COMPLEX ORIENTATIONS, FORMAL GROUP LAWS, AND MU

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LiveTEX by Howard Beck

Abstract. **Speaker**: Oakley Edens (Harvard)

In the previous talk, we saw that the behaviour of the complex cobordism spectrum MU controls homotopy groups of spheres via the Adams-Novikov spectral sequence. In this talk, I will further discuss how MU can be used to relate topology and algebra via formal group laws. This will lead to defining the moduli stack of formal groups, which will be a central focus of the remaining talks. Finally, via the theory of Hopf algebroids, we will show how one can associate, to any spectrum, certain quasi-coherent sheaves on the moduli stack of formal groups.

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

Notation 1. Let E be a commutative ring spectrum (E_2-algebra should be enough, but everything we'll do is $\mathbb{E}_\infty$). ◁

Definition 2. A **complex orientation** on E is a choice of class $x_E \in \widetilde{E}^2(\mathbb{C}P^\infty)$ such that the restriction map:

$$\widetilde{E}^2(\mathbb{C}P^\infty) \to \widetilde{E}\left(\mathbb{C}P^1\right) = \widetilde{E}^0\left(S^0\right) = \pi_0(E)$$

◁

These cohomology classes classify complex line bundles, which is why these are complex orientations.

Example 3. For Eilenberg-MacLane spectra HR, we have:

$$\widetilde{\mathrm{H}R}^2(\mathbb{C}P^\infty) = \widetilde{\mathrm{H}}^2(\mathbb{C}P^\infty, R) = R$$

The map $R \xrightarrow{\sim} R$ is an isomorphism, so we have one choice of orientation. ◁

Example 4. E is a spectrum with $\pi_* E$ even, then we have:

$$\Sigma^{\infty-2}\mathbb{C}P^1 \to E$$

The obstruction to lifting to $\mathbb{C}P^2$ lives in $\pi_3 E$, so we have a lift:

$$\Sigma^{\infty-2}\mathbb{C}P^\infty \to E$$

For example, for $E = \mathrm{KU}$ (complex K-theory), we have:

$$\mathrm{KU}^2(\mathbb{C}P^\infty) = \mathrm{KU}(\mathbb{C}P^\infty) = [O(1)] - 1$$

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Proposition 5. *If (E, x_E) is a complex-oriented ring spectrum, then:*

(1) $E^*(\mathbb{C}P^\infty) = E^*[\![x_E]\!]$
(2) $E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = E^*[\![1 \otimes x_E, x_E \otimes 1]\!]$

Remark 6. This is kind of a lie, it is not a power series ring. What we mean is:

$$E^*[\![x_E]\!] \cong \lim E^*[x_E]/\left(x_E^n\right)$$

in GrRing.

However, we will treat these objects as honest power series rings. ◁

Proof. We wil use the Atiyah-Hirzebruch Spectral Sequence.

$$E_2^{pq} = \mathrm{H}^p(\mathbb{C}P^\infty, E^q) = E^q \otimes y^{p/2}\mathbb{Z} \implies E^{p+q}(\mathbb{C}P^\infty)$$

The powers of $1 \otimes y$ generate the E_2 page, and y remains on the spectral sequence forever due to the fact it lives on by definition. [Proposition 5] □

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Reminder 7. $\mathbb{C}P^\infty = BU(1)$ is the classifying space for complex line bundles. The multiplication map:

$$m : B(U(1) \times U(1)) \simeq BU(1) \times BU(1) \to BU(1)$$

classify tensor products of line bundles. ◁

Therefore, for E a complex-oriented ring spectrum, we have, pulling back along m,

$$m^* E^*[[t]] \to E^*[[u, v]]$$

Tensor products of line bundles are commutative, so we get:

$$F = m^*(t) \implies F(u, v) = F(v, u)$$

The fact there is a unit means

$$F(0, u) = F(u, 0) = u$$

Associativity of the tensor product gives us:

$$F(u, F(v, w)) = F(F(u, v), w)$$

Definition 8. For R a ring, then a **formal group law** on R over R is a formal power series $F \in R[[x, y]]$ such that:

(1) $F(x, 0) = F(0, x) = x$
(2) $F(x, y) = F(y, x)$
(3) $F(x, F(y, z)) = F(F(x, y), z)$

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Example 9. Here are some examples of formal group laws:

(1) The complex orientation of E induces a formal group law
(2) $F = x + y$ (additive formal group law)
(3) $F = x + y + xy$ (multiplicative formal group law)

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Remark 10. The multiplicative formal group law never will come from topology, as we would require homogenous degrees of 0. ◁

Remark 11. The axioms of formal group laws give us some equations in coefficients, so we can form the **Lazard Ring**:

$$L = \mathbb{Z}[\{a_{i,j}\}_{i,j \geq 0}]/(\text{equations coming from axioms})$$

with the formal group law:

$$F = \sum_{i,j \geq 0} a_{i,j} x^i y^j$$

Then, the set of formal group laws over R is corepresented by L:

$$\mathsf{FGL}(R) = \{\text{formal group laws over } R\} = \mathrm{Hom}(L, R)$$

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We can give the Lazard ring a grading, by imagining $\deg x = \deg y = -2$ and F homogenous of degree -2. Then,

$$\deg\left(a_{i,j}\right) = 2(a + j - 1)$$

You can check this is consistent with the axioms: $F(x, F(y, z)) = F(F(x, y), z)$ have homogenous degree -2, so that the coefficient of $x^i y^j z^k$ is $2(i + j + k - 1)$. Therefore, the grading descends to a grading on L, which is non-negatively graded with $L_0 = \mathbb{Z}$.

Theorem 12 (Lazard)**.** *There is an isomorphism of graded rings:*

$$L \to \mathbb{Z}[t_1, t_2, \dots]$$

with $\deg(t_i) = 2i$.

Proof idea. One way to get any formal group law is to take a power series h. If we take the additive formal group law, another one is given by:

$$h^{-1}(h(x) + h(y))$$

In fact, this is an isomorphism of formal group laws over $\mathbb{Q}$. We need to show it is an isomorphism over $\mathbb{Z}$ for L. Therefore, we will need some good choice of coordinates. □

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Definition 13. Let X be a topological space and

$$\xi : V \to X$$

be a real vector bundle that admits a Riemannian metric. Then, we have two fiber bundles associated to it:

(1) $\mathbb{S}(\xi) = \{x \in V \mid \|x\| = 1\}$
(2) $\mathbb{D}(\xi) = \{x \in V \mid \|x\| \leq 1\}$

We define the **Thom space** of ξ as:

$$\mathrm{Th}(\xi) = \mathbb{D}(\xi)/\mathbb{S}(\xi)$$

In fact, this is well-defined and functorial. ◃

Proposition 14. *If we have two vector bundles* $\xi : V \to X$ *and* $\eta : W \to Y$*, then we can take:*

$$\mathrm{Th}(\xi \times \eta) = \mathrm{Th}(\xi) \wedge \mathrm{Th}(\eta)$$

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For each $n \geq 0$, let $\xi^n : E(n) \to BU(n)$ denote the universal complex vector bundle of rank n.
Let $MU(n) = \mathrm{Th}(\xi^n)$. We want a suspension spectrum with spaces:

$$MU = \{MU(0), \Sigma MU(0), MU(1), \Sigma MU(1), \dots\}$$

The even maps are obvious, for odd maps we need:

$$\Sigma^2 MU(n-1) \to MU(n)$$

Using the proposition, we can take add a factor of $\mathbb{C}$ and have:

$$\begin{array}{ccc} E(n-1) \oplus \mathbb{C} & \longrightarrow & E(n) \\ {\scriptstyle \xi^{n-1} \oplus 1_{\mathbb{C}}} \big\downarrow & & {\scriptstyle \xi^n} \big\downarrow \\ BU(n-1) & \longrightarrow & BU(n) \end{array}$$

(the bottom map is induced by the inclusion of $\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$). After taking Thom spaces, we get a map:

$$\begin{array}{ccc} \mathrm{Th}\left(\xi^{n-1}\right) \wedge \mathrm{Th}(1_{\mathbb{C}}) & \longrightarrow & \mathrm{Th}(\xi^n) = MU(n) \\ \Vert & & \\ MU(n-1) \wedge S^2 & & \\ \Vert & & \\ \Sigma^2 MU(n-1) & & \end{array}$$

Therefore, we get a spectrum MU. We can check that the map $BU(n) \times BU(m) \to BU(n+m)$ induces a map:

$$\mathrm{MU} \times \mathrm{MU} \to \mathrm{MU}$$

Further, we have $BU(0) = *$, so that $MU(0) = S^0$, so we get a unit map:

$$\mathbb{S} \to \mathrm{MU}$$

Therefore, MU is a commutative ring spectrum (in fact, $\mathbb{E}_\infty$-ring).

We will see that MU gives complex orientations.

We have a unit section $BU(1) \to MU(1)$, and we have the associated bundle construction:

$$\mathbb{S}(\xi^n) = EU(n) \underset{U(n)}{\times} S^{2n-1}$$

Specifically,

$$\mathbb{S}\left(\xi^1\right) = EU(1) \underset{U(1)}{\times} S^1 = EU(1) \simeq *$$

Therefore, we have $\mathbb{C}P^\infty \simeq BU(1) \simeq MU(1)$, and we have a map:

$$\mathbb{C}P^\infty \to MU(1) = MU_2$$

which gives a complex orientation:

$$x_{\mathrm{MU}} \in \widetilde{\mathrm{MU}}^2(\mathbb{C}P^\infty)$$

Denote $F_{\mathrm{MU}} \in \mathrm{MU}^*[\![x,y]\!]$ associated to the induced formal group law. We get an induced map:

$$L \to \mathrm{MU}^*$$

However, the grading is wrong since MU^* is cohomologically graded. As ungraded rings, it is the same as MU_*, so we have a map:

$$L \to \mathrm{MU}_*$$

Theorem 15 (Quillen)**.** *The map $L \to \mathrm{MU}_* \simeq \pi_*\mathrm{MU}$ is a graded isomorphism.* ◃

By a computation of Milnor, there are generators $\mathbb{Z}[t_1, t_2, \dots]$ that have $\deg(t_i) = 2i$.

Theorem 16 (Adams)**.** MU *completely determines complex orientations:*

$$\{\phi : \mathrm{MU} \to E \text{ map of commutative ring spectra}\} \to \{\text{complex orientations on } E\}$$
$$\phi \mapsto \pi_* x_{\mathrm{MU}}$$

where $\phi_* : \widetilde{\mathrm{MU}}^2(\mathbb{C}P^\infty) \to \widetilde{E}^2(\mathbb{C}P^\infty)$. This map is a bijection. ◃

Definition 17. A **homomorphism of formal group laws** $h : F \to G$ over R is a power series $h \in R[\![t]\!]$ with:

$$h(F(x,y)) = G(h(x), h(y)) \text{ such that } h(0) = 0$$

It is a **strict isomorphism** is such a map such that $h'(0) = 1$. ◃

Remark 18 (Obvious)**.** A homomorphism is invertible if and only if the coefficient of t is invertible. ◃

If $h : F \to G$ is a (strict) isomorphism, then G is completly determined by being:

$$G = h\left(F\left(h^{-1}(x), h^{-1}(y)\right)\right)$$

Specifying h from $F \to G$ is equivalent to specifying a formal group, so a map $L \to R$, and a map

$$B^+ = \mathbb{Z}[b_0^{\pm 1}, b_1, b_2, \dots] \to R \text{ such that } h = \sum_{i \geq 0} h_i t^{i+1}$$

For a strict isomorphism, the corepresenting object is $B = \mathbb{Z}[b_1, \dots]$. This is all the same as a map:

$$LB^{(+)} \to R \qquad LB^{(+)} = L \otimes B^{(+)}$$

That is,

$$\mathrm{mor}\left(\mathsf{FGL}_{(\mathrm{st})}/\simeq\right) = \mathrm{Hom}\left(LB^{(+)}, -\right)$$

The source is given by:

$$s : L \to LB$$

(inclusion, representing) source formal group law.

For the target map,

$$t : L \to LB$$

There is a formal group law on $LB[\![x,y]\!]$ given by $h\left(F\left(h^{-1}(x), h^{-1}(y)\right)\right)$, corepresenting t.

There is an identity:

$$1 : LB \to L$$

that is like projection.

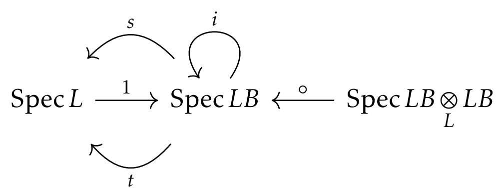
We will also want inverses. We have $h^{-1}(t) \in B[\![t]\!]$ which is corepresented by a map $i : B \to B$ which, by tensoring, gives a map:

$$i : LB \to LB$$

Also, $h(h'(t)) \in (B \otimes B)[\![t]\!]$ which is corepresneted by a map $B \to B \otimes B$. This gives a map:

$$\circ : LB \to LB \underset{L}{\otimes} LB$$

Taking Spec everywhere:



Definition 19. A **Hopf algebroid** (A, B) is a groupoid object in affine schemes where A are the objects, and B are morphisms. The associated groupoid object is denoated as $\mathrm{Spec}(A, B)$. ◃

Example 20. Here are some examples:

(1) (L, LB) is a Hopf algebroid
(2) For E a ring spectrum, then: (E_*, E_*E) where $E_*E = \pi_*(E \otimes E)$. If E is a **flat** ring spectrum, then this is a Hopf algebroid.

$$s : E = \mathbb{S} \otimes E \to E \otimes E$$

$$t : E = E \otimes \mathbb{S} \to E \otimes E$$

flat here means $\pi_* s$ is flat.
(3) (case of above), $E = \mathrm{MU}$.

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We want to show that B for MU is in fact LB.

Remark 21. Let E be a complex-oriented ring spectrum. Then, $E \otimes \mathrm{MU}$ has two complex orientations given by x_E and x_{MU}.

Fact 22. *There exists a strict isomorphism h in $(E \otimes \mathrm{MU})_*[\![t]\!] = E_*\mathrm{MU}[\![t]\!]$ such that $x_{\mathrm{MU}} = h(x_E)$.* ◃

If we take:

$$h = 1 + \sum_{i \geq 1} b_i t^{i+1}$$

Therefore, we get a map:

$$E_* \otimes B \to E_*\mathrm{MU}$$

For MU, we get a map:

$$LB \to \mathrm{MU}_*\mathrm{MU}$$

One can check with the Atiyah-Hirzebruch Spectral Sequence that this is an isomorphsim. ◃

Theorem 23. *The map $(L, LB) \to (\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ is an isomorphism of Hopf algebroids.*

Proof idea. The source has to be the same. For the target, you pick up a formal group law, with the first component giving you the source and the second is the target. □

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Definition 24. Let (A,B) be a Hopf algebroid. Then a **comodule** is an A-module M with a map (a coaction) $M \to B \underset{A}{\otimes} M$. It must have a unit coming from the unit map in the Hopf algebroid. The action should be compatible with the composition. Explicitly,

$$\begin{array}{ccc} M & \xrightarrow{\rho} & B \underset{A}{\otimes} M \\ \big\downarrow & & \big\downarrow \ell \\ B \underset{A}{\otimes} M & \xrightarrow[m]{} & B \underset{A}{\otimes} B \underset{A}{\otimes} M \end{array}$$

where we tensor along the source. $\triangleleft$

Proposition 25. *If B is a flat A-module (over the source map), then $\mathsf{CoMod}_{(A,B)}$ is an abelian category and has enough injectives.* $\triangleleft$

Proposition 26. *If (A,B) is a Hopf algebroid then we have a prestack $\mathcal{M}_{(A,B)}$ given by s points of the groupoid object in affine schemes. Also, $\mathrm{QCoh}\big(\mathcal{M}_{(A,B)}\big) = \mathsf{CoMod}_{(A,B)}$.* $\triangleleft$

By this construction, if X is any spectrum and E is a flat ring spectrum, we have a quasi-coherent sheaf:

$$\mathcal{F}_X \in \mathrm{QCoh}\big(\mathcal{M}_{(A,B)}\big)$$

and E_*X carries a comodule structure.

Definition 27. A **formal group** (commutative, one-dimensional) over R is a Zariski sheaf G on $\operatorname{Spec} R$ such that locally on $\operatorname{Spec} R$, $G = G_F$ for a formal group law F over R. Here, G_F is the Zariski sheaf given by $\operatorname{Spf} R[\![t]\!]$ with group multiplication given by $(a,b) \to F(a,b)$. $\triangleleft$

Definition 28. $\mathcal{M}_{\mathrm{fg}}$, the **moduli stack of formal groups**, is the functor that sends:

$$\mathcal{M}_{\mathrm{fg}} : R \to \{\text{formal groups over } R \text{ with isomorphisms}\}$$

$\triangleleft$

Remark 29. For a formal group, you cannot globally choose a coordinate, whereas you can for formal group laws. $\triangleleft$

Construction 30. Some equivalent constructions:

(1) If you sheafify FGL up to isomorphism, we get $\mathcal{M}_{\mathrm{fg}}$: $\mathsf{FGL}^{\#}_{\simeq} = \mathcal{M}_{\mathrm{fg}}$
(2) $\operatorname{Spec} B^+ = [\operatorname{Spec} L/G^+] = \mathcal{M}_{\mathrm{fg}}$
(3) $[\mathcal{M}_{\mathrm{fgl}}/\mathbb{G}_m] = \mathcal{M}_{\mathrm{fg}}$ (strict isomorphisms have invertible coefficients, so we quotient out scaling)
(4) If we stackify, $\mathcal{M}^{\#\#}_{\mathrm{fgl}} = \mathcal{M}_{\mathrm{fg}}$

$\triangleleft$

Theorem 31. *$\mathcal{M}_{\mathrm{fg}}$ is a stack for the fpqc topology. It is also "algebraic" (representable qcqs diagonal and a flat surjective qc cover by a scheme).* $\triangleleft$

Proposition 32. *Sheaves on $\mathcal{M}_{\mathrm{fg}}$ should be $\mathbb{G}_m$-equivariant sheaves on $\mathcal{M}_{\mathrm{fgl}}$. Because of even grading, we have:*

$$\mathrm{QCoh}\big(\mathcal{M}_{\mathrm{fg}}\big) \simeq \mathsf{CoMod}(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})^{\mathrm{ev}}$$

$\triangleleft$

Given X a spectrum, we have two quasi-coherent sheaves on $\mathcal{M}_{\mathrm{fg}}$:

$$\mathcal{F}_X^{\mathrm{ev}} = (\mathrm{MU}_* X^{\mathrm{ev}})$$
$$\mathcal{F}_X^{\mathrm{odd}} = (\mathrm{MU}_* X)[1]^{\mathrm{ev}}$$