LA COHOMOLOGIE MOD 2 DE CERTAINS ESPACES HOMOGÈNES
MOD 2 COHOMOLOGY OF CERTAIN HOMOGENEOUS SPACES

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ABSTRACT. In this talk, we will study the mod 2 cohomology of some classifying spaces of certain compact Lie groups. Our topological groups will come from automorphism groups of $\mathbb{R}^n$ instead of the usual $\mathbb{C}^n$, the latter of which is an easier story. The study of these spaces will be done primarily through looking at homogeneous spaces once we quotient out a "maximal torus," so these intermediate spaces will also be of interest.

1. Introduction

All cohomology will be taken with $\mathbb{F}_2$ coefficients.

Theorem 1. $\mathrm{H}^\bullet(BO(n);\mathbb{Z}/2) \simeq S_{\mathbb{Z}/2}[y_1,\ldots,y_n]$ *with* $|y_i| = 1$. *Further, the map* $\iota : (\mathbb{Z}/2)^n \hookrightarrow O(n)$ *induces inclusion of the* Σ_n*-fixed points of* $\mathbb{Z}[x_1,\ldots,x_n]$ *with*

$$\mathrm{H}^\bullet(B(\mathbb{Z}/2)^n;\mathbb{Z}/2) \simeq \mathbb{Z}/2[x_1,\ldots,x_n]$$

Further, it sends the i*-th Stiefel-Whitney class* w_i *to the* i*-th elementary symmetric function* σ_i*.*

All cohomology will be taken mod 2.

Why should we care about this? The story is much easier when we work over $\mathbb{C}$. Before we get to that, let's review some fibrations:

1.1. Fibrations.

Pick a (topological) group G and subgroup H. Then, pick some EG (up to H-equivariant homotopy). One model for BH is EG/H. Therefore, we get the following:

$$G/H \to EG/H \to EG/G$$

We will write this thing as:

$$G/H \to BH \to BG$$

Note this is unexpected: there is a map $BH \to BG$ induced by the inclusion $H \subset G$. This goes in the opposite direction.

1.2. The complex case.

Pick a field – for this section alone we will work over coefficients in your favorite field (yes, yours specifically!) We will try to calculate $\mathrm{H}^\bullet(BU(n))$ using the fibration sequence. The claim is that it is given by:

$$\Bbbk[x_1,\ldots,x_n] \quad |x_i| = 2i$$

When $n = 1$, we have $\mathrm{H}^\bullet(BU(1)) = \mathrm{H}^\bullet(BS^1) = \mathrm{H}^\bullet(\mathbb{CP}^\infty) = \Bbbk[x_1]$ with $|x_1| = 2$. Now, we proceed inductively. By including $U(n-1) \subset U(n)$, we have a fibration sequence:

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

Draw spectral sequence

We can deduce $BU(n)$ from $BU(n-1)$ given we know what has to survive. We can also say a lot using Lie theory, which I won't get into since I know nothing about it. However, it is utterly useless for $BO(n)$.

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2. Poincaré series

For a space X, we define its Poincaré series as a formal power series:

$$P_K(X) = \sum_i t^i \dim_{\Bbbk} \mathrm{H}^i(X; \Bbbk)$$

Notice that Poincaré series are multiplicative according to the Künneth theorem.

We can actually generalize this:

Proposition 1. *For a fibration, the Serre spectral sequence collapses at the E_2 page and has trivial local coefficient system if and only if the Poincaré series is multiplicative. We will need F to be compact and everything to be connected.* ***We will only prove the only if direction***

Proof. Let $C^q(F)$ be the largest subgroup of $\mathrm{H}^q(F)$ acted on trivially by $\pi_1(B)$. Notice that:

$$E_2^{0,q} = \mathrm{H}^0(X; \mathrm{H}^q(F)) = C^q(F)$$

Why? Here is my best explanation. The E_2 page is full of honest groups, not local coefficients. Therefore, $\mathrm{H}^0(...)$ picks out the "global" bits – cohomology classes from the local coefficient system that can be globally extended. These are precisely the $\pi_1(B)$-invariant ones. If I say the words **sheaf** and **global section** this might resonate more with some of you.

With that aside, we will induce over q to show $C^q = H^q$. Note that:

$${}^kE_2 = \bigoplus_i \mathrm{H}^i(B; \mathrm{H}^{k-i}(F_b))$$

At $i = 0$, we get C^q, and elsewhere we get a trivial local coefficient system, and thus $\mathrm{H}^i(B) \otimes \mathrm{H}^{k-i}(F)$ So that:

$$\dim^k E_2 = \dim \mathrm{H}^k(B \times F) - \dim \mathrm{H}_k(F) + \dim C^q(F)$$

If the spectral sequence collapses at E_2, then this is:

$$\dim^k E_2 = \dim^k E_\infty - \dim H_k(F) + \dim C^q \geq \dim^k E_\infty$$

Therefore proving that $C^q(F) = H^q(F)$. Therefore, we have multiplicativity of the Poincaré series. □

3. $BO(n)$

Let $Q(n) = (\mathbb{Z}/2)^n \subset O(n)$ and $F_n = O(n)/Q(n)$ as groups.

Grad student enrichment activities:

$$B\mathbb{Z}/2 = \mathbb{RP}^\infty \implies \mathrm{H}^\bullet(B\mathbb{Z}/2) = \mathbb{Z}/2[x]$$

Because B is multiplicative, we have:

$$\mathrm{H}^\bullet(B(\mathbb{Z}/2)^n) = \otimes^n_{\mathbb{Z}/2}\mathbb{Z}/2[x_i] = \mathbb{Z}/2[x_1, \dots, x_n]$$

where all $|x_i| = 1$. The Poincaré polynomial of $\mathbb{RP}^\infty$ is $1 + t + t^2 + \cdots = (1-t)^{-1}$. By multiplicativity (products collapse at E_2), we have:

$$P(BQ(n)) = (1-t)^{-n}$$

3.1. Some fibrations.

$$F_n \to BQ(n) \to BSO(n)$$

$$F_{n-1} = \frac{O(n-1)}{Q(n-1)} \to F_n = \frac{O(n)}{Q(n)} \to \frac{O(n)}{\mathbb{Z}/2 \times O(n-1)} \simeq \mathbb{RP}^{n-1}$$

$$BQ(n) \to BO(n) \to O(n)/Q(n)$$

3.2. **Some results.** To study $\mathrm{H}^\bullet(BO(n))$, we will need to understand the cohomology of F_n. Here's a proposition:

Proposition 2. *$\mathrm{H}^\bullet(F_n)$ is generated by its elements in degree ≤ 1, and has Poincaré polynomial:*

$$P(F_n) = (1-t^2)(1-t^3)\dots(1-t^n)(1-t)^{1-n}$$

Proof. Fact: $\dim \mathrm{H}^1(F_n) \geq n-1$. Why? Look at the fibration:

$$F_n \to BQ(n) \to BSO(n)$$

In fact, a consequence of the Poincaré polynomial is that this is an equality.

Also fact: $F_2 \simeq S^1$, so the proposition is true at $n = 2$. We can embed $\mathbb{Z}/2 \times O(n-1) \subset O(n)$ by putting ± 1 on the upper-left corner. We have the following fibration:

$$\frac{O(n-1)}{Q(n-1)} \to \frac{O(n)}{Q(n)} \to \frac{O(n)}{\mathbb{Z}/2 \times O(n-1)} \simeq \mathbb{RP}^{n-1}$$

At the E_2 page, we have, at dim 1:

$$\dim^1 E_2 = 1 \text{ (from RP)} + (\leq n-2)\text{(from C1)} \leq n-1$$

But in fact $\mathrm{H}^1(F_n) \geq n-1$, so this is the 1-dim part of the E_∞ page. Therefore, we have equality. Further, C^1 must be everything, and everything in dim 1 persists until E_∞. The image of $\mathrm{H}^\bullet(F_n) \to \mathrm{H}^\bullet(F_{n-1})$ contains $H^1(F_{n-1})$ and thus everything. Therefore, the image surjects onto $\mathrm{H}^\bullet(F_{n-1})$ and no other differentials happen – it collapses at the E_2 page. Therefore, the local coefficient system is trivial, and

$$E_2 = \mathrm{H}^\bullet(F_n) \otimes \mathrm{H}^\bullet(\mathbb{RP}^{n-1})$$

Both of these things are generated by deg ≤ 1 elements, and thus so is the left, and therefore so is $\mathrm{H}^\bullet(F_n)$. We can check the Poincaré polynomial inductively. I don't want to do that. □

Lets go back to the Main Theorem now.

Proof.

a. We will show that the Poincaré series for $BO(n)$ is:

$$P(BO(n), t) = (1-t)^{-1}(1-t^2)^{-1}\dots(1-t^n)^{-1}$$

Further, we can include $O(n-1) \hookrightarrow O(n)$. $(B\iota)^*$ will be shown to be injective.

Consider the following fibration sequence:

$$F_n = \frac{O(n)}{Q(n)} \to BQ(n) = \frac{EO(n)}{Q(n)} \to \frac{EO(n)}{O(n)}$$

Spectral sequence:

$$E_2^{p,q} = \mathrm{H}^p(BO(n), \mathrm{H}^q(F_n))$$

At dimension 1, we have rank:

$$\dim^1 E_2 = \dim H^1(BO(n)) + \dim H^0(BO(n); H^1(F_n)) \leq n$$

(the second bit is $\leq n-1$ and the first bit is equal to 1 as $\pi_1(BO(n)) = \pi_0(O(n))$). But in fact, we know $\mathrm{H}^\bullet(BQ(n))$ – at dim 1, its rank is n exactly. Therefore, the thing on the right is just $H^1(F_n)$, and these remain cocycles forever. Because the degree ≤ 1 elements generate all of it, we actually get that this spectral sequence collapses at E_2. In this case, it is known that p^* is injective – the differentials are the only thing generating the kernel, and there aren't any.

b. Let $N(Q(n))$ be the normalizer of $Q(n)$ in $O(n)$ and Ψ_n be the Weyl group:

$$\Psi_n = \frac{N(Q(n))}{Q(n)} \simeq \Sigma_n$$

It acts on $(\mathbb{RP}^\infty)^n$ by permuting the factors, and therefore on cohomology permutes x_i.

The fibration $F_n \to BQ(n) \to BO(n)$ can be thought of Σ_n-equivariantly. What is the action of Σ_n on $BO(n)$? Σ_n acts on $O(n)$ by conjugation. For a group G, conjugation by g gives a functor $\mathbf{B}G \to \mathbf{B}G$. In fact, g defines a natural transformation $g(-)g^{-1} \implies \mathrm{id}$. This becomes a homotopy equivalence when passing to BG: conjugation acts trivially on BG up to homotopy. Therefore, it acts trivially on its cohomology. The upshot is that $\mathrm{H}^\bullet(BO(n)) \to \mathrm{H}^\bullet(BQ(n))$ is injective (from earlier) and the image lands in the Σ_n-invariants. In fact, by a dimension argument using the Poincaré polynomials which we know, we also get surjectivity. Therefore,

$$\mathrm{H}^\bullet(BO(n)) \simeq S(x_1, \dots, x_n)$$

[typed notes ended here. I had handwritten notes I did not get to during the presentation. Here is a summary of those handwritten notes]

c. We want to show $w_i \to \sigma_i$, the i-th elementary symmetric polynomials. We will need the fact that w_{i+1} is the only non-zero element of degree $i+1$ killed by $\mathrm{H}^\bullet(BO(n)) \to \mathrm{H}^\bullet(BO(i))$, which I will not prove.

$$\begin{array}{ccc} \mathrm{H}^\bullet(BQ(i)) & \xleftarrow{\alpha^*} & \mathrm{H}^\bullet(BQ(n)) \\ \rho_i^* \uparrow & & \rho_n^* \uparrow \\ \mathrm{H}^\bullet(BO(i)) & \xleftarrow[\beta^*]{} & \mathrm{H}^\bullet(BO(n)) \end{array}$$

Using the previous fact: note that $\alpha^* \circ \rho_n^*(w^{i+1}) = \rho_i^* \circ \beta^*(w^i) = 0$. We are looking for a symmetric polynomial in degree $i+1$ killed by the right things – therefore, this is precisely the image of w_{i+1}. □

4. Fun facts

Theorem 2.

$$Sq^i w_j = \sum_{0 \le t \le i} \binom{j-i+t-1}{t} w_{i-t} w_{i+t}$$

Proof idea: We know what it is for $BQ(n)$, and we know the Cartan formula + Main Theorem

Proposition 3. $\mathrm{H}^\bullet(BO(n)) \to \mathrm{H}^\bullet(BSO(n))$ *has kernel* w_1.

Proposition 4. $\mathrm{H}^\bullet(BQ(n)) \to \mathrm{H}^\bullet(BSQ(n))$ *has kernel* $x_1 + \dots + x_n$.

Proposition 5. $\mathrm{H}^\bullet(BSO(n)) \to \mathrm{H}^\bullet(BSQ(n))$ *is injective, and its image if the quotient* $S(x_1, \dots, x_n)/(x_1 + \dots + x_n)$.

Theorem 3. $\mathrm{H}^\bullet(V_{n,n-k})$ *(Stiefel variety, total space above Grassmanian) is generated by* $h_k, \dots, h_{k-1}$.

$$Sq^i h_j = \binom{j}{i} h_{i+j} \quad i + j \le n-1$$