# THE STEENROD ALGEBRA AND FURTHER Spec(MU)LATION

MIT/Harvard Babytop Seminar, Spring 2025
Tuesday, February 18th, 2025
LiveTEX by Howard Beck

Abstract. **Speaker**: Keita Allen (MIT)

The theory of (faithfully flat) descent provides a way of understanding a ring in terms of an algebra over it, which can be very useful when the base ring is complicated. An analogue of these ideas in the stable homotopy category, where we consider ring spectra and algebras over them, leads to the Adams spectral sequence; we obtain a systematic way of studying the sphere spectrum by "resolving" it in terms of another, easier to understand, ring spectrum. A surprising discovery of Quillen tells us that when we use the complex cobordism spectrum MU to resolve the sphere, we obtain a spectral sequence with input coming from the purely algebraic theory of formal groups. This provides another plan of attack for understanding the homotopy groups of spheres, and stable homotopy theory more broadly; understanding how formal groups are organized can in turn help us understand how homotopy theory is organized. This turns out to be very effective, and is the basis for what has become known as chromatic homotopy theory. In my talk, I will give an overview of how some of these ideas are developed, emphasizing the analogy with algebra, and highlight some important characters who we will be seeing later.

**Disclaimer**: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

Let $k \in \mathsf{CRing}$ and $R$ a $k$-algebra.

**Definition 1.** The **cobar complex** is a cosimplicial $k$-algebra:

$$\mathrm{CB}_k(R) = R \rightrightarrows R \underset{k}{\otimes} R \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

◃

**Question 2.** How much information about base ring $k$ does the cobar complex $\mathrm{CB}_k(R)$ recover? ◃

**Theorem 3** (Faithfully flat descent, Grothendieck)**.** *Let $k \to R$ be faithfully flat. We look at the Dold-Kan correspondence:*

$$\mathrm{DK}(\mathrm{CB}_k(R)) = R \to R \underset{k}{\otimes} R \to \cdots$$

*which can be extended to an exact sequence:*

$$0 \to k \to R \underset{k}{\otimes} R \to \cdots$$

◃

**Remark 4.** Equivalently, we have the map $k \to \mathrm{DK}(\mathrm{CB}_k(R))$ is a quasisomorphism in the derived category $D(k)$. ◃

**Question 5.** Can we use a similar notion to study ring spectra? ◃

Generalizations:

Given a map of $\mathbb{E}_\infty$-ring spectra $A \to B$, can form a cobar complex:

$$\mathrm{CB}_A(B) = B \rightrightarrows B \underset{A}{\otimes} B \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

**Example 6.** $R \in \mathsf{CRing}$, can take Eilenberg-MacLane spectrum $\mathrm{H}R$. Then, given $k \to R$, we have a map $\mathrm{H}k \to \mathrm{H}R$.

$$\mathrm{CB}_{\mathrm{H}k}\,\mathrm{H}R = \mathrm{H}R \rightrightarrows \mathrm{H}R \underset{\mathrm{H}k}{\otimes} \mathrm{H}R \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

◃

**Example 7.** $\mathbb{S}$ is a monoidal unit in $\mathsf{Sp}$, initial in $\mathsf{CAlg}(\mathsf{Sp})$, so we get a unique map $\mathbb{S} \to E$ for any other $E \in \mathsf{CRing}(\mathsf{Sp})$:

$$\mathrm{CB}(E) = E \rightrightarrows E \otimes E \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

◁

**Definition 8.** For a cosimplicial spectrum $X_*$, we define the totalization:

$$\mathrm{Tot}\, X_* = \lim_{\mathbb{\Delta}} X_* = \lim(X_1 \rightrightarrows X_2 \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots) \in \mathsf{Sp}$$

(homotopy limit in $\infty$-category $\mathsf{Sp}$) In fact, it is a ring spectrum ◁

**Construction 9.** Given a cosimplicial spectrum $X_*$, there is an $\infty$-category version of Dold-Kan so we get a filtered spectrum:

$$\mathrm{DK}(X_*) = \cdots \to D(2) \to D(1) \to D(0)$$

Associated to this filtration we get a spectral sequence:

$$E_1^{s,t} = \pi_s X_t \implies \pi_{s+t}\,\mathrm{Tot}\, X_*$$

◁

**Remark 10** (Natalie)**.** This is a cosimplicial filtration, and it should maybe be a cofiltered spectrum. It is a Bousfield-Kan spectral sequence. ◁

**Example 11.** Take $k \to R$ faithfully flat, and consider $\mathrm{CB}_{\mathrm{H}k}\,\mathrm{H}R$:

$$E_1^{s,t} = \pi_s\left(\mathrm{H}R^{\otimes(t+1)}_{\mathrm{H}k}\right) \implies \pi_{s+t}\,\mathrm{Tot}(\mathrm{CB}_{\mathrm{H}k}\,\mathrm{H}R)$$

We have:

$$\pi_s\left(\mathrm{H}R^{\otimes t+1}_{\mathrm{H}k}\right) \cong H_s R^{\otimes t+1}_{k} = \begin{cases} R^{\otimes t+1} & s = 0 \\ 0 & \text{otherwise} \end{cases}$$

In particular, the $E_1$ page is concentrated on $s = 0$ where it is $\mathrm{DK}_k\, R = R \to R \underset{k}{\otimes} R \to \cdots$. On $E_2$ page, we take homology.

**Faithfully flat descent** tells us that when we take homology,

$$E_2^{s,t} = \begin{cases} k & s = t = 0 \\ 0 & \text{otherwise} \end{cases}$$

Obviously it collapses, so that $E_2 = E_\infty$ so we get:

$$\pi_*\,\mathrm{Tot}(\mathrm{CB}_{\mathrm{H}k}\,\mathrm{H}R) = \begin{cases} k & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

So that:

$$\mathrm{H}k \xrightarrow{\sim} \mathrm{Tot}(\mathrm{CB}_{\mathrm{H}k}\,\mathrm{H}R)$$

◁

**Remark 12.** This is a restatement of faithfully flat descent. ◁

How do we exploit this to study the sphere spectrum $\mathbb{S}$? We can get maps out of it into any other ring spectrum.

**Construction 13.** For a ring specturm $E$, we take the $E$-nilpotent completion of $\mathbb{S}$:

$$\mathbb{S}_E^\wedge = \mathrm{Tot}(\mathrm{CB}(E))$$

This gives a spectral sequence:

$$E_1^{s,t} = \pi_s E^{\otimes t+1} \implies \pi_{s+t}\mathbb{S}_E^\wedge$$

known as the $E$-Adams spectral sequence ◁

**Example 14.** If $E = \mathrm{H}\mathbb{F}_p$, then:

$$\mathbb{S}^\wedge_{\mathrm{H}\mathbb{F}_p} \cong \mathbb{S}^\wedge_p$$

◁

**Example 15.** If $E$ is connective, and $\mathbb{Z} \simeq \pi_0\mathbb{S} \to \pi_0 E$ is an isomorphism, then:

$$\mathbb{S}_E^\wedge \simeq \mathbb{S}$$

(such as $E = \mathrm{H}\mathbb{Z}$, or MU) ◁

The MU-based Adams Spectral Sequence is often called the Adams-Novikov Spectral Sequence (ANSS), although this is sometimes based on BP.

**Example 16.** $\mathrm{H}\mathbb{F}_p$-based A.S.S. is the classical A.S.S. ◁

**Question 17.** Can we simplify the input to the A.S.S.? ◁

**Definition 18.** A ring spectrum $E$ is **flat** or **descent flat** if $\pi_*(E \otimes E) = E_*E$ is flat as a $\pi_*(E) = E_*$-module. ◁

We can ask if either map $E_* \rightrightarrows E_*E$ is flat (in fact, they both are or both aren't).

**Example 19.** $\mathrm{H}\mathbb{F}_p$ is flat, because $\mathbb{F}_p$ is a field so $\left(\mathrm{H}\mathbb{F}_p\right)_* X$ is a free $\mathbb{F}_p$-module ◁

**Example 20.** MU is flat:

$$\mathrm{MU}_*\mathrm{MU} = \mathrm{MU}_*[b_1, b_2 \ldots]$$

◁

**Non-Example 21.** $\mathrm{H}\mathbb{Z}$ is *not* flat ◁

**Theorem 22.** *If $E$ is flat, then the $E_2$-page is*

$$E_2^{s,t} = \mathrm{Ext}_{E_*E}^{s,t}$$

*where* Ext *is computed in* $\mathsf{CoMod}_{(E_*, E_*E)}$ ◁

**Proposition 23.** *If $E$ is flat, then $E_* \rightleftarrows E_*E$ is a* ***Hopf algebroid****. A Hopf algebroid presents a stack:*

$$Y_E : \mathsf{Ring} \to \mathsf{Grpd}$$

*And we have:*

$$E_2^{s,t} = \mathrm{H}^s(Y_E; \mathcal{O}(t))$$

*This is a graded stack and graded sheaf. The rings you get in $\pi_*$ are* GrRing. ◁

We will interpret some computations in algebraic topology in the language of algebraic geometry now.

**Example 24** (Milnor)**.** The **Dual Steenrod algebra**:

$$\mathcal{A}_* = (\mathrm{H}\mathbb{F}_2)_*\mathrm{H}\mathbb{F}_2$$

The stack associated to $\mathbb{F}_2 \rightleftarrows \mathcal{A}_*$ is:

$$Y_{\mathrm{H}\mathbb{F}_2} = \mathrm{B}\,\mathrm{Aut}\left(\widehat{\mathbb{G}}_a\right) \to \mathrm{Spec}\,\mathbb{F}_2$$

You send it to automorphisms of formal groups. ◁

It takes an $\mathbb{F}_2$-algebra $R$ and sends it to one object groupoid where the maps are $\mathrm{Aut}\left(\widehat{\mathbb{G}}_a \underset{\mathbb{F}_2}{\times} \mathrm{Spec}\, R\right)$. This is written down more carefully by Piotr.

**Example 25** (Quillen)**.** $E = \mathrm{MU}$ then $\mathrm{MU}_*$ is the Lazard ring, corepresenting formal group laws. It is an honest commutative ring.

$\mathrm{MU}_* \rightleftarrows \mathrm{MU}_*\mathrm{MU}$ presents the moduli stack $\mathcal{M}_{\mathrm{fg}}^s$ with strict isomorphisms. We have $\mathcal{M}_{\mathrm{fg}}^s \to \mathrm{Spec}\,\mathbb{Z}$. There is a $\mathbb{G}_m$-action coming from the grading, and we can quotient it out:

$$\mathcal{M}_{\mathrm{fg}} \to \mathrm{B}\,\mathbb{G}_m$$

which classifies a line bundle $\omega$. ◁

ANSS:

$$E_2^{s,t} = \mathrm{H}^{s/2}\left(\mathcal{M}_{\mathrm{fg}}; \omega^{\otimes t}\right) \implies \pi_*\mathbb{S}$$

Keita: this doesn't make it easier, it just makes it someone else's job.

**Question 26.** Can we use the structure on $\mathcal{M}_{\mathrm{fg}}$ to study $\pi_*\mathbb{S}$ or stable homotopy theory more broadly? ◁

**Answer 27.** Chromatic homotopy theory! ◁