GALOIS THEORY OF RING SPECTRA

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Abstract. **Speaker**: Thomas Brazelton (Harvard)

We'll provide a leisurely introduction to the theory of descent for maps of ring spectra, and Rognes' theory of Galois extensions of ring spectra. We will also present a profinite Galois correspondence in this setting due to Mathew, and time pending discuss applications to the $K(n)$-local setting.

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

1. Descent for (vintage) rings

Let $R \to S$ be a ring homomorphism. Alternatively, it is a one object cover of $\operatorname{Spec} S \to \operatorname{Spec} R$. We can form the Čech nerve or cobar complex:

$$R \to S \overset{\leftarrow}{\underset{\leftarrow}{\rightrightarrows}} S \underset{R}{\otimes} S \to S \underset{R}{\otimes} S \underset{R}{\otimes} S \to \cdots$$

If $\mathcal{F} : \mathsf{CRing} \to \mathcal{C}$ is any presheaf valued in any ∞-category, we can ask if $\mathcal{F}$ descends along the one-object cover.

We say it descends if for

$$\mathcal{F}(R) \xrightarrow{*} \lim\left(\mathcal{F}(S) \rightleftarrows \mathcal{F}\left(S \underset{R}{\otimes} S\right) \overset{\rightarrow}{\underset{\leftarrows}{\leftrightarrows}} \cdots\right)$$

we have that $*$ is an equivalence.

Example 1.1. If $\mathcal{F}$ takes values in Set, Ab, Mod_A, etc, then this condition truncates to the sheaf condition. ◃

Example 1.2. If $\mathcal{F}(R)$ is a 1-category, then $*$ truncates, but you need triple overlaps. This means the stack condition. ◃

Example 1.3. We take

$$\begin{aligned} \mathcal{F} : \mathsf{CRing} &\to \mathsf{Cat} \\ R &\mapsto \mathsf{Mod}_R \end{aligned}$$

when we do get descent? ◃

Theorem 1.4 (Grothendieck). *If $R \xrightarrow{f} S$ is faithfully flat, then:*

$$\mathsf{Mod}(R) \xrightarrow{\sim} \lim\left(\mathsf{Mod}(S) \rightleftarrows \mathsf{Mod}\left(S \underset{R}{\otimes} S\right) \overset{\rightarrow}{\underset{\leftarrows}{\leftrightarrows}} \mathsf{Mod}\left(S \underset{R}{\otimes} S \underset{R}{\otimes} S\right)\right)$$

is an equivalence of categories. This is a limit in 2-categories. The rightmost term is often labeled $\operatorname{Desc}(f)$, *the category of descent data.*

Question 1.5. Is this if and only if? ◃

We have a restriction and extension of scalars adjunction:

$$\mathsf{Mod}_R \rightleftharpoons \mathsf{Mod}_S$$

that gives a comonad $\Omega : \mathsf{Mod}_S \to \mathsf{Mod}_S$.

Theorem 1.6 (Joyal-Tierney). *The following are equivalent:*

(1) $\mathsf{Mod}(R) \xrightarrow{\sim} \mathrm{Desc}(f)$
(2) Extension of scalars along f is comonadic.
(3) Extension of scalars is faithful.
*(4) S is **pure** as an R-module. This means $R \underset{R}{\otimes} - \to S \underset{R}{\otimes} -$ is monic.*

Note that the last two conditions are more obviously equivalent by definitions.

Also, if the map f is faithfully flat, then extension of scalars is faithful and we recover Grothendieck's result.

2. Galois Extensions

Definition 2.1. A finite field extension $k \subset L$ is **Galois** if k is the fixed subfield of some subgroup $G \subseteq \mathrm{Aut}(L)$. $\triangleleft$

Note that we are allowing inseparable extensions to be Galois.

Question 2.2. Is there a notion of Galois extensions for rings? $\triangleleft$

Definition 2.3 (Auslander-Goldman)**.** Let $S \in \mathsf{CRing}$. Let $G \subseteq \mathrm{Aut}(S)$ be some finite subgroup, and $R = S^G$ be the fixed subring. We say $R \to S$ is **Galois** if we have a normal basis theorem:

$$S \underset{R}{\otimes} S \to \prod_{g \in G} S$$
$$s_1 \otimes s_2 \mapsto (s_1 \cdot g(s_2))_{g \in G}$$

$\triangleleft$

Note we can do some of this in non-commutative rings, but we don't really want to.

Remark 2.4. This is equivalent (at least if S is connected - 0 and 1 are the only idempotents, or $\mathrm{Spec}\, S$ is connected as a space) to $R \to S$ being finite étale. Namely, S is finitely generated and projective as an R-module. $\triangleleft$

We can also have that a trace form is non-generate.

Example 2.5. If $K \subset L$ is an unramified G-Galois extension of number fields, then $\mathcal{O}_K \subset \mathcal{O}_L$ is a G-Galois extension of their rings of integers. $\triangleleft$

Example 2.6. There are no nontrivial Galois extensions to $\mathbb{Z}$.

Sketch. If k is a number field, then $\mathbb{Z} \to \mathcal{O}_K$ ramifies. $\triangleleft$

$\triangleleft$

Proposition 2.7. *If $R \to S$ is a G-Galois ring extension, then it is faithfully flat.*

IN the case of a Galois ring extension, the category of descent data admits two more cool descriptions:

Setup 2.8. Let $f : R \to S$ be a G-Galois extension. $\triangleleft$

Proposition 2.9. *There is an equivalence of categories:*

$$\mathrm{Desc}(f) = \mathsf{Mod}(S)^{\mathrm{h}G} = \mathsf{Fun}(\mathcal{E}G, \mathsf{Mod}(S))^G$$

where $\mathcal{E}G$ is a free contractible G-groupoid (a 1-category), that geometrically realizes to the topological EG. We get a group action via extension of scalars.

For example, $\mathcal{E}C_2$ is the category of a free isomorphism:

$$\begin{tikzcd} \bullet \arrow[d, bend left, "\simeq"] \\ \bullet \arrow[u, bend left, "\simeq"] \end{tikzcd}$$

If $\theta : G \to \mathrm{Aut}(S)$, define: $S_\theta[G]$ to be the **twisted group ring**:

$$(s_1, g_1) \cdot (s_2, g_2) = s_1 \theta_{g_1}(s_2) g_1 g_2$$

Proposition 2.10. $\mathsf{Mod}(S)^{\mathrm{h}G} \xrightarrow{\sim} \mathsf{Mod}_{S_\theta[G]}$

Note that this is no longer commutative.

3. Homotopic Galois Stuff

Definition 3.1 (Rognes)**.** Let G be a finite group, and $A \to B$ be a map of $\mathbb{E}_\infty$-rings, and let $G \subset \mathrm{Aut}_{\mathsf{CAlg}_A}(B)$. We say this is **Galois** if there are equivalences:

(1) $A \xrightarrow{\sim} B^{\mathrm{h}G}$
(2) $B \underset{A}{\otimes} B \xrightarrow{\sim} \prod_{g \in G} B$

◁

In standard rings, we have:

$$B \underset{A}{\otimes} B \to B$$
$$(b_1, b_2) \mapsto (b_1, g b_2)$$

Upshot 3.2. If we understand $\pi_* B$ well, we can get access to $\pi_* A$. ◁

Example 3.3 (Rognes)**.** KO $\to$ KU given by changing scalars from reals to complex numbers is a C_2-Galois extension.

Sketch. Use the homotopy fixed point spectral sequence to show $\mathrm{KO} = \mathrm{KU}^{\mathrm{h}G}$:

$$H^p\Big(C_2; \pi_q \mathrm{KU}\Big) \implies \pi_{q-p} \mathrm{KU}^{hC_2}$$

(Atiyah) We also need that $\mathrm{KU} \underset{\mathrm{KO}}{\otimes} \mathrm{KU} \xrightarrow{\sim} \underset{C_2}{\otimes} \mathrm{KU}$, and that requires Bott periodicity and nilpotence in some way. ◁

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Non-Example 3.4. ko $\to$ ku is <u>not</u> Galois. ◁

Sanity Check 3.5. If $R \to S$ is a G-Galois extension of vintage rings, then $\mathrm{H}R \to \mathrm{H}S$ is a G-Galois extension of ring spectra. ◁

Sanity Check 3.6. If $A \to B$ is a Galois extension of ring spectra, then B is dualizable as an A-module. ◁

Remark 3.7. Galois extensions of ring <u>spectra</u> are not necessarily faithful, unlike vintage rings. ◁

Proposition 3.8. *For a map $f : A \to B$, the following are equivalent:*

*(1) B is **faithful** as an A-module*
(2) ... by faithful, we mean that extension of scalars $\mathsf{Mod}(A) \to \mathsf{Mod}(B)$ *is conservative (tensoring resulting in zero means the original was zero)*

Further, if $A \to B$ is G-Galois, then this is further equivalent to:

(3) $\mathsf{Mod}(A) \to \mathsf{Mod}(B)^{\mathrm{h}G}$ *is conservative*
(4) $B^{\mathrm{t}G} \xrightarrow{\sim} 0$.
(5) $\mathsf{Mod}(A) \to \mathsf{Mod}(B)^{\mathrm{h}G} \xrightarrow{\sim} \mathsf{Mod}_{\mathsf{Fun}(BG,\mathsf{Sp})}(B)$

The Tate construction $B^{\mathrm{t}G}$ is the cofiber of the norm map:

$$B^{\mathrm{t}G} := \mathrm{cofib}\Big(B_{\mathrm{h}G} \xrightarrow{\mathsf{Nm}} B^{\mathrm{h}G}\Big)$$

Non-Example 3.9 (Wieland, "unfaithful.pdf" on Rognes' website)**.** There exists Galois ring extensions which aren't faithful. `https://www.mn.uio.no/math/personer/vit/rognes/papers/unfaithful.pdf`

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Example 3.10. If $|G|$ is invertible in $\pi_0 A$, then $A \to B$ is a faithful Galois extension. ◁

Remark 3.11. If $A \to B$ is faithful and B is dualizable, then $A \to A^\wedge_B$ is an equivalence. This is very far from being an if and only if statement. This completion is the totalization of the cobar complex. It is also the nilpotent completion in the category of modules. ◁

Example 3.12. $\mathbb{S} \to \mathrm{MU}$ is not Galois and not faithful, even though $\mathbb{S} \to \mathbb{S}^\wedge_{\mathrm{MU}}$ is an equivalence. ◁

Remark 3.13 (Dhilan)**.** If $\mathrm{H}R \to \mathrm{H}S$ is a Galois extension, then $R \to S$ is too. Take π_0. ◃

Remark 3.14 (Natalie)**.** Non-faithfulness was needed in the disproof of the telescope conjecture. You show that the separable closure of the $K(n)$-local sphere is not faithful, but not for $T(n)$. ◃

Remark 3.15. Meier has a definition of twisted group ring spectra, and analogous theorems like:

$$\mathsf{Mod}(S)^{\mathrm{h}G} \xrightarrow{\sim} \mathsf{Mod}(S_\theta[G])$$

where again $S_\theta[G]$ is like an $\mathbb{E}_1$-ring, but not $\mathbb{E}_\infty$. ◃

The motivation for all of this was the following, with a very much not finite group:

Theorem 3.16. *There is a weak equivalence between:*

$$L_{K(n)}\mathbb{S} \to E_n^{\mathrm{h}\Gamma_n}$$

where this is the homotopy fixed points with respect to the Morava stabilizer group.

We would love for this to be a Galois extension.

4. Profinite Galois Theory

Let k be a field and G_s be its absolute Galois group.

Theorem 4.1. *There is an equivalence of categories between:*

$$\mathrm{F\acute{E}t}_k \simeq \mathsf{Fin}_{G_s}^{\mathrm{op}}$$

finite étale k-algebras and continuous actions of G_s on finite groups.

Namely, if G is a finite group then any continuous homomorphism $G_s \to G$ corresponds to a G-Galois extension of k.

For rings:

$$\mathrm{F\acute{E}t}_R \simeq \mathsf{Fin}_{\pi_1^{\acute{e}t}(R)}^{\mathrm{op}}$$

What is the analogue of $\pi_1^{\acute{e}t}(R)$ for R a ring spectrum?

Answer 4.2 (Mathew)**.** It is a profinite group denoted $\pi_1\mathsf{Mod}(R)$. ◃

Question 4.3 (for audience)**.** How related is this to taking π_1 of the space underlying $\mathsf{Mod}(R)$ and profinitely completing? ◃

Theorem 4.4 (Mathew)**.** *If R is an $\mathbb{E}_\infty$-ring and G is a finite group, then any continuous homomorphism $\pi_1\mathsf{Mod}(R) \to G$ corresponds to a G-Galois extension of R.*

Note that we make no claims about it being faithful.

There is always a surjective map:

$$\pi_1\mathsf{Mod}(R) \twoheadrightarrow \pi_1^{\acute{e}t}(\pi_0 R)$$

where by the right we mean taking the underlying discrete ring.

Theorem 4.5 (Mathew)**.** *If R is even periodic and $\pi_0 R$ is regular Noetherian, then:*

$$\pi_1\mathsf{Mod}(R) \simeq \pi_1^{\acute{e}t}(\pi_0 R)$$

Theorem 4.6 (Mathew)**.** $\pi_1\mathsf{Mod}\left(L_{K(n)}\mathbb{S}\right) = \Gamma_n$.

Example 4.7 (Rognes)**.** Every Galois extension of $L_{K(n)}\mathbb{S}$ is faithful. ◃

Again, this fails $T(n)$-locally, which led to the disproof of the telescope conjecture.

Theorem 4.8 (Rognes, in Mathew's language)**.** $\pi_1(\mathsf{Sp}) = 0$. *In other words, the Galois group of the sphere spectrum is trivial.*

Slogan 4.9. $\mathbb{S}$ is seperably closed. ◃

Proof sketch. Suppose $\mathbb{S} \to B$ is a finite G-Galois ring extension. Then B is dualizable in $\mathsf{Mod}(\mathbb{S}) = \mathsf{Sp}$. Thus, B has the homotopy type of a retract of a finite CW spectrum. Then, $\mathrm{H}\mathbb{Z}^*B$ is finitely generated and non-zero in only finitely many degrees. Recall that $B \otimes B \xrightarrow{\sim} \prod_{g \in G} B$, which tells us that homology is concentrated in degree 0 given the last bit. Using Hurewicz, $H_0 B = \pi_0 B$, which is some finitely generated free abelian group of rank equal to $|G|$. This tells us that a map $\mathbb{Z} = \pi_0 \mathbb{S} \to \pi_0 B$ is a Galois ring extension, and now we are in business. The only Galois extension of $\mathbb{Z}$ is trivial, so $G = e$. There are some more details to work out to make sure B is not $\mathrm{H}\mathbb{Z}$-flavored. □