \quad H \in \mathcal{F}$$
$$(E\mathcal{F})^H = \emptyset \quad H \notin \mathcal{F}$$

In fact, we are able to construct $E\mathcal{F}$ in a very slick way. Consider the functor:

$$\underline{\mathcal{F}} : \mathcal{O}_G \to \mathrm{Top}$$
$$G/H \mapsto \begin{cases} \star & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases}$$

This gives us a presheaf on the orbit category, and we may apply Elmendorf's right-adjoint Ψ from Theorem 2.2.6 to get $E\mathcal{F}$:

$$E\mathcal{F} = \Psi\underline{\mathcal{F}}$$

We also form its cone:

Definition 2.5.9. Denote by $\widetilde{E\mathcal{F}}$ the mapping cone of $E\mathcal{F} \to \star$.

We are now ready to define the geometric fixed point functor with respect to a family, which will be the homotopically correct way to think about fixed points of spectra:

Definition 2.5.10. We define the geometric fixed points functor $\Phi^{\mathcal{F}}$ as:

$$\Phi^{\mathcal{F}} : G\text{-Spectra}_U \to G\text{-Spectra}_U$$
$$E \mapsto (E \wedge \widetilde{E\mathcal{F}})^{\mathcal{F}}$$

2.6. **Borel-Equivariant Genuine Spectra.** Here, we fix a group G, and for simplicity require it be finite and abelian. For some family $\mathcal{F}$, we may form a genuine G-spectrum $A_{\mathcal{F}}$:

Definition 2.6.1 ([MNN17], Definition 6.1)**.**

$$A_{\mathcal{F}} = \prod_{H \in \mathcal{F}} \mathrm{Fun}(G/H_+, \mathbb{S}_G)$$

In fact, this will be an $\mathbb{E}_\infty$ spectrum, so that $A_{\mathcal{F}} \in \mathrm{CAlg}(G\text{-Sp})$.

Now, for a genuine G-spectrum, we have a notion of locality with respect to another genuine G-spectrum, and we have Bousfield localization functors. Specifically, we will take a look at the Borel equivariant case:

Definition 2.6.2. For a genuine G-spectrum E, we say E is Borel-complete if it is $A_{\mathcal{F}}$-local with the Borel family: $\mathcal{F} = \{\{e\}\}$.

We also can "Borelify" any G-spectrum:

Definition 2.6.3 ([Yua23], 2.3)**.** For a genuine G-equivariant spectrum E, applying the adjoints from Proposition 2.5.4 gives a Borelification functor β, which can be thought of as a Bousfield localization $L_{A_{\mathcal{F}}}$:

$$\begin{aligned} \beta : G\text{-Sp} &\to G\text{-Sp} \\ E &\mapsto i_*(i^*E) \end{aligned}$$

We may also regard any standard spectrum E as naive G-spectrum $\tilde{E}$ with trivial actions on each underlying space, as per the earlier discussion. Then, taking $i^*(\tilde{E})$ lets us regard E as a Borel-equivariant genuine G-spectrum.

2.7. **Tate Construction.** For a standard spectrum E, there is a special case of its geometric fixed points. We take the trivial family $\mathcal{F} = \{\{e\}\}$, and "define",

Definition 2.7.1. For $E \in \text{Sp}$, let its Tate construction E^{tC_p} be given by:

$$\begin{aligned} (-)^{tC_p} : \text{Sp} &\to \text{Sp} \\ E &\mapsto \Phi^{\{\{e\}\}}\tilde{E} \end{aligned}$$

where $\tilde{E} \in G\text{-Sp}_{UG}$ is the naive G-spectrum version of E, with trivial actions.

3. Chromatic Blueshift

3.1. **Introduction.**

Definition 3.1.1. We define the chromatic support $\text{supp}(R)$ of a ring spectrum R as:

$$\text{supp}(R) = \{n \in \mathbb{N} \mid T(n) \otimes R \neq 0\}$$

In other words, these are the n such that R is *not* $T(n)$-acyclic. However, the following result of [Hah16] greatly simplifies what $\text{supp}(R)$ can look like:

Theorem 3.1.2 ([Hah16] 1.3)**.** *Let R be a spectrum that admits a D_p-algebra structure for a prime p. Then, for any $n \geq 0$, if R is $K(n)$-acyclic, then R is also $K(n+1)$-acylcic.*

Remark 3.1.3. I have no idea what a D_p-algebra is, but any $\mathbb{E}_\infty$ ring spectrum $R \in \text{CAlg}(\text{Sp})$ is a D_p-algebra, as mentioned in [Hah16].

This helps us as, in the case of ring spectra, being $K(n)$-acyclic is precisely the same as being $T(n)$-acyclic:

Theorem 3.1.4 ([LMMT24], 2.3)**.** *Any ring spectrum R is $K(n)$-acyclic if and only if it is $T(n)$-acyclic.*

Therefore, restricting our study to $\mathbb{E}_\infty$-rings, we get that if R is $T(n)$-acylic, it is also $T(n+1)$-acyclic. Namely, this says either $\text{supp}(R) = \emptyset$ or $\text{supp}(R) = [0, n]$. This means we can define the height of a p-local ring spectrum R, as the maximum n where R is not $T(n)$-acyclic:

$$\text{height}(R) = \max\{n \geq -1 \mid T(n) \otimes R \neq 0\}$$

Note that we take $T(-1) = \mathbb{S}$ so that $\mathbb{S} \otimes R \neq 0$.

3.2. **Chromatic Blueshift Conjecture.** Fix some finite abelian group A:

Definition 3.2.1 ([BSY22], Definition 9.6)**.** We define the rank of A at p as:

$$\text{rk}_p(A) = \dim_{\mathbb{F}_p}(A \otimes_{\mathbb{Z}} \mathbb{F}_p)$$

When A is a p-group, for some family $\mathcal{F}$ of subgroups of A, we define the corank of A:

$$\text{cork}_p(A) = \min\{\text{rk}_p(A') \mid A' \subset A \text{ and } A' \notin \mathcal{F}\}$$

Now, we may state the converse of chromatic blueshift:

Theorem 3.2.2 ([BSY22], Theorem 9.8)**.** *Let A be a finite abelian p-group and $\mathcal{F}$ be a family of proper subgroups. Take an $\mathbb{E}_\infty$ ring spectrum $R \in \text{CAlg}(\text{Sp})$. Regard R as a Borel-equivariant A-spectrum. Then, if $\Phi^{\mathcal{F}}R$ is $T(n - \text{cork}_p(\mathcal{F}))$-acyclic, then R itself is $T(n)$-acyclic.*

And now, the point of these notes. The Chromatic Blueshift Conjecture is that this theorem's converse is true too. The phenomenon is called chromatic blueshift since $T(n)$ detects peridiocity at different chromatic heights through the v_n-self maps. The claim is that by passing to the geometric fixed points of R, after thinking of it as a Borel-equivariant genuine spectrum, you are lowering the chromatic height, thus decreasing the periodicity - resulting in "blueshift."

Conjecture 3.2.3 (Chromatic Blueshift Conjecture - [BSY22], Conjecture 9.9)**.** *Let A be a finite abelian p-group and $\mathcal{F}$ be a family of proper subgroups. Take an $\mathbb{E}_\infty$ ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$. Regard R as a Borel-equivariant A-spectrum. Then, if R is $T(n)$-acyclic, then $\Phi^{\mathcal{F}} R$ is $T(n - \mathrm{cork}_p(\mathcal{F}))$-acyclic.*

3.3. **Some Thoughts on Calculations.** For some $\mathbb{E}_\infty$ spectrum R as in Conjecture 3.2.3, $\Phi^{\mathcal{F}} R$ also inherits a ring spectrum structure. Therefore, by Theorem 3.1.4, the claim is that if $R \otimes K(n) \simeq 0$ then $\Phi^{\mathcal{F}} R \otimes K(n - \mathrm{cork}_p(\mathcal{F})) \simeq 0$.

In the case of $A = C_p$ and $\mathcal{F} = \{\{e\}\}$, this asks whether $\Phi^{\mathcal{F}} R \otimes K(n-1) \simeq 0$. If we further assume R is complex oriented, then we have a commutative square, where f and φ come from complex orientations:

$$\begin{tikzcd}
\mathrm{MU} \arrow[r, "\varphi"] \arrow[d, "f"'] & K(n) \arrow[d] \\
R \arrow[r] & R \otimes K(n) \simeq 0
\end{tikzcd}$$

The central calculation we need to understand is $\pi_\bullet(R \otimes K(n))$. As an intermediate step, we may form the pushout of $R \leftarrow \mathrm{MU} \to K(n)$ as the relative smash product, and realize the standard smash product as the pushout of $R \leftarrow \mathbb{S} \to K(n)$.

$$\begin{tikzcd}
\mathbb{S} \arrow[dr] \arrow[drr, dashed, bend left] \arrow[ddr, dashed, bend right] & & & \\
& \mathrm{MU} \arrow[r] \arrow[d] & K(n) \arrow[d] \arrow[ddr, bend left] & \\
& R \arrow[r] \arrow[drr, bend right] & R \otimes_{\mathrm{MU}} K(n) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "\exists!"'] & \\
& & & R \otimes K(n)
\end{tikzcd}$$

While $\pi_\bullet$ does not in general preserve pushouts, we do have that

$$\pi_\bullet(R \otimes_{\mathrm{MU}} K(n)) \simeq \pi_\bullet(R) \otimes_{\pi_\bullet(\mathrm{MU})} \pi_\bullet(K(n)) \simeq v_n^{-1} \pi_\bullet(R)/p$$

[**TODO**: check - is this true?] where we identify v_n as the image of x_{p^n-1} from $\pi_\bullet(\mathrm{MU})$. Therefore, we still have a pushout-type square:

$$\begin{tikzcd}
\mathbb{Z}[x_1, \ldots] \arrow[r, "\varphi_\bullet"] \arrow[d, "f_\bullet"'] & \mathbb{F}_p[v_n^{\pm 1}] \arrow[d] \arrow[ddr, bend left] & \\
\pi_\bullet(R) \arrow[r] \arrow[drr, bend right] & v_n^{-1}\pi_\bullet(R)/(p, v_1, \ldots, v_{n-1}) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists!"'] & \\
& & \pi_\bullet(R \otimes K(n))
\end{tikzcd}$$

Note that $\varphi_\bullet$ sends $1 \mapsto 1$ and $x_i \mapsto 0$ for $i < p^n - 1$ and $x_{p^n-1} \mapsto v_n$. [**TODO**: There's other relations coming from $v_{>n} \in \mathbb{F}_p[v_n^\pm]$ – what happens to them? But it receives a surjective map from the given ring and is a pushout]

Claim 3.3.1. $\left(v_n^{-1}\pi_\bullet(R)\right)/(p, v_1, \ldots, v_{n-1}) \simeq 0$ *if and only if* $\pi_\bullet(R \otimes K(n)) \simeq 0$.

Proof attempt. The only if is easy: if the first ring is 0, then there's no maps into any other non-zero ring.

(This is the sketchy section.) Note that we didn't use that there is a unique map. The map gets induced from the map $R \otimes_{\mathrm{MU}} K(n) \to R \otimes$. However, we also get a map going backwards from the pushout $R \leftarrow \mathbb{S} \to K(n)$. Therefore, there is a map $\pi_\bullet(R \otimes K(n)) \simeq 0 \to \left(v_n^{-1}\pi_\bullet(R)\right)/(p, v_1, \ldots, v_{n-1})$. This cannot exist unless the target ring is 0. □

Here's the upshot: we can tell if an $\mathbb{E}_\infty$ MU-algebra R is $K(n)$-acyclic by calculating $\left(v_n^{-1}\pi_\bullet(R)\right)/(p, v_1, \ldots, v_{n-1})$ and checking if it is 0.

Proposition 3.3.2. $\left(v_n^{-1}\pi_\bullet(R)\right)/(p, \ldots, v_{n-1}) = 0$ *if and only if* $v_n^k = 0 \mod (p, \ldots, v_{n-1})$ *for some* $k \geq 0$.

Proof. Let $\mathcal{I} = \{p, \ldots, v_{n-1})$.

$$\begin{aligned}
\left(v_n^{-1}\pi_\bullet(R)\right)/(p, \ldots, v_{n-1}) = 0 &\iff xv_n^{-k} \in (\mathcal{I}) \subset v_n^{-1}\pi_\bullet(R) && \forall x \in \pi_\bullet(R), k \geq 0 \\
&\iff xv_n^{-k} = \sum_{e \in \mathcal{I}} x_e v_n^{-k_e} e && \text{for some } x_e \in \pi_\bullet(R), k_e \geq 0 \\
&\iff xv_n^{\ell} = \sum_{e \in \mathcal{I}} x_e v_n^{\ell_e} e && \text{for some } \ell, \ell_i \geq 0 \\
&\iff xv_n^{\ell} = \sum_{e \in \mathcal{I}} y_e e && \text{for some } \ell \geq 0, y_e \in \pi_\bullet(R) \\
&\iff xv_n^{\ell} \in (\mathcal{I}) && \forall x \in \pi_\bullet(R) \\
&\iff v_n^{p} \in (\mathcal{I}) && \text{at } x = 1
\end{aligned}$$

□

The next step is to identify the homotopy groups of R^{tC_p}. [**TODO**: ????]

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