THE UNREASONABLE EFFECTIVENESS OF NILPOTENCE IN STABLE HOMOTOPY THEORY

You Will Care About Morava K-theory

MIT/Harvard Babytop Seminar, Spring 2025

Tuesday, March 11th, 2025

LiveTEX by Howard Beck

Abstract. **Speaker**: Natalie Stewart (Harvard)

It's a classical result due to Nishida that $\mathrm{H}\mathbb{F}_p$ detects nilpotence of simple p-torsion elements in the homotopy groups of a ring spectrum – as a corollary, one finds that all elements of $\pi_*\mathbb{S}$ are nilpotent. In this talk, we'll sketch Devanitz-Hopkins-Smith's more advanced proof of this fact: MU detects arbitrary nilpotence. We'll also discuss various corollaries in stable homotopy theory.

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

References

- Douglas C Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106(2):351–414, 1984
- Ethan S Devinatz, Michael J Hopkins, and Jeffrey H Smith. Nilpotence and stable homotopy theory i. *Annals of Mathematics*, 128(2):207–241, 1988
- Michael J Hopkins and Jeffrey H Smith. Nilpotence and stable homotopy theory ii. *Annals of mathematics*, 148(1):1–49, 1998

0. Facts and Notations

- $MU = \mathrm{Th}(BU)$
- BP := indecomposable component of MU
  $\pi_*(\mathrm{BP}) \simeq \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, $|v_i| = 2\left(p^i - 1\right)$
- $E(n)_* := \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, kill of higher v_is and invert the top degree. $E(n)_*$ are the topological lifts of the moduli stack of p-typical formal groups. These are spectra whose Bousfield classes tell you about support in these strata.
- $P(n+1)_* := \mathbb{F}_p[v_{n+1}, v_{n+2}, \ldots]$ – these correspond to higher strata.
- $K(n)_* := \mathbb{F}_p[v_n^{\pm 1}]$ – these correspond to locally closed strata. This is constructed by killing all other v_is and inverting v_n.

All of these are ring spectra. By $\mathrm{Sp}_{(p)}^{\omega}$ we will mean compact p-local spectra. By compact, we will mean in the categorical sense, which will also mean (perfect) finite in the sense of their CW structure.

Remark 0.1 (Andy Senger)**.** Ps and Ks depend on your choice of v_is. We will ignore this. ◃

0.1. Ravenel's Correct Conjectures.

Nilpotence

a. BP detects nilpotence (we will implicitly assume p-locality – if not, put in MU)
b. $X \to Y \to Z \xrightarrow{f} \Sigma X$ such that $\mathrm{BP} \otimes f \simeq 0$, then $\langle Y \rangle = \langle X \rangle$
c. $x \in \langle K(n) \rangle^c \cap \langle K(n-1) \rangle^{\omega}$ is equivalent to X has "type n". That is, X has a "v_n-self map".

Realizability

There exists finite type n spectra.

Class invariance and boolean algebra

The lattice of Bousfield of finite p-local spectra is:

$$0 \subset \ldots \subset \langle K(n) \rangle^{\omega} \subset \langle K(n-1) \rangle^{\omega} \subset \ldots \subset \langle K(0) \rangle^{\omega} \subset \mathsf{Sp}_{(p)}^{\omega}$$

Definition 0.2. The **Bousfield class** of X is given by $\langle X \rangle = \{E\text{-acyclic objects (such as spectra)}\}$. Two spectra are **Bousfield equivalent** if they have the same Bousfield class. ◁

Remark 0.3. The Bousfield class measures how much your cohomology theory can see. ◁

1. Pop Quiz

Exercise 1.1. Prove or disprove: $\pi_n\left(S^k\right)$ is torsion when $n \neq k$. ◁

Answer 1.2. The Hopf fibration $\eta \in \pi_3\left(S^2\right)$ generates $\pi_3\left(S^2\right) \simeq \mathbb{Z}$. ◁

Theorem 1.3 (Serre '53)**.** *The rank*

$$\pi_k(S^n) = \begin{cases} 1 & n = k \text{ or } k \text{ even and } n = 2k-1 \\ 0 & \text{otherwise} \end{cases}$$

Proof idea. Apply the Serre spectral sequence to the path space fibration to compute rational homotopy of Eilenberg-Maclane spaces.

$$\begin{array}{ccccc} \Omega K(\mathbb{Z},k) & \longrightarrow & \mathrm{Map}_*(I, K(\mathbb{Z},k)) & \longrightarrow & K(\mathbb{Z},k) \\ \| & & \big\downarrow \sim & & \| \\ K(\mathbb{Z},k+1) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z},k) \end{array}$$

We start at $K(\mathbb{Z},1)$, and by chasing differentials around:

$$\mathrm{H}^*(K(\mathbb{Z},k);\mathbb{Q}) \simeq \begin{cases} \Lambda[i_k] & k \text{ odd} \\ \mathbb{Q}[i_k] & k \text{ even} \end{cases}$$

where i_k is a generator in degree k.

When k is odd, $S^k \to K(\mathbb{Z},k)$ is a $\mathbb{Q}$-equivalence. When k is even, $F \to S^k \to K(\mathbb{Z},k)$ has the attaching map be null. By rotating it once, we get the fiber sequence:

$$K(\mathbb{Z},k-1)_{\mathbb{Q}} \xrightarrow{0} K(\mathbb{Z},2k-1)_{\mathbb{Q}} \to S^k_{\mathbb{Q}}$$

□

The additive structure cannot then give us too much.

Corollary 1.4. *The stable homotopy groups $\pi_{\geq 1}\mathbb{S}$ are torsion.*

How far can we go by multiplication? For example, we can ask if η is nilpotent. In fact, the strongest possible thing is true:

Theorem 1.5 (Nishida)**.** *$\pi_{\geq 1}\mathbb{S}$ is nilpotent.*

Proof. This is inspired by Jeremy Hahn's notes from a previous Juvitop: `https://math.mit.edu/juvitop/pastseminars/notes_2016_Fall/Nishida.pdf`.

We will want $\mathrm{H}\mathbb{F}_p$ to "detect nilpotence" among simple p-torsion elements in π_*R for a ring spectrum. That is, we have a unit map:

$$\mathbb{S} \xrightarrow{\eta} \mathrm{H}\mathbb{F}_p$$

which induces a Hurewicz map:

$$\pi_*(R) \xrightarrow{\eta_*} \mathrm{H}_*\left(R;\mathbb{F}_p\right)$$

that is a ring map. If x is in the kernel of this map has simple p-torsion $px = 0$, then x is nilpotent. This is proven using power operations on $\mathrm{H}\mathbb{F}_p$. We might need power operations on R, so we may need to require it to have an H_∞ structure. □

How do we get rid of the simple p-torsion requirement? This doesn't happen in general, $\mathbb{S}$ is special.

2. Nilpotence

Definition 2.1. A commutative ring spectrum $E \in \mathsf{CAlg}\big(\mathsf{HoSp}_{(p)}\big)$ **detects nilpotence** if $\forall R \in \mathrm{Alg}\big(\mathsf{HoSp}_{(p)}\big)$, $\ker(\pi_* R \to E_* R)$ consists of nilpotent elements. We will not require R to be commutative – it will be important to allow tensor algebras, for example.

E is **cool** if it detects smash nilpotence: if for all maps $f : F \to Y$ for F finite, if we have:

$$E \otimes f : E \otimes F \to E \otimes Y \text{ is nullhomotopic}$$

$$\Downarrow$$

$$f^{\otimes n} : F^{\otimes n} \to Y^{\otimes n} \text{ is nullhomotopic for large } n$$

◃

Remark 2.2. The hard thing in this implication is the word "ring". Y does not need to be a ring, so we have to get a ring structure out of nowhere that lets us say something about maps $F \to Y$. ◃

Proposition 2.3 (DHS = Devinatz-Hopkins-Smith)**.**

a. Cool things detect nilpotence.
b. If $E_* R$ is always torsion free, and E detects nilpotence, then

Proof.

a. Set $F = \Sigma^k \mathbb{S}$.
b. For $f : F \to Y$, let

$$T = \bigoplus_{n \in \mathbb{N}} (\mathbb{D}F \otimes Y)^{\otimes n}$$

where $\mathbb{D}E$ is the $\mathbb{S}$-linear dual – the Spanier-Whitehead dual $\mathbb{D}E = \mathrm{Map}(E, \mathbb{S})$.

$$(f^{\otimes} : \mathbb{S} \to T) \longrightarrow (\mathbb{D}F \otimes Y)^{\otimes n} \quad \text{(Mate } f)^{\otimes n}$$

so that $f^{\otimes n} \sim 0$ is equivalent to $f^{\otimes}$ is nilpotent.

$$f^{\otimes} \text{ is nilpotent}$$

$$\Uparrow$$

$$f^{\otimes} \in \ker(\pi_* T \to E_* T)$$

$$\Uparrow$$

$$E \otimes f \simeq 0$$

□

Theorem 2.4 (DHS)**.** BP *detects nilpotence of p-local finite spectra.*

Remark 2.5. p-localized MU is a bunch of copies of shifted BPs. ◃

Corollary 2.6 (DHS)**.** *Let Y be a finite p-local spectra, $f \in \mathrm{End}_*(Y)$ be a graded self-map. That is, f is some map $Y \to \Sigma^k Y$. $\mathrm{BP} \otimes f \simeq 0$ is equivalent to $f^{-1} Y = 0$, by which we mean:*

$$f^{-1} Y := \mathrm{colim}\left(Y \xrightarrow{f} \Sigma^k Y \xrightarrow{\Sigma^k f} \Sigma^{2k} Y \to \cdots \right)$$

This is the localization to invert f.

Proof Idea. We will start with this claim: if X is a p-local spectrum such that $H^*\left(X;\mathbb{Z}_{(p)}\right)$ is finite-dimensional and torsion-free, then $\langle X\rangle = \{0\} = \langle S\rangle$. To see this, by shifting we will assume without loss of generality that $H_*\left(X;\mathbb{Z}_{(p)}\right) \simeq 0$. Then, we have $H_0\left(X;\mathbb{Z}_{(p)}\right) \simeq \mathrm{BP}_0(X)$. We choose an element of $\delta \in \pi_0(X)$ that is non-zero in $H_0\left(X;\mathbb{F}_p\right)$. Then, we have a fiber sequence:

$$\mathbb{S} \xrightarrow{\delta} \mathbb{S} \to X$$

We get that $\mathrm{BP} \otimes \delta \sim 0$ if and only if $\mathrm{H}\mathbb{Z}_{(p)} \otimes \delta \sim 0$. If we choose things correctly, we can choose δ to be tensor-null. We find then that every X-acyclic has to be 0.

Once that is established, we need to construct X satisfying $f^{-1}Y \otimes X = 0$. We can find this with a vanishing line in the BP-based Adams spectral sequence (ANSS) that has arbitrarily small slope ε. The telescope acts on the ANSS by sending things to zero faster than ε, after tensoring with X. □

Upshot 2.7. BP detects nilpotence in 3 different ways. It does so via smash nilpotence, on homotopy groups of ring spectra, and on endomorphism rings. ◁

Definition 2.8. Let $\mathrm{supp}(X) = \{n \text{ such that } K(n) \otimes X \neq 0\}$. ◁

Proposition 2.9. *If X is finite, then there a* $\mathrm{type}(X) \in [0,\infty]$ *such that:*

$$\mathrm{supp}(X) = [\mathrm{type}(X),\infty]$$

Theorem 2.10 (Hopkins-Smith '98)**.** *The fact that* BP *detects nilpotence implies that* $\bigoplus_n K(n)$ *detects nilpotence.*

Remark 2.11. The claim is that the Bousfield class $\langle \mathrm{BP}\rangle = \langle K(0)\rangle \cup \ldots \cup \langle K(n)\rangle \cup \langle P(n+1)\rangle$. There is a compactness argument where you can check acylicity against finitely many of these. ◁

Theorem 2.12 (stronger result, also HS)**.** *Knowing that* BP *detects nilpotence (we haven't proved it yet), then we have that E detects nilpotence for arbitrary E if and only if E is fully supported: $E \otimes K(n) \neq 0$ for all n.*

3. Field Theory

Definition 3.1. $E \in \mathrm{Alg}(\mathsf{HoSp})$ is a **field** if E_* is a graded field. That is, all the homogenous elements of $\pi_*(E)$ are invertible. ◁

Proposition 3.2.
1. *Free E-modules for a field E in spectra are free in graded sets.*
2. *That is, as an E_*-module, they split into copies of shifted E_*s.*
3. *Retracts of free E-modules are free*
4. *We have a Kunneth isomorphism, $E_*(X) \underset{E_*}{\otimes} E_*(Y) \simeq E_*(X \times Y)$.*

Proposition 3.3. *If E is a field then $\exists h$ such that E is a free $K(h)$-module.*

Proof. $K(h)$ jointly detect nilpotence, so that the non-nilpotent map that is the unit is not zero. Then,

$$\bigoplus_j \Sigma^{n_j} K(n) \simeq E \otimes K(n) \simeq \bigoplus_i \Sigma^{m_i} E \leftrightarrows E$$

The proposition before gives us that $E \simeq \bigoplus \Sigma^{p_k} K(h)$, so we are done. □

Proof of Stronger Theorem. If E detects nilpotence, then if $x \otimes E \simeq 0$, then $X \otimes E \otimes K(n) \simeq 0$ for all n. Then $X \otimes K(n) = 0$ for all n, so X is nilpotent. This is one direction, the other isn't too hard. □

Upshot 3.4. $K(n)$ are prime fields, they are positive height versions of $\mathbb{F}_p$ or $\mathbb{Q}$. We can study spectra by studying heights at each prime. They also completely classify nilpotence, and it is relatively easy to compute $K(n)$ homology due to its field properties. ◁

4. Thickness

Definition 4.1. We call a subcategory $\mathcal{C} \subset \mathsf{Sp}_{(p)}^{\omega}$ **thick** if it is closed under:

- fibers
- cofibers
- extensions

◃

Theorem 4.2 (Hopkins-Smith). *The lattice of thick subcategories of* $\mathsf{Sp}_{(p)}^{\omega}$ *is:*

$$0 \subset \ldots \subset \langle K(n) \rangle^{\omega} \subset \langle K(n-1) \rangle^{\omega} \subset \ldots \subset \langle K(0) \rangle^{\omega} \subset \mathsf{Sp}_{(p)}^{\omega}$$

Proof. Mitchell showed these subcategories are different from each other. If we assume $\mathrm{type}(Y) \geq \mathrm{type}(X)$, then Y is in the thick subcategory generated by X.

Take the coevaluation map and its fiber:

$$\begin{array}{ccccc} F & \xrightarrow{f} & \mathbb{S} & \xrightarrow{\mathrm{coev}} & \mathbb{D}X \otimes X \\ & & \downarrow & & \\ F^{\otimes n} & \longrightarrow & \mathbb{S} & \longrightarrow & C_{(n)} \end{array}$$

We know that $C_{(1)}$ is in the thick subcategory generated by X, so that $Y \otimes C_{(1)}$ also is. We claim that we have a cofiber sequence

$$Y \otimes C_{(n)} \to C_{(n)} \to Y^{\otimes n} \otimes C_{(1)}$$

Inductively, $C_{(n)}$ and $C_{(n+1)}$ are in the thick subcategory generated by X. Due to type considerations, $f \otimes K(n) \otimes Y = 0$ for all n – if not for $K(n)$, then for Y. Then, $f \otimes Y$ is tensor-nilpotent. Thus, for n large, the cofiber of the tensor power has:

$$\mathrm{cofib}\left(f^{\otimes n} \otimes Y\right) = Y \oplus \Sigma\left(Y \otimes F^{\otimes n}\right)$$

This is proven with homological algebra.

□

5. Sketch of Nilpotence

Define $X(n) := \mathrm{Th}(\Omega SU(n) \to \Omega \mathrm{SU} \simeq BU)$. Observe that $X(0) \simeq \mathbb{S}$, and we have maps:

$$X(0) \to X(1) \to X(2) \to \cdots$$

Further, the colimit $\mathrm{colim}(X(0) \to X(1) \to \cdots) \xrightarrow{\sim} MU$. You prove it going down, using a compactness argument.

We need a refinement of this for the inductive step.

Recall 5.1. There is an equivalence $\Omega\Sigma S^{2n} = \Omega S^{2n+1}$ and $\mathrm{colim}_{k \to \infty} \prod_{\ell < k} \left(S^{2n}\right)^{\ell}$. ◃

We get a filtration $\mathrm{Fil}^k \Omega S^{2n+1}$. We then get a pullback

$$\begin{array}{ccccc} F'_k & \longrightarrow & \Omega SU(n+1) & \longrightarrow & BU \\ \downarrow & \lrcorner & \downarrow & & \\ \mathrm{Fil}^k \Omega S^{2n+1} & \longrightarrow & \Omega S^{2n+1} & & \end{array}$$

We have a filtration $\mathrm{Fil}^k X(n+1)$ as the Thom spectrum of the top. Observe that we have a filtration of $X(n+1)$ by $X(n)$-modules that is exhaustive and whose zero-th component is $X(n)$.

Step II. Show that $X(n+1)_* \alpha$ is nilpotent, so $\mathrm{Fil}^{p^k-1} X(n+1)_{(p)} \otimes \alpha^{-1} R = 0$ for large k.

Step III. Show $\left\langle \mathrm{Fil}^{p^k-1} X(n+1)_{(p)} \right\rangle = \left\langle X(n)_{(p)} \right\rangle$.

References

Ethan S Devinatz, Michael J Hopkins, and Jeffrey H Smith. Nilpotence and stable homotopy theory i. *Annals of Mathematics*, 128(2):207–241, 1988.

Michael J Hopkins and Jeffrey H Smith. Nilpotence and stable homotopy theory ii. *Annals of mathematics*, 148(1):1–49, 1998.

Douglas C Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106(2):351–414, 1984.