18.906 - Classifying spaces, simplicial sets, and Čech categories

Howard Beck*

Department of Mathematics, Massachusetts Institute of Technology

October 25th, 2024

These are notes for one of my presentations from Prof. Haynes Miller's 18.906 (Algebraic Topology II) reading group at MIT in Fall 2024, and follows chapters 57-59 from his book [Mil22]. As they are presentation notes, they are not meant to be complete, as some material is cut for time. However, they are also not a subset of the book, as there are additional remarks and details here. 18.906 was not taught this academic year, and the reading group sought out to address this gap. These notes were updated afterwards to address confusions during the talk, and also to include insightful remarks and correct errors pointed out by Miller.

1 Universal bundles and compact Lie groups

1.1 Representability of Bun_G

Remember: Bun_G is a functor:

$$\begin{aligned} \mathrm{Bun}_G : \mathrm{Top} &\to \mathrm{Set} \\ X &\mapsto \{\text{isomorphism classes of principal } G\text{-bundles}\} \end{aligned}$$

Further, we have I-invariance, which implies homotopy invariance, so it descends onto the homotopy category:

$$\mathrm{Bun}_G : \mathrm{Ho}(\mathrm{Top}) \to \mathrm{Set}$$

Theorem 1.1.1. *Bun_G is representable on $\mathrm{Ho}(\mathrm{CW})$: that is, we have a universal bundle $\xi : EG \downarrow BG$ that every G-bundle is pulled back from via a unique homotopy class of maps.*

$$\begin{aligned} [X, BG] &\xrightarrow{\sim} \mathrm{Bun}_G(X) \\ [f] &\mapsto f^*\xi \end{aligned}$$

for $x \in \mathrm{CW}$

Proof idea:

0. Check it is well-defined: we did this last lecture - homotopic maps induce isomorphic pullbacks.
1. Find a (not $G-$) weakly contractible G-space EG, BG is the quotient by the group action. This will be our universal bundle. We defer the construction to later.
2. With it, prove that homotopic maps induce isomorphic pullbacks (well-defined)
3. Prove injectivity and surjectivity

*hbeck@mit.edu

Remark 1.1.2. *EG is only contractible as a standard space, not a G-space. For example, we will soon see, $B\mathbb{Z} = \mathbb{S}^1 = K(\mathbb{Z}, 1)$, and then EG is the universal cover $EG = \mathbb{R}$. $\mathbb{R}$ is clearly contractible and has the standard $\mathbb{Z}$-action. This comes from the homotopy:*

$$\begin{aligned} f : \mathbb{R} &\to * \\ x &\mapsto * \\ g : * &\to \mathbb{R} \\ * &\mapsto 0 \\ g \circ f = \{x \mapsto 0\} &\simeq \mathrm{id}_{\mathbb{R}} \quad \textit{via } h(t, x) = tx \end{aligned}$$

But h is certainly not a G-homotopy. Therefore, EG is (weakly) contractible in Top*, but certainly not in* $\mathbb{Z}$-Top*, even though I treat it as a $\mathbb{Z}$-space so that is the category we're working with, and its notion of contractability is more complicated and considers the G-action (I haven't told you what it is, but it's beyond the scope of these notes anyways). However, I will still say EG is a weakly contractible G-space. To make this more explicit: EG is a G-space, and for the forgetful functor $F : G\text{-Top} \to \text{Top}$ that discards the G-action, $F(EG)$ is weakly contractible.*

The difficult is that G may have a topology we want to respect. When G has a discrete topology, this is easy: we take $BG = K(G, 1)$ and EG to be its universal cover, which has a $\pi_1(BG) = \pi_1(K(G, 1)) = G$-action by construction. However, for example, we will want the action $G \curvearrowright EG$ to be continuous.

We will give a very "geometric" construction of EG and BG for compact Lie groups, and then later on we talk about how to construct these objects for any group. However, we won't always get that $EG \downarrow BG$ is a bundle, but it will be for a very broad class of topological groups.

Useful proposition:

Proposition 1.1.3. *Let:*

- *E is a weakly contractible G-space*
- *$(P,\ A)$ is a relative G-CW complex pair and P has a free G-action (all the cells are free, that is, constructed from $D^n \times G$, as $H \backslash G$ has fixed points)*

Then an equivariant map $f : A \to E$ extends to an equivariant map $\tilde{f} : P \to E$, that is unique up to equivariant homotopy relative to A

$$\begin{tikzcd} P \arrow[dr, dashed, "\tilde{f}"] & \\ A \arrow[u, hook] \arrow[r, "f"'] & E \end{tikzcd}$$

Proof. Same idea as proof of I-invariance of Bun_G. □

Now, back to the proof of Theorem 1.1.1:

Surjectivity
We use Proposition 1.1.3 to a bundle $\xi : P \to X$, using the pair $(\emptyset, P)$ and $E = EG$:

$$\begin{tikzcd} P \arrow[dr, dashed] & \\ \emptyset \arrow[u, hook] \arrow[r] & EG \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ X \arrow[r, dashed] & BG \end{tikzcd}$$

So for a principal G-bundle P, we get a (unique up to equivariant homotopy) map $P \to EG$, which descends onto a map $X \to BG$ via quotient by the G-action on both sides. By construction, this is a pullback diagram, determining P via the map $X \to BG$, that is now unique up to normal homotopy.

The upshot is that P determines a unique (up to homotopy) map $X \to BG$, from which ξ is pulled back.

Injectivity
Suppose there are two maps $f_1, f_2 : X \to BG$ that induce the same pullback, along with functions upstairs $\tilde{f}_1, \tilde{f}_2 : P \to EG$:

$$\begin{tikzcd} P \arrow[r, "\tilde{f}_1, \tilde{f}_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ X \arrow[r, "f_1, f_2"'] & BG \end{tikzcd}$$

We can think of the top as a function $P \times \partial I \to EG$, and we apply Proposition 1.1.3 to the pair $(P \times \partial I, P \times I)$

$$\begin{tikzcd} P \times I \arrow[dr, dashed, "\tilde{h}"] & \\ P \times \partial I \arrow[u, hook] \arrow[r, "\tilde{f}"'] & EG \end{tikzcd}$$

We now get that f_1, f_2 are equivariantly homotopic. After quotienting out by the G-action, we get a standard homotopy $h : X \times I \to BG$ between f_1, f_2, which completes this part of the proof.

We've proved injectivity and surjectivity, so the game is now to actually construct EG and BG.

Also, as a note for terminology:

Definition 1.1.4. *We call BG the classifying space of G, and $EG \downarrow BG$ is the universal bundle.*

1.2 Universal bundles of compact Lie groups

So how do we construct BG? When G is a compact Lie group, we have:

Theorem 1.2.1 (Peter-Weyl)**.** *Any compact Lie group admits a faithful unitary representation.*

That is, there is continuous and injective homomorphism:

$$\rho : G \to U(\mathbb{C}^n)$$

Of course then, $G \simeq \mathrm{im}(\rho)$ which is a subgroup of $U(\mathbb{C}^n)$. If E is a weakly contractible, free $U(\mathbb{C}^n)$-space, then it is also a principal G-space. By quotienting out the G-action, we would get BG.

Lemma 1.2.2. *Any (n-dimensional) vector bundle over a compact Hausdorff space embeds into a trivial bundle.*

Proof. Look at the book, it's not too interesting. Uses a finite cover with k open sets and a partition of unity subordinate to this cover (guaranteed as compact Hausdorff spaces are also paracompact). The upshot is you have the following embedding:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & B \times \mathbb{R}^{nk} \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

□

In particular, we have complements:

Corollary 1.2.3. *A vector bundle ξ over a paracompact space has a complement $\xi^\perp$ where $\xi \oplus \xi^\perp = (nk)\epsilon$ (trivial bundle).*

Proof. Use inner product on $\mathbb{R}^{nk}$. $\square$

The fibers have dimension n in $\mathbb{R}^{nk}$, which gives a map, so that B embeds as n-planes in $\mathbb{R}^{nk}$. This let's us express ξ as a pullback of the tautological bundle:

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(\xi_{nk,n}) \arrow[d] \\ B \arrow[r] & \mathrm{Gr}_n(\mathbb{R}^{nk}) \end{tikzcd}$$

We will try to find a principal $U(n)$-bundle over $\mathrm{Gr}_n(\mathbb{R}^{nk})$. Warmup:

Definition 1.2.4. *A Stiefel variety/manifold is*

$$V_n(\mathbb{C}^{n+k}) = \{\textit{linear isometric embeddings } \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+k}\}$$

We can imagine a transitive right-action by $U(\mathbb{C}^{n+k})$ via post-composition. Note however that everything is in the same $U(\mathbb{C}^k)$-orbit, where $U(\mathbb{C}^k) \subset U(\mathbb{C}^{n+k})$. In fact, for two elements to be in different orbits, they must differ by some $g \in U(\mathbb{C}^k)$. This means we can identify:

$$V_n(\mathbb{C}^{n+k}) = U(\mathbb{C}^{n+k})/U(\mathbb{C}^k)$$

We also get a fiber bundle

$$\begin{aligned} p : V_n(\mathbb{C}^{n+k}) &\to \mathrm{Gr}_n(\mathbb{C}^{n+k}) \\ \{f : \mathbb{C}^n \to \mathbb{C}^{n+k}\} &\mapsto \mathrm{im}(f) \end{aligned}$$

that gives us a principal $U(\mathbb{C}^n)$-bundle.

$U(\mathbb{C}^q)$ acts transitively on the unit sphere in $\mathbb{C}^q$, which is $\mathbb{S}^{2q+1}$. The isotropy group of the basis vector e_q is $Uq - 1$

$$\begin{tikzcd} \mathbb{S}^{2k+1} \arrow[r] & U(\mathbb{C}^{n+k})/U(\mathbb{C}^k) \arrow[r, equal] \arrow[d] & V_n(\mathbb{C}^{n+k}) \\ \mathbb{S}^{2k+3} \arrow[r] & U(\mathbb{C}^{n+k})/U(\mathbb{C}^{k+1}) \arrow[r, equal] \arrow[d] & V_{n-1}(\mathbb{C}^{n+k}) \\ & \vdots \arrow[d] & \\ \mathbb{S}^{2(n+k)-1} \arrow[r, equal] & U(\mathbb{C}^{n+k})/U(\mathbb{C}^{n-1+k}) \arrow[r, equal] & V_1(\mathbb{C}^{n+k}) \end{tikzcd}$$

Long exact homotopy sequence: $V_n(\mathbb{C}^{n+k})$ is $(2k)$ -connected:

$$V_n(\mathbb{C}^\infty) = \varinjlim V_n(\mathbb{C}^{n+k})$$

is a contractible CW complex with a principal $U(\mathbb{C}^n)$-action, with a quotient map $\mathrm{Gr}_n(\mathbb{C}^\infty)$. This gives us:

$$EU(\mathbb{C}^n) = V_n(\mathbb{C}^\infty), BU(\mathbb{C}^n) = \mathrm{Gr}_n(\mathbb{C}^\infty)$$

and we can construct EG, BG for any compact Lie group via quotients.

2 Simplicial sets

2.1 Simplicial objects

Definition 2.1.1. *Simplex category* $\mathbb{\Delta}$*:*
Objects are the n*-simplices:* $[n] = \{0, 1, \ldots, n\}$*, morphisms are order-preserving maps*

Morphisms generated by the (injective) coface maps:

$$d^i : [n] \to [n+1], \text{forgets } i$$

and (surjective) codegeneracy maps:

$$s^i : [n+1] \to [n], \text{doubles up on } i$$

Definition 2.1.2. *Simplicial objects in a category* $\mathcal{C}$*, is:*

$$\mathrm{s}\,\mathcal{C} = \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$$

i.e. functors between the simplex category and your category.
The objects in $\mathrm{s}\,\mathcal{C}$ are called **simplicial objects** - these are specific functors $X : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$. Namely, for a simplicial object, we can define the n-simplices, which are the images $X_n = X([n])$. Of course, I just need to tell you where the simplices go, as those are all the objects of $\mathbb{\Delta}$.

Example 2.1.3 (Important)**.** *Simplicial sets:*

$$\mathrm{s}\,\mathrm{Set} = \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Set})$$

The objects in $\mathrm{s}\,\mathrm{Set}$ *are called simplicial sets.*

Example 2.1.4. *Each simplex gives you a simplicial set:*

$$\mathrm{Hom}_{\mathbb{\Delta}}(-, [n]) : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$$

Example 2.1.5 (Important)**.** *Any space* X *gives us a simplicial object of* Top*, or equivalently, an object of* $\mathrm{s}\,\mathrm{Top}$ *via:*

$$\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$$

These form the basis for singular homology.

Note that Top is locally small, so that in fact, $\mathrm{Sing}(X)$ is a simplicial set. We can also think of it as a covariant functor:

$$\mathrm{Sing} : \mathrm{Top} \to \mathrm{s}\,\mathrm{Set}$$

If you've heard of words such as ∞-category or Kan complex, we use simplicial objects to encode such higher information. For example, in Ho(Top), we would like to encode information about objects and homotopies, but also homotopies of homotopies, etc.

2.2 Geometric realization

We want a way to go the other way from Sing - to get a topological space out of a simplicial set. This is geometric realization.

Let K be a simplicial set (an object of $\mathrm{s}\,\mathrm{Set}$).

Definition 2.2.1. *The geometric realization functor is given by:*

$$\begin{aligned} |-| : \mathrm{s\,Set} &\to \mathrm{Top} \\ K &\mapsto \left(\coprod_{n \geq 0} \Delta^n \times K_n \right) / \sim \end{aligned} \tag{1}$$

with equivalence relation

$$\Delta^m \times K_m \ni (v, \phi^* x) \sim (\phi_* v, x) \in \Delta^n \times K_n$$

here $\phi : [m] \to [n]$ induces maps $\phi_* : \Delta^m \to \Delta^n$ and $\phi^* : K_n \to K_m$. You've probably seen it before when calculating homology of (geometrically realized) simplicial sets.

We have a full subcategory of Top given by the simplices: $\mathbb{\Delta}_{\mathrm{Top}}$. This lets us think of geometric realization as a balanced product:

$$|K| = \mathbb{\Delta}_{\mathrm{Top}} \times_{\mathbb{\Delta}} K$$

from the functors $\Delta : \mathbb{\Delta} \to \mathrm{Top}$ (covariant) and $K : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ (contravariant). This should remind you of the Borel construction, taking a fiber product of a principal bundle with a right G-action with a set with a left G-action.

Exercise: think about why $|K|$ is a CW complex. Hint: K_n just has the discrete topology, it's just a set... for now...

Remark 2.2.2. *Recall the simplicial set from Example 2.1.4. The geometric realization of* $\mathrm{Hom}_{\mathbb{\Delta}}(, [n])$ *is homeomorphic to the n-simplices Δ^n.*

Upshot:

Theorem 2.2.3. $|-| : \mathrm{s\,Set} \rightleftharpoons \mathrm{Top} : \mathrm{Sing}$ *is an adjoint pair.*

Proof. Left out for the sake of time - look at details in book. ☐

Remark 2.2.4. *We have an incredibly inefficient CW approximation,*

$$|\operatorname{Sing}(X)| \to X$$

The map comes from the counit of the adjunction.

2.3 Nerves and classifying spaces

We've seen how to embed $\mathbb{\Delta}$ into Top via the functor $\Delta : \mathbb{\Delta} \to \mathrm{Top}$, mapping $[n]$ to the topological n-simplex Δ^n. We can ask about how to do this in general, which is what we do now:

We can think of any simplex as a poset, with order-preserving maps. A poset, as a category, has morphisms $x \to y$ when $x \leq y$. This gives a functor $i : \mathbb{\Delta} \to \mathrm{Cat}$, the category of small categories. Simplices get sent to posets. Morphisms (order-preserving maps) get sent to functors between the posets.

Definition 2.3.1 (Nerves)**.** *For a small category $\mathcal{C}$, the nerve is a simplicial set:*

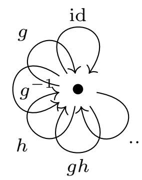
$$N_n\,\mathcal{C} = (N\,\mathcal{C})_n = \mathrm{Fun}(i([n]), \mathcal{C})$$

This is a simplicial set: it is a functor from $\mathbb{\Delta}$ to Set. It sends $[n]$ to the set of functors from $i([n])$ to $\mathcal{C}$. We can also think of it as a covariant functor:

$$N : \mathrm{Cat} \to \mathrm{s\,Set}$$

It feels like categorical nonsense, but it's pretty neat. $N_0\,\mathcal{C}$ are the objects, $N_1\,\mathcal{C}$ are morphisms. $N_n\,\mathcal{C}$ are n-chains of morphisms. d_0 sends a morphism to its target, d_1 to its source.

Reminder: a group can be thought of as precisely being a monoidal groupoidal category. 18.701 has a different definition. However, you get to look smart now and throw around fancy words that sound like an alien language. Unfortunately, this new definition is actively unhelpful for learning what a group is for the first time.



Then,

$$N_n G = G^{\times n}$$

as you can compose all morphisms, so that n-chains are just n ordered choices of morphisms in G (that is, elements of the underlying set of G). This comes with the face maps:

$$d_i(g_1, \ldots, g^n) = (g_1, \ldots, g_{i-1}, \overbrace{g_i g_{i+1}}, g_{i+2}, \ldots)$$

Definition 2.3.2. *We can now form the classifying space of a small category, which is a functor:*

$$\begin{aligned} B : \mathrm{Cat} &\to \mathrm{Top} \\ \mathcal{C} &\mapsto |N\,\mathcal{C}\,| \end{aligned} \tag{2}$$

from small categories to topological spaces.

Normal examples:

Example 2.3.3.

$$BC_2 \cong \mathbb{R}\,P^\infty$$

$$B\,\mathbb{Z} \cong \mathbb{S}^1$$

Messed up example:

Example 2.3.4.

$$B\,\mathbb{N} \cong \mathbb{S}^1$$

Therefore, we certainly cannot get injectivity.

Theorem 2.3.5. *$B(\mathcal{C} \times \mathcal{D}) \cong B\,\mathcal{C} \times B\,\mathcal{D}$, and naturally so.*

Note: this is not just "categorical nonsense." You need certain limits to commute with colimits. It's moreso a fact of topology than of category theory, based on decomposition of the simplices Δ^n.

An important corollary:

Corollary 2.3.6. *B sends natural transformations to homotopies. That is, for functors: $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta : F \Rightarrow G$, we get maps $BF, BG : B\,\mathcal{C} \to B\,\mathcal{D}$. However, this is not just a map, but a homotopy. That is, $B\eta$ is a homotopy between*

$$BF : B\,\mathcal{C} \to B\,\mathcal{D}$$

$$BG : B\,\mathcal{C} \to B\,\mathcal{D}$$

Proof. A natural transformation between $F, G : \mathcal{C} \to \mathcal{D}$ determines a functor

$$\mathcal{C} \times [1] \to \mathcal{D}$$

with F at 0 and G at 1. Applying B to both sides gives a map:

$$B(\mathcal{C} \times [1]) \cong B\mathcal{C} \times B[1] = B\mathcal{C} \times I \to B\mathcal{D}$$

This is a homotopy between BF and BG. □

Remark 2.3.7. *This is something quite remarkable. Natural transformations do not go in both directions, but homotopies are, by definition, symmetric. We can then think of homotopy theory as a symmetric analog of category theory.*

In particular,

Corollary 2.3.8. *If $\mathcal{C}$ contains an initial object, then $B\mathcal{C}$ is contractible.*

Proof. If $o \in \mathcal{C}$ is initial, then $F : [0] \to \mathcal{C}$ sending $0 \mapsto o$ is a left-adjoint to $G : \mathcal{C} \to [0]$. The unit of the adjunction is a natural transformation:

$$\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$$

Of course, BG is forced to be a map $B\mathcal{C} \to *$, so that $B(FG)$ factors as $B\mathcal{C} \to * \to B\mathcal{C}$. $B\,\mathrm{id}_C$ is of course the trivial isomorphism $B\mathcal{C} \cong B\mathcal{C}$. Corollary 2.3.6 tells us then that

$$\{B\mathcal{C} \to * \to B\mathcal{C}\} \simeq \{B\mathcal{C} \overset{\mathrm{id}_{\mathrm{Top}}}{\to} B\mathcal{C}\}$$

In other words, $B\mathcal{C}$ is contractible. □

Note: this also works if $\mathcal{C}$ has a final object, you just use the counit instead.

2.4 Translation groupoid

Definition 2.4.1. *Let $G \curvearrowright X$ be a G-set. We have the translation groupoid GX:*

$$\mathrm{ob}(GX) = X$$
$$\mathrm{Hom}_{GX}(x, y) = \{g \in G : gx = y\}$$

Then, we can form GG based on the left-action $G \curvearrowright G$ by multiplication. Here, there is only one morphism between two objects. In other words, all objects in GG are both initial and final. Therefore, $B(GG)$ is contractible, by Corollary 2.3.8. Also, it still has a G-action from the right, via the right action on GG. And, $B(GG)/G = BG$.

This gives up the upstairs part, so that $EG = B(GG)$ and BG is... well... BG. What remains is to know that this respects topology when G is a topological group, and that the projection map is in fact a fiber bundle map.

3 Enriched and internal categories, Čech categories

3.1 Topologically enriched and internal categories

Definition 3.1.1. *A (small) category enriched in* Top *has topologies on the hom-sets, and composition is continuous.*

For example, we require that the following be a continuous map between topological spaces, for any morphism $f : y \to z$:

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \overset{f:y \to z}{\to} \mathrm{Hom}_{\mathcal{C}}(x, z)$$

There is a construction parallel to that of the category of small categories:

Definition 3.1.2. *The category of small categories enriched in* Top *is called* Top - Cat.

Example 3.1.3. *When G is a topological group, the category G is enriched in* Top. $\mathrm{ob}(G) = \{\bullet\}$, *and* $\mathrm{mor}(G) = G$ *which has a topology, and composition is continuous.*

A slight generalization:

Definition 3.1.4. *A category internal to* Top *has spaces C_0 and C_1 (objects and morphisms) with continuous maps:*

$$s, t : C_1 \to C_0$$
$$\mathrm{id} : C_0 \to C_1$$
$$\textit{composition} : C_1 \times_{C_0} C_1 \to C_1$$

with the category axioms

Definition 3.1.5. *The category of small categories internal to* Top *is called* Cat(Top).

Note: you can make similar constructions with other categories. You can think of a Top-enriched category as a category internal to Top where the object space has the discrete topology.

For a topological group, NG is now a simplicial space:

$$NG \in \mathrm{s\,Top}$$

We can still apply geometric realization without modification from Equation (1). It won't have a (canonical) CW structure, as the simplices will be topological spaces, not sets - we used the discrete topology on N_nG to make it a CW complex. However, it's still a perfectly good classifying space functor. We can do this in general:

Definition 3.1.6 (Classifying space functor for categories internal to Top)**.**

$$\begin{aligned} B : \mathrm{Cat}(\mathrm{Top}) &\to \mathrm{Top} \\ \mathcal{C} &\mapsto |NG| \end{aligned} \tag{3}$$

Again, we take $NG \in \mathrm{s\,Top}$, and its n-simplices are not sets, but rather spaces.

Example 3.1.7. *A topological group G is a category internal to* Top, *with the single point as an object, and the space of morphisms given by the group elements of G with their prescribed topology. We can also think of it as a* Top*-enriched category with the discrete topology (not that we have a choice - there is one object).*

Example 3.1.8. *For a group action of a topological group G on a topological space $X \in$* Top, *we get GX as a category internal to* Top. *The object space is X, and the morphisms are group elements G. The hom-sets carry the subspace topology from G.*

Example 3.1.9 (silly, but useful)**.** *For a space X, cX is the category internal to* Top *with $C_0 = C_1 = X$. That is, there are only identity morphisms $x \to x$. Also, $B(cX) = X$, where use B from Equation (3).*

Remark 3.1.10. *This example seems to tell us that every space is realized as a classifying space. But we have to be careful about which classifying space functor we're talking about. As a standard category, cX has a classifying space from Equation (2) that is the set X with the discrete topology. We work here with a topologically-compactible generalization of the classifying space functor, from Equation (3).*

Note that if G is a topological group and $X \in \mathrm{Top}$ is a space, then also GX is a category internal to Top.

Putting everything together:

Proposition 3.1.11. *$B(GG) \to BG$ is a principal G-bundle when G is a Lie group (and more generally too)*

This also lets us construct Eilenberg-MacLane spaces in another way. If π is an abelian group (not a topological one - or alternatively, a topological group with the discrete topology), then this means precisely that multiplication is a homomorphism $\pi \times \pi \to \pi$. Applying B, since it's a functor:

$$B(\pi \times \pi) = B\pi \times B\pi \to B\pi$$

This gives $B\pi$ the structure of a topological abelian group. We can then repeat this process:

$$B(B\pi) = K(\pi, 2), \dots, B^n\pi = K(\pi, n), \dots$$

3.2 Čech categories

What we would like to do now is explicitly construct a map $X \to BG$ classifying some principal G-bundle $P \downarrow X$. This will require considering a numerable trivializing cover $\mathcal{U}$ and using it to construct such a map. We can construct a map on each open set $U \in \mathcal{U}$, but we need to glue them together in a consistent way. Therefore, we need to encode the information from all the intersections inside of $\mathcal{U}$ to do this properly.

We start with a more general case. Let $\pi : Y \to X$.

Definition 3.2.1. *We define the descent/Čech category $\check{C}(\pi)$ as a* Top*-category with objects and morphisms:*

$$\mathrm{ob}(\check{C}(\pi)) = Y$$

$$\mathrm{mor}(\check{C}(\pi)) = Y \times_X Y$$

To be more explicit, the morphism set is a fiber product - it's the subspace of the product Y^2 that agrees over preimages of X. More explicitly:

$$Y \times_X Y = \{(y_1, y_2) \in Y^2 \mid \pi(y_1) = \pi(y_2)\}$$

The structure maps look like:

$$\mathrm{id} = \Delta : Y \to Y \times_X Y$$

$$s, t = \mathrm{pr}_1, \mathrm{pr}_2 : Y \times_X Y \to Y$$

Composition is a bit more complicated:

$$(Y \times_X Y) \times_X (Y \times_X Y) \to Y \times_X Y$$

which sends $((y_1, y_2), (y_2, y_3)) \mapsto (y_1, y_3)$.

Note 3.2.2. *Because we're working over categories internal to* Top*, we can require our functors be continuous. An important case is that we get a continuous functor between categories internal to* Top*:*

$$\check{\pi} : \check{C}(\pi) \to cX$$

This ties back into the discussion of open covers. Consider $\mathcal{U}$ an open cover for X. This determines a map:

$$Y = \coprod_{U \in \mathcal{U}} U \to X$$

sending $\{x \in U\} \mapsto \{x \in X\}$

$$Y \times_X Y = \coprod U \cap V$$

We form the Čech category and denote it by $\check{C}(\mathcal{U})$.

By applying B to 3.2.2, we get a map:

$$B\check{\pi} : B\check{C}(\mathcal{U}) \to X$$

This map has a homotopy inverse in good cases:

Proposition 3.2.3. *When the cover $\mathcal{U}$ has a subordinate partition of unity, we recover a map going in the opposite direction, and these form a homotopy equivalence.*

Remark 3.2.4. *In constructing $B\check{C}(\mathcal{U})$, we first need to take the nerve of the Čech category. This nerve $N\check{C}(\mathcal{U})$ is sometimes called the Čech nerve, and its geometric realization is $B\check{C}(\mathcal{U})$.*

3.3 Classifying maps

How do we form $X \to BG$ classifying $\xi : P \to X$? Pick a cover $\mathcal{U}$ of B with trivializations $\phi_U : p^{-1}U \xrightarrow{\sim} GU \times G$. This gives us a functor:

$$\check{C}(\mathcal{U}) \to G$$

where G is again a moinoidal category. Objects get sent to $\bullet$. Morphisms use the transition functions: $U \cap V$ is a subspace of the morphisms in $\check{C}(\mathcal{U})$. Functors correspond to points in $U \cap V$ for any $U, V \in \mathcal{U}$: We map these to $\mathrm{mor}(G) = G$ (as a set of group elements) by sending:

$$\{x \in U \cap V\} \mapsto \varphi_V(x)\varphi_U^{-1}(x) \in G$$

You've probably seen the Čech cocycle condition: it precisely says that this assignment constitutes a functor.

Applying B gives us:

$$\begin{tikzcd} B\check{C}(\mathcal{U}) \arrow[r] \arrow[d, "\simeq"', shift left] & BG \\ X \arrow[u, hook, shift left] & \end{tikzcd}$$

and this gives the classifying map $X \to BG$.

3.4 Mayer-Vietoris sequences

This is not spelled out explicitly in the book, but I want to make some comments about the connection between this construction and one we've seen before, for the Mayer-Vietoris sequence. In this section, we will assume $X = A \cup B$, where the interiors of A, B also cover X. Further, we will assume there's a partition of unity subordinate to the cover $\{A, B\}$.

We've run into the problem of properly gluing maps out of subsets $\mathcal{U}$ together before. For some coefficient ring R, we can regard $R\,\mathrm{Sing}(X)$ as the standard singular chain complex for R-valued singular homology, with differentials defined using the face maps of $\mathbb{\Delta}$. We would like to make a short exact sequence of chain complexes:

$$0 \to R\,\mathrm{Sing}(A \cap B) \to R\,\mathrm{Sing}(A) \oplus R\,\mathrm{Sing}(B) \to R\,\mathrm{Sing}(X) \to 0 \qquad \text{This is FALSE}$$

But this fails, as:

$$\mathrm{Sing}(X) \neq \mathrm{Sing}(A) \times_{\mathrm{Sing}(A\cap B)} \mathrm{Sing}(B)$$

In other words, not all maps $\Delta^n \to X$ come from gluing together maps $\Delta^n \to A, B$ that agree on the intersection. However, the following is a short exact sequence, for $\mathcal{U} = \{A, B\}$, which you should check for yourself:

$$0 \to R\,\mathrm{Sing}(A \cap B) \to R\,\mathrm{Sing}(A) \oplus R\,\mathrm{Sing}(B) \to R\,\mathrm{Sing}\left(B\check{C}(\mathcal{U})\right) \to 0$$

The snake or zig-zag give us a a long exact sequence in homology:

$$\cdots \to H_*(A \cap B; R) \to H_*(A; R) \oplus H_*(B; R) \to H_*\left(B\check{C}(\mathcal{U}); R\right) \to H_{*-1}(A \cap B; R) \to \cdots$$

$B\check{\pi}$ is a homotopy equivalence, which gives an isomorphism $B\check{\pi}_* : H_*\left(B\check{C}(\mathcal{U}); R\right) \xrightarrow{\sim} H_*(X; R)$ as desired.

References

[Mil22] Haynes R. Miller. *Lectures on algebraic topology.* World Scientific Publishing Co, New Jersey, 2022.