STACK TO THE FUTURE

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ABSTRACT. **Speaker**: Tyler Lane (Harvard)

As the title suggests, this talk is all about the moduli stack of formal groups. I'll begin by presenting the moduli stack of formal groups as a quotient stack. Then I'll discuss the height filtration and some of its basic properties. One of my main goals is to present some theorems about quasi-coherent sheaves on the stack of formal group which will serve as inspiration for some corresponding results in chromatic homotopy theory. Finally I will discuss the Landweber exact functor theorem.

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

1. Formal Group Laws

Reminder 1.1. Last time, we saw that the functor:

$$\begin{aligned}\mathrm{Fgl} : \mathsf{Aff} &\to \mathsf{Set}\\ R &\mapsto \{\text{formal group laws over } R\}\end{aligned}$$

is representable by the Lazard ring. ◁

Question 1.2. Consider the category fibered in groupoids $\mathsf{FGL} \to \mathsf{Aff}$ whose objects are pairs (S, F) (ring, formal group law over S) and whose morphisms $(S, F) \to (S', F')$ are pairs (f, ϕ) where $S' \xrightarrow{f} S$ is a ring map an $\phi : F \to f^*F'$ is an isomorphism of formal group laws. <u>Is this a stack?</u> ◁

Answer 1.3. No, descent fails. We will see two fixes ◁

Construction 1.4. Let $F \in R[\![x]\!]$ be a formal group law. We define a functor:

$$\begin{aligned}S_F : \mathsf{Aff}_{/R} &\to \mathsf{Ab}\\ S &\mapsto \mathrm{Nil}(S)\end{aligned}$$

where the group law on $\mathrm{Nil}(S)$ is given by $x + y = F(x, y)$. ◁

Note 1.5. This is an fpqc sheaf. ◁

Definition 1.6. We say that a sheaf of abelian groups $\mathcal{G}$ on $(\mathsf{Aff}_{/R})_{\mathrm{fpqc}}$ is a **formal group law** if $\mathcal{G} \simeq \mathcal{G}_F$ for some formal group law $F(x, y) \in R[\![x, y]\!]$.

We say that $\mathcal{G}$ is a **formal group** if it is fpqc-locally isomorphic to a formal group law, and we say that a formal group $\mathcal{G}$ is **coordinatizable** if $\exists F$ such that $\mathcal{G} \simeq \mathcal{G}_F$. ◁

Remark 1.7. This all works if you replace fpqc with the Zariski topology (Lurie does this). An fpqc-sheaf is a Zariski sheaf. Being Zariski-locally a formal group law is a priori stronger, but we will see that these are equivalent. ◁

Remark 1.8. There are a lot of definitions of formal groups. This definition will be exactly what we need to turn the category from (Question 1.2) into a sheaf. ◁

Definition 1.9 (Fix 1)**.** The **moduli stack of formal groups** $\mathcal{M}_{\mathrm{fg}}$ is the category fibered in groupoids, whose objects are pairs $(S, \mathcal{G})$ (ring, formal group) and whose morphisms "are isomorphisms." ◁

Remark 1.10. This will be the stackification of the category from (Question 1.2) from earlier. ◁

Definition 1.11 (Fix 2)**.** $\mathcal{M}_{\mathrm{fg}}^s$ is the category fibered in groupoids whose objects are formal group laws and whose morphisms are strict isomorphisms. ◁

Remark 1.12. The stackification of $\mathcal{M}_{\mathrm{fg}}^s$ is *not* $\mathcal{M}_{\mathrm{fg}}$, these are two different stacks. ◁

Theorem 1.13. *$\mathcal{M}_{\mathrm{fg}}$ and $\mathcal{M}_{\mathrm{fg}}^s$ are stacks in the* fpqc *topology. The natural map $\mathcal{M}_{\mathrm{fg}}^s \to \mathcal{M}_{\mathrm{fg}}$ is a $\mathbb{G}_m$-bundle. $\mathcal{M}_{\mathrm{fg}}$ are $\mathcal{M}_{\mathrm{fg}}^s$ are both quotient stacks.*

Remark 1.14. There are not algebraic stacks, we are quotienting out by a group scheme that is not locally in finite presentation. ◁

Let G^+ be the group scheme whose group of R-points is

$$\{g \in R[\![t]\!] \mid g(t) = b_0 t + b_1 t^2 + \ldots, b_0 \in R^\times\}$$

and:

$$(f \cdot g)(t) = f(g(t))$$

Then, G^+ acts on $\operatorname{Spec} L$

$$(g \in G(R), f \in R[\![y]\!]) \mapsto \Big(g\Big(f\Big(g^{-1}(x), g^{-1}(y)\Big)\Big)\Big)$$

We can view $\mathbb{G}_m$ as the subgroup of G^+ consisting of those power series such that $b_i = 0$ for $i > 0$.

Let G be the subgroup of those power series such that $b_0 = 1$.

Note 1.15. G^+ is the semidirect product of $\mathbb{G}_m$, G. ◁

Theorem 1.16. $\mathcal{M}_{\mathrm{fg}} \simeq [\mathsf{FGL}/G^+]$

Proof. We can construct a map $\mathcal{M}_{\mathrm{fg}} \to [\mathsf{FGL}/G^+]$. Let S/R be a formal group. We can define an fpqc sheaf:

$$\mathrm{Coord}_{\mathcal{G}}(S) = \{F \in \mathsf{FGL}(S) : S_F \simeq S\}$$

G^+ acts on $\mathrm{Coord}_{\mathcal{G}}$ making the inclusion map FGL-equivariant. This section makes $\mathrm{Coord}_{\mathcal{G}}$ into a G^1-torsor in the fpqc topology, so it is a scheme.

Now, we have a diagram

$$\begin{array}{ccc} \mathrm{Coord}_{\mathcal{G}} & \longrightarrow & \mathsf{FGL} \\ \downarrow & & \\ \operatorname{Spec} R & & \end{array}$$

We want a morphism in the other direction: $[\mathsf{FGL}/G^+] \to \mathcal{M}_{\mathrm{fg}}$. Let P/R be a G^+-torsor with an equivariant map to FGL (i.e., a point of $[\mathsf{FGL}/G^+](R)$).

Let $f : U \to \operatorname{Spec}(R) := S$ be an fpqc morphism trivializing P. Then, the fiber $P(U,f)$ of $P(U) \to R$ over f is a free $G^+(U)$-set.

We now get:

$$\begin{array}{ccccc} P(U,f) & \longrightarrow & \mathsf{FGL}(U) & & \\ \downarrow & & \downarrow & & \\ \frac{P(U,f)}{G^+(U)} = * & \longrightarrow & \mathbb{F}_G(U) & = & \{\text{set of formal groups}\} \end{array}$$

The right arrow kills the G^+-action, so it factors through the bottom-left. This specifies a formal group G_f over U, so G_f is a formal group law. The fiber of the right arrow over G_f is the free $G^1(U)$-set $\mathrm{Coord}_{G_f}(U)$.

The top arrow defines a $G^+(U)$-equivariant morphism $P(U,f) \to \mathrm{Coord}_{G_f}(U)$. They both have transitive G-actions, so it is an isomorphism.

We can use descent, glue results together, and get the desired results. □

Note 1.17. Does anyone want to see descent?
(No answer)
Exactly. ◁

Theorem 1.18. $[\mathcal{M}_{\mathrm{fg}}^s] = [\mathsf{FGL}/G]$

Definition 1.19. Let $\mathcal{G}$ be a formal group law over R. The Lie algebra of $\mathcal{G}$ is:

$$\mathfrak{g} = \ker\Big(\mathcal{G}\Big(R[t]/t^2\Big) \to \mathcal{G}(R)\Big)$$

(1) $\mathfrak{g}$ is an R-module. Let $\lambda \in R$. λ acts on $R[t]/t^2$ by $t \mapsto \lambda t$. This gives the action on $\mathfrak{g}$.
(2) Suppose $\mathcal{G} = \mathcal{G}_F$. $\mathcal{G}\Big(R[t]/t^2\Big) = \{\lambda t : \lambda \in R\}$. So when $\mathcal{G} \simeq \mathcal{G}_F$, $\mathfrak{g} \simeq R$. That is to say, $\mathfrak{g}$ is an invertible R-module which is trivial if $\mathcal{G}$ is coordinatizable.

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Remark 1.20. There may be other nilpotents in R, but taking the kernel gives only those that are multiples of t. ◃

Theorem 1.21. *A formal group is coordinatizable if and only if its Lie algebra is trivial.*

Sending a formal group to its Lie algebra defines an invertible sheaf ω^{-1} on $\mathcal{M}_{\text{fg}}$. It is the inverse of a line bundle.

Note 1.22. $\mathcal{M}^s_{\text{fg}} \to \mathcal{M}_{\text{fg}}$ is the $\mathbb{G}_m$-bundle associated to the line bundle ω. A point in $\mathcal{M}^s_{\text{fg}}$ is a formal group law which has a trivialization of the Lie algebra. The isomorphisms preserve the trivializations. ◃

Remark 1.23. This is why it doesn't matter if we use the Zariski topology. This happens for line bundles.

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2. Height

Theorem 2.1. *Let R be a $\mathbb{Q}$-algebra. Any formal group laws over R are isomorphic (namely, to the additive one). In fact, $\mathcal{M}_{\text{fg}} \otimes \operatorname{Spec}\mathbb{Q} \simeq \mathrm{B}\,\mathbb{G}_m$.*

The multiplicative formal group law $x + y + xy$ is isomorphic, over $\mathbb{Q}$, to the additive formal group law via the power series:

$$g(t) = t + \frac{t^2}{2} + \frac{t^3}{6} + \dots$$

This isomorphism doesn't work in positive characteristic.

Motivation: We want an invariant to tell formal group laws in positive characteristic apart. One such invariant is **height**.

Definition 2.2. Let $F(x,y) \in R[\![x,y]\!]$ be a formal group law. For each $n \geq 0$, define its n-**series** $[n](t) \in R[\![t]\!]$ as:

$$[n](t) = \begin{cases} 0 & n = 0 \\ F([n-1]t, t) & \text{otherwise} \end{cases}$$

In other words, F-add t to itself n times.

$$[n] = t +_F t +_F +_F \dots +_t$$

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Proposition 2.3. *Let $R \in$ CRing in which $p = 0$, and F be a formal group law over R. Either $[p](t) = 0$, or:*

$$[p](t) = \lambda t^{p^n} + O\Big(t^{p^n+1}\Big)$$

for some $\lambda \neq 0$ and for some n.

Definition 2.4. Let $R \in$ CRing, and fix a prime p, and F be a formal group law over R. Let v_n defnote the coefficient of t^{p^n} in $[p](t)$. We say F has height n if $v_n \neq 0 \bmod p$ and $v_i = 0 \bmod p$ for all $i < n$. ◃

Example 2.5. Here are two examples:

(1) $F(x,y) = x + y + xy$ has height 1. In this case, $[p](t) = (1+t)^p - 1 = t^p \mod p$.
(2) $F(x,y) = x + y$, then $[p](t) = 0$ when $p = 0$. The height is ∞.

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We will now look at the height filtration. We can extend the notion of height to arbitrary formal groups by saying:

$$\text{height}(\mathcal{G}) = \text{height}(\mathcal{G}_F)$$

where $\mathcal{G}\Big|_S = \mathcal{G}_F$ where $S \to \text{Spec}(R)$ is a trivializing fpqc cover.

Fix a prime p. For every n, let $v_n \in L$ be the coefficient of t^{p^n} in the p-series of the universal formal group law. over the Lazard ring. For each n, let $\mathcal{M}_{\text{fg}}^{\geq n} = [\text{Spec}(L/(p, v_1, \ldots, v_n))/G^+]$ be the moduli stack of formal group laws over height $\geq n$. We should check this is a G^+-invariant closed subscheme of $\text{Spec}\, L$. The v_ns in another ring R are pulled back from the universal ones.

$$\mathcal{M}_{\text{fg}} \geq \mathcal{M}_{\text{fg}}^{\geq 1} \geq \mathcal{M}_{\text{fg}}^{\geq 2} \geq \ldots$$

Theorem 2.6 (Thick Subcategory Theorem)**.** *We say a full abelian subcategory $\mathcal{C} \subset \text{QCoh}\Big(\mathcal{M}_{\text{fg}} \otimes \text{Spec}\,\mathbb{Z}_{(p)}\Big)$ is **thick** if $A \oplus B \in \mathcal{C}$ implies $A \in \mathcal{C}$ or $B \in \mathcal{C}$. The thick subcategories of $\text{QCoh}\Big(\mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}\Big)$ are*

(1) itself
(2) $\mathcal{C}_n = \{F : \text{supp}(F) \subset \mathcal{M}_{\text{fg}}^{\geq n}\}$
(3) It is the trivial subcategory (0).

There is a famous analog of this in homotopy theory.
Let X be a spectrum. There is a spectrum BP such that:

(1) p-localized MU, $\text{MU}_{(p)}$ is the sum of copies of BP.
(2) $\text{Spec}(\text{BP}_*)$ is the moduli space of p-typical formal group laws.
(3) $\mathcal{M}_{\text{fg}}^s \otimes \mathbb{Z}_{(p)}$ is the satck associated to the flat Hopf algebroid $(\text{BP}_*, \text{BP}_*\text{BP})$.

By $(-)_*$ we mean $\pi_*(-)$. p-localized means Bousfield localization at the Moore specturm $S\mathbb{Z}_{(p)}$.
For any spectrum, $X \to \text{BP}_{\text{even}}(X)$ is a quasi-coherent sheaf on $\mathcal{M}_{\text{fg}} \otimes \mathbb{Z}_{(p)}$.
We can look at the full subcategory of finite spectra such that the associated sheaf lives in $\mathcal{C}_n$.

Theorem 2.7. *These are all the thick triangulated subcategories of Sp^{ω}.*