THE $K(1)$-LOCAL SPHERE

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Abstract. **Speaker**: Kush Singhal (Harvard)

We will compute the homotopy groups of the Bousfield localisation of the sphere at $K(1)$ and (time permitting) $E(1)$. On the way, we will encounter Adams operations on complex K-theory. If time permits, we will also discuss how the J-homomorphism appears in this picture.

Disclaimer: Do not take these notes too seriously, sometimes half-truths are told in exchange for better exposition, and there may be errors in my liveTEXing

0. Motivation

Let:

$$L_n X := L_{E(n)} X = L_{K(1)\oplus \ldots \oplus K(n)} X$$

Theorem 0.1 (Chromatic Convergence Theorem, Hopkins-Ravenal). *If X is a finite spectrum, then:*

$$X_{(p)} = \lim(\cdots \to L_n X \to \cdots \to L_1 X \to L_0 X)$$

That is, to understand the homotopy groups of the p-localized X, we should understand its homotopy groups at each chromatic layer. $n = 0$ is easy, as $E(0) = K(0) = \mathrm{H}\mathbb{Q}$. Today we will try to understand the $n = 1$ layer. Specifically, we will want to understand $L_{K(1)}\mathbb{S}$ and $L_1\mathbb{S}$.

1. Bousfield Localization

Let E be a spectrum.

Definition 1.1. For X a spectrum, X is E-**acyclic** if $X \otimes E \simeq 0$. We define the **Bousfield class** of E:

$$\langle E\rangle = \{E\text{-acyclic spectra}\} \subset \mathsf{Sp}$$

which is a full subcategory of spectra. ◃

There is a dual notion:

Definition 1.2. We say X is E-local if $\forall Y \to X$ with $Y \in \langle E\rangle$, then $Y \to X$ is nulhomotopic. ◃

Example 1.3. $E = \mathrm{H}\mathbb{Q}$.

- → X is $\mathrm{H}\mathbb{Q}$-acyclic if and only if $\pi_n(X) \otimes \mathbb{Q} = 0$ which is true iff $\pi_n(X)$ is torsion, for all n.
- → X is $\mathrm{H}\mathbb{Q}$-local iff $\pi_n(X)$ are all rational vector spaces.

◃

Example 1.4. For $E = \mathrm{H}\mathbb{F}_p$ and $X = \mathrm{H}C^\bullet$ for some bounded chain complex $C^\bullet$,

- → X is $\mathrm{H}\mathbb{F}_p$-acyclic iff $H^n(C^\bullet)$ are $\mathbb{Z}[1/p]$-modules.
- → X is $\mathrm{H}\mathbb{F}_p$-local iff $H^n(C^\bullet)$ are all p-complete.

The p-completion of another spectrum X is $L_{\mathrm{H}\mathbb{F}_p} X$. ◃

We see that $\langle E\rangle \subset \mathsf{Sp}$ is closed under all colimits. By the adjoint functor theorem, there is a right adjoint $G_E : \mathsf{Sp} \to \langle E\rangle$. This takes a spectrum and returns the closest E-acyclic.

There is a counit map $G_E X \to X$. We take the cofiber:

Definition 1.5. We define the Bousfield localization of X with respect to E as:

$$L_E X := \mathrm{cofib}(G_E X \to X)$$

◁

For example, Bousfield localization with respect to $\mathrm{H}\mathbb{F}_p$ is p-completion.

Lemma 1.6. *$L_E X$ is E-local for all spectra.*

Example 1.7.

$$L_{\mathrm{H}\mathbb{Q}}\mathbb{S} = \mathrm{H}\mathbb{Q}$$

Recall that $\pi_0(\mathbb{S}) = \mathbb{Z}$ and $\pi_n(\mathbb{S})$ is finite for $n > 0$. ◁

Definition 1.8.

$$L_n := L_{E(n)}$$

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Recall 1.9. $E(n)$ is the Landweber exact spectrum associated to $\mathbb{Z}_p[\![v_1, \ldots, v_{n-1}]\!][\beta^{\pm 1}]$. As usual, $|v_i| = 2\left(p^i - 1\right)$ and $|\beta| = 2$. We use the formal group law of height n over $\mathbb{F}_p$. ◁

If X is $E(n)$-acyclic, then v_is are going to be non-zero on the homotopy groups. In other words, $\mathcal{F}_X$ and $\mathcal{F}_{\Sigma X}$ on $\mathcal{M}_{\mathrm{fg}}$ are supposed on $\mathcal{M}_{\mathrm{fg}}^{\geq n+1}$. So really $L_{E(n)}$ behaves like restriction to the open substack of height at most n, $\mathcal{M}_{\mathrm{fg}}^{\leq n}$. The suspension ΣX has to do with even homotopy groups, so that you look at the even and odd ones.

Proposition 1.10. *We have the following Cartesian square:*

$$\begin{tikzcd}
L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{n-1} X \arrow[d] \\
L_{K(n)} X \arrow[r] & L_{n-1} L_{K(n)} X
\end{tikzcd}$$

Therefore, to understand the L_n-localizations, we should understand $K(n)$-localizations.

Goal 1.11. Compute $L_1 \mathbb{S}_{(p)}$ by computing $L_{K(1)} \mathbb{S}_{(p)}$. ◁

2. What are $E(1)$ and $K(1)$ explicitly?

Recall 2.1. $\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$ ◁

Heuristic 2.2. $L_{K(n)}$ behaves like completion along the locally closed substack $\mathcal{M}_{\mathrm{fg}}^n$. ◁

The height 1 formal group over $\mathbb{F}_p$ is $\widehat{\mathbb{G}}_m(x, y) = x + y + xy$. Recall from Daishi's talk that KU is attached (as a Landweber exact spectrum) to $x + y + \beta xy$. Therefore, we may expect them to be very closely related, and this is true.

Proposition 2.3. $E(1) = \widehat{\mathrm{KU}}$*, the p-adic completion of complex K-theory. By this we mean a Bousfield localization.*

Recall the n-th Morava stabilizer group Γ_n sits in the exact sequence:

$$1 \to \mathrm{Aut}\Big(\text{height } n \text{ formal group laws over } \mathbb{F}_p\Big) \to \Gamma_n \to \mathrm{Gal}\Big(\bar{\mathbb{F}}_p/\mathbb{F}_p\Big) \to 1$$

At $n = 1$, we have:

$$1 \to \mathrm{Aut}\Big(\widehat{\mathbb{G}}_m\Big) \to \Gamma_1 \to G_{\mathbb{F}_p} \to 1$$

The automorphism group is:

$$\mathrm{Aut}\Big(\widehat{\mathbb{G}}_m\Big) = \{f(t) \in \mathbb{Z}_p[\![t]\!] \mid f(x + y + xy) = f(x) + f(y) + f(xy)\} = \Big(\{[n]_{\widehat{\mathbb{G}}_m} : n \in \mathbb{Z}\}_p^{\wedge}\Big)^{\times} \mathbb{Z}_p^{\times}$$

Question 2.4. What is the $\mathbb{Z}_p^{\times} \subset \Gamma_1$ action on $E(1) = \widehat{\mathrm{KU}}$? ◁

Construction 2.5 (Adams Operations on KU)**.** For $k \in \mathbb{Z}$, we have operations:

$$\Psi^k : \mathrm{KU}\left[\frac{1}{k}\right] \to \mathrm{KU}\left[\frac{1}{k}\right]$$

which are unique ring maps. We can define them on line bundles, as $\Psi^k([L]) = [L^{\otimes k}]$. ◁

When we p-adically complete, we get an action $\mathbb{Z}_p^\times \curvearrowright \widehat{\mathrm{KU}}$.

Fact 2.6. *$\mathbb{Z}_p^\times \curvearrowright E(1)$ is precisely the Adams operations.*

Note 2.7.

$$\begin{gathered}\Psi^k : \pi_*\widehat{\mathrm{KU}} = \mathbb{Z}_p[\beta^{\pm 1}] \to \mathbb{Z}_p[\beta^{\pm 1}] = \pi_*\widehat{\mathrm{KU}} \\ \beta \mapsto k\beta\end{gathered}$$

◁

Construction 2.8. We have the $p-1$-th roots of units inside: $\mu_{p-1} \subset \mathbb{Z}_p^\times \curvearrowright \widehat{\mathrm{KU}}$. We can then define:

$$\widehat{\mathrm{KU}}^{\mathrm{Ad}} := \widehat{\mathrm{KU}}^{\mathrm{h}\mu_{p-1}}$$

◁

For $\alpha \in \mu_{p-1}$, *beta* gets sent to $\alpha\beta$. Then, β^{p-1} gets sent to $\alpha^{p-1}\beta^{p-1} = \beta^{p-1}$. This is everything that is fixed, so we get, remembering that $v_1 = \beta^{p-1}$.

$$\pi_*\widehat{\mathrm{KU}}^{\mathrm{Ad}} = \mathbb{Z}_p[v_1^{\pm 1}]$$

If we take the cofiber of the multiplication by p, we get:

$$\pi_*\left(\widehat{\mathrm{KU}}^{\mathrm{Ad}}/p\right) = \mathbb{F}_p[v_1^{\pm 1}] = \pi_*K(1)$$

Remark 2.9. It is a summand because:

$$\widehat{\mathrm{KU}} = \bigoplus_{i=0}^{p-2} \Sigma^{2i}\widehat{\mathrm{KU}}^{\mathrm{Ad}}$$

Proof.

$$\pi_*\left(\Sigma^{2i}\widehat{\mathrm{KU}}^{\mathrm{Ad}}\right) = \beta^i\mathbb{Z}_p[\beta^{\pm(p-1)}]$$

◁

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Fact 2.10.

$$K(1) \simeq \widehat{\mathrm{KU}}^{\mathrm{Ad}}/p$$

Proof idea. Use the fact that $K(1)$ is a field. □

3. $L_{K(1)}\mathbb{S}$

Theorem 3.1 (Devinatz-Hopkins, 2004)**.**

$$L_{K(n)}\mathbb{S} \simeq E(n)^{\mathrm{h}\Gamma_n}$$

This now becomes a question about group (co)homology.

For $p > 2$ and $n = 1$, we can give a low-brow proof using $K(1) = \widehat{\mathrm{KU}}_p^{\mathrm{Ad}}$.

Proof. Remember that $\mathbb{Z}_p^\times = \mu_{p-1} \times \left(1 + p\mathbb{Z}_p\right)^\times$. The first term is cyclic of order $p-1$, generated by ζ. The second term is pro-p-cyclic, in that it is abstractly isomorphic to $\mathbb{Z}_p$, with topological generator $1+p$. By the Chinese Remainder Theorem, we have that $\mathbb{Z}_p^\times$ is pro-cyclic, and is topologically generated by $g := \zeta(1+p) \in \mathbb{Z}_p^\times$. Then,

$$E(1)^{h\Gamma_1} = \mathrm{fib}\left(E(1) \xrightarrow{1-\Psi^g} E(1)\right) = \mathrm{fib}\left(\widehat{\mathrm{KU}} \xrightarrow{1-\Psi^g} \widehat{\mathrm{KU}}\right)$$

We want to show that $L_{K(1)}\mathbb{S}$ is this fiber F. That is equivalent to asking that the natural map $\mathbb{S} \to F$ induces an isomorphism on $K(1)$-homology. But $K(1)$ is a retract of $\widehat{\mathrm{KU}}/p$, so we we need $\mathbb{S} \to F$ to induce an isomorphism on $\widehat{\mathrm{KU}}/p$-homology. By retract, we mean we have maps:

$$\begin{tikzcd}
K(1) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \widehat{\mathrm{KU}}/p \arrow[r] & K(1)
\end{tikzcd}$$

Therefore, equivalently we want:

$$\pi_*\Big(\widehat{\mathrm{KU}}/p\Big) \simeq \Big(\widehat{\mathrm{KU}}/p\Big)_* F$$

We have implicitly used:

Fact 3.2. *$E(n)$ is $K(n)$-local*

Claim 3.3. $\widehat{\mathrm{KU}}_*\Big(\widehat{\mathrm{KU}}/p\Big) = \mathrm{Cont}\Big(\mathbb{Z}_p^\times, \mathbb{F}_p\Big)[\beta^{\pm 1}]$, *continuous maps between the two. Further, under this identification, Φ^g acts on these continous maps by the translation action of g on $\mathbb{Z}_p^\times$.*

We continue the earlier proof assuming the claim, which we will return to later. We have a long exact sequence

$$\cdots \to \Big(\widehat{\mathrm{KU}}/p\Big)_* F \to \Big(\widehat{\mathrm{KU}}/p\Big)_* \widehat{\mathrm{KU}} \xrightarrow{1-\Psi^g} \Big(\widehat{\mathrm{KU}}/p\Big)_* \widehat{\mathrm{KU}} \to \cdots$$

Therefore,

$$\ker\Big(1 - \Psi^g \mid \mathrm{Cont}\Big(\mathbb{Z}_p^\times, \mathbb{F}_p\Big)\Big) = \text{constant functions} = \mathbb{F}_p$$

$$\mathrm{coker}\Big(1 - \Psi^g \mid \mathrm{Cont}\Big(\mathbb{Z}_p^\times, \mathbb{F}_p\Big)\Big) = 0$$

Therefore, we are done once we identify:

$$\pi_*\Big(\widehat{\mathrm{KU}}/p\Big) \simeq \mathbb{F}_p[\beta^{\pm 1}]$$

□

Proof of (Claim 3.3). Remember that:

$$\mathcal{M}_{\mathrm{fg}} := [(\mathrm{Spec}\, L/\mathrm{Spec}(\mathrm{MU}_*\mathrm{MU}))/\mathbb{G}_m]$$

$\widehat{\mathrm{KU}}$ is the Landweber exact spectrum attached to $\widehat{\mathbb{G}}_m$. The map that classifies $\widehat{\mathbb{G}}_m$ is:

$$\Big[\Big(\mathrm{Spec}\,\widehat{\mathrm{KU}}/\mathrm{Spec}\,\widehat{KU}_*\widehat{\mathrm{KU}}\Big)/\mathbb{G}_m\Big] \xrightarrow{\widehat{\mathbb{G}}_m} \mathcal{M}_{\mathrm{fg}}$$

So $\mathrm{Aut}_{\mathbb{Z}_p}\Big(\widehat{\mathbb{G}}_m\Big)$, the automorphism scheme of the formal group law, is given by:

$$\mathrm{Aut}_{\mathbb{Z}_p}\Big(\widehat{\mathbb{G}}_m\Big) = \mathrm{Spec}\,\widehat{\mathrm{KU}}_*\widehat{\mathrm{KU}}$$

This is what it means to be a Hopf algebroid. On the other hand,

$$\mathrm{Aut}\Big(\widehat{\mathbb{G}}_m\Big) \simeq \mathbb{Z}_p^\times = \mathrm{Spec}\Big(\mathrm{Cont}\Big(\mathbb{Z}_p^\times, \mathbb{Z}_p\Big)\Big)$$

If we base change to mod p, we get:

$$\Big(\widehat{\mathrm{KU}}_0\Big(\widehat{\mathrm{KU}}/p\Big)\Big)[\beta^{\pm 1}]/\widehat{\mathbb{G}}_m \simeq \Big(\widehat{\mathrm{KU}}_*\widehat{\mathrm{KU}}/p\Big)/\widehat{\mathbb{G}}_m \simeq \mathrm{Cont}\Big(\mathbb{Z}_p^\times, \mathbb{F}_p\Big)$$

□

Corollary 3.4 (p odd)**.**

$$\pi_n L_{K(1)}\mathbb{S} = \begin{cases} \mathbb{Z}_p & n = 0, \text{-}1 \\ \mathbb{Z}/p^{k+1}\mathbb{Z} & n+1 = 2(p-1)p^k m', \quad p \nmid m' \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Using $\Psi^g\beta = g\beta$,

$$\cdots \to \pi_n\widehat{\mathrm{KU}} \xrightarrow{1-\Psi^g} \pi_n\widehat{\mathrm{KU}} \to \pi_{n-1}L_{K(1)}\mathbb{S} \to \pi_{n-1}\widehat{\mathrm{KU}} \xrightarrow{1-\Psi^g} \pi_{n-1}\widehat{\mathrm{KU}} \to \cdots$$

We also need that $\pi_{2m-1}L_{K(1)}\mathbb{S} = \mathbb{Z}_p/(1-g^m)$, with $m \geq 1$. Remembering that $g = \zeta(1+p)$. Thus,

$$\pi_{2m-1}L_{K(1)}\mathbb{S} = \begin{cases} 0 & p-1 \nmid m \\ \mathbb{Z}/\left(\frac{m}{p-1}\right) & p-1 \mid m \end{cases}$$

□

We can do similar things for $p = 2$ with real K-theory. However, we won't.

4. $L_{E(1)}\mathbb{S}$

Recall the fracture square from earlier:

$$\begin{tikzcd} L_1\mathbb{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{\mathrm{H}\mathbb{Q}}\mathbb{S} \arrow[r, "\sim"] \arrow[d] & \mathrm{H}\mathbb{Q} \\ L_{K(1)}\mathbb{S} \arrow[r] & L_{\mathrm{H}\mathbb{Q}}L_{K(1)}X & \end{tikzcd}$$

The bottom right has homotopy:

$$\pi_*\left(L_{\mathrm{H}\mathbb{Q}}L_{K(1)}\mathbb{S}\right) = \begin{cases} \mathbb{Q}_p & * = 0, -1 \\ 0 & \text{otherwise} \end{cases}$$

$$\cdots \to \pi_{n+1}\left(L_{K(1)\oplus L_{\mathrm{H}\mathbb{Q}}\mathbb{S}}\right) \to \pi_{n+1}L_{\mathrm{H}\mathbb{Q}}L_{K(1)}\mathbb{S} \to \pi_n L_1\mathbb{S} \to \pi_n\left(L_{K(1)}\mathbb{S} \oplus L_{\mathrm{H}\mathbb{Q}}\mathbb{S}\right) \to \pi_n L_{\mathrm{H}\mathbb{Q}}L_{K(1)}\mathbb{S} \to \cdots$$

For $n \neq -2, -1, 0$, we then have $\pi_n L_1\mathbb{S} = \pi_n L_{K(1)}\mathbb{S}$. Otherwise, we have an exact sequence:

$$0 \to \pi_0 L_1(\mathbb{S}) \to \mathbb{Z}_p \oplus \mathbb{Q} \to \mathbb{Q}_p \to \pi_{-1}L_1\mathbb{S} \to \mathbb{Z}_p \to \mathbb{Q}_p \to \pi_{-2}L_1\mathbb{S} \to 0$$

Proposition 4.1 (p prime)**.**

$$\pi_n L_1\mathbb{S} = \begin{cases} \mathbb{Z}_{(p)} & n = 0 \\ \mathbb{Q}_p/\mathbb{Z}_p & n = -2 \\ \mathbb{Z}/p^{k+1}Z & n+1 = 2(p-1)p^k m', \quad p \nmid m' \\ 0 & \textit{otherwise} \end{cases}$$