EQUIVARIANT RECOGNITION PRINCIPLE

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Abstract. **Speaker**: Branko Juran (Copenhagen)

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

1. Introduction

X based space, we can attach homotopy groups $\pi_n(X)$, and it is a group when $n \geq 1$ and is further abelian when $n \geq 2$. The group structure comes from a structure $\Omega^n X = \mathrm{Map}_*(S^n, X)$. This space is called an $\mathbb{E}_n$-algebra.

There is a category Disk_n of n-dimensional disks and framed embeddings, with disjoint unions.

Definition 1.1. An $\mathbb{E}_n$-algebra is:

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) = \mathsf{Fun}^{\otimes}(\mathsf{Disk}_n, \mathcal{C})$$

where $\mathcal{C}$ is a symmetric monoidal $\otimes$-category. ◁

The recognition principle says that the E_n-algebra structure retains the deloopings.

We can ask whether all E_n-algebras in spaces are n-th loop spaces. Not all are - in fact, the only obstruction is that we need π_0 to be a group, instead of just a monoid.

Definition 1.2. An E_n-algebra in spaces A is called group-like if $\pi_0(A)$ is a group. ◁

Theorem 1.3 (Recognition Principle, May-Segal)**.** *There is an equivalence of categories:*

$$(\mathsf{Spaces}_*)_{\geq n} \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{gp}}_{\mathrm{E}_n}(\mathsf{Spaces})$$

of n-th connected based spaces and group-like E_n-algebras in spaces.

2. Equivariant Setting

Let G be a finite group.

In equivariant homotopy theory, we have loop spaces indexed not by integers but by representations. Thus, we also fix a finite-dimensional real G-representation.

We have the V-fold loop space of a G-space $X \in \mathsf{Spaces}^G$:

$$\Omega^V X = \mathrm{Map}_*\left(S^V, X\right)$$

where S^V is the representation sphere of V - its one-point compactification. π_0 of this is the V-th homotopy group.

To make an analogous theory, we will need G-symmetric monoidal G-categories.

We want to construct G-spaces.
Idea: we want to model topological spaces with an action of G.

Here is an attempt: Spaces^{BG}. This is called Borel-equivariant, but is not good enough – we cannot take arbitrary fixed points.

Definition 2.1. Let the **orbit category** $\mathcal{O}_G$ be the category of finite transitive G-sets. In other words, these are G/H for some subgroups $H \subset G$. G-**spaces** are given by:

$$\mathsf{Spaces}^G = \mathsf{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, S\right)$$

where we send G/H to X^H. ◃

Definition 2.2. A G-category is $\mathcal{O}_G^{\mathrm{op}} \to \mathsf{Cat}$. ◃

Example 2.3. $G/H \mapsto \mathsf{Spaces}^H$ ◃

Example 2.4. For V a G-representation, we can take Disk_V^H to be the category of H-disks framed in $\mathrm{Res}_H^G V$ and embeddings.

For example, you can take the free embedding of two disjoint disks (with the C_2 action) into the disk with a C_2 action via the sign representation. ◃

For a symmetric monoidal structures, we need **norms**

Definition 2.5. For a commutative monoid $X \in \mathsf{CMon}(\mathsf{Top})$ with a G-action, we get a map:

$$\begin{aligned} M^e &\to M^G \\ m &\mapsto \sum_{g \in G} g \cdot m \end{aligned}$$

This should be part of the definition. ◃

Recall 2.6. The category of symmetric monoidal categories:

$$\mathsf{Cat}^{\otimes} = \mathsf{CMon}(\mathsf{Cat}) = \mathsf{Fun}^{\prod}(\mathsf{Span}(\mathsf{FinSet}), \mathsf{Cat})$$

$\mathsf{Span}(\mathsf{FinSet})$ has objects finite sets and morphisms $S \leftarrow T \to S'$. Because this is product preserving:

$$\langle 1 \rangle \mapsto \mathcal{C}$$

$$\langle n \rangle \mapsto \mathcal{C}^n$$

We have a special morphism $\langle 2 \rangle = \langle 2 \rangle \to \langle 1 \rangle$. The map $\mathcal{C}^2 \to \mathcal{C}$ is multiplication. ◃

Definition 2.7. A G-**symmetric monoidal** G-**category** is this but with G-sets:

$$\mathsf{Fun}^{\prod}(\mathsf{Span}(\mathsf{FinSet}_G), \mathsf{Cat})$$

G/H goes to some category $\mathcal{C}^H$, and these can be chosen freely. The rest are built from products (disjoint unions). ◃

There is the span $G/e = G/e \to G/G$ that gives you:

$$\mathcal{C}^e \xrightarrow{\mathsf{Nm}_e^G} \mathcal{C}^G$$

This is called a **norm map** and is just part of the data.

We can also compose spans like this:

$$\begin{tikzcd}
 & & G/e \times G/e \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & G/e \arrow[dl, equal] \arrow[dr] & & G/e \arrow[dl] \arrow[dr, equal] & \\
G/e \arrow[rr, dotted, "\mathsf{Nm}_e^G"'] & & G/G \arrow[rr, "\mathrm{Res}_e^G"'] & & G/e
\end{tikzcd}$$

We get that:

$$\mathrm{Res}_e^G \mathsf{Nm}_e^G(x) \simeq \bigotimes_{g \in G} gx$$

(in $\mathcal{C}^e$)

Example 2.8 ($\underline{\mathsf{Spaces}}$)**.** The G-category with $\underline{\mathsf{Spaces}}^H = \mathsf{Spaces}^H$ has a G-symmetric monoidal structure for $K \subset H \subset G$:

$$\begin{aligned} \mathsf{Nm}_K^H : \mathsf{Spaces}^K &\to \mathsf{Spaces}^H \\ X &\mapsto \mathrm{Map}^K(H, X) \end{aligned}$$

◁

Example 2.9. We take Disk_V with norms.

$$\begin{aligned} \mathsf{Disk}_V^K &\to \mathsf{Disk}_V^H \\ D &\mapsto H \underset{K}{\times} D \end{aligned}$$

This is a topological induction, not a representation-theoretic one. Note that D is a manifold ◁

Definition 2.10. An $\mathbb{E}_V$-algebra in $\mathcal{C}$ (a G-symmetric monoidal G-∞-category) is a G-symmetric monoidal G-functor (preserves all this data) from Disk_V:

$$\mathsf{Disk}_V \to \mathcal{C}$$

◁

Example 2.11. $\Omega^V X \in \mathrm{Alg}_{\mathbb{E}_V}\left(\underline{\mathsf{Spaces}}\right)$

For example, take $G = C_2$ and $V = \mathrm{sgn}$. For $X \in \mathsf{Spaces}^{C_2}$, $\Omega^{\mathrm{sgn}} X$ is an $\mathbb{E}_{\mathrm{sgn}}$-algebra.

- On underlying:
$$(\Omega^{\mathrm{sgn}} X)^e = \Omega(X^e)$$
is an $\mathbb{E}_1$-algebra.
- On C_2-fixed points:
$$(\Omega^{\mathrm{sgn}} X)^{C_2} = \mathrm{Map}_*^{C_2}(S^{\mathrm{sgn}}, X)$$
This is not an $\mathbb{E}_1$-algebra, as sgn has no valid pinch map. However, in $\mathsf{Disk}_{\mathrm{sgn}}^{C_2}$, we have an object $D(\mathrm{sgn})$ with the C_2 swap action which gets sent to $\Omega^{\mathrm{sgn}} X$. We also have $C_2 \underset{e}{\times} D(\mathbb{R})$, which has two disks with the swap action between them. $D(\mathbb{R})$ gets sent to non-equivariant loops: $\Omega X^e \times \Omega X^e$ with the C_2-swap. There is also the disjoint union $D(\mathrm{sgn}) \coprod \left(C_2 \underset{e}{\times} D(\mathbb{R})\right)$ which gets sent to the product $\Omega X^e \times \Omega X^e \times \Omega^{\mathrm{sgn}} X$. There is an embedding of this into $D(\mathrm{sgn})$. You sent the fixed point to the fixed point, and free orbits to free orbits. We thus get a map $\Omega X^e \times \Omega X^e \times \Omega^{\mathrm{sgn}} X \to \Omega^{\mathrm{sgn}} X$.

  If we take C_2-fixed points, we get
$$\Omega X^e \times (\Omega^{\mathrm{sgn}} X)^{C_2} \to (\Omega^{\mathrm{sgn}} X)^{C_2} = \mathrm{Map}_*^{C_2}(S^{\mathrm{sgn}}, X)$$
This is the structure the sign loop spaces have - it is a module over ΩX^e.

◁

Definition 2.12. An $\mathbb{E}_V$-algebra in G-spaces is called **grouplike** if $\pi_0\left(A^H\right)$ is a group for all $H \subset G$ such that $\dim V^H \geq 1$ (whenever it makes sense). ◁

Theorem 2.13.

$$\mathrm{Alg}_{\mathbb{E}_V}^{\mathrm{gp}}\left(\underline{\mathsf{Spaces}}\right) \simeq \left(\mathsf{Spaces}_*^G\right)_{\geq V}$$

If $\mathbb{R} \subset V$, this is proven by Guillou-May, Hausach, Rouke-Sanderson.

Why $\mathbb{E}_V$-algebras?

- V-fold loop spaces
- G-manifolds
- Equivariant factorization homology
- "real" homotopy theory cares about $G = C_2$ and sgn, such as $\mathrm{KU}_{\mathbb{R}}$, $\mathrm{MU}_{\mathbb{R}}$, and friends.

Proof. Reduce to, for $X \in \mathsf{Spaces}_*^G$:

$$\mathrm{Free}_{\mathbb{E}_V}^{\mathrm{gp}} X \simeq \Omega^V \Sigma^V X$$

The hard part is the group-like part - without it, it is much easier.

Non-equivariantly, we have group-completion $(-)^{\mathrm{gp}} \simeq \Omega B(-)$ of a monoid. □