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# 1 Charge and Coulomb's law

This class is about electromagnetism, one of the four fundamental interactions of nature. Let's see how it compares to the other three:

Force	Range	Strength	Acts on	Carried by
Electromagnetic	"∞"	$10^{-2}$	Charged particles	Photons
Gravitational	"∞"	$10^{-45}$	Mass/Energy/Momentum	Gravitons
Strong	$10^{-15}\text{m}$	1	Color (quarks/gluons)	Gluons
Weak	$10^{-18}\text{m}$	$10^{-6}$	Left-handedness (quarks/leptons)	W/Z/Higgs bosons

- When we say that electromagnetism and gravity have infinite range, we mean that the potential energy between two charges/masses falls off like  $1/r$ , which is rather slow. For comparison the potential energy for the strong force falls off like  $e^{-r/(10^{-15}\text{m})}$  and the potential energy for the weak force falls off like  $e^{-r/(10^{-18}\text{m})}$ . Electromagnetism and gravity are how we obtain information about the world around us.
- Gravity is much weaker than electromagnetism - the electromagnetic force is thus perhaps the "most important" of the four forces for understanding our everyday experiences.
- The only reason we see gravity at all is that electric charge can have either sign, and thus the electromagnetic force is often canceled out.

Electromagnetism was known as an empirical phenomenon already in antiquity. For example the Greek word for amber is  $\eta\lambda\epsilon\kappa\tau\rho\nu$ , and there are records of compasses in China during the Han dynasty (4th century BCE). The modern formalism of classical electromagnetism you will learn in this class was developed in Europe the 19th century, although the full explanatory power of electromagnetism was not realized until it was combined with quantum mechanics. Indeed our current understanding of atoms, light, chemistry, solids, and biology is all based on **quantum electrodynamics**, developed by many people (including Heisenberg, Dirac, Pauli, Feynman, Schwinger, and Tomonaga) during first half of the 20th century. For example one of the great triumphs of quantum electrodynamics is its successful prediction of something called the anomalous magnetic moment of the electron:

$$\begin{aligned} \text{Theory :} & \quad a_e = 0.001159652181643(764) \\ \text{Experiment :} & \quad a_e = 0.00115965218073(28). \end{aligned} \tag{1.1}$$

We won't get all the way there in this class, but we will make a good beginning. Let's get started!

## 1.1 Charge

The first basic axiom of electromagnetism is that every object carries some quantity of "electric charge", which describes how strongly the object feels the electromagnetic force.

- The amount of charge an object carries can be positive or negative, and as far as we know it is also **quantized**: every object we've seen has an amount of charge which is an integer multiple of that of the proton.<sup>1</sup>
- In a reasonable world we would therefore measure charge in units of the charge carried by a proton. A glance at the news however will convince you that we do not live in a reasonable world, and we instead measure charge in "**Coulombs**". By definition the charge of a proton is  $1.602176634 \times 10^{-19}$  Coulombs, which we write as  $1.602176634 \times 10^{-19}$  C.
- To the best of our knowledge charge is **conserved**: it can move around from here to there, and be transferred from one object to another, but it can never be created or destroyed.

<sup>1</sup>Inside of protons and neutrons there are particles called "quarks" which have charges which are  $2/3$  or  $-1/3$  times of the proton, but they cannot be removed without creating more quarks in such a way that the total charge of whatever comes out is an integer multiple of the proton charge.

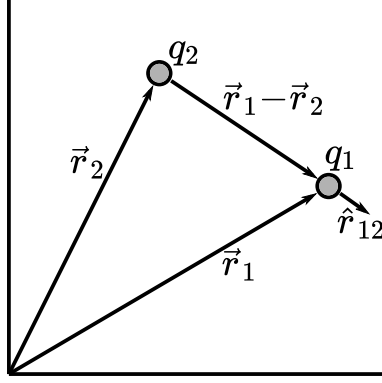


Figure 1: The Coulomb force between two charges: if  $q_1$  and  $q_2$  have the same sign they repel, while if they have opposite signs they attract.

## 1.2 Coulomb's law

Coulomb's law is our first illustration of the electromagnetic force: it is the observation that there is a force between any two charged objects which is proportional to the product of their charges and inversely proportional to the square of the distance between them. Its direction is determined by the principle that “opposite charges attract, while similar charges repel” (see figure 1). In terms of equations, if we have an object of charge  $q_1$  at position  $\vec{r}_1$  and an object of charge  $q_2$  at position  $\vec{r}_2$ , and we define the separation vector

$$\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2 \quad (1.2)$$

and its unit version

$$\hat{r}_{12} \equiv \frac{\vec{r}_{12}}{|\vec{r}_{12}|} \quad (1.3)$$

then the Coulomb force on object one is

$$\vec{F}_1 = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_{12}|^2} \hat{r}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_{12}|^3} \vec{r}_{12} \quad (1.4)$$

and the Coulomb force on object two is

$$\vec{F}_2 = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_{21}|^2} \hat{r}_{21} = \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_{21}|^3} \vec{r}_{21}. \quad (1.5)$$

Here

$$|\vec{r}_{12}| \equiv \sqrt{\vec{r}_{12} \cdot \vec{r}_{12}} \quad (1.6)$$

is the length of  $\vec{r}_{12}$ . We can check that  $\vec{F}_1 = -\vec{F}_2$ , as required by Newton's third law. The quantity  $\epsilon_0$  is called the **permittivity of free space**, and in SI units it is given by

$$\epsilon_0 = 8.8541878128(13) \times 10^{-12} \frac{C^2 s^2}{kg m^3}. \quad (1.7)$$

Here the “(13)” represents experimental uncertainty, since ultimately strength of the Coulomb force is something that we measure. Looking at equations (1.4)-(1.5), it is tempting to redefine the electric charges to be  $\frac{q_1}{\sqrt{\epsilon_0}}$  and  $\frac{q_2}{\sqrt{\epsilon_0}}$  (or perhaps  $\frac{q_1}{\sqrt{4\pi\epsilon_0}}$  and  $\frac{q_2}{\sqrt{4\pi\epsilon_0}}$ ), but, alas, the arbiters of SI units do not allow this:

they insist on measuring charge in Coulombs, so we are unfortunately forced to spend the rest of the semester constantly writing useless factors of  $\frac{1}{4\pi\epsilon_0}$ .<sup>2</sup>

It is important to emphasize that in general the Coulomb force is *not* the only electromagnetic interaction between these two objects. It will be dominant however provided that

- (1) The separation between the objects is large compared to their size. Otherwise “tidal” effects which depend on how the charge is distributed on the objects become important.
- (2) The velocities of the objects are both small compared to the speed of light. Otherwise “magnetic” forces become important.
- (3) Their accelerations are small enough that light can travel back and forth between them before their velocities change appreciably. Otherwise we have to worry about the time delay as “news” about one travels to the other.

For the first third of this class we will always assume that conditions (2) and (3) are satisfied, which puts us in the realm of **electrostatics**. We will now discuss what to do when (1) does not hold.

### 1.3 Superposition and charge density

What do we do if there are more than two charges? In principle it could be that the force of particle two on particle one depends on the location of particle three, and so on, but no such dependence has ever been seen. Indeed there is excellent evidence that to find the net Coulomb force on an object, we simply add up the Coulomb forces due to all of the other objects which are around. In equations, if we label the particles by  $i = 1, 2, \dots$ , then we have

$$\vec{F}_i = \sum_{j \neq i} \frac{q_i q_j}{4\pi\epsilon_0 |r_{ij}|^2} \hat{r}_{ij}. \quad (1.8)$$

This is called the **principle of superposition**.

The principle of superposition gives us a nice way to understand how the Coulomb force generalizes to objects whose size is not small compared to the separation between them. The idea is just to break each object up into very small pieces, each of which *is* small compared to the separation between the objects, and then add up their contributions to the total force. In practice this is most easily accomplished by using infinitely small pieces, with the charge distribution being described by a continuous function of position called a **charge density** and the sums becoming integrals. There are three kinds of charge density we will consider in this class:

- **Linear charge density:** If an object is very thin in two directions but extended in the third, such as a wire, we can describe the charge distribution on it via a function  $\lambda(s)$ , where  $s$  parametrizes length along the object.  $\lambda(s)$  is called the linear charge density, and it tells us that at position  $s$  the amount of charge in an infinitesimal length  $ds$  is

$$dq(s) = \lambda(s)ds. \quad (1.9)$$

$\lambda$  has units of Coulombs per meter. To compute the amount of charge contained between  $s_0$  and  $s_1$ , we evaluate

$$q(s_0, s_1) = \int_{s_0}^{s_1} ds' \lambda(s'). \quad (1.10)$$

---

<sup>2</sup>There *are* more enlightened systems of units where these redefinitions are done. They are called **Heaviside-Lorentz** or **Gaussian** units depending on whether or not we include the  $4\pi$  in the redefinition. Somewhat confusingly, people who use Heaviside-Lorentz or Gaussian units also tend to use centimeters and grams instead of meters and kilograms, leading to some people calling one or the other **cgs** units, but the question of how we define charge is independent from the question of whether or not we prefer centimeters to meters and grams to kilograms. Unfortunately neither Heaviside-Lorentz nor Gaussian units corresponds to measuring charge in units of the proton charge: for quantum mechanical problems using the proton charge is indeed the most natural, but it requires the introduction of Planck’s constant  $\hbar$  and so would be distracting for purely classical applications. For my money the best choice for classical electromagnetism is to use Heaviside-Lorentz units for charge together with meters and kilograms for length and mass.

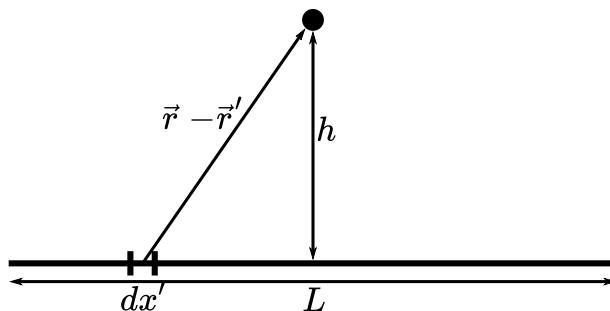


Figure 2: Computing the force of a rod of charge on a point charge above the midpoint of the rod.

- **Surface charge density:** If an object is very thin in one direction but extended the other two, we can describe the charge distribution on it via a function  $\sigma(x, y)$ , where  $x$  and  $y$  are coordinates that tell us where we are on the surface.  $\sigma(x, y)$  is called the surface charge density, and has units of Coulombs per meter squared. It tells us that at location  $(x, y)$  on the surface, the amount of charge in an infinitesimal area  $dA$  is

$$dq(x, y) = \sigma(x, y)dA. \quad (1.11)$$

To compute the charge contained in a region  $R$  of the surface, we integrate

$$q(R) = \int_R dA' \sigma(x', y'). \quad (1.12)$$

The form of  $dA$  depends on what kind of coordinates we use, for example in cartesian coordinates  $dA = dx dy$  while in polar coordinates  $dA = r dr d\theta$ . We will see later that in electrostatics the charge on a conductor is always distributed on its surface, and thus should be described using a surface charge density.

- **Volume charge density:** If an object has no narrow directions, we can describe the charge distribution on it using a function  $\rho(x, y, z)$ , where  $x, y, z$  are coordinates in space.  $\rho(x, y, z)$  is called the charge density function, and has units of Coulombs per meter cubed. The charge in an infinitesimal volume  $dV$  located at position  $(x, y, z)$  is

$$dq(x, y, z) = \rho(x, y, z)dV, \quad (1.13)$$

so to compute the total charge in a spatial volume  $D$  we evaluate

$$q(D) = \int_D dV' \rho(x', y', z'). \quad (1.14)$$

The form of  $dV$  also depends on our choice of coordinates, for example in cartesian coordinates  $dV = dx dy dz$  while in spherical coordinates  $dV = \rho^2 \sin \theta d\rho d\theta d\phi$ .

## 1.4 Force due to a line of charge

We can get a sense of how to use these charge densities by computing the Coulomb force of a finite rod with constant linear charge density  $\lambda$  on a point object with charge  $q$ . Namely we break the rod up into pieces of thickness  $dx'$ , each located at

$$\vec{r}' = x' \hat{x} \quad (1.15)$$

between  $-L/2$  and  $L/2$  on the  $x$ -axis, compute the Coulomb force of each piece on a point charge at point

$$\vec{r} = h \hat{y} \quad (1.16)$$

on the  $y$ -axis, and then integrate. The separation vector is

$$\vec{r} - \vec{r}' = h\hat{y} - x'\hat{x}, \quad (1.17)$$

so we want to compute

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{q\lambda dx'}{(x'^2 + h^2)^{3/2}} (h\hat{y} - x'\hat{x}). \quad (1.18)$$

The reflection symmetry of the problem suggests that the  $x$ -component of this force should vanish, and indeed we see that this is so since the integrand is odd. For the  $y$ -component we can consult Mathematica (free for MIT students) to find the indefinite integral

$$\int^x \frac{dx'}{(x'^2 + h^2)^{3/2}} = \frac{x}{h^2\sqrt{x^2 + h^2}}, \quad (1.19)$$

which tells us that the force is

$$\vec{F} = \frac{q\lambda L}{4\pi\epsilon_0 h\sqrt{h^2 + L^2/4}} \hat{y}. \quad (1.20)$$

There are two interesting limits of this expression. The first is when  $h \gg L$ : then we find

$$\vec{F} = \frac{q\lambda L}{4\pi\epsilon_0 h^2} \hat{y}, \quad (1.21)$$

which is just Coulomb's law for an object of charge  $q$  and an object of charge  $\lambda L$ . The second is when  $L \gg h$ : then we find

$$\vec{F} = \frac{q\lambda}{2\pi\epsilon_0 h} \hat{y}. \quad (1.22)$$

Apparently the force is finite in the limit  $L \rightarrow \infty$ ; we will soon derive this in another way.

## 2 Electric field and Gauss's law

We saw last time that if we have a set of objects at positions  $\vec{r}_1, \vec{r}_2, \dots$  carrying charges  $q_1, q_2, \dots$ , the Coulomb force on an additional object of charge  $q$  at position  $\vec{r}$  is

$$\vec{F} = \sum_i \frac{qq_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i). \quad (2.1)$$

There is a seemingly trivial way of rewriting this equation:

$$\vec{F} = q\vec{E}(\vec{r}), \quad (2.2)$$

where the vector function  $\vec{E}(\vec{r})$  is called the **electric field**: it is given by<sup>3</sup>

$$\vec{E}(\vec{r}) \equiv \sum_i \frac{q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i). \quad (2.3)$$

We can think of the electric field as describing the “electrostatic environment at  $\vec{r}$ ” which is created by the charges at  $\vec{r}_1, \vec{r}_2, \dots$ ; it tells us how any new charge we put at  $\vec{r}$  will respond to the others which are already there. At the moment this may seem like mere book-keeping, but we will eventually see that this definition is one of the best ideas in the history of physics. Indeed our current theories of physics are all **field theories**, where it is fields like the electric field which are the fundamental degrees of freedom and the charged objects are re-interpreted as excitations of those fields.

<sup>3</sup>The electric field is an example of something called a **vector field**, which is a rule that assigns a vector to each point in space. We will say more about the mathematics of vector fields in a few lectures.



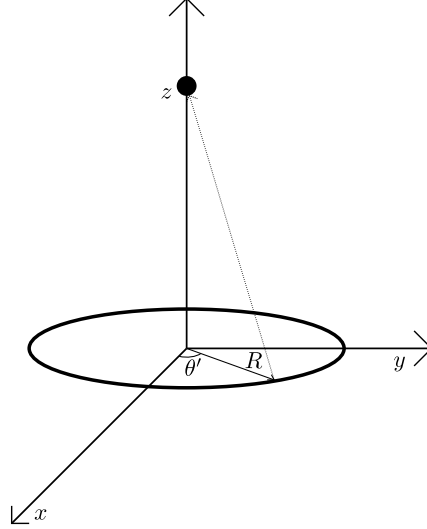


Figure 3: Computing the electric field on the  $z$ -axis from of a charged ring of radius  $R$  in the  $xy$  plane. The separation vector (2.7) is shown as a dotted line.

We can generalize our formula for the electric field to continuous charge distributions. A line density  $\lambda(\vec{r})$  of charge produces an electric field

$$\vec{E}(\vec{r}) = \int \frac{\lambda(\vec{r}') ds'}{4\pi\epsilon_0 |r - r'|^3} (\vec{r} - \vec{r}'), \quad (2.4)$$

a surface density  $\sigma(\vec{r})$  of charge produces an electric field

$$\vec{E}(\vec{r}) = \int \frac{\sigma(\vec{r}') dA'}{4\pi\epsilon_0 |r - r'|^3} (\vec{r} - \vec{r}'), \quad (2.5)$$

and a volume density  $\rho(\vec{r})$  of charge produces an electric field

$$\vec{E}(\vec{r}) = \int \frac{\rho(\vec{r}') dV'}{4\pi\epsilon_0 |r - r'|^3} (\vec{r} - \vec{r}'). \quad (2.6)$$

## 2.1 Electric field of a ring or a disk

Let's see how this works by computing the electric field in a few examples. The first example is to compute the electric field everywhere on the  $z$ -axis created by a circular ring of radius  $R$  and constant linear charge density  $\lambda$  which is lying in the  $xy$  plane and centered at the origin (see figure 3). We want to add up the contributions to the electric field at the point  $(0, 0, z)$  from all points on the ring, which we will do using an integral as in equation (2.4). We can label these points using the polar angle  $\theta'$ , in terms of which we have separation vector

$$\vec{r} - \vec{r}' = z\hat{z} - R(\cos\theta'\hat{x} + \sin\theta'\hat{y}). \quad (2.7)$$

The contribution to the electric field from the piece of the wire at angle  $\theta'$  and with thickness  $d\theta'$  is

$$d\vec{E}(z) = \frac{\lambda R d\theta'}{4\pi\epsilon_0 |r - r'|^3} (\vec{r} - \vec{r}') = \frac{\lambda R d\theta'}{4\pi\epsilon_0 (R^2 + z^2)^{3/2}} (z\hat{z} - R(\cos\theta'\hat{x} + \sin\theta'\hat{y})). \quad (2.8)$$

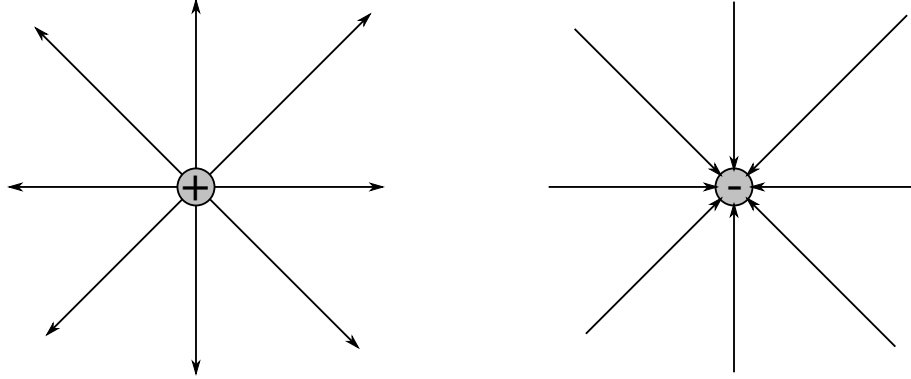


Figure 4: Electric field lines for a single positive charge on the left, and for a single negative charge on the right.

We then integrate  $\theta'$  from zero to  $2\pi$ : the  $x$  and  $y$  components will vanish, as they must by symmetry, so we are left with just the  $z$  component:

$$\vec{E}(z) = \frac{\lambda R z}{2\epsilon_0(R^2 + z^2)^{3/2}} \hat{z}. \quad (2.9)$$

It is worth emphasizing that no calculation is necessary to see the vanishing of the  $x$  and  $y$  components: if we think of adding up the charges in pairs that are on opposite sides of the ring, each pair will only create a  $z$  component for the electric field!

Our second example is to compute the electric field everywhere on the  $z$  axis for a disk of radius  $R$  and *surface* charge density  $\sigma$ . This is simplified if we make use of the calculation we just did: if we think of splitting up the disk into strips of thickness  $dr'$ , then we can view each strip as a wire with linear charge density  $\lambda = \sigma dr'$ ! We then just need to make this replacement in our previous result: each strip contributes an electric field

$$d\vec{E}(z) = \frac{\sigma z r' dr'}{2\epsilon_0(r'^2 + z^2)^{3/2}} \hat{z}, \quad (2.10)$$

and to compute the total electric field we integrate this from  $r' = 0$  to  $r' = R$ . Noting that

$$\frac{d}{dr'} \left( -\frac{1}{\sqrt{r'^2 + z^2}} \right) = \frac{r'}{(r'^2 + z^2)^{3/2}}, \quad (2.11)$$

we find

$$\vec{E}(z) = \frac{\sigma z}{2\epsilon_0} \hat{z} \left( -\frac{1}{\sqrt{r'^2 + z^2}} \right) \Big|_{r'=0}^{r'=R} = \frac{\sigma z}{2\epsilon_0} \left( \frac{1}{|z|} - \frac{1}{\sqrt{R^2 + z^2}} \right) \hat{z}. \quad (2.12)$$

You may already be getting tired of these integrals. There is good news and bad news: the good news is that we will soon learn a powerful technique that in some situations allows us to avoid doing them. The bad news is that sometimes we still need to do them.

## 2.2 Electric field lines

There is a nice way to qualitatively visualize the electric field: we draw directed lines, called **electric field lines**, which at each point are tangent to it and which can begin only at positively-charged objects and end only at negatively charged objects. At each point the direction of the lines tells us the direction of the electric field and the density of lines gives us a rough sense of its magnitude. A few examples are shown in figures 4, 5.

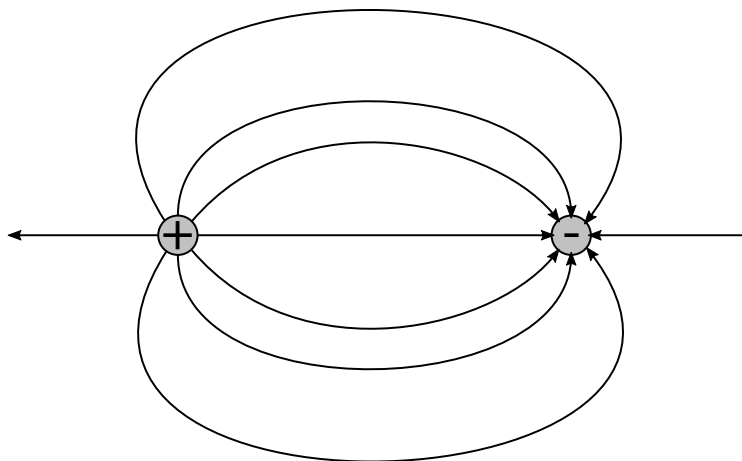


Figure 5: Electric field lines for two charges of equal magnitude and opposite sign. This kind of system is called an **electric dipole**.

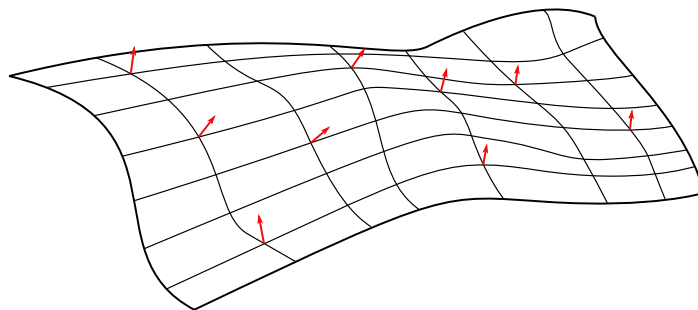


Figure 6: The normal vector to a surface in space, shown at a few points using red arrows. At each point on the surface the normal vector is orthogonal to the surface there, and has length one, and points to the outside of the surface. From the figure it may not be clear which direction is outwards, since we have only shown part of the surface, but every closed surface in space splits it into an inside and an outside.<sup>5</sup>

### 2.3 Electric flux and Gauss's law

One nice feature of electric field lines is that since they can only begin and end at charges, the number of lines which pass through any closed surface (meaning a surface with no boundary) depends only on how many charges are enclosed within it. For example if we consider a sphere surrounding a single point charge, the number of lines which pass through it is independent of its radius. This tells us that the density of the lines must decrease like  $1/r^2$  as we increase  $r$ , which is just what happens to the magnitude of the electric field. We can formalize this intuition using the idea of **electric flux**. To define electric flux, we first need to recall that at each point on any closed surface there is a **normal vector**, which is orthogonal to the surface at that point, has length one, and points outwards (see figure 6). We can then obtain a scalar function on the surface by taking the dot-product of the electric field with the normal vector, and the integral of this

<sup>5</sup>This seemingly obvious statement is not actually so obvious once you think more carefully, and it depends on how you in define a surface. To see an example of the trouble you can get into, you can try googling “Klein bottle”. Since this is not a class in topology, we will not worry about this - all the surfaces we consider will certainly have an inside and an outside!

function over the surface is defined to be the electric flux  $\Phi(S)$  through the surface:

$$\Phi(S) \equiv \int_S dA (\hat{n} \cdot \vec{E}). \quad (2.13)$$

This definition is often simplified by introducing a vector version of the area element,

$$d\vec{A} \equiv \hat{n}dA, \quad (2.14)$$

in terms of which we have

$$\Phi(S) \equiv \int_S \vec{E} \cdot d\vec{A}. \quad (2.15)$$

We can think of  $\Phi(S)$  as being a more precise version of the notion of “how many electric field lines are passing through  $S$ ”. Indeed this is formalized in what is surely the most remarkable result in electrostatics, **Gauss’s law**:

- For any distribution of charges and any closed surface  $S$ , the electric flux through  $S$  is just the charge enclosed by  $S$  divided by  $\epsilon_0$ :

$$\int_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}. \quad (2.16)$$

No matter how complicated a charge distribution inside of some surface might be, the total charge inside is determined entirely by the electric field on the surface!

To prove Gauss’s law, the first step is to consider the case of the electric field of a point charge,

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0|r|^2}\hat{r}, \quad (2.17)$$

where we have chosen to locate the point charge at the origin of our coordinates so  $\hat{r}$  is just a unit vector at each point in space which points radially outwards. Let’s compute the electric flux through a sphere of radius  $R$  surrounding this charge. The normal vector for such a sphere is just  $\hat{r}$ , so on the surface we have

$$\hat{n} \cdot \vec{E} = \frac{q}{4\pi\epsilon_0 R^2}. \quad (2.18)$$

This is constant over the sphere, so the flux is easy to compute: the integral over the area element just multiplies  $\hat{n} \cdot \vec{E}$  by  $4\pi R^2$ , giving us

$$\Phi(S) = \frac{q}{\epsilon_0}. \quad (2.19)$$

This of course is just the charge enclosed divided by  $\epsilon_0$ , as expected from Gauss’s law. In particular notice that the flux is independent of  $R$ . Now the miracle: this answer also holds for surfaces which are not spherical! The reason is that we can approximate any non-spherical surface by a bunch of pieces which locally look like part of a sphere and a bunch of pieces whose normal vector is orthogonal to  $\vec{E}$  and thus do not contribute to the flux integral (see figure 7 for an illustration). Moreover once we decide that Gauss’s law holds for any surface in the field of a single point charge, then superposition immediately tells us it will work for any charge distribution! We will give a more careful derivation of Gauss’s law in a few lectures once we know a bit more vector calculus.

## 2.4 Electric field of a charged solid ball or hollow sphere

Gauss’s law can be very useful in helping us determine the electric field created by symmetric charge configurations. We will spend the rest of this lecture doing a few examples of this. The first we will consider is a solid ball of radius  $R$  carrying a constant charge density  $\rho$ . We could try to compute this by adding up the contributions to the electric field from each point in the ball, but this integral would be somewhat difficult to evaluate. Gauss’s law however gives us a powerful shortcut. Due to the spherical symmetry, if we choose

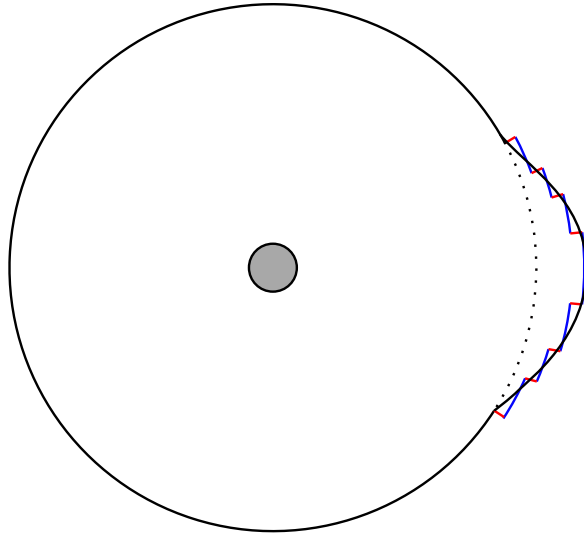


Figure 7: Extending Gauss's law to non-spherical surfaces. The non-spherical part of the black surface is approximated by the red and blue pieces: the red pieces do not contribute to the flux since their normal vectors are orthogonal to  $\vec{E}$  while the flux through the blue pieces depends only on the angle they sweep out (and not their distance from the charge). Thus the flux through the blue pieces can be replaced by the flux through the dotted line, which returns us to the spherical case we have already checked.

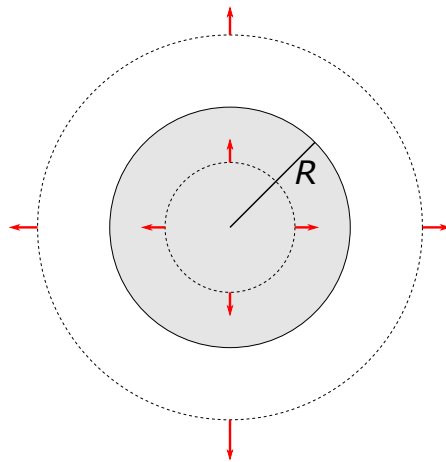


Figure 8: The dashed lines represent Gaussian surfaces for determining the electric field inside and outside of a charged ball or a charged hollow sphere of radius  $R$ . The electric field on these surfaces always points in the normal direction and is independent of angle, making the computation of the flux easy.

the center of the ball to be the origin of coordinates then the electric field has no choice but to be pointing in the radial direction and to depend only on the radial coordinate  $r$ :

$$\vec{E}(\vec{r}) = E_r(r)\hat{r}. \quad (2.20)$$

This form of the electric field makes it easy to compute the electric flux through a spherical surface of radius  $r$ ,  $S_r$ , centered at the center of the charged sphere. Since the normal vector to any such surface is again  $\hat{r}$ , and  $E_r$  is independent of the angular directions on the sphere, the electric flux is simply (see figure 8 for an illustration)

$$\Phi(S_r) = \int_{S_r} \vec{E} \cdot d\vec{A} = 4\pi r^2 E_r. \quad (2.21)$$

Let's first consider the case where  $r > R$ : then the charge enclosed is  $Q_{\text{enclosed}} = \frac{4\pi R^3}{3}\rho$ , so Gauss's law tells us that we must have

$$4\pi r^2 E_r = \frac{4\pi R^3 \rho}{3\epsilon_0}. \quad (2.22)$$

Thus we see that the electric field outside is merely

$$E_r = \frac{\rho R^3}{3\epsilon_0 r^2} \quad (r > R), \quad (2.23)$$

which is the same as we would have found for a point charge of strength  $\frac{4\pi R^3}{3}\rho$ . On the other hand if  $r < R$  then we instead have  $Q_{\text{enclosed}} = \frac{4\pi r^3}{3}\rho$ , so Gauss's law instead tells us that

$$4\pi r^2 E_r = \frac{4\pi r^3 \rho}{3\epsilon_0}, \quad (2.24)$$

or in other words that

$$E_r = \frac{\rho r}{3\epsilon_0} \quad (r < R). \quad (2.25)$$

Thus the electric field of a uniformly-charged ball starts out at zero in the center (as it must by symmetry), grows linearly until we reach the edge of the ball, and then falls off like  $1/r^2$  just like a point charge.

Our second example will be a hollow spherical shell of radius  $R$ , carrying a constant surface charge density  $\sigma$ . We may again assume that the electric field points in the radial direction and depends only on the radial coordinate, and for  $r > R$  the analysis proceeds as before except that the charge enclosed is now  $Q_{\text{enclosed}} = 4\pi R^2 \sigma$  so we find

$$E_r = \frac{\sigma R^2}{\epsilon_0 r^2} \quad (r > R). \quad (2.26)$$

The electric field inside is more interesting: now for  $r < R$  we have  $Q_{\text{enclosed}} = 0$ , so in fact we have

$$E_r = 0 \quad (r < R). \quad (2.27)$$

The electric field is zero everywhere inside! This may seem surprising: if we look at an interior point which is very close to the shell it may seem like it is quite close to some of the charges, and that the field created by the rest of the charges can't possibly compete, but apparently it can.

## 2.5 Electric field of a plane

Our last example will be the electric field of an infinitely large sheet carrying a constant surface charge density  $\sigma$ . Let us take this sheet to lie in the  $xy$  plane. This setup is invariant under rotations about the  $z$  axis, so the only possible direction for the electric field to point is the  $z$  direction. Similarly it is invariant under arbitrary translations of  $x$  and  $y$ , so it can depend only on  $z$ :

$$\vec{E} = E_z(z)\hat{z}. \quad (2.28)$$

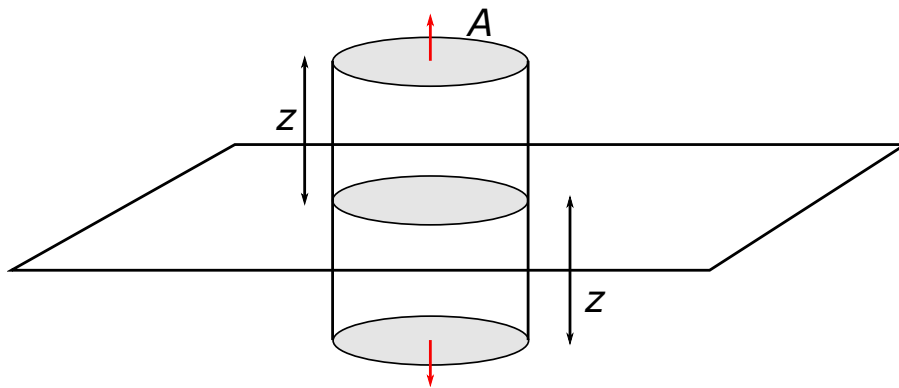


Figure 9: A cylindrical Gaussian surface for an infinite sheet of charge. The cylinder has cross-sectional area  $A$ , and extends a distance  $z$  above and a distance  $z$  below the sheet. Only the top and bottom disks contribute to the electric flux.

Moreover it is also invariant under the reflection  $z \rightarrow -z$ , so the electric field for  $z < 0$  must be minus the electric field for  $z > 0$ . We can determine the electric field by computing the electric flux through the cylindrical surface  $S_z$  shown in figure 9: the sides of the cylinder do not contribute to the flux, while the top and bottom each contribute  $|E_z|A$ , so we have

$$\int_{S_z} \vec{E} \cdot d\vec{A} = 2A|E_z|. \quad (2.29)$$

The charge enclosed by this surface is just a disc of area  $A$ , so Gauss's law then tells us that we must have

$$2A|E_z| = \frac{\sigma A}{\epsilon_0}. \quad (2.30)$$

In other words the electric field is

$$E_z = \begin{cases} \frac{\sigma}{2\epsilon_0} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} & z < 0 \end{cases}. \quad (2.31)$$

There are two things worth noting about this result:

- (1) The electric field changes as we cross the sheet, but otherwise is independent of  $z$ . You might have thought that the field would get smaller as we get farther away from the sheet, but since the sheet is infinitely big we can never get far away from it! A real charged sheet will of course not be infinitely big, and eventually we *will* see the field begin to fall off as we move away from it. Near the sheet and away from the edges however the result (2.31) will be an excellent approximation.
- (2) The electric field jumps discontinuously across the sheet. This is typical for surface charge densities. In the real world there is of course not really such a thing as an infinitely thin layer of charge, so this jump will always be smoothed out very close to surface.

### 3 Energy in electrostatics

In this lecture we will discuss various aspects of the role of energy in electrostatics.

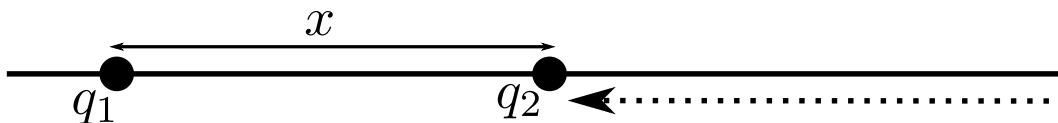


Figure 10: Assembling a pair of charges by bringing the second one in from infinity along the positive  $x$ -axis, doing work against the Coulomb force due to the first.

### 3.1 Energy in a pair of charges

We begin by considering the energy  $U$  stored in a distribution of charges. We will first consider a pair of point charges  $q_1, q_2$  located at positions  $\vec{r}_1$  and  $\vec{r}_2$ . There are various ways we could try to define the energy of this configuration, the definition we will adopt is that its energy is equal to the amount of work we need to do to assemble it by starting out with the second charge being infinitely far away from the first and then moving it in to the position  $\vec{r}_2$ . It is not obvious that this is the “right” definition, for example why shouldn’t we also consider the amount of energy it took to create the charges in the first place? In fact we could, but we will see at the end of the lecture that doing so leads to an embarrassing result so for now we won’t.

To simplify things we can take the first charge to be at the origin of coordinates and the second to ultimately lie on the  $x$ -axis at a position  $x > 0$ . We will first bring it there by moving it slowly inwards along the  $x$ -axis from  $x = +\infty$  (see figure 10). You may remember from your mechanics class that the work we need to do to move any particle along a curve  $C$  is

$$W = \int_C \vec{F}_{ext} \cdot d\vec{s}, \quad (3.1)$$

where  $\vec{F}_{ext}$  is the force we apply to the particle. In this case we can split this force up into two parts: a part which cancels the Coulomb force, and a part which gets the particle moving and then stops it once we get to position  $x$ :

$$\vec{F}_{ext}(x) = -\frac{q_1 q_2}{4\pi\epsilon_0 x^2} \hat{x} + \vec{F}_{start/stop}. \quad (3.2)$$

You might worry that the work we do will depend in detail on how we start and stop the particle, but in fact  $\vec{F}_{start/stop}$  will always cancel in the computation of  $W$ ! The reason is that once we have canceled the Coulomb force, the particle feels no other forces except for  $\vec{F}_{start/stop}$ ; thus any change in energy due to  $\vec{F}_{start/stop}$  must result in a change of the kinetic energy of the particle. Since we are starting with the particle at rest at  $+\infty$  and stopping with it at rest at  $x$ , there is no change in the kinetic energy and thus no net contribution to  $W$  from  $\vec{F}_{start/stop}$ . Noting that  $d\vec{s} = \hat{x}dx$ , we thus have

$$W = - \int_{\infty}^x dx' \frac{q_1 q_2}{4\pi\epsilon_0 x'^2} = \frac{q_1 q_2}{4\pi\epsilon_0 x}. \quad (3.3)$$

Rewriting this in a way that does not depend on our choice of coordinates, we have therefore found that the stored energy is

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 |r_{12}|}. \quad (3.4)$$

This result makes some sense: if both charges have the same sign then we need to do work to bring them together, while if they have opposite signs then we can extract work from them. Either way the amount of work is larger if the charges end up closer together.

What if we instead bring in  $q_2$  along a different path? After all if the amount of work depended on the path we chose, then it wouldn’t really be fair to call it *the* energy contained in the charge distribution. Fortunately for us, it doesn’t! To see this, we note that any path we choose can be approximated as well as we like by one which consists only of strictly radial segments and segments which are at fixed radius (see figure



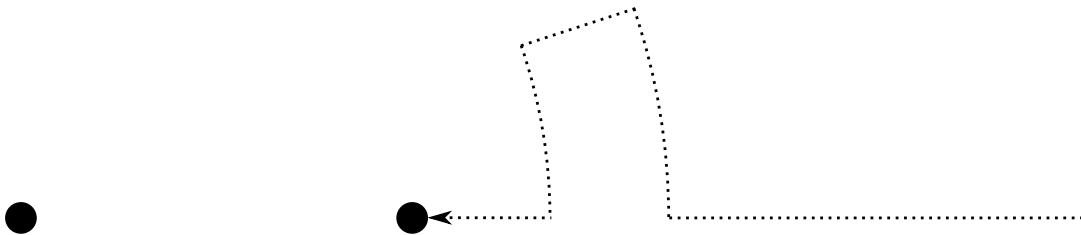


Figure 11: Bringing in charge two along a distorted trajectory: the circular parts do not contribute since there the electric field is orthogonal to the line element, while the radial parts contribute the same regardless of angle. Any trajectory can be approximated as well as we like by one made out of radial and circular parts in this way.

11). The segments at fixed radius do not contribute since they are orthogonal to the Coulomb force, and the radial segments will just add up to give the same result as before since we can move them all to lie in a single radial line without changing the answer.

This path-independence of the work tells us something quite interesting about the electric field. Let's first recall that we can rewrite the Coulomb force on the second charge as  $q_2 \vec{E}$ , where  $\vec{E}$  is the electric field created by the first charge. Thus the work to bring in the charge  $q_2$  from infinity to position  $\vec{r}_2$  is

$$W = -q_2 \int_{\infty}^{\vec{r}_2} \vec{E} \cdot d\vec{s}, \quad (3.5)$$

where we can perform the line integral along any path we like. This then implies a rather remarkable result: if  $\vec{E}$  is the electric field produced by a point charge, we have

$$\int_L \vec{E} \cdot d\vec{s} = 0 \quad (3.6)$$

for any closed loop  $L$ . This is because we could imagine moving in a charge from infinity to any point on the loop, and then moving it around the loop. Since the energy does not depend on the path, it cannot depend on whether or not we went around the loop. Therefore the loop must contribute nothing to the integral. Moreover once we have established this for the electric field of a point charge, superposition tells us that (3.6) must hold for the electric field of *any* charge distribution! We will make use of this result many times this semester, so I encourage you to remember it.

### 3.2 Multiple charges

Let's now consider the work we need to do to assemble a collection of point charges  $q_1, q_2, \dots$  and positions  $\vec{r}_1, \vec{r}_2, \dots$ . We will compute it by bringing in the charges one at a time from infinity. The first one we bring in does not feel any Coulomb force, so the energy after bringing it in is zero. We just saw that bringing in the second charge requires us to do an amount of work

$$W_2 = -q_2 \int_{\infty}^{\vec{r}_2} \vec{E}_1 \cdot d\vec{s} = \frac{q_1 q_2}{4\pi\epsilon_0 |r_{12}|}, \quad (3.7)$$

where  $\vec{E}_1$  is the electric field created by the first charge. Now let's consider bringing in the third charge. We now need to do work against the Coulomb force sourced by each of the first two charges, so we have

$$W_3 = -q_3 \int_{\infty}^{\vec{r}_3} (\vec{E}_1 + \vec{E}_2) \cdot d\vec{s} = \frac{q_1 q_3}{4\pi\epsilon_0 |r_{13}|} + \frac{q_2 q_3}{4\pi\epsilon_0 |r_{23}|}. \quad (3.8)$$

Here  $\vec{E}_2$  is the electric field sourced by charge two, and we've used superposition. Continuing in this way we can bring in the fourth charge, doing work

$$W_4 = -q_4 \int_{\infty}^{\vec{r}_4} \left( \vec{E}_1 + \vec{E}_2 + \vec{E}_3 \right) \cdot d\vec{s} = \frac{q_1 q_4}{4\pi\epsilon_0 |r_{14}|} + \frac{q_2 q_4}{4\pi\epsilon_0 |r_{24}|} + \frac{q_3 q_4}{4\pi\epsilon_0 |r_{34}|}, \quad (3.9)$$

and so on. The final energy contained will then be  $U = W_2 + W_3 + W_4 + \dots$ , and it is hopefully clear that this will be given by a sum of terms  $\frac{q_i q_j}{4\pi\epsilon_0 |r_{ij}|}$ , with each pair appearing once. The easiest way to write this is

$$U = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\epsilon_0 |r_{ij}|}, \quad (3.10)$$

where the sum is over all  $i$  and  $j$  such that  $i \neq j$  (we include the factor of  $1/2$  since otherwise we will double-count each pair). When we write the formula this way it is clear that it does not depend on which order we brought in the charges, which is reassuring since the energy stored in the configuration should not depend on this. We can also write a version of (3.10) which is appropriate for continuous charge distributions:

$$U = \frac{1}{2} \int dV dV' \frac{\rho(\vec{r})\rho(\vec{r}')}{4\pi\epsilon_0 |r - r'|}. \quad (3.11)$$

### 3.3 Potential energy of a point charge moving in the presence of a charge distribution

Our formula for the energy of a charge distribution gives us a nice way to think about the motion of a point charge  $q$  in the presence of a charge distribution. Let's first recall that we can describe the trajectory of the point charge as a vector function  $\vec{r}(t)$  saying where it is at time  $t$ . Its velocity vector  $\vec{v} = \frac{d\vec{r}}{dt}$ . Now say that we have a collection of other charges  $q_1, q_2, \dots$  at positions  $\vec{r}_1, \vec{r}_2, \dots$ . If we hold the locations of these other charges fixed, then in determining the motion of the point charge we can ignore the energy stored between the other charges. We then have the expression

$$U_{particle} = \frac{1}{2} m |v|^2 + \sum_i \frac{q q_i}{4\pi\epsilon_0 |r - r_i|} \quad (3.12)$$

for the energy of the particle. In other words we can think of

$$V(\vec{r}) \equiv \sum_i \frac{q q_i}{4\pi\epsilon_0 |r - r_i|} \quad (3.13)$$

as a kind of potential energy! For future reference we note that we can write this as

$$V(\vec{r}) = q\phi(\vec{r}), \quad (3.14)$$

where

$$\phi(\vec{r}) \equiv \sum_i \frac{q_i}{4\pi\epsilon_0 |r - r_i|} \quad (3.15)$$

is called the *potential* of the charge distribution. As a simple application, let's consider the motion of a point charge  $q$  in the presence of a single other point charge  $Q$ . Moreover let's say that we place  $q$  at rest at a distance  $r_0$  from  $Q$ . How fast will it be moving when it reaches radius  $r_f$ ? Since the energy  $U_{particle}$  must be conserved (remember we are not allowing  $Q$  to move), we must have

$$\frac{qQ}{4\pi\epsilon_0 r_0} = \frac{qQ}{4\pi\epsilon_0 r_f} + \frac{1}{2} m |v|^2. \quad (3.16)$$

In particular if the charges have the same sign then in the limit  $r_f \rightarrow \infty$  we have

$$|v| = \sqrt{\frac{qQ}{2\pi\epsilon_0 r_0 m}}. \quad (3.17)$$

We can think of this as extracting the potential energy to accelerate the particle, which gives back to us the work we did to set up the charges in the first place.

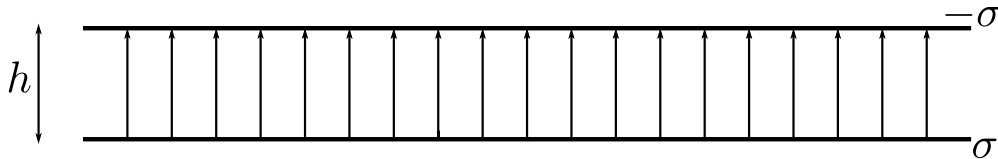


Figure 12: The electric field between two parallel plates carrying opposite charge densities.

### 3.4 Where is the energy stored?

Now that we have seen that any charge distribution contains an energy (3.10), you might wonder “where” the energy is stored. There doesn’t necessarily need to be a nice answer to that question, but in fact there is: it is stored in the electric field! Let’s understand this using an example: two large parallel plates of charge, one with surface charge density  $\sigma$  and another with surface charge density  $-\sigma$ . Let’s put the first plate at  $z = 0$ , and the second at  $z = h$ , with  $h > 0$ , and let’s work in the approximation where  $h$  is very small compared to the size of the plates so that we can think of them as infinite. For  $z < 0$  and  $z > h$  the electric field will be zero, since in those regions each plate contributes an electric field of magnitude  $\frac{\sigma}{2\epsilon_0}$  in the  $z$  direction but with opposite signs. In between the plates however they both contribute a field which points up, giving a net electric field (see figure 12)

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z}. \quad (3.18)$$

Let’s now consider moving the upper plate upwards from height  $h$  to height  $h'$ . In doing this we need to do work against the Coulomb force, which wants to pull the upper plate down onto the lower plate. The force we need to exert on an area element  $dA$  of the upper plate to cancel the Coulomb force from the lower plate is

$$d\vec{F} = -(-\sigma dA) \times \frac{\sigma}{2\epsilon_0} \hat{z} = \frac{\sigma^2 dA}{2\epsilon_0} \hat{z}, \quad (3.19)$$

where we have been careful to only include the electric field due to the lower plate (the upper plate cannot exert a force on itself!) Adding up all the area elements, we get a total net force of

$$\vec{F} = \frac{\sigma^2 A}{2\epsilon_0} \hat{z}, \quad (3.20)$$

where  $A$  is the area of the plates (there will be corrections to this formula due to edge effects which will vanish in the limit that  $\sqrt{A}/h \rightarrow \infty$ ). The work we need to do to move the plate from  $h$  to  $h'$  will then be

$$W = \int_h^{h'} \vec{F} \cdot d\vec{s} = \frac{\sigma^2 A (h' - h)}{2\epsilon_0}. \quad (3.21)$$

The point is then that we can rewrite this formula in a nice way:

$$W = \frac{\epsilon_0}{2} A (h - h') \left( \frac{\sigma}{\epsilon_0} \right)^2 = \frac{\epsilon_0}{2} \int_R dV |E|^2, \quad (3.22)$$

where  $R$  is the region between  $z = h$  and  $z = h'$ . The electric field in this region was zero when the upper plate was at  $z = h$ , so in moving the plate from  $z = h$  to  $z = h'$  we created it. What this formula tells us is that we can think of this electric field as carrying an energy density

$$u = \frac{\epsilon_0}{2} |E|^2, \quad (3.23)$$

which “stores” the energy we put into the system by moving the plate. In fact this result is completely general: we can always think of the electric field as having an energy density given by (3.23)! The proof

of this statement will have to wait for another few lectures, but for now let's use it to compute the energy stored in a ball of radius  $R$  carrying a constant charge density  $\rho$ . Indeed we can recall from last time that

$$\vec{E} = \begin{cases} \frac{\rho r}{3\epsilon_0} \hat{r} & r < R \\ \frac{\rho R^3}{3\epsilon_0 r^2} \hat{r} & r > R \end{cases} \quad (3.24)$$

Therefore the energy is

$$\begin{aligned} U &= \frac{\epsilon_0}{2} \left[ \int_0^R dr' (4\pi r'^2) \left( \frac{\rho r'}{3\epsilon_0} \right)^2 + \int_R^\infty dr' (4\pi r'^2) \left( \frac{\rho R^3}{3\epsilon_0 r'^2} \right)^2 \right] \\ &= \frac{\epsilon_0}{2} \frac{4\pi\rho^2}{9\epsilon_0^2} (R^5/5 + R^5) \\ &= \frac{4\pi\rho^2 R^5}{15\epsilon_0} \\ &= \frac{3Q^2}{20\pi\epsilon_0 R}, \end{aligned} \quad (3.25)$$

where  $Q$  is the total charge on the ball. In the homework you will confirm that this is the right answer by computing the work you need to do to build up the ball by bringing in charge from infinity.

There is something puzzling about this result. If we hold  $Q$  fixed and take  $R$  to zero, then our charged ball will become a point charge. The expression (3.25) for its energy however will diverge in this limit! This is *not* consistent with our result (3.10) for the energy stored in a charge distribution: that required at least two charges to have a nonzero energy. The resolution of this puzzle is that we have to remember how we defined  $U$  in equation (3.10): it was the amount of work we needed to do to assemble the charge distribution *given* that we already had all the point charges lying around at infinity. The integral of  $\frac{\epsilon_0}{2} |E|^2$  over space however also includes the energy from the work we did to construct the point charges in the first place: apparently this is infinite! We did not have to worry about this in our discussion of the parallel plates, since we only computed the change in energy as we moved the plates around so this did not include the energy to assemble the plates. It is worrisome that we find the energy of a point charge to be infinite, for example does this mean that an electron creates a gravitational field which is infinitely strong? The answer had better be no, and it is, but to understand this we need to study quantum field theory so you will have to wait for another class.

## 4 Vector calculus

This is a primarily mathematical lecture, where we review some properties of scalar and vector fields that we will use throughout the course.

### 4.1 Scalar and vector fields

A **scalar field**  $f(\vec{r})$  is by definition a rule which assigns a real number to each point in space. A mathematician would describe this by saying that a scalar field is a map

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (4.1)$$

which is a way of indicating the domain ( $\mathbb{R}^3$ ) and the range ( $\mathbb{R}$ ) of  $f$ . For example the temperature and pressure of the air in this room are described by a pair of scalar fields  $T(\vec{r})$  and  $P(\vec{r})$ .

A **vector field**  $\vec{V}(\vec{r})$  is by definition a rule which assigns a vector to each point in space. A mathematician would describe it as a map

$$\vec{V} : \mathbb{R}^3 \rightarrow \mathbb{R}^3. \quad (4.2)$$



Figure 13: A varying temperature field. Here the gradient points in the  $\hat{x}$  direction; derivatives of  $T$  in other directions will be smaller in magnitude.

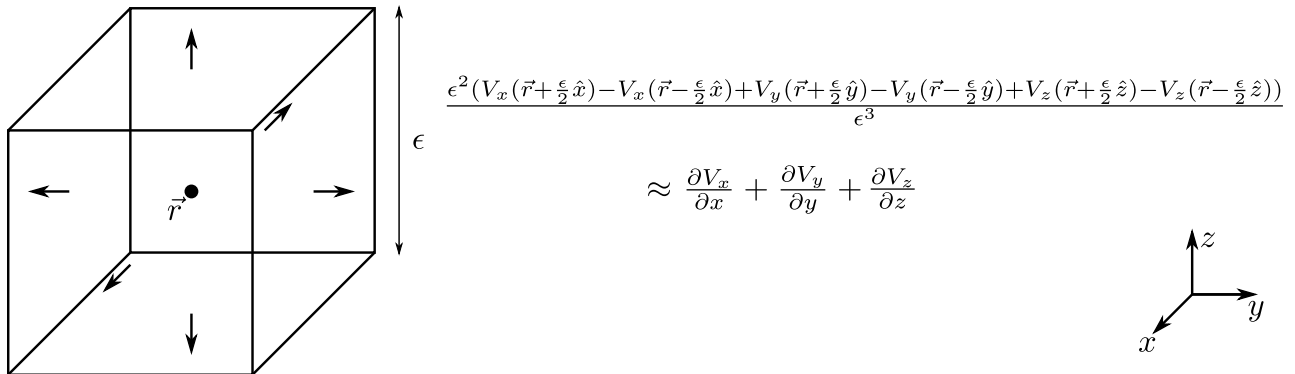


Figure 14: The geometric meaning of the divergence: it tells us how much flux is coming out of an infinitesimal cube centered at  $\vec{r}$ .

Our main example so far of a vector field is the electric field  $\vec{E}(\vec{r})$ . Another example is the average velocity  $\vec{v}(\vec{r})$  of the gas particles making up the air in this room. The former tells us what force a charged particle will feel at any point  $\vec{r}$ , while the latter tells us the local wind direction and magnitude.

## 4.2 Gradient, divergence, and curl

In single-variable calculus we study functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and one of the fundamental operations we apply to them is differentiation. In vector calculus there are *three* differential operations which relate scalar and vector fields: the gradient, the divergence, and the curl. We will define these operations in cartesian coordinates, and then describe their geometric interpretations.

We begin with the **gradient**, which is an operation that turns a scalar field  $f$  into a vector field  $\vec{\nabla}f$  via the formula

$$\vec{\nabla}f \equiv \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}. \quad (4.3)$$

At each point we can compute the derivative of the function  $f$  in the direction of a unit vector  $\hat{n}$  via  $\hat{n} \cdot \nabla f$ , so in particular we can interpret the direction of the gradient at that point as giving the direction of steepest increase of  $f$  and the magnitude of the gradient as giving the ordinary derivative of  $f$  in that direction (see figure 13 for a simple example).

Our second differential operation is the **divergence**, which turns a vector field  $\vec{V}$  into a scalar field  $\vec{\nabla} \cdot \vec{V}$  via

$$\vec{\nabla} \cdot \vec{V} \equiv \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \quad (4.4)$$

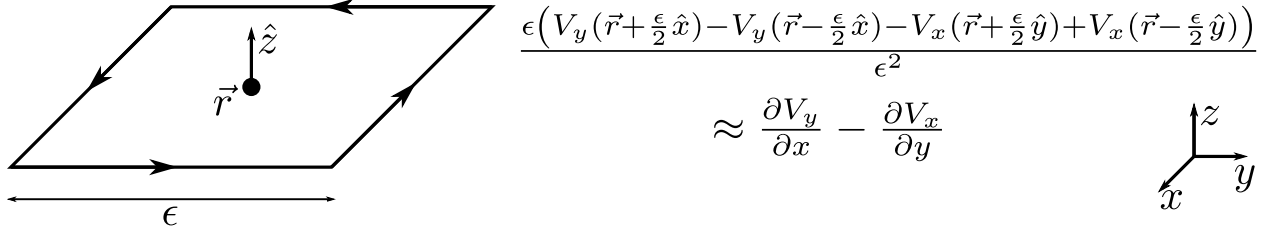


Figure 15: The geometric meaning of the curl: it tells us how much the vector field is wrapping around any particular direction. You can remember the orientation by using the right-hand rule: if you wrap the fingers of your right hand around the direction of the circulation, your thumb should point in the direction of the normal vector.

The geometric interpretation of the divergence is that it tells us how much the vector field  $\vec{V}$  is “flowing out from (or in to)” each point  $\vec{r}$ . More precisely, if we take a cube  $C_\epsilon(\vec{r})$  of size  $\epsilon$  centered at  $\vec{r}$  then the divergence of  $\vec{V}$  is the ratio of the outward flux of  $V$  through the boundary of  $C_\epsilon(\vec{r})$  to the volume of  $C_\epsilon(\vec{r})$  in the limit that  $\epsilon$  goes to zero:

$$\vec{\nabla} \cdot \vec{V} = \lim_{\epsilon \rightarrow 0} \frac{\int_{\partial C_\epsilon(\vec{r})} \vec{V} \cdot d\vec{A}}{\int_{C_\epsilon(\vec{r})} dV}. \quad (4.5)$$

This equivalence is explained in figure 14. As you might expect, we will soon see that the divergence of the electric field at  $\vec{r}$  is related to the amount of charge which is present at  $\vec{r}$ .

Our third differential operation is the **curl**, which turns a vector field  $\vec{V}$  into *another* vector field  $\vec{\nabla} \times \vec{V}$ , via the formula

$$\vec{\nabla} \times \vec{V} \equiv \left( \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{x} + \left( \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \hat{y} + \left( \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{z}. \quad (4.6)$$

One way to remember this formula is that we can obtain the right-hand side by computing the determinant of the “matrix”

$$\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix}. \quad (4.7)$$

The geometric interpretation of the curl is that for any unit vector  $\hat{n}$ , we can think of  $\hat{n} \cdot (\vec{\nabla} \times \vec{V})$  as telling us how much the vector field  $\vec{V}$  “winds around”  $\hat{n}$ . More precisely, if we take a square  $S_{\epsilon,n}(\vec{r})$  centered at  $\hat{r}$  with side length  $\epsilon$  and normal vector  $\hat{n}$ , then the  $\hat{n}$  component of the curl is the ratio of the line integral of  $\vec{V}$  around the boundary of  $S_{\epsilon,n}(\vec{r})$  to the area of  $S_{\epsilon,n}(\vec{r})$  in the limit of small  $\epsilon$ :

$$\hat{n} \cdot (\vec{\nabla} \times \vec{V}) = \lim_{\epsilon \rightarrow 0} \frac{\int_{\partial S_{\epsilon,n}(\vec{r})} \vec{V} \cdot d\vec{s}}{\int_{S_{\epsilon,n}(\vec{r})} dA}. \quad (4.8)$$

Here the notation  $\partial S_{\epsilon,n}(\vec{r})$  indicates the boundary of  $S_{\epsilon,n}(\vec{r})$ , with the orientation of the line integral chosen according to the “right-hand rule” shown in figure 15.

### 4.3 Integration theorems for gradient, divergence, and curl

In single-variable calculus, differentiation and integration are related by the fundamental theorem of calculus:

$$\int_a^b f'(x) dx = f(b) - f(a). \quad (4.9)$$

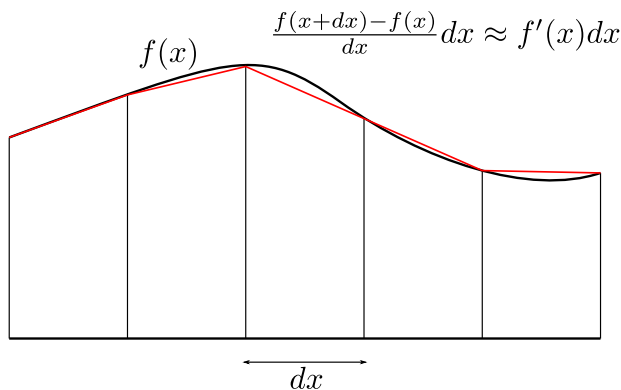


Figure 16: Proving the fundamental theorem of calculus: we split up the integral into pieces of thickness  $dx$ , approximate  $f(x)$  by the red linear segments, and then note that  $f(a + dx) - f(a) + f(a + 2dx) - f(a + dx) + \dots + f(b) - f(b - dx) = f(b) - f(a)$ .

The intuition for this formula is simple: if we split up the integration into small pieces, each with size  $dx$ , then  $f'(x)dx$  just tells us how much  $f$  changes from  $x$  to  $x + dx$ . Summing over all pieces then just gives us that net change of  $f$  from  $a$  to  $b$  (see figure 16). We will now see that this result has nice generalizations to the gradient, divergence, and curl.

We can think of the fundamental theorem of calculus as telling us that the integral of a derivative gives us something which depends only on what the function is doing at the boundary of the region of integration. Let's first see how this generalizes to the line integral of a gradient:

$$\int_{C(\vec{r}_0, \vec{r}_f)} \vec{\nabla} f \cdot d\vec{s} \equiv \int_0^L (\hat{\ell}(s) \cdot \vec{\nabla} f) ds, \quad (4.10)$$

where  $C(\vec{r}_0, \vec{r}_f)$  is a path from  $\vec{r}_0$  to  $\vec{r}_f$ ,  $s$  is the arc length along the curve,  $\hat{\ell}(s)$  is the unit tangent vector as a function of  $s$ , and  $L$  is the total length. By the relationship between the gradient and the directional derivative we have

$$\int_0^L (\hat{\ell}(s) \cdot \vec{\nabla} f) ds = \int_0^L ds \frac{df(\vec{r}(s))}{ds}, \quad (4.11)$$

where  $\vec{r}(s)$  gives us the motion along the curve as a function of arc length. This however is precisely something to which we can apply the fundamental theorem of calculus, so apparently we have

$$\int_{C(\vec{r}_0, \vec{r}_f)} \vec{\nabla} f \cdot d\vec{s} = f(\vec{r}_f) - f(\vec{r}_0). \quad (4.12)$$

Similarly we can consider the integral of the divergence of a vector field  $\vec{V}$  over a spatial volume  $R$ .<sup>6</sup> The **divergence theorem**, also called Gauss's theorem or Ostrogradsky's theorem, tells us that this volume integral is just given by the flux of  $\vec{V}$  out through the boundary  $\partial R$  of  $R$ :

$$\int_R (\vec{\nabla} \cdot \vec{V}) dV = \int_{\partial R} \vec{V} \cdot d\vec{A}. \quad (4.13)$$

<sup>6</sup>Don't worry if this is the first time you have seen a multiple integral: such integrals are defined in a similar way as single-variable integrals. Indeed we split up the region of integration into small cubes (or squares for a surface integral), and then sum over the value of the function in the center of each cube (or square) times the volume of the cube (or the area of the square). To evaluate such integrals, we can usually integrate over the coordinates one after another (this is called Fubini's theorem). For more details I recommend the wikipedia page "multiple integral". In this class most multiple integrals we evaluate will end up being the integral of a constant function, in which case we just get the volume (or area) of the region times the value of the function.

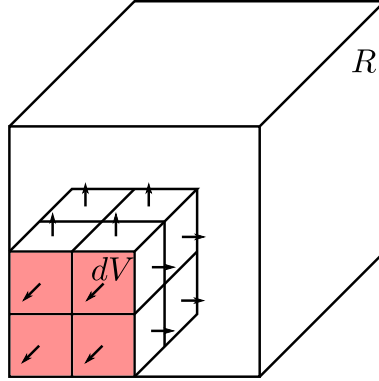


Figure 17: Proving the divergence theorem: we split up the region  $R$  into small boxes of volume  $dV$ , and then add up the fluxes through them. The flux through all sides of the boxes which are in the interior of  $R$  will cancel, so we are left only with the flux through those sides (shaded red) which are part of its surface  $\partial R$ .

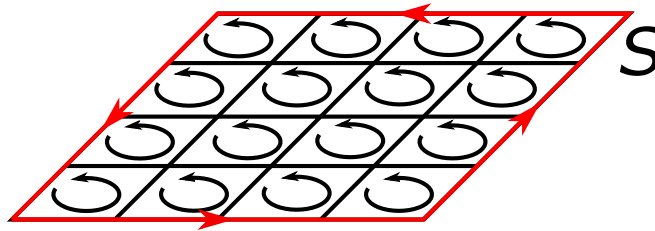


Figure 18: Proving Stokes' theorem: we split up the surface into small squares of area  $dA$ , and then add up the flux of the curl through them. This is given by the circulation of the vector field around the squares as in figure 15, and the internal pieces of the circulations cancel in the sum. We are left only with the net circulation around the boundary of  $S$ .

The proof of this theorem is simple if we make use of the geometric formulation of the divergence shown in figure 14: we split the region  $R$  up into small boxes of volume  $dV$ , and then add up all of their divergences.

Finally we can also consider the flux of the curl of  $\vec{V}$  through a surface  $S$ . We then have **Stokes' theorem**, which tells us that this is equivalent to the line integral of  $\vec{V}$  around the boundary  $\partial S$  of  $S$ :

$$\int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \int_{\partial S} \vec{V} \cdot d\vec{S}. \quad (4.14)$$

The orientation of the line integral around  $\partial S$  is again given by the right-hand rule, as in figure 15. The proof of Stokes' theorem proceeds similarly to those for the fundamental theorem of calculus and the divergence theorem: we split the surface  $S$  up into small squares, and then add up the flux of the curl through each of these (see figure 18).

#### 4.4 Vector antiderivatives

In single-variable calculus, functions whose derivative vanishes everywhere are very special: they are constants! Scalar and vector fields with vanishing gradient, divergence, or curl are also very special. The simplest case is a scalar whose gradient is zero: such a scalar must be a constant.

$$\vec{\nabla} f = 0 \iff f \text{ constant.} \quad (4.15)$$



This may seem quite obvious, but as a warmup let's prove it. We can compute the difference between the values of  $f$  at any two points  $\vec{r}_1$  and  $\vec{r}_2$  by using the line integral version (4.12) of fundamental theorem of calculus, so apparently we have

$$f(\vec{r}_2) - f(\vec{r}_1) = \int_C \vec{\nabla} f \cdot d\vec{s} = 0. \quad (4.16)$$

Thus the value of  $f$  is the same for any two points, so  $f$  is constant.

How about a vector field  $\vec{V}$  whose curl is vanishing? By Stokes' theorem this is a vector field whose line integral is zero about any closed loop:

$$\int_{\partial S} \vec{V} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = 0. \quad (4.17)$$

In fact such a vector field is always the gradient of a scalar field  $f$ :

$$\vec{\nabla} \times \vec{V} = 0 \iff \vec{V} = \vec{\nabla} f. \quad (4.18)$$

This  $f$  is clearly not unique, we can add to it any constant without changing  $\vec{V}$ . To prove (4.18), we need to show two things. The first is that the curl of any gradient is zero: you will show this on the problem set. The second is that given a  $\vec{V}$  with vanishing curl, we need to construct an  $f$  such that  $\vec{V} = \vec{\nabla} f$ . I claim that one which works is

$$f(\vec{r}) \equiv \int_{\vec{r}_0}^{\vec{r}} \vec{V} \cdot d\vec{s}, \quad (4.19)$$

where the line integral can be taken along any curve from  $\vec{r}_0$  to  $\vec{r}$ . Due to Stokes' theorem, the result of this integral does not depend on which path we pick: the difference of the integral along two different paths is just the line integral around a closed loop. To see that the gradient of this function is  $\vec{V}$ , let's compute its derivative in the  $x$  direction. This is given by

$$\frac{\partial f}{\partial x} = \lim_{\epsilon \rightarrow 0} \frac{\int_{\vec{r}}^{\vec{r} + \epsilon \hat{x}} \vec{V} \cdot d\vec{S}}{\epsilon} = V_x, \quad (4.20)$$

and similar arguments hold for  $V_y$  and  $V_z$ .

Finally let's consider a vector field  $\vec{V}$  whose divergence vanishes. By the divergence theorem, this is a vector field whose surface integral over any closed surface vanishes:

$$\int_{\partial R} \vec{V} \cdot d\vec{A} = \int_R \vec{\nabla} \cdot \vec{V} = 0. \quad (4.21)$$

In fact such a vector field is always the curl of another vector field  $\vec{U}$ :

$$\vec{\nabla} \cdot \vec{V} = 0 \iff \vec{V} = \vec{\nabla} \times \vec{U}. \quad (4.22)$$

$\vec{U}$  is again not unique, since we can add to it the gradient of any scalar function  $f$  and still get the same  $\vec{U}$  (since the curl of the gradient is zero). To establish (4.22), we need to show two things. The first is that the divergence of the curl is always zero, you will again show this on the homework. The second is to actually construct  $\vec{U}$ : we will postpone this for a few lectures since it is a bit involved and will feel more natural once we discuss the magnetic field.

We'll close by mentioning two interesting generalizations of these results, which are not important for this class but are very important in other physics applications. The first is that if we consider spatial dimensions other than three, all statements about the gradient and divergence go through unchanged. The curl however is specific to three spatial dimensions: it is only in three dimensions that we can make a vector out of antisymmetrized derivatives of another vector. In other dimensions the curl must be generalized to produce an object which has more components than a vector: a "differential form". The other comment is that we could consider spaces other than  $\mathbb{R}^3$  (or  $\mathbb{R}^n$ ), for example a torus or a sphere. In such spaces it is not necessarily true that a vector field with vanishing divergence is a curl or a vector field with vanishing curl is a gradient: the question of which are and which aren't leads to a beautiful branch of mathematics called de Rham cohomology, which has many applications in high energy and condensed matter physics.

## 5 Electric potential

A few lectures ago, we saw that the electric field produced by any static charge distribution has the property that for any closed loop  $L$  we have

$$\int_L \vec{E} \cdot d\vec{s} = 0. \quad (5.1)$$

This property was essential for our definition of the energy stored in a collection of charges to be independent of the paths along which we brought them in from infinity. Using our interpretation of the curl as a limit of the line integral around a small loop divided by the area of the loop, we see that apparently in electrostatics we must have

$$\vec{\nabla} \times \vec{E} = 0. \quad (5.2)$$

We can check this directly from our expression

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^3} \vec{r} = \frac{q}{4\pi\epsilon_0 (x^2 + y^2 + z^2)^{3/2}} (x\hat{x} + y\hat{y} + z\hat{z}) \quad (5.3)$$

for the electric field of a point charge, for example the  $z$  component of the curl is

$$\begin{aligned} \partial_x E_y - \partial_y E_x &= \frac{q}{4\pi\epsilon_0} \left( \partial_x \left( \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \right) - \partial_y \left( \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) \right) \\ &= 0. \end{aligned} \quad (5.4)$$

We learned at the end of the last lecture that any vector field with zero curl must be equal to the gradient of a scalar: in particular this must apparently be the case for the electric field. It is conventional to include a minus sign in the definition of this scalar, so we say that in electrostatics we must have

$$\vec{E} = -\vec{\nabla}\phi, \quad (5.5)$$

where  $\phi$  is called the **electric potential**. Moreover we can give a formula for  $\phi$ :

$$\phi(\vec{r}) \equiv - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{s}. \quad (5.6)$$

The line integral may be evaluated along any path from  $\vec{r}_0$  to  $\vec{r}$ , since by (5.1) the answer does not depend on which path we pick. It *does* however depend on our choice of  $\vec{r}_0$ : different choices of  $\vec{r}_0$  change  $\phi(\vec{r})$  by an  $\vec{r}$ -independent constant. More concretely

$$- \int_{\vec{r}'_0}^{\vec{r}} \vec{E} \cdot d\vec{s} = - \int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{s} - \int_{\vec{r}'_0}^{\vec{r}_0} \vec{E} \cdot d\vec{s}, \quad (5.7)$$

where the second term on the right gives an  $\vec{r}$ -independent constant. This constant is removed when we take the gradient to find  $\vec{E}$ , so we are free to choose it to be whatever we like. One nice way to think about this is that if we define  $\phi(\vec{r})$  by (5.6), then we are guaranteed that  $\phi(\vec{r}_0) = 0$ . We can thus think of the choice of  $\vec{r}_0$  as the choice of at which point we want to say that the electric potential is zero. Nothing measurable will ever depend on the *value* of the electric potential at any particular point, it is only the electric potential difference between different points which is meaningful. One way to see this is to observe that

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = - \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s}, \quad (5.8)$$

so the choice of  $\vec{r}_0$  cancels on the left hand side and the right hand side just depends on  $\vec{E}$ , which is physical.

It gets a bit tedious constantly saying “electric potential”, so from here on we will usually just say “potential”.

## 5.1 Potential for a charge configuration

We will now write down expressions for the potential of various charge configurations, starting with that of a single point charge. For distributions with finite total charge, it is convenient to remove the ambiguity of shifting  $\phi$  by a constant by choosing  $\vec{r}_0 = \infty$ : we can then interpret  $\phi(\vec{r})$  as the difference of the potential at  $\vec{r}$  and the potential at infinity. Thus we want to evaluate

$$\phi(\vec{r}) = - \int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{s}. \quad (5.9)$$

In fact when  $\vec{E}$  is the electric field of a point charge  $q$  sitting at the origin of space, this is precisely the integral we evaluated a few lectures ago in computing the work to assemble a pair of charges by bringing them in from infinity! The result is

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}. \quad (5.10)$$

Moreover by superposition if we have multiple charges  $q_1, q_2, \dots$  at locations  $\vec{r}_1, \vec{r}_2, \dots$ , then apparently we must have

$$\phi(\vec{r}) = \sum_i \frac{q_i}{4\pi\epsilon_0 |r - r_i|}. \quad (5.11)$$

We can think of the potential as describing a “landscape” which has peaks at positive charges and holes at negative charges (this is why the minus sign is included in the definition). For a continuous volume charge distribution, we can rewrite this expression as

$$\phi(\vec{r}) = \int dV' \frac{\rho(\vec{r}')}{4\pi\epsilon_0 |r - r'|}. \quad (5.12)$$

These formulas may look familiar, in fact they are very closely related to our expression  $V(\vec{r})$  for the potential energy of a charge  $q$  moving in the electric field sourced by a bunch of other charges:

$$V(\vec{r}) = q\phi(\vec{r}). \quad (5.13)$$

This relationship is the origin of the term “electric potential”.

It is worth mentioning that for infinite charge configurations we may not be able to choose the potential to vanish at infinity. For example the electric field of an infinite sheet of charge in the  $xy$  plane is for  $z > 0$  just given by  $\frac{\sigma}{2\epsilon_0} \hat{z}$ . If we try to compute the line integral of this electric field from some  $z > 0$  to  $z = \infty$ , we get

$$- \int_{\infty}^z \vec{E} \cdot d\vec{s} = \int_z^{\infty} \vec{E} \cdot d\vec{s} = \int_z^{\infty} \frac{\sigma dz}{2\epsilon_0} = \infty. \quad (5.14)$$

## 5.2 Sample computation of potential

Equations (5.5) and (5.6) tell us that the electric field and the potential contain the same information (up to an unphysical constant shift). Why then do we bother to introduce the potential? The deep reason is that by doing so we automatically build in the requirement that the curl of the electric field is zero. There is also however a practical reason: it is easier to do computations with scalar fields than it is to do them with vector fields! As an illustration, we’ll compute the potential throughout space from a wire of length  $L$  carrying a constant linear charge density  $\lambda$ . We can choose the line to lie on the  $x$  axis between  $-L/2$  and  $L/2$ , and restrict to points in the  $xy$  plane. We want to evaluate

$$\phi(\vec{r}) = \int_{-L/2}^{L/2} \frac{\lambda dx'}{4\pi\epsilon_0 |r - r'|}, \quad (5.15)$$

with

$$\begin{aligned}\vec{r} &= x\hat{x} + y\hat{y} \\ \vec{r}' &= x'\hat{x}.\end{aligned}\tag{5.16}$$

Thus we have

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dx'}{\sqrt{(x' - x)^2 + y^2}}.\tag{5.17}$$

We can simplify this using the change of variables

$$(x' - x) = |y|u,\tag{5.18}$$

after which we have

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \int_{-(L/2+x)/|y|}^{(L/2-x)/|y|} \frac{du}{\sqrt{u^2 + 1}}.\tag{5.19}$$

This indefinite integral is  $\operatorname{arcsinh}(u)$ , so we at last have

$$\phi = \frac{\lambda}{4\pi\epsilon_0} \left[ \operatorname{arcsinh}\left(\frac{L/2 - x}{|y|}\right) + \operatorname{arcsinh}\left(\frac{L/2 + x}{|y|}\right) \right].\tag{5.20}$$

Doing this computation directly for the electric field would have been considerably more annoying; we would have had to keep track of the direction of the vector components at every step!

### 5.3 Differential form of Gauss's law

We can use the geometric interpretation of the divergence and Gauss's law to derive a very important relationship between the electric field and the charge density  $\rho$ . Consider any continuous volume charge distribution  $\rho$ : over small enough regions, we can think of  $\rho$  as basically being constant. If we again denote by  $C_\epsilon(\vec{r})$  a cube centered at  $\vec{r}$  with side-length  $\epsilon$ , then we have

$$(\vec{\nabla} \cdot \vec{E})(\vec{r}) = \lim_{\epsilon \rightarrow 0} \frac{\int_{\partial C_\epsilon(\vec{r})} \vec{E} \cdot d\vec{A}}{\epsilon^3} = \lim_{\epsilon \rightarrow 0} \frac{\int_{C_\epsilon(\vec{r})} dV \rho}{\epsilon_0 \epsilon^3} = \frac{\rho(\vec{r})}{\epsilon_0}.\tag{5.21}$$

In the first equality we have used the geometric interpretation of the divergence, in the second we have used Gauss's law, and in the third we have used that  $\rho$  is approximately constant over sufficiently small volumes. We thus have arrived at what is typically called the **differential form of Gauss's law**:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.\tag{5.22}$$

In the homework you will verify this for the electric field of a charged ball.

In the way we have presented things, the differential form of Gauss's law is the result of a mathematical argument. But in fact we also could have taken it as the starting point for the class! Indeed by the divergence theorem we have

$$\int_{\partial R} \vec{E} \cdot d\vec{A} = \int_R dV \vec{\nabla} \cdot \vec{E} = \int_R dV \frac{\rho}{\epsilon_0} = \frac{Q_{\text{enclosed}}}{\epsilon_0},\tag{5.23}$$

so our original integrated version of Gauss's law is a direct consequence of the differential form. Assuming spherical symmetry, we then could have used the integrated version to determine the electric field of a point charge and everything else would proceed from there. What if we don't assume spherical symmetry however? Then we need to supplement the differential form of Gauss's law with the condition that the curl of  $\vec{E}$  vanishes everywhere:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= 0.\end{aligned}\tag{5.24}$$

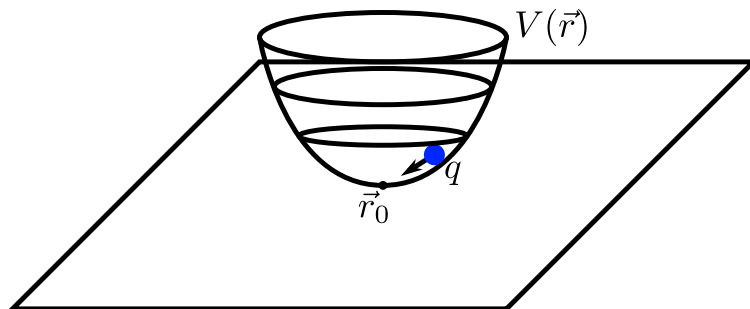


Figure 19: Earnshaw's theorem: in order for the location of a charge to be stable, it must be sitting at a local minimum of the electrostatic potential energy  $V = q\phi$ . By Laplace's equation however, no such minima can exist.

A famous theorem called the Helmholtz theorem, which we will prove later in the class, tells us that any vector field which goes to zero at infinity is completely determined by its divergence and curl: thus these equations are sufficient to determine  $\vec{E}$  everywhere. Moreover since the electric fields we constructed from Coulomb's law do obey these two equations, they are indeed the right ones from this point of view as well. Once these two equations are corrected for the possibility of moving charges they constitute half of **Maxwell's equations**, which are the four fundamental equations of electrodynamics (the other two equations tell us the divergence and curl of the magnetic field).

We can re-express the differential form of Gauss's law in terms of the electric potential  $\phi$ :

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \vec{\nabla} \phi \equiv -\nabla^2 \phi = \frac{\rho}{\epsilon_0}. \quad (5.25)$$

Here we have defined the symbol  $\nabla^2$  to mean the divergence of the gradient, leading to **Poisson's equation**

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \quad (5.26)$$

In particular away from any charges we must have  $\nabla^2 \phi = 0$ , which is called **Laplace's equation**. The operation  $\nabla^2$  is called the **Laplacian**, and it appears all over the place in physics, engineering, and beyond.

## 5.4 Earnshaw's theorem

There is an immediate consequence of the differential form of Gauss's law which is especially interesting: Earnshaw's theorem. Earnshaw's theorem is one of the great "no-go" theorems of engineering, it says that it is impossible to build a stable configuration of charges interacting only via Coulomb forces. To see this, let's suppose that we *could* build a stable configuration. This would mean that if we move any single charge  $q$  in the configuration away from its current location, it would feel a restoring force pulling it back to where it was. We can think about this in terms of its potential energy

$$V(\vec{r}) = q\phi(\vec{r}) : \quad (5.27)$$

in order for the charge to always feel a restoring force if we displace it, it must be resting in a local minimum of its potential energy (see figure 19). This however is impossible: if  $V$  had a local minimum, then close enough to it  $q\vec{E}$  would have to be pointing inwards everywhere. Therefore if we consider a ball  $B_\epsilon$  of radius  $\epsilon$  and centered at the minimum, for sufficiently small  $\epsilon$  we would have

$$q \int_{\partial B_\epsilon} \vec{E} \cdot d\vec{A} < 0. \quad (5.28)$$

By the divergence theorem and the differential form of Gauss's law however, we must instead have

$$q \int_{\partial B_\epsilon} \vec{E} \cdot d\vec{A} = q \int_{B_\epsilon} \vec{\nabla} \cdot \vec{E} = 0. \quad (5.29)$$

Therefore no minimum can exist!

Earnshaw's theorem was very puzzling to the physicists of the 19th century: the only forces they knew were electric and gravitational, and they knew that the gravitational force was too weak to be important in understanding the physics of materials. How then could matter be stable? It certainly appears to be both stationary and stable (e.g. if you sneeze on your desk it does not fall apart, and it seems to be quite stationary if you do not do anything to it), but what could be responsible for this? As with many of the other puzzles we will meet in this class, the resolution ultimately involves quantum mechanics.

## 5.5 Energy stored in the electric field

Two lectures ago we claimed that the energy stored in any finite electrostatic charge configuration is given by

$$U = \frac{\epsilon_0}{2} \int dV |E|^2. \quad (5.30)$$

We now have the tools necessary to derive this. Let's begin with our expression for the energy of a continuous charge distribution:

$$U = \frac{1}{2} \int dV dV' \frac{\rho(\vec{r})\rho(\vec{r}')}{4\pi\epsilon_0|r-r'|}. \quad (5.31)$$

The first step is to realize that we can rewrite this using the potential:

$$U = \frac{1}{2} \int dV \rho(\vec{r}) \left( \int dV' \frac{\rho(\vec{r}')}{4\pi\epsilon_0|r-r'|} \right) = \frac{1}{2} \int dV \rho(\vec{r})\phi(\vec{r}). \quad (5.32)$$

Note that in making this replacement it is essential that we define the zero of the potential to be at infinity, which wouldn't have been possible were the charge distribution infinite. Moreover since the charge distribution is finite,  $\rho$  must be zero beyond some large radius  $R$  so we can rewrite this as

$$U = \frac{1}{2} \int_{B_R} dV \rho\phi, \quad (5.33)$$

where  $B_R$  is a solid ball of radius  $R$ . We then rewrite this using the differential Gauss's law:

$$U = \frac{\epsilon_0}{2} \int_{B_R} dV (\vec{\nabla} \cdot \vec{E})\phi. \quad (5.34)$$

Now the key trick: we note that there is a vector calculus identity

$$\vec{\nabla} \cdot (\phi\vec{E}) = \vec{\nabla}\phi \cdot \vec{E} + \phi\vec{\nabla} \cdot \vec{E}, \quad (5.35)$$

which is a generalization of the product rule for differentiation. I'll leave it to you to check this by writing everything out in Cartesian coordinates (you will also have to use it on the homework), now just using it we can rewrite our expression for the energy as

$$U = \frac{\epsilon_0}{2} \int_{B_R} dV \left( \vec{\nabla} \cdot (\phi\vec{E}) - \vec{\nabla}\phi \cdot \vec{E} \right). \quad (5.36)$$

We can simplify the first term using the divergence theorem and the second using that  $\vec{E} = -\vec{\nabla}\phi$ , leading to:

$$U = \frac{\epsilon_0}{2} \int_{B_R} dV |E|^2 + \frac{\epsilon_0}{2} \int_{\partial B_R} \phi\vec{E} \cdot d\vec{A}. \quad (5.37)$$

In the limit that  $R \rightarrow \infty$  the first term gives us what we want, so we need to argue that the second vanishes in the same limit. In fact it does: the area of the surface of  $B_R$  is  $4\pi R^2$ , but  $\vec{E}$  falls off like  $1/R^2$  and  $\phi$  falls off like  $1/R$ , so this term falls off at least as fast as  $1/R$  and thus (5.30) holds.

You might wonder what happened to the “self-energy” of the charges, which we claimed earlier was included in the formula (5.30). After all our formula (5.31) for  $U$  came from computing the work we had to do to assemble the charges but what about the energy to make them? The resolution of this is that when we consider continuous charge distributions, the self-energy of the charge in any particular region vanishes in the limit of vanishing region size. For example we showed that the energy of a charged ball of radius  $R$  is

$$U = \frac{4\pi\rho^2 R^5}{15\epsilon_0}, \quad (5.38)$$

which vanishes as  $R \rightarrow 0$  with fixed  $\rho$ . It is only if we want to include objects with vanishing size but finite charge that we need to worry about self-energy.

## 6 Electric fields around a conductors

So far in this class we have considered only situations where we have complete control over where all charges are located. Indeed we specified either a set of point charges  $q_1, q_2, \dots$  at known positions  $\vec{r}_1, \vec{r}_2, \dots$  or a known continuous charge distribution  $\rho(\vec{r})$ . This is a reasonable description of what is going on if we are in a situation where it is difficult for charge to move about.

For example consider the air in this room: it is made out of neutral molecules, so despite the various drafts and eddies blowing around there is very little opportunity for charge to move from one place to another. Indeed the only way for it to happen is to create a strong enough electric field that we blow some electrons away from their host molecules, after which both the electrons and the ionized molecules they leave behind can move independently until they recombine: this how lightning works! A material such as air in which it is difficult for charge to move around is called an **insulator**. Another example of a good insulator is glass.

There are also materials in which it is *easy* for charge to move around: these are called **conductors**. The canonical example of a conductor is a metal: in a metal we have a lattice of ionized atoms arranged in a periodic array, with the remaining valence electrons free to move around from ion to ion. Another reasonably good conductor is seawater, which is water that has various ionized salt molecules dissolved in it that can move around freely.

You may be wondering how we decide which materials have charges which are “mobile enough” to be called conductors. After all seawater is a better conductor than air, but not so good a conductor as copper. And even air tends to have a few ions floating around which can eventually transport whatever charge we need. We won’t really be able to answer this until we go beyond electrostatics, but I will mention now that to each of these materials we can associate a certain number  $\sigma$ , called its **conductivity**, which has units of  $\frac{\text{C}^2\text{s}}{\text{kg m}^3}$ . It is not so obvious why these somewhat bizarre units should be a measure of conductivity; this is yet another tragic consequence of using Coulombs to measure charge. We can get a better understanding of the meaning of  $\sigma$  by noting that the quantity

$$\tau \equiv \frac{\epsilon_0}{\sigma} \quad (6.1)$$

has units of seconds, and this timescale turns out to give a good estimate of how long it takes any localized charge within a material to dissipate. Therefore higher conductivity means faster charge transport, while lower conductivity means slower charge transport. Roughly speaking, if the timescales at which we are adjusting the locations of objects and the voltage of batteries and so on are long compared to  $\tau$  then we can view a material as a conductor, while if they are short compared to it then we should view it as an insulator. Here is a table of  $\sigma$  (in  $\frac{\text{C}^2\text{s}}{\text{kg m}^3}$ ) and  $\tau$  (in s) for various materials to give you a sense of the possibilities.<sup>7</sup>

<sup>7</sup>The values become less precise as  $\sigma$  decreases: this is because in very strong insulators, the primary source of conductivity is stray defects in the material. These understandably vary quite a bit depending on how exactly the material was made (or where it was found). For example the electrical conductivity of air depends significantly on the humidity level, and also on the rate at which air molecules are ionized by cosmic rays.

$$\int_S \vec{E} \cdot d\vec{A} = (\vec{E} \cdot \hat{n})A = \frac{\sigma A}{\epsilon_0}$$

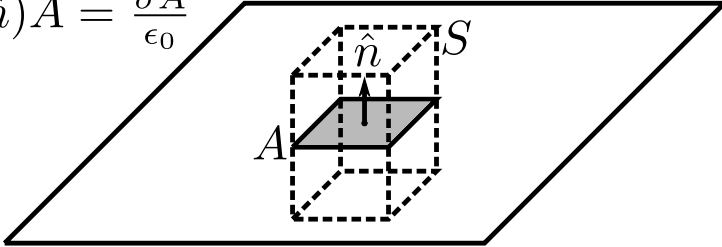
$$\Rightarrow \vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$$


Figure 20: Using Gauss's law to relate the normal electric field and the surface charge density at the surface of a conductor. There is no flux through the sides of the box since the electric field is normal to the surface, and there is no flux through the bottom since the bottom lies inside the conductor and  $\vec{E} = 0$  there. We are left with only the flux through the top of the box, which for a small enough box is just  $(\vec{E} \cdot \hat{n})A$ .

Material	$\sigma \left( \frac{\text{C}^2 \text{s}}{\text{kg m}^3} \right)$	$\tau$ (s)
Silver	$6.3 \times 10^7$	$1.4 \times 10^{-19}$
Copper	$6.0 \times 10^7$	$1.5 \times 10^{-19}$
Seawater	4.8	$1.8 \times 10^{-12}$
Drinking water	$\sim 10^{-3}$	$\sim 10^{-8}$
Air	$\sim 10^{-12}$	$\sim 10$
Glass	$\sim 10^{-13}$	$\sim 100$
Fused quartz	$\sim 10^{-18}$	$\sim 10^7$

The range of possibilities here is remarkable, over twenty-five orders of magnitude! Typically for any particular experimental setup, each material on this list will either be an excellent conductor or excellent insulator.

## 6.1 Basic properties of conductors

The goal of this lecture is to understand the electrostatic properties of objects made of conducting material. Somewhat confusingly, such objects are also referred to as conductors: for example silver is a conductor, but so is a ball made out of silver. When we say that we want to understand the electrostatic properties of conductors, we mean that we want to describe how the charges in an object made out of conducting material are distributed once we wait for them to finish moving.<sup>8</sup>

The key observation from which all electrostatic properties of conductors can be derived is the following:

**In electrostatics, the electric field inside of a conductor is zero!**

To understand why this must be true, say that we had  $\vec{E} \neq 0$  somewhere inside of a conductor. The mobile charges would then feel a force, causing them to move. But by assumption we have already waited until all charges have finished moving, so we thus must not have had an electric field in the first place. However the charges distribute themselves, we must end up in a situation with  $\vec{E} = 0$  everywhere inside the conductor. From this result we can immediately deduce the four essential features of conductors in electrostatics:

- **The charge density  $\rho$  vanishes everywhere in the interior of a conductor.** This follows from the vanishing of  $\vec{E}$  together with our differential version of Gauss's law,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \tag{6.2}$$

<sup>8</sup>Our table tells us that for a good conductor we do not have to wait very long!



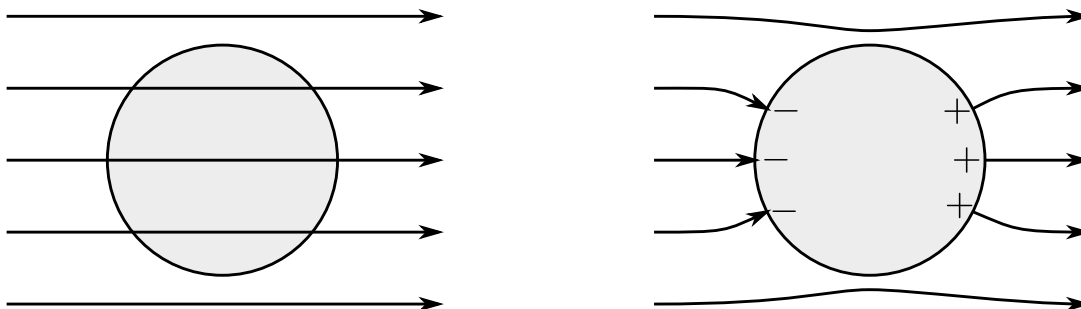


Figure 21: Insulating and conducting balls in the presence of an external electric field. In the insulating case the electric field has no effect, so the field is nonzero in the interior. In the conducting case the electric field causes positive charge to move to the right. Eventually negative charge accumulates on the left side of the ball's surface to cancel the incoming electric field, while positive charge accumulates on the right side of the ball's surface to reproduce the outgoing field, and the electric field in the interior is zero.

Therefore the only possible location for charge density on a conductor is its surface, where it is described via a surface charge density  $\sigma(\vec{r})$ .<sup>9</sup>

- **The electric potential  $\phi$  is constant throughout the interior of a conductor.** This follows from our expression

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = - \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s} \quad (6.3)$$

for the potential difference: since the right-hand side is path independent, we can make sure to choose a path connecting  $\vec{r}_1$  and  $\vec{r}_2$  which stays entirely inside the conductor. Since  $\vec{E}$  is zero everywhere on the path, we must have  $\phi(\vec{r}_1) = \phi(\vec{r}_2)$  for all pairs of points in the conductor.

- **The electric field can be nonzero at the surface of a conductor, but it must always be parallel to the normal vector  $\hat{n}$ .** A normal electric field is allowed because the mobile charges are not able to leave the conductor: any normal electric force they feel there will be canceled by whatever forces are keeping them within the material. A tangential electric field however would cause them to move, so it must not exist.
- **The normal electric field at the surface of a conductor is proportional to the surface charge density.** This follows from an application of Gauss's law to a small box surrounding each point  $\vec{r}$  in the surface, as shown in figure 20, which tells us that

$$\vec{E}(\vec{r}) = \frac{\sigma(\vec{r})}{\epsilon_0} \hat{n}(\vec{r}). \quad (6.4)$$

To get some intuition for what these rules imply, let's imagine that we have a neutral insulating ball sitting in a constant electric field. The electric field is the same both inside and outside of the ball, as shown in the left diagram of figure 21, and in particular it does not vanish inside. Now let's say that we gradually tune some parameter (say the temperature) such that the insulating material becomes conducting (materials where this can happen do exist, they are called semiconductors). The charges must then move to cancel the electric field inside, leading to the situation in the right diagram of figure 21.

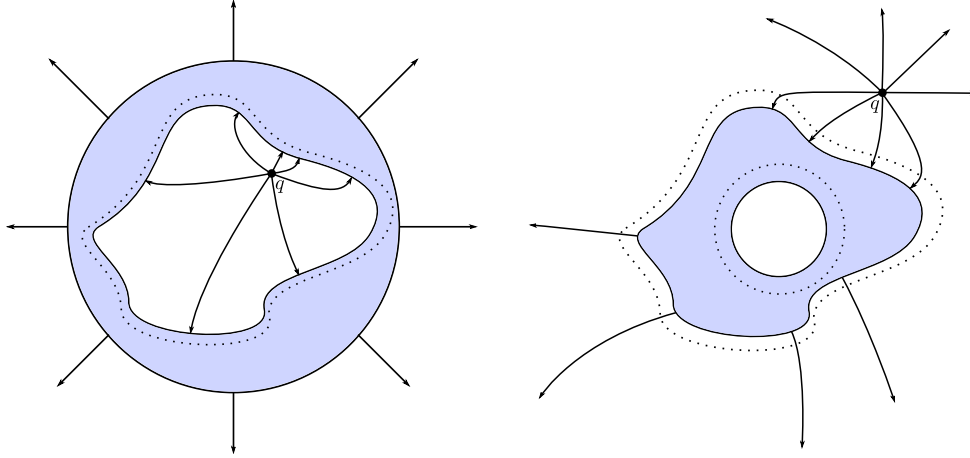


Figure 22: Electric field lines for a point charge in the vicinity of two differently shaped conductors. On the left, we have a ball-shaped conductor with an arbitrarily-shaped hole, and the charge is inside the hole. On the right, we have a ball shaped hole in an arbitrarily-shaped conductor and a point charge on the outside. The endpoints of the electric field lines show where the surface charges are accumulated on the inside and outside surfaces of the conductors. The dotted lines show the various Gaussian surfaces referenced in the text.

## 6.2 A point charge and a shell

One of the most important properties of conductors is that we can use them to *screen* electric fields: for example we can protect sensitive electronics or fragile organic life forms from potentially dangerous electric effects. Let's first see how this works in two simple examples, shown in figure 22.

In the first example we have a ball-shaped conductor with an arbitrarily-shaped hole, and a point charge of strength  $q$  inside the hole. To understand what is going on, we can surround the hole with a Gaussian surface lying within the conductor, as in the dotted line in the left diagram of figure 22. Since  $\vec{E} = 0$  inside the conductor, this surface must enclose zero charge. Therefore the net charge on inner surface of the conductor must be  $-q$ . Since the conductor is neutral, the net charge on the outside surface therefore must be  $q$ . Now I make the following claim: the surface charge density on the inner surface will depend on position in a complicated way depending on where the point charge is, but the surface charge density on the outer surface will be spherically symmetric and the electric field outside of the conductor will just be that of a point charge  $q$  sitting in the center of the sphere! I will justify this by showing that such a charge distribution is able to produce an vanishing electric field inside the conductor; we will see in the following subsection that there can only be one such charge distribution so the one we find must be the right one. Indeed let's first consider the limit where the radius of the outer surface of the shell is very large; any charge distributed on it cannot have any significant effect on the charges on the inner surface. Therefore if we include only the contributions to the electric field coming from the point charge and the inner surface charge, we still get an electric field which vanishes in the interior of the conductor. In fact it also vanishes in the exterior, since if we remove the charges on the outer surface then there is nowhere for electric field lines outside of the conductor to begin. Now let us take the charge distribution on the interior surface which we found for the case of large radius, and use it also for our finite-radius ball. The electric field sourced by this distribution and the point charge will still vanish everywhere outside the inner surface. But we now must restore the charge  $q$  on the outer surface: let's do this in a spherically symmetric way. We've already seen that inside a hollow spherical shell with a uniform surface charge density the electric field vanishes, so after

<sup>9</sup>I'm really REALLY sorry that the symbol  $\sigma$  is used both for surface charge density and conductivity. Both uses are completely standard and can't be avoided :-).

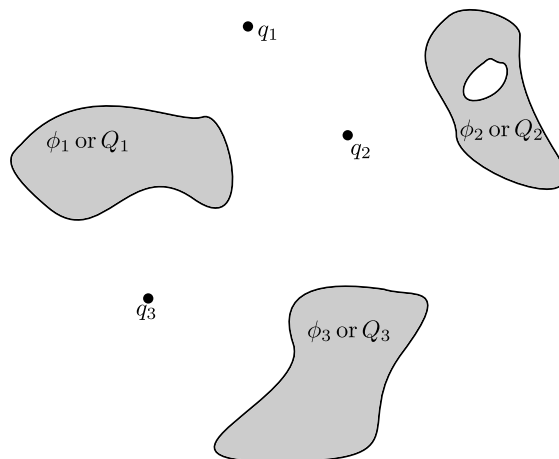


Figure 23: A setup where we can apply the uniqueness theorems for conductors: we have a set of conductors, shaded grey, and we specify either the potential  $\phi_i$  or the total charge  $Q_i$  on each of them. We further specify the locations of any other external charges, here the point charges  $q_i$ . The theorems then tell us that the electric field and potential everywhere, and thus the surface charge distributions on the conductors, are uniquely determined.

we add these charges the electric field inside the conductor will still vanish!

Our second example is similar: we have an arbitrarily-shaped conductor with a ball-shaped hole, and we introduce a point charge on the outside as shown in the right-hand diagram of figure 22. The inner Gaussian surface shown in the figure tells us that there must be no net charge on the inner surface, and the outer Gaussian surface then tells us that there must also be no net charge on the outer surface. In fact something more remarkable is true: all surface charge density lies on the outer surface of the conductor, the surface charge density of the inner surface vanishes, and the electric field inside the hole is zero. In other words somebody living inside of the hole would have *no idea* that there is a point charge outside! We again will argue that this charge distribution is consistent with the vanishing of the electric field inside of the conductor, and then appeal to the uniqueness theorem we prove momentarily to ensure that it is only such distribution. The idea is to consider what would happen if we “filled in” the hole: we would then just have a solid conductor and there would be no choice except to have all the charge density on the outer surface. There is therefore a possible charge distribution on the outer surface which together with the point charge produces zero electric field everywhere inside the conductor and also inside the hole. By uniqueness, this must be the right one also in the presence of the hole. In fact this argument does not even require our assumption that the hole is ball-shaped, but if we use this then we can get some sense of why the distribution is unique. Say we now tried to put some charge density  $\sigma$  on the inner surface. Since the field near this surface produced by the point charge and the outer surface charge density is zero, there will be no force on the charges so they will want to distribute themselves in a spherically-symmetric manner. But the only spherically symmetric surface charge density with zero total charge is  $\sigma = 0$ .

### 6.3 Uniqueness theorems

The arguments in the previous subsection relied on assuming that there is only one charge distribution on the surface of a conductor which is consistent with the potential or total charge on the conductor and the vanishing of the electric field inside. Let’s now study this more carefully. We will prove that either fixing the potential or the charge on each of a collection of conductors is enough to uniquely determine the field everywhere.

We first consider the case where we have a collection of conductors  $C_1, C_2, \dots$ , each of which is at a fixed

potential (relative to infinity)  $\phi_1, \phi_2, \dots$ . Moreover outside of the conductors we allow there to be some fixed charge distribution  $\rho(\vec{r})$  (such as the point charges in the examples of figure 22). See figure 23 for an illustration. The potential everywhere outside the conductors must obey Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}. \quad (6.5)$$

Our claim is that for a given set of  $C_1, C_2, \dots, \phi_1, \phi_2, \dots$ , and  $\rho$ , there can only be one such solution. Since knowing  $\phi$  determines  $\vec{E}$  and knowing  $\vec{E}$  determines the surface charge distribution, the surface charge distributions must also be unique. The proof is as follows. Say that there were two solutions  $\phi$  and  $\phi'$ . Everywhere outside of the conductors we then would have

$$\nabla^2(\phi - \phi') = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0, \quad (6.6)$$

so  $\phi - \phi'$  obeys Laplace's equation outside of the conductors (it may not obey it right at the surfaces, since the surface charge distributions could be different for  $\phi$  and  $\phi'$ ). Moreover at each conductor  $C_i$  we have  $\phi - \phi' = \phi_i - \phi_i = 0$ , and  $\phi - \phi'$  also must vanish at infinity since  $\phi$  and  $\phi'$  both do. There is certainly one solution of Laplace's equation outside of the conductors which fits the bill,

$$\phi - \phi' = 0. \quad (6.7)$$

This of course just says that  $\phi = \phi'$ . In fact there can be no other solution: the reason is that any nonzero function  $\phi' - \phi$  which vanishes at infinity must have a minimum or a maximum somewhere at which  $\phi \neq 0$ , but no solution of Laplace's equation which vanishes at infinity and also at the locations of the conductors can have this property. We already gave one argument for this in our discussion of Earnshaw's theorem, but since it is important I'll now give another. It is based on a somewhat surprising property of solutions of Laplace's equation: if  $f$  is a solution of Laplace's equation, then the average of  $f$  over the surface of any ball is equal to the value of  $f$  at its center! The proof proceeds by showing that the average is independent of the radius of the ball:

$$\begin{aligned} \frac{d}{dr} \left( \frac{\int_{\partial B_r} f dA}{4\pi r^2} \right) &= \frac{d}{dr} \left( \int d\theta d\phi \sin \theta \frac{f}{4\pi} \right) \\ &= \int d\theta d\phi \sin \theta \frac{\partial_r f}{4\pi} \\ &= \frac{\int d\theta d\phi \sin \theta r^2 \partial_r f}{4\pi r^2} \\ &= \frac{\int_{\partial B_r} \vec{\nabla} f \cdot d\vec{A}}{4\pi r^2} \\ &= \frac{\int_{B_r} \nabla^2 f dV}{4\pi r^2} \\ &= 0. \end{aligned} \quad (6.8)$$

Therefore we may perform the average over a very small ball, which will just give us the value of  $f$  at the center. Applying this result now to a ball centered at the claimed maximum or minimum of  $\phi - \phi'$ , we can take the ball to be large enough that it is close to intersecting one of the conductors but doesn't quite. Since  $\phi - \phi' = 0$  at each conductor, the average over this ball must then be less than the value of  $\phi - \phi'$  at the maximum (or greater than the value of  $\phi - \phi'$  at the minimum), but neither is allowed.

This theorem has various applications, in particular one is to justify our solution of the second example of the previous section. Here is the argument: whatever the charge distribution is, the conductor will be at some potential  $\phi_0$ . One solution where the conductor is at this potential is obtained by filling in the hole and obtaining a solution where the filled-in conductor is at  $\phi_0$ . If we then restore the hole, keeping  $\phi = \phi_0$  inside, then this is still a good solution in the presence of the hole; by uniqueness it must be the right one!

Note that this works for any shape of hole, the presence of charge outside of a conductor is always invisible for someone living in a cavity in the conductor.

To justify our treatment of the first example, we need to instead consider the situation where we fix the total charges on the conductors instead of their potentials. A somewhat similar argument then shows that in fact the solution is also unique in this situation: there can only be one charge distribution for a fixed  $C_1, C_2, \dots, Q_1, Q_2, \dots$ , and  $\rho(\vec{r})$ . The proof goes like this: say that we had two electric fields  $\vec{E}$  and  $\vec{E}'$  which both obeyed these conditions. The field  $\vec{E} - \vec{E}'$  would then obey

$$\vec{\nabla} \cdot (\vec{E} - \vec{E}') = 0 \quad (6.9)$$

outside of the conductors and

$$\int_{S_i} (\vec{E} - \vec{E}') \cdot d\vec{A} = Q_i - Q_i = 0, \quad (6.10)$$

where  $S_i$  is a Gaussian surface just outside of  $C_i$ . The idea is then to compute the integral of the quantity

$$\vec{\nabla} \cdot \left( (\phi - \phi')(\vec{E} - \vec{E}') \right) = \vec{\nabla}(\phi - \phi') \cdot (\vec{E} - \vec{E}') + (\phi - \phi')\vec{\nabla} \cdot (\vec{E} - \vec{E}') = -|E - E'|^2 \quad (6.11)$$

over the region  $R$  which lies to the exterior of all conductors:

$$\begin{aligned} - \int_R dV |E - E'|^2 &= \int_R dV \vec{\nabla} \cdot \left( (\phi - \phi')(\vec{E} - \vec{E}') \right) \\ &= - \int_{\partial R} (\phi - \phi')(\vec{E} - \vec{E}') \cdot d\vec{A} \\ &= \sum_i (\phi_i - \phi'_i) \int_{S_i} (\vec{E} - \vec{E}') \cdot d\vec{A} \\ &= 0. \end{aligned} \quad (6.12)$$

Therefore we must have  $\vec{E} = \vec{E}'$  everywhere! In the second to last equality, we used that  $\phi$  and  $\phi'$  are both constant at the surface of each conductor, so we could pull them out of the integral.

One thing we learn from this theorem is that in first example above, even if the outer surface is not spherical the field outside will still have no knowledge of where the internal charge is located. The reason is that on the inner surface we can use the charge distribution from the situation where the outer surface is infinitely big, while on the outer surface we can use the charge distribution from the situation where we fill in the hole and just put a net charge  $q$  on the conductor. Only the latter charge distribution sources the outer electric field, so it cannot have any information about the shape of the cavity or the location of the charge.

## 7 Images charges and capacitors

In the previous lecture we discussed some general properties of conductors, in this lecture we discuss a few examples.

### 7.1 The classic image problem

Our first problem will be to determine the potential and electric field for a situation where a conductor is filling the lower half of space (say the region  $z < 0$ ) and we have a point charge  $q$  sitting at a height  $d$  above the conductor (see figure 24). Let's think about this as a boundary value problem: what conditions does the potential  $\phi$  need to obey? The first thing we can say is that it must be constant for all points with  $z \leq 0$ , so for simplicity let's define the potential so that  $\phi = 0$  for all such points. We then are left looking for a solution of Poisson's equation in the region  $z \geq 0$  which has  $\phi = 0$  at  $z = 0$  and is consistent with having a point charge on the  $z$ -axis at  $z = d$ . We may then make a very clever observation: we *already know* a

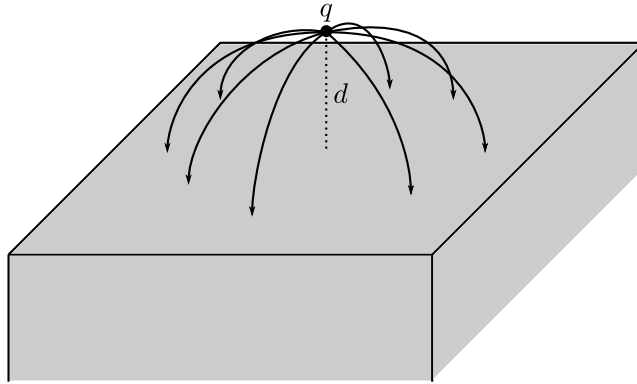


Figure 24: Electric field lines for the classic image problem

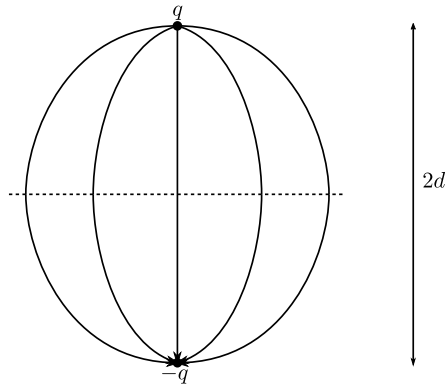


Figure 25: The method of images: we note that for  $z > 0$  the conditions required of  $\phi$  in the conductor problem are also obeyed by the upper half of the potential for the charge distribution shown here. By the first uniqueness theorem, this must actually be the potential for the conductor problem!

function  $\phi$  which satisfies all of these requirements! It is given by the restriction to  $z \geq 0$  of the potential sourced by *two* point charges on the  $z$  axis, our charge  $q$  at  $z = d$  and another charge  $-q$  at  $z = -d$  (see figure 25). Writing out explicitly in cartesian coordinates,

$$\phi(x, y, z) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + d)^2}} \right), \quad (7.1)$$

which indeed vanishes when  $z = 0$ . Moreover we can take the gradient and add a minus sign to compute the electric field:

$$\vec{E}(x, y, z) = \frac{q}{4\pi\epsilon_0} \left[ \left( \frac{1}{(x^2 + y^2 + (z - d)^2)^{3/2}} - \frac{1}{(x^2 + y^2 + (z + d)^2)^{3/2}} \right) (x\hat{x} + y\hat{y} + z\hat{z}) - d \left( \frac{1}{(x^2 + y^2 + (z - d)^2)^{3/2}} + \frac{1}{(x^2 + y^2 + (z + d)^2)^{3/2}} \right) \hat{z} \right]. \quad (7.2)$$

At  $z = 0$  this simplifies to

$$\vec{E}(x, y, 0) = -\frac{qd}{2\pi\epsilon_0(x^2 + y^2 + d^2)^{3/2}}\hat{z}, \quad (7.3)$$

which points downwards for  $q > 0$  and is normal to the surface of the conductor as expected. It is absolutely essential to remember that in the setup of figure 24, the expressions (7.1) and (7.2) are *only* valid for  $z \geq 0$ . For  $z < 0$  we have  $\phi = 0$  and  $\vec{E} = 0$ . In other words, the “image charge” at  $z = -d$  in figure 25 **does not exist!** It is merely a mathematical trick to determine the potential and electric field for  $z > 0$ .

Now let’s recall that the surface charge density  $\sigma$  at the surface of a conductor is related to the normal electric field there via

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}. \quad (7.4)$$

Therefore apparently the surface charge density on the surface of the conductor for this problem is

$$\sigma(x, y) = -\frac{qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}. \quad (7.5)$$

Note in particular that this is negative, and it is largest directly below the charge. It then goes to zero as we move further and further away. As a consistency check, we can compute the total charge on the conductor:

$$\begin{aligned} Q &= -\frac{qd}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty \frac{r dr}{(r^2 + d^2)^{3/2}} \\ &= qd \frac{1}{\sqrt{r^2 + d^2}} \Big|_{r=0}^{r=\infty} \\ &= -q. \end{aligned} \quad (7.6)$$

Thus the induced charge on the surface precisely cancels that of the point charge, giving a configuration which looks neutral from far away.

## 7.2 Capacitance

Let’s now consider an isolated conductor of finite size. It is clear that the potential of the conductor (relative to infinity) will depend on how much charge we put onto the conductor. For example if we put no charge onto it, then the electric field will be zero everywhere and we’ll have

$$\phi(\vec{r}_{\text{conductor}}) = -\int_\infty^{\vec{r}_{\text{conductor}}} \vec{E} \cdot d\vec{s} = 0, \quad (7.7)$$

where  $\vec{r}_{\text{conductor}}$  is any point in the conductor. More generally, if we double the charge on the conductor it is not hard to see that this will just double the potential as well. Indeed say that we begin with a charge

$Q_0$  on the conductor. This charge will place itself in a surface charge distribution  $\sigma_0$  at the boundary of the conductor, which will then source an electric field  $\vec{E}_0$ . We can do the line integral of this  $\vec{E}_0$  in from infinity to determine the potential  $\phi_0$  on the conductor. Now say that we double the charge, instead putting charge  $2Q_0$  on the conductor. By the uniqueness theorem, we know that any charge distribution we come up with which integrates to a charge  $2Q_0$  on the conductor and produces an electric field which vanishes inside it must be the right one. And indeed there is a simple proposal: we just take the surface charge density to be  $2\sigma_0$ . This certainly integrates to  $2Q_0$ , and by superposition the electric field it produces is  $2\vec{E}_0$ . This will of course vanish inside of the conductor since  $\vec{E}_0$  does, and thus this must be the correct charge distribution. We may then compute the potential:

$$-\int_{\infty}^{\vec{r}_{\text{conductor}}} 2\vec{E}_0 \cdot d\vec{s} = 2\phi_0. \quad (7.8)$$

In fact this argument still works if we replace 2 by any other number; therefore we see that the charge and the potential on any conductor are linearly related:

$$Q = C\phi, \quad (7.9)$$

where the quantity  $C$  is called the **capacitance** of the conductor.<sup>10</sup>  $C$  depends on the size and shape of the conductor, but it does not depend on the amount of charge or the potential. In SI units the units of  $C$  are called **Farads**, and are defined as

$$1 \text{ F} = 1 \frac{\text{C}^2 \text{s}^2}{\text{kg m}^2}. \quad (7.10)$$

As is usually the case in SI, the Farad turn out to be a rather silly unit: conductors one encounters in the wild tend to have capacitances which are of order of  $10^{-12}$  Farads! As with conductivity, if we throw in an  $\epsilon_0$  we can make the units more reasonable: the quantity  $C/\epsilon_0$  has units of meters, which makes a lot of sense for something that depends only on the spatial geometry of the conductor. Either way you can think of the capacitance as a measure of how easy it is to charge up the conductor by plugging it into a battery.

As a simple example, let's compute the capacitance of a conducting ball of radius  $R$ . We know that if we put a charge  $Q$  onto it it will distribute itself in a thin shell of surface charge density  $\sigma = \frac{Q}{4\pi R^2}$  at the surface of the conductor, and we further know that the electric field outside will just look like that of a point charge of strength  $Q$  at the center of the ball. The potential at the edge of the conductor is therefore

$$\phi = \frac{Q}{4\pi\epsilon_0 R}, \quad (7.11)$$

so apparently the capacitance is

$$C = 4\pi R\epsilon_0. \quad (7.12)$$

In particular if  $R$  is one meter, then we have  $C = 1.1 \times 10^{-10}$  F (and  $C/\epsilon_0 = 12.6$  m).

### 7.3 Capacitors

There is also a kind of capacitance we can define for *pairs* of conductors, and it turns out that this one is much more important in practice than the capacitance of a single conductor. Let's call the conductors  $C_1$  and  $C_2$ . The idea is that if we put a charge  $Q$  on  $C_1$  and a charge  $-Q$  on  $C_2$ , the potential difference and charge are related as

$$Q = C(\phi_1 - \phi_2), \quad (7.13)$$

where  $C$  is a constant depending only on the shapes and relative locations of the conductors and which again is called capacitance. When  $Q$  is positive electric field lines go from  $C_1$  to  $C_2$ , so we expect  $\phi_1 > \phi_2$ .

<sup>10</sup>You might wonder why our ancestors did not define the capacitance so that  $\phi = CQ$ . After all computing the potential given a fixed amount of charge seems more intuitive than computing the charge given a fixed potential. The reason is that although fixing the amount of charge makes more sense theoretically, it is much easier experimentally to fix the potential on a conductor: we just plug it into a battery with a fixed voltage.



Similarly if while if  $Q$  is negative the field lines go from  $C_2$  to  $C_1$ , so we expect  $\phi_2 < \phi_1$ ; either way  $C$  will be a positive number. The argument that  $C$  does not depend on the charge or potential difference is the same as before: if we double  $Q$  then we can get a valid charge distribution by just doubling the surface charge density everywhere, and this just doubles the electric field and therefore the potential difference.

Let's compute the capacitance for two parallel conducting plates of surface area  $A$  and separation  $d$  (in the limit where  $A \gg d^2$ ). If we put the plate with positive charge at  $z = 0$  and the plate with negative charge at  $z = d$ , then the electric field between the plates is

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z}, \quad (7.14)$$

with  $\sigma = Q/A > 0$ . The potential difference is therefore

$$\phi(0) - \phi(d) = \int_0^d \vec{E} \cdot d\vec{s} = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A}, \quad (7.15)$$

so apparently the capacitance is

$$C = \frac{\epsilon_0 A}{d}. \quad (7.16)$$

Apparently to get a "better" capacitor, meaning a capacitor which stores more charge for a given potential difference, we want very large plates which are very close together.

## 7.4 Energy stored in a capacitor

A capacitor which is charged up sources an electric field, so we should think of it as storing energy. One way to think about this energy is that it is the work we have to do starting with the pair of conductors being both neutral and then moving charge from one to the other to build up the charges  $Q$  and  $-Q$ . Let's say that so far we have built up a charge  $Q'$ . The potential difference is currently  $Q'/C$ , so in moving a charge  $dQ'$  from one to the other we have to do an amount of work

$$dW = \frac{Q' dQ'}{C}. \quad (7.17)$$

The total amount of work we need to do is thus

$$W = \int_0^Q \frac{Q' dQ'}{C} = \frac{Q^2}{2C} = \frac{C(\phi_1 - \phi_2)^2}{2}. \quad (7.18)$$

Thus if we fix the potential difference, a capacitor with a large capacitance can store quite a lot of energy!

Let's check this formula for our two parallel plates. We can compute the energy via

$$U = \frac{\epsilon_0}{2} \int dV |E|^2 = \frac{\epsilon_0}{2} \times dA \times \frac{(Q/A)^2}{\epsilon_0^2} = \frac{Q^2}{2(\epsilon_0 A/d)} = \frac{Q^2}{2C}, \quad (7.19)$$

so indeed it works!

## 8 Current and ohmic materials

We are now at long last ready to begin describing charges in motion. The most direct way to discuss this would be to simply try to work out the location of each charge as a function of time:  $\vec{r}_1(t), \vec{r}_2(t), \dots$ . In most applications however, there are far too many moving charges present for this to be a viable strategy. A much more useful notion is that of **current**, where instead of asking what the charges are doing individually we ask instead how frequently charge is passing through a surface in space.

Let's begin with a simple example, shown in figure 26. We have two rooms, each of which has some number of charged particles in it. We will call the total charge in the left room  $Q_L$  and the total charge in

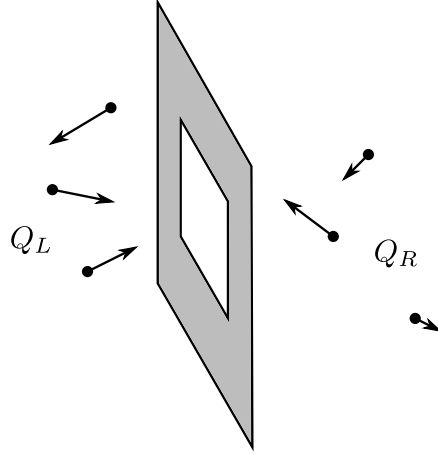


Figure 26: Charges passing through a window. The rate of charge passing from the left to the right defines the current to the right, while the rate of charge passing from right to left defines the current to the left. Since charge is conserved, the two currents are opposites of each other.

the right room  $Q_R$ . Moreover let's say that the only way for charge to enter or leave either room is through a single window that connects the two rooms. Typically there will be some charges doing this, so  $Q_L$  and  $Q_R$  will be functions of time. The **current through the window to the right** is defined to be

$$I_R \equiv \frac{dQ_R}{dt} \equiv \dot{Q}_R. \quad (8.1)$$

Similarly the **current through the window to the left** is defined to be

$$I_L \equiv \frac{dQ_L}{dt} \equiv \dot{Q}_L. \quad (8.2)$$

These two currents are related in a simple way by charge conservation, which is the statement that

$$\frac{d(Q_R + Q_L)}{dt} = 0. \quad (8.3)$$

Indeed we have

$$I_L = \dot{Q}_L = -\dot{Q}_R = -I_R. \quad (8.4)$$

The situation is more complicated if there are other ways the current could have gone between the rooms. For example what if there are two windows? Then  $\dot{Q}_R$  will be the sum of two terms, one describing the amount of charge which is arriving (or leaving) through the first window and the other describing the amount which is arriving (or leaving) through the second window. These two terms we then call the currents through the first and second windows respectively. In SI the units of current are Coulombs per second, a combination which is called the **Ampere**:

$$1\text{A} \equiv 1 \frac{\text{C}}{\text{s}}. \quad (8.5)$$

## 8.1 Current density

We can get some more intuition for currents by considering the special case of a gas of particles of charge  $q$ , constant number density  $n$ , and all moving at velocity  $\vec{v}$ . We then ask in a time  $dt$  how much charge passes

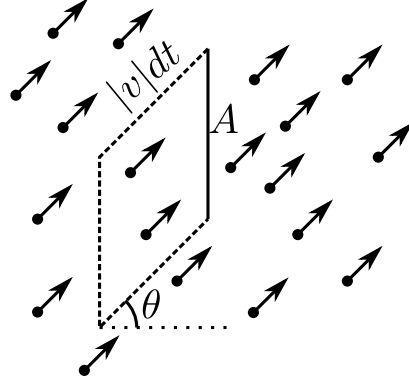


Figure 27: A uniform charge distribution moving with uniform velocity through a window of area  $A$  whose normal vector is at angle  $\theta$  relative to the direction of motion. The dashed parallelogram encloses the set of particles which will pass through the window in time  $dt$ .

through a flat window of surface area  $A$  and with normal vector  $\hat{n}$ ? The answer is that it is  $q$  times the number of particles in the parallelepiped shown in figure 27. In other words,

$$dQ = qn|v|dt \cos \theta A. \quad (8.6)$$

We can rewrite this as

$$I \equiv \frac{dQ}{dt} = qn|v| \cos \theta A = qn\vec{v} \cdot \hat{n}A \equiv qn\vec{v} \cdot \vec{A}. \quad (8.7)$$

The quantity

$$\vec{J} \equiv qn\vec{v} = \rho\vec{v} \quad (8.8)$$

here is called the **current density**, and we can use it to compute the current through any flat surface by computing  $\vec{J} \cdot \vec{A}$ .

Let's now think about how to generalize this example to more realistic situations. Several things need to be improved. The first is that inside of a real conductor, the charge carriers are flying all over the place in all sorts of directions; claiming they all are moving in the same direction would be absurd. But in fact if we instead talk about the *average velocity* of the charge carriers, most of the crazy motion at the atomic scale will wash out, leaving a net drift of charge which is relatively small and varies relatively slowly with position. This is similar to the fact that in air the individual molecules are moving at  $\sim 500\text{m/s}$  but typical wind speeds are only  $5\text{m/s}$ . If the number density and average velocity are independent of position, then we still have

$$I = \vec{J} \cdot \vec{A}, \quad (8.9)$$

but now with

$$\vec{J} = qn\vec{v}_{\text{average}}. \quad (8.10)$$

The second thing we need to change is that there may be more than one kind of charge carrier, for example in salt water there are both salt ions and water ions. If we use  $i$  to label the type of charge carrier, we now will have

$$\vec{J} = \sum_i q_i n_i \vec{v}_{i,\text{average}}. \quad (8.11)$$

Finally we need to deal with the fact that  $n_i$  and  $\vec{v}_{i,\text{average}}$  will in general depend on position. Our final formula for the current density will thus be

$$\vec{J}(\vec{r}) = \sum_i q_i n_i(\vec{r}) \vec{v}_{i,\text{average}}(\vec{r}), \quad (8.12)$$

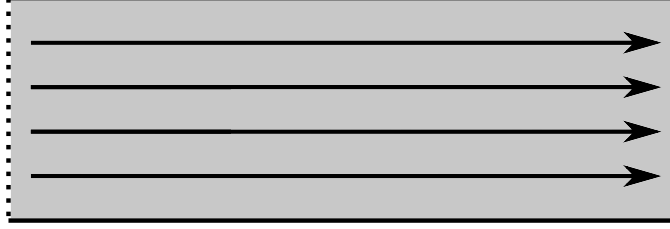


Figure 28: A steady current. The flux arriving on the left side is the same as the flux departing on the right, so there is no accumulation of charge in the grey region.

and to compute the current through a surface  $S$  we now need to compute the flux integral

$$I = \int_S \vec{J} \cdot d\vec{A}. \quad (8.13)$$

This last equation is the key one to remember for the current density: the flux of the current density through a surface gives the amount of current flowing through the surface. This is quite similar to the charge density  $\rho(\vec{r})$ , which we can integrate over any volume region  $R$  to find the total charge in that region.

## 8.2 Current conservation

One of the fundamental properties of electric charge is that it is conserved. But in fact more is true: it is *locally* conserved. This means that not only does the total amount of charge in an isolated system not change, it is not possible to have charge disappear in one place and reappear in another without moving through the space in between. In other words, the only thing that can happen to charge is that we can move it around from place to place. We can use this to derive a very important relationship between the charge density  $\rho$  and the current density  $\vec{J}$ . Indeed consider a volume region  $R$ . The net charge in this region is

$$Q = \int_R dV \rho. \quad (8.14)$$

By charge conservation, the time derivative of  $Q$  must also be equal to the rate which charge is entering the region:

$$\dot{Q} = \int_R dV \dot{\rho} = - \int_{\partial R} \vec{J} \cdot d\vec{A}. \quad (8.15)$$

Now let's take  $R$  to be a small cube of side length  $\epsilon$ . If  $\epsilon$  is small enough, we can think of  $\rho$  as being constant over  $R$  (but still time dependent). We then have:

$$\dot{Q} \approx \epsilon^3 \dot{\rho} \approx -\epsilon^3 \vec{\nabla} \cdot \vec{J}, \quad (8.16)$$

where in the last step we used our geometric interpretation of the divergence. Taking the limit  $\epsilon \rightarrow 0$ , we have the **continuity equation**:

$$\vec{\nabla} \cdot \vec{J} = -\dot{\rho}. \quad (8.17)$$

This equation, also sometimes called the current conservation equation, is the mathematical expression of local charge conservation.

## 8.3 Steady currents

There is an important special case of current flow, where there is no net charge accumulating or dissipating anywhere. Mathematically this means that

$$\dot{\rho} = 0, \quad (8.18)$$

and by the continuity equation this implies

$$\vec{\nabla} \cdot \vec{J} = 0. \quad (8.19)$$

A current density obeying (8.19) is called a **steady current**. You can think of it as being similar to the flow of a river: the water is moving by, but the water level is not changing since water is arriving just as fast as it is departing. An example is shown in figure 28: we can compute the flux through the boundary of the shaded region as

$$\int_{\partial R} \vec{J} \cdot d\vec{A} = \int_R dV \vec{\nabla} \cdot \vec{J} = 0, \quad (8.20)$$

where there is no flux through the top, bottom, front, or back, and the fluxes through the left and right boundaries cancel since one points inward and one outward.

## 8.4 Ohm's law

We've learned how to describe how charges move. Now the real question: *why* do they move? Clearly it will be because they feel some force. There are various options for what this force will be, for example a charge might move because I pick it up and carry it across the room. Since this is electromagnetism class however, we will begin by considering situations where the only force on the charges is the Coulomb force: they move because they feel an electric field!

If our charges were just floating around in vacuum, it would be relatively easy to describe their motion in an electric field. Namely for each charge we would just solve the differential equation

$$m\ddot{\vec{r}} = q\vec{E}(\vec{r}) \quad (8.21)$$

to determine  $\vec{r}(t)$ . As we have already discussed however, in a realistic situation like a copper wire the charge carriers are constantly banging around with each other and also the copper ions, and tracking each one individually is hopeless. It turns out however that nature has decided to be quite kind to us: in most materials we encounter the current density produced by an electric field  $\vec{E}$  is just given by

$$\vec{J}(\vec{r}) = \sigma\vec{E}(\vec{r}), \quad (8.22)$$

where the constant  $\sigma$  is a property of the material called its **conductivity**. We already mentioned the conductivity briefly a few lectures ago, equation (8.22), which is called **Ohm's law**, is what really defines it.

You may be confused that after making such a big deal out of the fact that  $\vec{E} = 0$  inside of conductors a few lectures ago, I'm now saying that it isn't. But actually I only said that is true in *static situations*. Once the charges are moving it doesn't need to be true, and Ohm's law tells us what happens instead. And if actually the charges aren't moving, we have  $\vec{J} = 0$  so Ohm's law indeed tells us that we have  $\vec{E} = 0$  as well.

Unlike Maxwell's equations and the Lorentz force law, Ohm's law is *not* a "fundamental" equation. Instead it is an "emergent" property of certain materials, arising from the way in which many individual particles interact with each other. In fact there are materials where it is not true, and even in those where it does work this is only if the electric field is not too large. After all, a sufficiently powerful electric field will just strip all of the electrons off of the protons and tear the material apart!

Using the continuity equation and Ohm's law, we can now understand better the meaning of the timescale

$$\tau \equiv \epsilon_0/\sigma \quad (8.23)$$

we introduced a few lectures ago. Indeed we have

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = \frac{\epsilon_0}{\sigma} \vec{\nabla} \cdot \vec{J} = -\frac{\epsilon_0}{\sigma} \dot{\rho}. \quad (8.24)$$

The solution of this equation is

$$\rho = \rho_0 e^{-t/\tau}, \quad (8.25)$$

so indeed  $\tau$  sets the timescale for local charge density to dissipate in an ohmic conductor!

## 8.5 “Derivation” of Ohm’s law

If you think a little, Ohm’s law is somewhat surprising. Current density basically is describing the velocity of the charged particles, while the Coulomb force is proportional to the electric field. In mechanics we say that it is *acceleration* which is proportional to force, not velocity. Indeed if we have a charged particle in a constant electric field, shouldn’t we expect it to accelerate faster and faster instead of move with a steady flow? To understand how this puzzle is resolved, we need to think more about what is actually going on inside of the material.

The key intuition is that you should think of the motion of charges inside of a material as being something the motion of a skydiver through air: for a little while he/she accelerates downwards, but eventually he/she reaches terminal velocity due to the viscosity of the air through which he/she is falling. Essentially this happens because the skydiver is constantly banging into air molecules, which prevent an excessive buildup of velocity. Let’s make this more quantitative: say that we have a charged particle of charge  $q$  moving in a material in the presence of a constant electric field  $\vec{E}$  (here we have zoomed in to the atomic scale, so any electric field can be viewed as constant). The particle is repeatedly crashing into other particles, and each time it does its velocity is “reset” to be in a random direction. Let’s say that this has just happened at  $t = 0$ . The momentum of the particle is  $m\vec{v}_0$ , where  $\vec{v}_0$  is pointing in a random direction. As the particle moves away from the collision it will feel an electric force  $q\vec{E}$ , which gives it an impulse

$$\Delta\vec{p} = q\vec{E}\Delta t. \quad (8.26)$$

Thus the *average* momentum per charged particle is

$$m\vec{v}_{average} = \frac{1}{N} \sum_{a=1}^N \left( m\vec{v}_a + q\vec{E}t_a \right), \quad (8.27)$$

where  $a$  labels all the charge-carrying particles (which for the moment we take to all have mass  $m$  and charge  $q$ ),  $\vec{v}_a$  is the velocity each of them happens to have from its latest collision, and  $t_a$  is the time since the last collision for each of them. To simplify this, we first note that the average of a bunch of randomly oriented vectors is zero! After all what else could it be, there is no preferred direction for the average to point so it must be the only vector that doesn’t have a direction. Moreover in a given material there will be an average time  $\tau$  between collisions of particles, so on average we can replace  $t_a \rightarrow \tau$ . Thus we find

$$m\vec{v}_{average} = q\vec{E}\tau. \quad (8.28)$$

Finally comparing to (8.10), we apparently have

$$\vec{J} = \left( \frac{n\tau q^2}{m} \right) \vec{E}, \quad (8.29)$$

which is nothing but Ohm’s law with conductivity

$$\sigma = \frac{n\tau q^2}{m}. \quad (8.30)$$

More generally if there is more than one kind of charge carrier, say labeled by  $i$ , then we have

$$\sigma = \sum_i \frac{n_i \tau_i q_i^2}{m_i}. \quad (8.31)$$

This model of Ohmic conductivity is called the **Drude model**, qualitatively it does a pretty good job but you need to include quantum effects to get things right in detail.

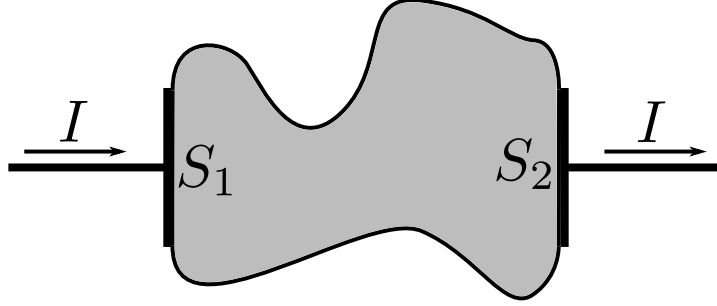


Figure 29: Current flowing in and out of a conductor at surfaces  $S_1$ ,  $S_2$  respectively.

## 9 Resistance and batteries

Let's now study in more detail how current flows through a conductor. More concretely, if we specify the rate at which charge is going in (or out) at each point on the boundary of a conductor, how does the charge flow distribute itself in the interior? Mathematically we can formulate this problem in the following way: if we fix the normal component of  $\vec{J}$  everywhere at the boundary of the conductor, what is  $\vec{J}$  everywhere inside? This mathematical problem is especially tractable for steady currents in Ohmic conductors, which fortunately for us is the situation of greatest practical interest (recall that steady currents are those with  $\vec{\nabla} \cdot \vec{J} = 0$  and Ohmic materials are those for which  $\vec{J} = \sigma \vec{E}$ ).

The first thing to note is that for steady currents in Ohmic materials, the current density inside of a conductor is unique. The proof is quite similar to that of our second uniqueness theorem for conductors, briefly it goes like this. Let  $\vec{J}'$  and  $\vec{J}$  be two different steady currents with the same normal component at the boundary. The quantity

$$\Delta \vec{J} \equiv \vec{J}' - \vec{J} \quad (9.1)$$

is also a steady current (meaning that  $\vec{\nabla} \cdot \Delta \vec{J} = 0$ ), and by assumption its normal component at the boundary is zero. We want to show that  $\Delta \vec{J} = 0$ . This follows from

$$\begin{aligned} \int_C dV |\Delta \vec{J}|^2 &= \sigma \int_C \Delta \vec{E} \cdot \Delta \vec{J} \\ &= -\sigma \int_C \vec{\nabla} \Delta \phi \cdot \Delta \vec{J} \\ &= -\sigma \int_C \left[ \vec{\nabla} \cdot (\Delta \phi \Delta \vec{J}) - \Delta \phi \vec{\nabla} \cdot \Delta \vec{J} \right] \\ &= -\sigma \int_{\partial C} \Delta \phi \Delta \vec{J} \cdot d\vec{A} \\ &= 0, \end{aligned} \quad (9.2)$$

where  $C$  is the volume region of the conductor and in the last step we used that at  $\partial C$  the normal component of  $\Delta \vec{J}$  vanishes.

Let's apply this theorem to the particular case where all of the current is being delivered to the conductor in one boundary region  $S_1$  and leaving at another boundary region  $S_2$  (see figure 29), and moreover let's assume that it is being delivered and removed from these regions by perfectly conducting wires. The potential will then have a constant value  $\phi_1$  in  $S_1$  and a constant value  $\phi_2$  in  $S_2$ . The total current will be

$$I = \int_{S_1} \vec{J} \cdot d\vec{A} = \int_{S_2} \vec{J} \cdot d\vec{A}, \quad (9.3)$$

where we've chosen the normal vector at  $S_1$  to point into the conductor and that at  $S_2$  to point out of it.<sup>11</sup> We then claim the following: if we double the current flow, we also double the potential difference  $\phi_1 - \phi_2$ . The argument is simple: if  $\vec{J}_0$  is a steady current density with current  $I_0$  flowing in through  $S_1$  and out through  $S_2$ , and with no current flowing in or out anywhere else, then  $2\vec{J}_0$  is a steady current density with current  $2I_0$  flowing in through  $S_1$  and out through  $S_2$ , again with no current flowing in or out anywhere else. And moreover by the uniqueness theorem it is the correct current density with current  $2I_0$ . But then by Ohm's law, the electric field inside the conductor is also doubled, and so thus is the line integral

$$\phi_1 - \phi_2 = \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{s}, \quad (9.5)$$

where  $\vec{r}_1$  and  $\vec{r}_2$  are any points in  $S_1$  and  $S_2$  respectively. Moreover this argument is true for any other constant multiple of  $I$  (not just two). Therefore, as in our discussion of capacitance, we have shown that the current and the potential drop along the current are linearly related:

$$\phi_1 - \phi_2 = IR, \quad (9.6)$$

where the constant  $R$  is called the **resistance** of the conductor. It depends on the shape and material of the conductor, and also on the regions  $S_1$  and  $S_2$ , but not on the current or the potential difference. In SI the units of resistance are called **Ohms**. Instead of writing "Ohms" it is conventional to instead write " $\Omega$ ", and in terms of the standard SI units we have<sup>12</sup>

$$1\Omega = 1 \frac{\text{kg m}^2}{\text{s C}^2}. \quad (9.7)$$

Equation (9.6) is in some sense just a re-expression of Ohm's law, and in fact it is itself often referred to as Ohm's law. We will sometimes use this terminology in what follows.

This is as good a place as any to mention that in SI the units of the electric potential are also given a new name, they are called **Volts**. In circuit applications one often hears the terms "potential" and "voltage" use interchangeably, and our new version of Ohm's law is sometimes rewritten as

$$V = IR, \quad (9.8)$$

with  $V$  just being defined as the potential (or voltage) drop across the conductor in the direction of current flow.

## 9.1 Resistance for a conductor of fixed cross-section

We now consider the special case of a long conductor of fixed cross-section, with the current being delivered and received uniformly at the two ends (see figure 30). You can think of it as a model for a long straight wire. We will call the cross-sectional area  $A$  and the length  $L$ . To understand the current density inside, we need to find a divergenceless vector field  $\vec{J}$  whose normal component vanishes on the sides and is constant at the ends. This is easy to find: we just take  $\vec{J}$  to be constant everywhere and pointing straight down the wire! Therefore the current is

$$I = \int_S \vec{J} \cdot d\vec{A} = |J|A = \sigma A |E| = (\phi_1 - \phi_2) \frac{\sigma A}{L}, \quad (9.9)$$

<sup>11</sup>To see that these expressions for  $I$  are equal, note that

$$\int_{S_2} \vec{J} \cdot d\vec{A} - \int_{S_1} \vec{J} \cdot d\vec{A} = \int_{\partial C} \vec{J} \cdot d\vec{A} = \int_C \vec{\nabla} \cdot \vec{J} = 0. \quad (9.4)$$

<sup>12</sup>As usual, I'll note that the units would simplify if we instead discussed the quantity  $\epsilon_0 R$ , which has units of  $\frac{\text{s}}{\text{m}}$  (inverse velocity).



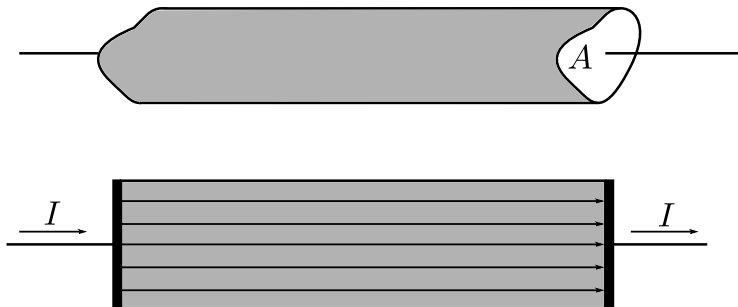


Figure 30: A long conductor of constant cross-section, otherwise known as a wire. If the current density is fed in uniformly at the two ends, then it will be constant throughout the wire.

where in the last step we used the fact that if  $\vec{J}$  is constant,  $\vec{E}$  also is easy to evaluate

$$\phi_1 - \phi_2 = \int_1^2 \vec{E} \cdot d\vec{s} = |E|L. \quad (9.10)$$

Comparing with (9.6), we see that the resistance is apparently given by

$$R = \frac{L}{\sigma A}. \quad (9.11)$$

This makes a good deal of sense: decreasing the conductivity increases the resistance, as does decreasing the area or increasing the length. You can think of this like a pipe of water: making the pipe wider allows more flow, while making it longer requires you to push harder to get the same rate of water through.

Given our assumptions of the wire being perfectly straight and the current being introduced uniformly at the ends, we were able to exactly determine the resistance. In fact as long as the wire is very long (meaning  $L \gg \sqrt{A}$ ), the formula (9.11) works pretty well even if the introduction of the current at the ends is not uniform and the wire is not straight. Indeed far away from the endpoints  $\vec{J}$  will always point along the direction of the wire and be close to uniform, since otherwise the line integral of  $\vec{E}$  from one end to the other would depend on which path we took through the cross section of the wire or the current would not be the same through each cross section.

I unfortunately need to mention that, unsatisfied with only confusing you between the conductivity and the surface charge density, it is fairly common for people to also introduce something called **resistivity**, which is defined as

$$\rho = \frac{1}{\sigma}. \quad (9.12)$$

Here  $\sigma$  is the conductivity, and  $\rho$  has nothing to do with charge density. I don't know why they do these things.

It is interesting to note that there are actually materials which in certain situations have  $\sigma = \infty$ : these are called **superconductors**. This cannot be understood without quantum mechanics, but you can probably already think of all sorts of amazing uses for such a material. Unfortunately so far all known superconductors require either low temperature or high pressure, and thus can only be used in situations where it is economical to produce these extreme conditions.

## 9.2 Power dissipation by conduction

One of the most important features of resistance is that it causes energy to be dissipated whenever a current is flowing. From our microscopic picture, this is because we keep accelerating the charge carriers using the electric field, and then they keep crashing into the ions in the conductor and dissipating their kinetic energy

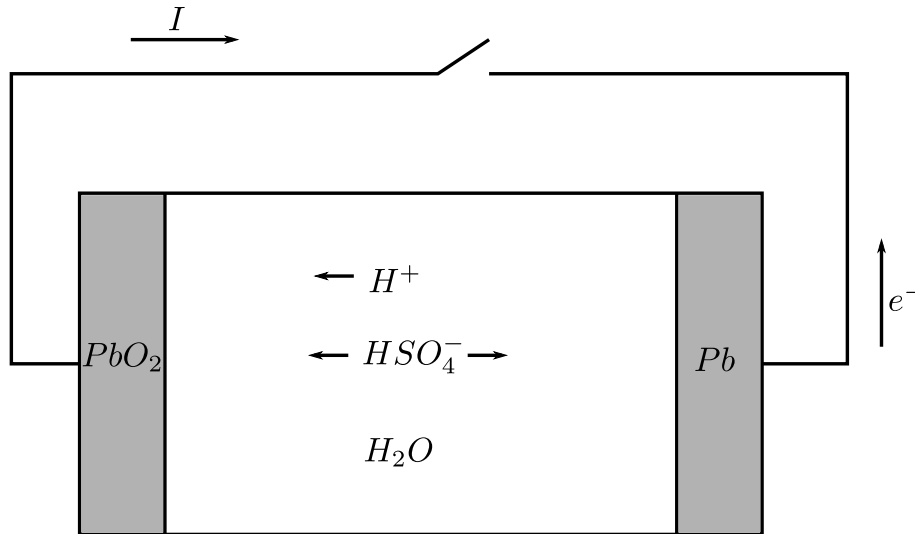


Figure 31: A lead-acid battery.  $HSO_4^-$  ions in the acid move symmetrically to the left and right, but  $H^+$  ions move preferentially to the left, leading to a net accumulation of positive charge at the “+” terminal and negative charge at the “-” terminal. This can be used to drive a circuit, with electrons moving counterclockwise and the current moving clockwise.

into heat. Let’s see how this works for the setup of figure 29. We would like to know how much energy  $dU$  is dissipated in a time  $dt$ . As each charge carrier  $q$  moves from one end of the conductor to the other, it experiences a change in potential energy  $q(\phi_2 - \phi_1) = -qV$ . The total amount of charge completing this journey in a time  $dt$  is  $dQ = Idt$ , so the total change in energy during this time is

$$dU = -IVdt. \quad (9.13)$$

Therefore the rate at which energy is being dissipated, called the **power**, is

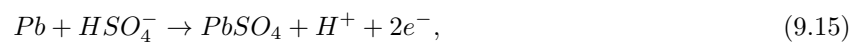
$$P \equiv -\frac{dU}{dt} = IV = I^2R. \quad (9.14)$$

This heat production by conductance is both good and bad. It is good because it gives a precisely controllable source of heat, for example in incandescent light bulbs. It is bad because our electronics are always heating up, so electrical engineers are in a constant struggle to get rid of this heat so that our circuits don’t fry!

### 9.3 Batteries

So far we have been discussing *how* current flows, it is time to understand *why*. We need some kind of device which transports charges *against* the electric field, establishing a potential difference where before none existed. There are various ways to do this. One is simply to pick up the charges and physically carry them from one place to another, this is the mechanism which operates in the Van de Graaff generators we’ve been playing with in class. In most practical applications however (at least those involving time-independent, or “direct” currents), the motion of the charges is established using chemical reactions.

This is not a chemistry class, so we will not go in any detail into the many options which are available for the construction of batteries. We will instead merely sketch how one particular kind of battery, a lead-acid battery, works (this is the kind of battery one finds in cars). The idea is that we have a canister (called a cell) of sulfuric acid ( $H^+ + HSO_4^- + H_2O$ ), with lead oxide ( $PbO_2$ ) stored at one side and metallic lead ( $Pb$ ) stored at the other (see figure 31). At the lead side, there is a tendency to produce electrons via the reaction



while at the lead oxide side there is a tendency to absorb electrons via the reaction



This leads to an accumulation of net positive charge at the lead oxide side and net negative charge at the lead side, which creates an electric field inside the cell pointing from the lead oxide side to the lead side. This electric field eventually is strong enough to prevent any further  $H^+$  particles from finding their way to the lead oxide, stopping the reaction and creating a fixed potential difference

$$\mathcal{E} = \phi_+ - \phi_-, \quad (9.17)$$

where  $\phi_+$  is the potential of the lead oxide and  $\phi_-$  is the potential of the lead. This potential difference  $\mathcal{E}$  is called the **electromotive force** of the battery, or EMF for short, although you should be careful since it isn't actually a force!

Now say that we connect the lead oxide and the lead together via a circuit, by closing the switch in figure 31. The electrons which are being created at the lead side can then flow through this circuit, neutralizing the positive charge which was accumulating at the lead oxide side. This removes the obstacle to the reactions continuing to run, so the electrons continue circulating around. Thus, roughly speaking, we can think of the battery as establishing a fixed potential difference  $\mathcal{E}$  which can then be used to drive an electronic circuit. This however is not quite right, since while the battery is in operation there is a current flowing through the sulfuric acid. We thus need to account for the resistance to this current, so more carefully the effect of the battery is to both provide a fixed potential difference  $\mathcal{E}$  together with a resistance  $R_{bat}$ . We can usually just include  $R_{bat}$  together with the resistance of whatever circuit we connect the battery to, so in practice we will usually just model a battery as establishing a potential difference  $\mathcal{E}$  from its “-” terminal to its “+” terminal. Notice that  $\mathcal{E}$  has units of volts; when speaking of e.g. a nine-volt battery, we mean a battery with  $\mathcal{E} = 9V$ .

As the reactions proceed, gradually the  $H^+$  and  $HSO_4^-$  ions will be used up and the acid converted to  $PbSO_4$  and water. As this happens, the EMF of the battery gradually decreases. If we wish the battery to stay useful, we have to occasionally plug it into some other stronger battery, which forces the electrons to run in the opposite direction. This then causes the chemical reactions to run backwards, recharging the battery. This is what happens every day when you plug in your cell phone at night!

## 10 Circuits and Kirchoff's rules

We've now spent some time studying both how electric current is created and how it flows through conducting materials. Most of modern technology is based on the idea that if we carefully configure a set of conducting objects, we can arrange for current to flow through them in a way that accomplishes something useful. For example it could lift up a heavy rock, or cook your food, or carry your voice far away for your mother to hear and then carry hers back for you to hear. In designing the machines through which electric current accomplishes such tasks, it is advantageous to forget as much as we can get away with about the details of how the charge is moving. We have already made some simplification in introducing the current density  $\vec{J}$ , but that still carries much more information than we typically need to understand what an electronic device will do. At least in the first glance, almost all such devices can be accurately described using the **circuit model** of electronics.

The idea of the circuit model is to replace any electronic device by a simpler one consisting of a set of idealized **circuit components**, which are then connected together in some pattern by perfectly conducting wires. We think about what we expect will happen in this idealized picture, make changes as needed, and only at the very end translate everything back into whatever actual hardware we are using. There are many types of circuit components one can consider, but today we will consider only three:

- A **Resistor** is any object through which current can flow with a specified resistance  $R$ .
- A **Battery** is any object which maintains a definite potential difference  $\mathcal{E}$ , regardless of how much current is flowing through it.

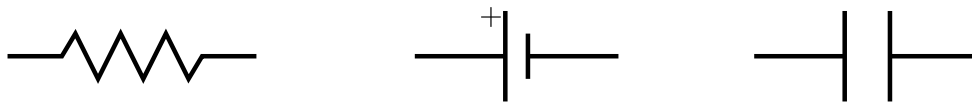


Figure 32: The symbols for a resistor, a battery, and a capacitor in the circuit model. The longer line on the battery indicates the end which is at higher potential.

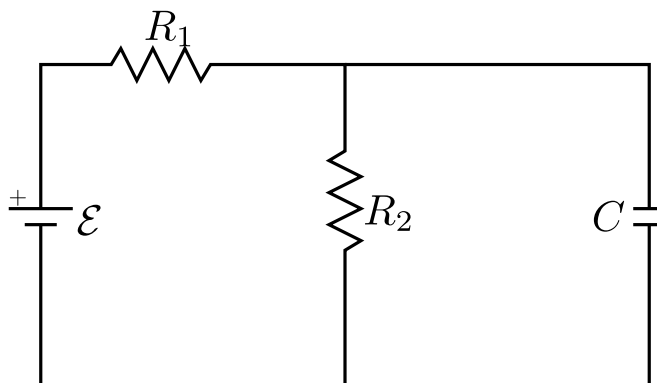


Figure 33: An example of a circuit diagram. The battery will produce a current, which if the initial charge on the capacitor is zero will at first flow through both of the other legs. Eventually the capacitor will charge up to match the potential difference across the battery and the first resistor, and there will be a steady current flowing through the two resistors with none onto the capacitor. We will soon learn how to derive this kind of story quantitatively.

- A **Capacitor** is any object which can accumulate charge  $Q$  on one part and  $-Q$  on another part, with the potential difference between the being  $\phi_1 - \phi_2 = Q/C$  for some specified capacitance  $C$ .

Real resistors, batteries, and capacitors come in all sorts of shapes and sizes, and in fact they are not always so distinct. For example a real battery or a real capacitor always has some resistance, and a real resistor may also have some capacitance for storing charge. And moreover the different components of a circuit might interact with each other through means other than the wires which connect them. In the circuit model however we ignore all of that: each resistor is described entirely by its resistance, each battery entirely by its potential difference, each capacitor entirely by its capacitance, and they talk to each other only by way of the perfectly-conducting wires we use to connect them.<sup>13</sup>

In the circuit model it is very useful to illustrate the layout of a circuit using a **circuit diagram**, which is a picture that shows all of the circuit components together with lines connecting them which represent the perfectly conducting wires. We show the symbols for resistors, batteries, and capacitors in figure 32, and we give an example of a circuit diagram in figure 33. Note that in the circuit model our perfectly conducting wires are allowed to be attached to each other, leading to the “Y junctions” shown in the figure.

## 10.1 Combining resistors

In analyzing circuit diagrams, there are two key principles to remember:

- **Charge conservation:** whatever current is flowing into a circuit component must also be flowing out of it (none of the components we consider can store a net charge).

<sup>13</sup>This is not so bad as it may seem, these realistic effects can usually be “simulated” within the circuit model by adding extra resistors, wires, and so on.

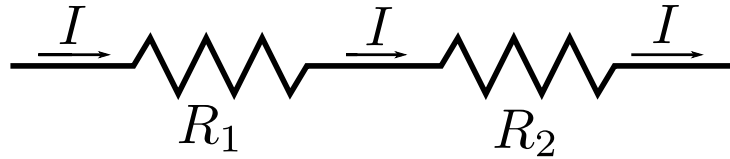


Figure 34: Two resistors in series.

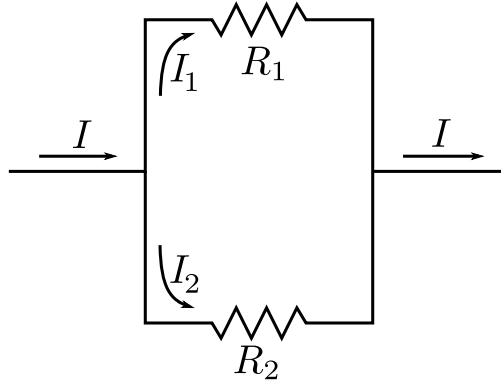


Figure 35: Two resistors in parallel.

- **Path independence of potential difference:** we can compute the potential difference between two points in the circuit using the line integral along any path through the circuit we like.

In the next subsection we will use these principles to formulate general rules for analyzing arbitrary circuit diagrams, here we first use them in two special cases to illustrate how they work.

Our first example is a circuit which contains two resistors linked together “in series”, as in figure 34. By charge conservation, if we feed in a current  $I$  from the left then the current flowing through each resistor must be  $I$ . The potential drop across the pair is just the sum of the potential drop across each separately, so we have

$$V_{total} = IR_1 + IR_2 = I(R_1 + R_2). \quad (10.1)$$

Therefore for the purposes of understanding how this pair of resistors functions when embedded into a larger circuit, we can think of it as being equivalent to a *single* resistor, with resistance

$$R_{eff} = R_1 + R_2. \quad (10.2)$$

This is quite natural from the point of view of our formula for the resistance of a conductor of uniform cross-section,

$$R = \frac{L}{\sigma A}. \quad (10.3)$$

Indeed we could just split up this conductor into two parts, whose total length is  $L$ , and then this formula says we get the total resistance by adding that for the two parts.

Our second example is a circuit which contains two resistors linked together “in parallel”, as in figure 35. Now if we send in a current  $I$  from the left, it is divided into a pair of currents  $I_1$  and  $I_2$  through the first and second resistors respectively. Charge conservation only tells us that

$$I_1 + I_2 = I, \quad (10.4)$$

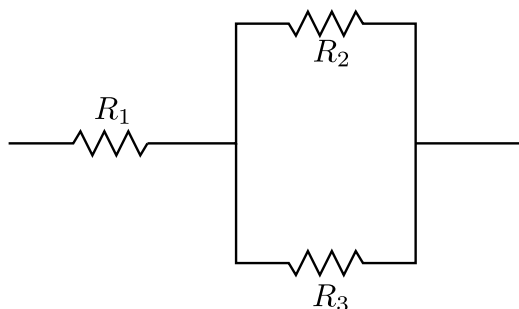


Figure 36: A combination of three resistors

which is now not enough to determine how much current goes each way. Fortunately however we can use our second principle: we can compute the potential drop from the first  $Y$ -junction to the second  $Y$ -junction by using the line integral of  $\vec{E}$  across either resistor, and we have to get the same answer. Therefore we have

$$I_1 R_1 = I_2 R_2. \quad (10.5)$$

Now we have two equations for two unknowns, which we can solve to determine the currents:

$$\begin{aligned} I_1 &= \frac{R_2 I}{R_1 + R_2} \\ I_2 &= \frac{R_1 I}{R_1 + R_2}. \end{aligned} \quad (10.6)$$

In particular notice that if we take  $R_2 \rightarrow \infty$  then all of the current goes through the first resistor, which makes quite a bit of sense. We can use these results to show that the potential drop from the first  $Y$ -junction to the second is

$$V = I_1 R_1 = \frac{R_1 R_2}{R_1 + R_2} I, \quad (10.7)$$

so from the point of view of the rest of whatever circuit we embed this in we again can replace the pair of resistors by a single resistor of resistance

$$R_{eff} = \frac{R_1 R_2}{R_1 + R_2}. \quad (10.8)$$

This result can also be written as

$$\frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_2}, \quad (10.9)$$

which again is quite natural from our formula for the resistance of a long conductor of constant cross-section.

These two results can be combined to simplify more complicated combinations of resistors, for example the configuration in figure 36 has effective resistance

$$R_{eff} = R_1 + \frac{R_2 R_3}{R_2 + R_3}. \quad (10.10)$$

## 10.2 Kirchoff's rules

In applying our principles of charge conservation and path-independence to more general circuits, it is useful to formulate them algorithmically. This is accomplished by **Kirchoff's rules**, which are implemented as follows:

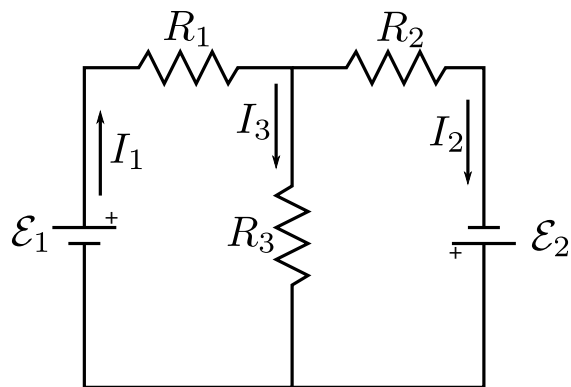


Figure 37: Applying Kirchoff's rules to a circuit.

- (1) Assign a direction to each wire in the circuit, and label the current through the wire in that direction as  $I_i$  (here  $i$  labels the wires).
- (2) For each Y-junction, write down the equation that the total current going in equals the total current going out (in figure 35 this gives two copies of the equation  $I = I_1 + I_2$ ).
- (3) For each loop in the circuit, write down the equation that the total potential drop around the loop equals zero (we get the same equation regardless of which direction we go around the loop). This is a consequence of our old result that the line integral of  $\vec{E}$  around any closed loop is zero, as the potential drop around a loop  $L$  is by definition just  $\int_L \vec{E} \cdot d\vec{s}$ .
- (4) Solve the equations from (2), (3) to determine the currents through all wires.

It is an exercise in linear algebra and graph theory to show that this algorithm always produces enough equations to determine all of the currents, I'll leave it to your imagination.

As a quick illustration of Kirchoff's rules in action, consider the setup of figure 37. From either junction we find that

$$I_1 = I_2 + I_3, \quad (10.11)$$

from the loop containing the first battery and the third resistor we find

$$I_1 R_1 + I_3 R_3 - \mathcal{E}_1 = 0, \quad (10.12)$$

and from the loop containing the second battery and the third resistor we find

$$I_2 R_2 - I_3 R_3 - \mathcal{E}_2 = 0. \quad (10.13)$$

The signs are very important in these equations, for example in the second loop equation  $I_3 R_3$  appears with a minus sign because we have written the potential drop as we go around in a clockwise direction but our orientation for  $I_3$  is counterclockwise so we need to flip it. Similarly in both equations our loop is crossing the battery from the minus to the plus terminal, so the potential *drop* is negative. Whenever you get confused about the signs, it helps to think about which direction the electric field should be pointing inside of each circuit component (this tells us how it will contribute to  $\int_L \vec{E} \cdot d\vec{s}$ ). These three equations can be solved for the three currents, but we won't bother.

There is an alternative presentation of Kirchoff's rules which often simplifies the algebra. The idea is to relabel the currents in a way that all the Y-junction equations are automatically satisfied. We do this by assigning a current to each *independent loop* instead of each wire. "Independent" here means that we keep labeling loops until every wire appears in at least one loop, and then we stop. This is best explained

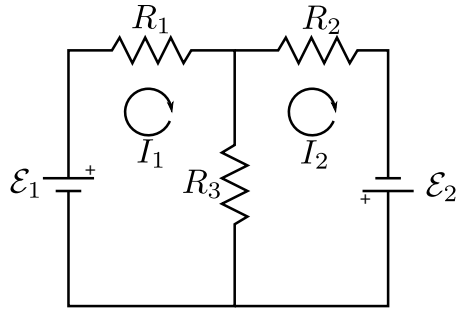


Figure 38: The loop parametrization of currents in a circuit.

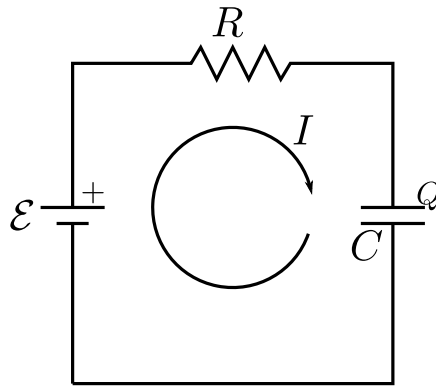


Figure 39: A circuit for charging a capacitor. We can then remove the battery to watch it discharge.

with an example, so let's use the same circuit as before, shown again in figure 38. This circuit has only two independent loops, and we have assigned currents  $I_1$  and  $I_2$  to them. The subtlety now is that for any wire which is contained in more than one loop, we need to include the contributions from both loops in determining the potential drop. Thus for this example we have the two loop equations

$$\begin{aligned} I_1 R_2 + (I_1 - I_2) R_3 - \mathcal{E}_1 &= 0 \\ I_2 R_2 + (I_2 - I_1) R_3 - \mathcal{E}_2 &= 0. \end{aligned} \tag{10.14}$$

These are the same as what we would have gotten from our previous equations after substituting  $I_3 = I_1 - I_2$ , but for more complicated circuits it is very convenient to decrease the number of variables and equations right at the beginning.

### 10.3 Charging and discharging a capacitor

As another illustration of Kirchoff's rules, we will study the process of charging and discharging a capacitor. The relevant circuit is shown in figure 39. We choose the current  $I$  to be oriented clockwise, and label the charge on the upper plate of the capacitor by  $Q$ . Charge conservation then tells us that we must have

$$I = \dot{Q}. \tag{10.15}$$

We will take the initial charge on the capacitor to be zero, and then see how it evolves with time.

The potential drop around the loop is

$$IR + Q/C - \mathcal{E} = \dot{Q}R + Q/C - \mathcal{E} = 0. \tag{10.16}$$



We can view this as a first-order differential equation for  $Q$ , which we can solve by observing that if we introduce a new quantity

$$\hat{Q} \equiv Q - C\mathcal{E} \quad (10.17)$$

then we just have

$$\dot{\hat{Q}} = -\frac{1}{RC}\hat{Q}. \quad (10.18)$$

Therefore the solution is

$$\hat{Q} = Ae^{-\frac{t}{RC}}, \quad (10.19)$$

for some constant  $A$ . We can determine  $A$  by requiring that  $Q(0) = 0$ , which gives us  $A = -C\mathcal{E}$  and thus

$$Q(t) = C\mathcal{E} \left(1 - e^{-\frac{t}{RC}}\right). \quad (10.20)$$

We can also compute the current

$$I(t) = \dot{Q} = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}}. \quad (10.21)$$

Therefore roughly speaking we see that the capacitor charges up to  $Q = C\mathcal{E}$  over a time

$$\tau \equiv RC, \quad (10.22)$$

which is called the time constant of the circuit. After this time the current rapidly drops to zero and the charged capacitor and the loaded battery just sit there staring at each other across the abyss.

We can use the same analysis to also answer the question of what happens if we remove the battery while the capacitor is charged. The loop equation then just reads

$$\dot{Q} = -\frac{1}{RC}Q, \quad (10.23)$$

so the solution for the charge is

$$Q = Q_0e^{-\frac{t}{RC}} \quad (10.24)$$

and the current is

$$I = -\frac{Q_0}{RC}e^{-\frac{t}{RC}}. \quad (10.25)$$

Here  $Q_0$  is the initial charge at  $t = 0$ , which rapidly decays away.

## 11 Lorentz force law

So far everything we have discussed in this class is ultimately a consequence of our expression for the Coulomb force between two charges. Beginning with the pioneering work of Oersted in the early 19th century however, it was understood that there is an additional force felt by charged particles: the **magnetic force**. Understanding the magnetic force quantitatively is our goal for the next few lectures.

### 11.1 Experimental motivation for the magnetic force

We'll begin with an experimental fact. This is that if we bring two parallel current-carrying wires near each other, there is a force between them. The magnitude of this force is proportional to the product of the currents, and it is attractive if the currents are running in the same direction and repulsive if they are running in the opposite direction (see figure 40 for an illustration). This force is present even if the wires are neutral, so it is *not* simply a Coulomb force between the wires. On the other hand this force clearly has something to do with charge: if we set either current to zero then the force disappears, so somehow this is a force which is produced (and felt) *only* by charges which are in motion.

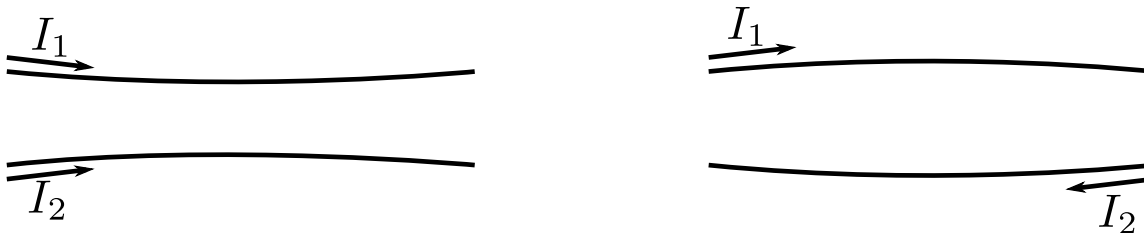


Figure 40: The magnetic force between parallel wires: they are attracted if the currents run in the same direction, while they are repulsed if the currents run in opposite directions. This force is present even if the wires are neutral, so it is not a result of the Coulomb force.

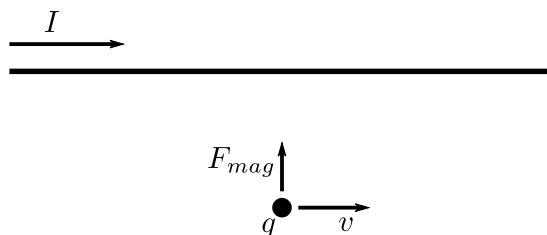


Figure 41: The magnetic force on a moving point charge in the vicinity of a current-carrying wire.

We can understand this more precisely through a related experiment which is harder to do in practice but easier to think about. For this experiment we replace one of the wires with a single point charge  $q$ , at distance  $d$  from the other wire and moving with velocity  $v$  in the direction which is parallel to the current flow (see figure 41). The result of this experiment is that the point charge is attracted to the wire, with a force  $F_{mag}$  whose magnitude is

$$|F_{mag}| = \frac{qv\mu_0 I}{2\pi d}. \quad (11.1)$$

Here the quantity  $\mu_0$  is called the **magnetic constant**, or sometimes the **permeability of free space**, and is given by<sup>14</sup>

$$\mu_0 = 4\pi \times 10^{-7} \frac{\text{kg m}}{\text{C}^2}. \quad (11.2)$$

You might wonder what such an absurd-looking constant is doing in an equation as important as the force on a charged particle from a current-carrying wire. The answer (of course) is that this is yet another tragic consequence of measuring charge in Coulombs: had we adopted a more reasonable unit system for charge, we would have had  $1/c^2$  appearing here instead of  $\mu_0$ , where  $c$  is the speed of light. This would be much more natural given the relativistic origin of the magnetic force which we are about to discuss.

## 11.2 Theoretical motivation for the magnetic force

The experimental results just described clearly demonstrate the existence of a new force which we therefore must try to understand. In fact it is also possible to make a theoretical argument for this: by combining the Coulomb force law with special relativity it is actually possible to *derive* the existence of this magnetic force! In Purcell's textbook this logic is developed to considerable depth in chapter 5, which culminates in a relativistic derivation of equation (11.1). In my view however the level of detail goes well beyond what is really needed to make the relevant conceptual point. In this subsection I will present a considerably simplified

<sup>14</sup>Actually this was the definition of  $\mu_0$  prior to May 2019. The new definition is different at one part in  $10^{10}$ , which is irrelevant for almost all purposes.

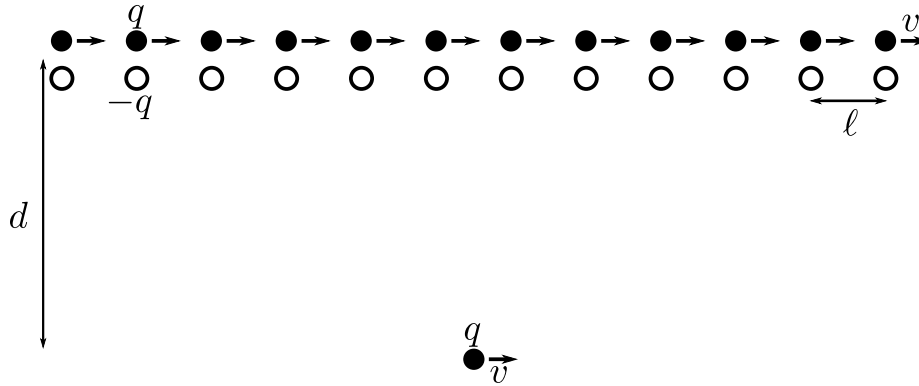


Figure 42: A microscopic model of a wire and a moving point charge. The wire consists of stationary ions of charge  $-q$  and charge carriers of charge  $q$ , with the latter moving to the right at velocity  $v$ . The point charge is at distance  $d$  from the wire, and moving with velocity  $v$  in the same direction as the charge carriers.

version of the argument in the context of a very specific example; we will return to the more general picture at the very end of the semester.<sup>15</sup>

The idea is to consider a “microscopic” version of the situation in figure 11.1, where we model the wire as a set of stationary ions of charge  $-q$ , equally-spaced with spacing  $\ell$ , and a set of mobile charge carriers of charge  $q$ , also equally-spaced with spacing  $\ell$ , but moving to the right with velocity  $v$  (see figure 42). Both the ions and the charge carriers are spaced by distance  $\ell$ , so the wire is neutral (we are here working in the approximation that  $d \gg \ell$ ). The idea is now to consider this same system in a different frame of reference: the “rest frame” of the charge carriers and the point charge. In this frame the charge carriers and the point charge are not moving, but the ions are now moving with a velocity  $-v$ . Moreover we can consider the spacing of the ions and charge carriers in this frame. Since the ions are now moving, their separation is length-contracted:

$$\ell_{ions} = \ell \sqrt{1 - \frac{v^2}{c^2}}. \quad (11.3)$$

How about the charge carriers? Here we have to be more careful, since they were *already* length contracted in the original frame of reference. To find their separation in the new frame, we need to *remove* that length contraction:

$$\ell_{carriers} = \frac{\ell}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (11.4)$$

But these results mean that the linear charge densities of the ions and the charge carriers are not opposites of each other in the new frame, so in this frame the wire is not electrically neutral! Moreover since it is the ions which are more closely spaced, the sign of the charge density on the wire will be opposite that of the point charge, leading to the point charge being attracted to the wire. Now we get to the key point: any acceleration of the point charge towards the wire in the new frame must *also* be present in the original frame. In other words even in the frame where wire is neutral, the point charge feels a force towards the wire: this is the magnetic force!

We can also work out the magnitude of this new force using a bit more special relativity, this will go a

<sup>15</sup>In my experience most 8.022 students are not particularly comfortable with special relativity. For now you can view this section as “recreational reading”, it may not be completely understandable until after we discuss special relativity in more detail at the end of the semester. Don’t worry if you can’t follow the argument in detail, it won’t be on the homework or the midterm!

bit fast so hold on to your hats! In the new frame the charge density on the wire is

$$\lambda = \frac{q}{\ell_{\text{carriers}}} - \frac{q}{\ell_{\text{ions}}} = \frac{q}{\ell} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( 1 - \frac{v^2}{c^2} - 1 \right) = -\frac{qv^2}{\ell c^2 \sqrt{1 - \frac{v^2}{c^2}}}. \quad (11.5)$$

We know that the electric field due to an infinite wire carrying a linear charge density  $\lambda$  is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}, \quad (11.6)$$

where  $r$  is distance from the wire, so apparently in the new frame our charged particle feels a force towards the wire of magnitude

$$|F'_{\text{mag}}| = \frac{q^2 v^2}{2\pi\epsilon_0 \ell c^2 d \sqrt{1 - \frac{v^2}{c^2}}}. \quad (11.7)$$

Finally to transform back to the original frame, we notice that force is defined as  $\vec{F} = \frac{d\vec{p}}{dt}$ , so due to time dilation (and using that here the force is orthogonal to the direction of motion so the relevant component of momentum does not transform) we have

$$\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{dt'} \frac{dt'}{dt} = \frac{d\vec{p}}{dt'} \sqrt{1 - \frac{v^2}{c^2}}, \quad (11.8)$$

where  $t'$  is the time in the new frame and  $t$  is the frame in the original frame (recall that time dilation means that  $t' = t\sqrt{1 - \frac{v^2}{c^2}}$ ). Thus the magnitude of the force towards the wire in the original rest frame is

$$|F_{\text{mag}}| = \frac{q^2 v^2}{2\pi\epsilon_0 \ell c^2 d}. \quad (11.9)$$

Finally we can note that since in the original frame one charge arrives over each time  $\frac{\ell}{v}$ , and thus the current is given by  $I = qv/\ell$ , we can rewrite this as

$$|F_{\text{mag}}| = \frac{qvI}{2\pi\epsilon_0 c^2 d}. \quad (11.10)$$

This would be compatible with (11.1) if only we had

$$\mu_0 = \frac{1}{\epsilon_0 c^2}. \quad (11.11)$$

It will take another 1.5 months for us to understand it, but in fact this relation is actually true! We have therefore indeed succeeded in deriving (11.1), at least for the special case where the charge carriers and the point charge are moving with the same velocity. For the case where they move with different velocities, I'll refer you to Purcell's book.

It is worth mentioning that this logic is almost completely the opposite of what happened historically. Electric and magnetic fields were understood in the 19th century, first as separate experimental phenomenon and then together through Maxwell's equations. Einstein was an expert in electromagnetism, and he was very puzzled about how Maxwell's equations could predict the same speed of light in any inertial frame. His resolution of this puzzle was the introduction of special relativity, which he presented in a famous 1905 paper that was revealingly titled *On the Electrodynamics of Moving Bodies*. In other words special relativity was derived from electromagnetism, while here we have gone the other way.

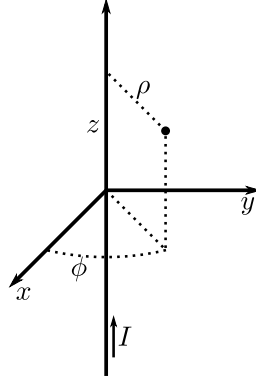


Figure 43: Cylindrical coordinates about a wire running on the  $z$ -axis.

### 11.3 The magnetic field and the Lorentz force law

So far we've discussed the magnetic force on a charged particle whose velocity is parallel to an infinitely-long current-carrying wire. What about the magnetic force on a charged particle whose velocity is in an arbitrary direction? This can also be worked out using a considerably more complicated relativistic argument, but we will instead just give the answer (you could also just go measure it if you don't feel like working out the relativity). The formula is easiest to describe in cylindrical coordinates  $(z, \rho, \phi)$ , where we take the wire to run along the  $z$  axis with the current oriented towards increasing  $z$  (see figure 43).<sup>16</sup> The force on a charged particle  $q$  moving with velocity  $\vec{v}$  at distance  $\rho$  from the wire is then given by

$$\vec{F}_{mag} = \frac{q\mu_0 I}{2\pi\rho} \vec{v} \times \hat{\phi}, \quad (11.12)$$

where  $\hat{\phi}$  is a unit vector which at each point in space wraps counter-clockwise around the wire as we look down from positive  $z$ . In Cartesian components,  $\hat{\phi}$  is given by

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}. \quad (11.13)$$

As in our discussion of the Coulomb force, it is convenient to split this expression into a part which is determined by the current configuration and a part which depends on the properties of the charged particle:

$$\vec{F}_{mag} = q\vec{v} \times \vec{B}(\vec{r}), \quad (11.14)$$

where  $\vec{r}$  is the location of the charged particle and  $\vec{B}(\vec{r})$  is the **magnetic field** produced by the current. Here it is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi\rho} \hat{\phi}. \quad (11.15)$$

As with the electric field, the utility of this definition may not be immediately obvious, but it turns out to be very convenient indeed! We show the field lines of this magnetic field in figure 44. In SI the units of the magnetic field are called **Teslas**, they are defined via

$$1\text{T} = 1 \frac{\text{kg}}{\text{sC}} \quad (11.16)$$

<sup>16</sup>I apologize for the conflict between  $\rho$  the cylindrical radius and  $\rho$  the charge density, hopefully it will be clear which is meant from the context. I could have instead used  $r$ , but this leads to a substantially worse clash with the radial distance in spherical coordinates.

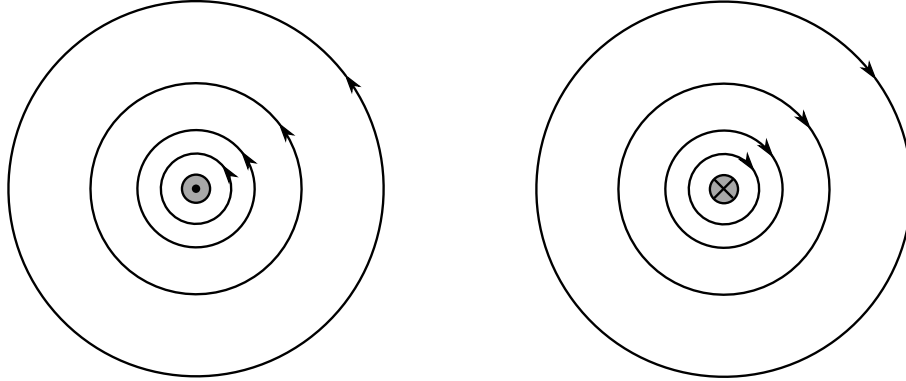


Figure 44: Magnetic field lines around a wire. On the left the wire points out of the page, while on the right it points in. The spacing of the lines grows with distance from the wire, reflecting the decreasing strength of the field. You can remember the direction of the field using the “right-hand rule”: if you wrap the fingers of your right hand around the magnetic field lines, your right thumb points in the direction of the current flow.

You presumably are already wondering how the magnetic force will generalize if we consider wires which are not infinitely long and perfectly straight. It will take us two more lectures to properly understand this, but the result is that a general configuration of current-carrying wires will always produce some magnetic field  $\vec{B}(\vec{r})$  whose effect on any charged particle is still just (11.14). More generally we could also have some charge density around producing some electric field  $\vec{E}(\vec{r})$ , so the full electromagnetic force on a charged particle  $q$  at location  $\vec{r}$  and with velocity  $\vec{v}$  is given by the **Lorentz force law**

$$\vec{F} = q \left( \vec{E}(\vec{r}) + \vec{v} \times \vec{B}(\vec{r}) \right). \quad (11.17)$$

There are two interesting properties of the magnetic force which are quite different from those of the electric force:

- (1) The magnetic force is **velocity dependent**: it only acts on charges which are moving. This is why we were able to ignore it in our discussion of electrostatics. You might worry that we were wrong to ignore it in our discussion of circuits, and indeed we have already seen that it causes a force between neutral wires. On the other hand typical drift velocities are quite small compared to  $c$ , so this force is usually fairly small. And moreover if we hold the positions of all wires fixed, as we usually do e.g. on a printed circuit board, then any magnetic force on the charge carriers will be canceled regardless by whatever forces are holding the wires in place. There is however a more subtle magnetic effect called induction, which we will study in a few lectures, and this *does* sometimes have an important effect on circuits.
- (2) The magnetic force **preserves the kinetic energy** of any particle it acts on. Since it is proportional to  $\vec{v} \times \vec{B}$ , it always points in a direction which is orthogonal to  $\vec{v}$ . Such a force can change the direction of the velocity, but not its magnitude. Therefore the kinetic energy  $\frac{1}{2}m|v|^2$  also cannot change. To see this mathematically, note that

$$\frac{d}{dt} \left( \frac{m}{2} |v|^2 \right) = \vec{v} \cdot (m\dot{\vec{v}}) = \vec{v} \cdot (q\vec{v} \times \vec{B}) = 0, \quad (11.18)$$

where we have used Newton’s second law  $\vec{F} = m\dot{\vec{v}}$ .

## 11.4 The force on a wire in a magnetic field

We'll now return to the first experiment we discussed: the force between two current-carrying wires. Indeed let's consider a wire segment carrying a current  $I$  in a magnetic field  $\vec{B}(\vec{r})$ . We can compute the force on the wire by adding up the forces on all of its charge carriers: for simplicity we consider the case where there is only one kind of charge carrier, with charge  $q$ , number density  $n$ , and average velocity  $\vec{v}_{average}$ . At each point on the wire, the current (in the direction of  $\vec{v}_{average}$ ) is given by

$$I = nq|v_{average}|A. \quad (11.19)$$

Here  $A$  is the cross-sectional area of the wire. The force on the wire is then given by adding up the forces on all charge carriers in each piece of length  $ds$ :

$$\vec{F} = \int ds nA \left( q\vec{v}_{average} \times \vec{B} \right) = \int I d\vec{s} \times \vec{B}, \quad (11.20)$$

where the line integral is done in the direction of  $\vec{v}_{average}$ .<sup>17</sup> Here we have used that  $|v_{average}|d\vec{s} = \vec{v}_{average}ds$ .

Especially simple is the case of a straight wire segment of length  $L$  in a constant magnetic field  $\vec{B}$ , if we denote the vector pointing from one end to the other of the wire by  $\vec{L}$  then we have

$$\vec{F} = I\vec{L} \times \vec{B}. \quad (11.21)$$

In particular if we have two parallel wires of length  $L$  and separated by distance  $d$ , carrying currents  $I_1$  and  $I_2$  which are pointing in the same direction, then either wire feels a force of magnitude

$$|F| = \frac{\mu_0 L I_1 I_2}{2\pi d} \quad (11.22)$$

in the direction of the other wire.

## 12 Ampere's law

In the last lecture we learned about the magnetic field created by a straight infinitely-long wire. We would obviously like to understand how to generalize this to an arbitrary current density  $\vec{J}$ , but it turns out that the full generalization is a bit complicated. For a few lectures we will therefore restrict to the case of **steady currents**, which I remind you are currents obeying

$$\vec{\nabla} \cdot \vec{J} = -\dot{\rho} = 0. \quad (12.1)$$

It is not possible to build a general steady current out of superposing straight infinitely-long wires, so we need to introduce some kind of new principle to tell us what happens.

Let's first recall that in the end our description of electrostatics was based on the following three equations:

$$\begin{aligned} \vec{F} &= q\vec{E} \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= 0. \end{aligned} \quad (12.2)$$

These three equations, together with the assumption that the electric field goes to zero at infinity, are equivalent to the Coulomb force law and the principle of superposition. Our goal in this lecture is to develop analogous equations for the magnetic field produced by steady currents. We've already met the analogue of the first equation, the Lorentz force law:

$$\vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right). \quad (12.3)$$

We now need to understand how to generalize the second two.

<sup>17</sup>Note that with this orientation  $I$  will be negative if  $q$  is. But in fact our final formula for  $\vec{F}$  is actually correct regardless of which orientation we choose for the current, since  $I$  and  $d\vec{s}$  will both change sign if we flip the orientation.

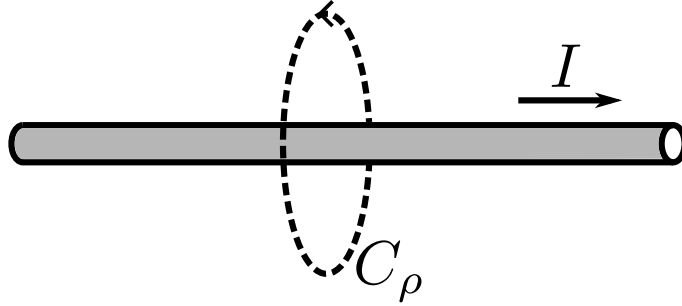


Figure 45: The line integral of  $\vec{B}$  around a circle surrounding a current: by Ampere's law this gives  $\mu_0 I$ .

## 12.1 Ampere's law

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$$\vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right). \quad (12.6)$$

We now need to understand how to generalize the second two.

## 12.2 Ampere's law

Let's begin by recalling our expression for the magnetic field around an infinite wire carrying a current  $I$  in cylindrical coordinates  $(z, r, \phi)$ :

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi r} \hat{\phi}, \quad (12.7)$$

where  $\hat{\phi}$  is a unit vector wrapping around the wire with an orientation specified by the right-hand rule and  $r$  is distance to the wire. Let's denote by  $C_r$  a circle of radius  $r$  surrounding the wire. We can easily compute the line integral around this circle in the direction of  $\hat{\phi}$  (see figure 45):

$$\int_{C_r} \vec{B} \cdot d\vec{s} = \int_0^{2\pi} \frac{rd\phi\mu_0 I}{2\pi r} = \mu_0 I. \quad (12.8)$$



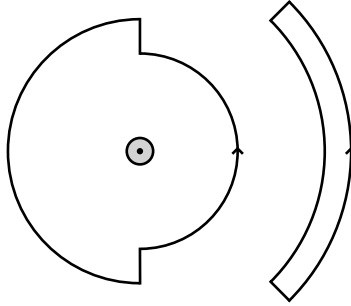


Figure 46: Examples of non-circular loops for the line integral of  $\vec{B}$  in the vicinity of an infinite straight wire. The wire is coming straight out of the page. The line integral of  $\vec{B}$  around the loop on the left gives  $\mu_0 I$ , while the line integral of  $\vec{B}$  around the loop on the right gives zero.

In other words this line integral just gives us the amount of current going through the circle, independent of the radius of the circle. Based on our experience with Gauss's law, we can guess that in fact this is true even for non-circular paths. Indeed it is! The argument is similar to that for Gauss's law: any non-circular loop around the wire can be approximated by one which is built out of circular, radial, and parallel segments, and the radial and parallel segments will not contribute. Each time the loop surrounds the wire in the direction of  $\hat{\phi}$  there will be contribution of  $\mu_0 I$ , so loops which do not enclose the wire will give zero. See figure 46 for some examples. We can rewrite this a more geometric way: if  $S$  is any two-dimensional surface, we have

$$\int_{\partial S} \vec{B} \cdot d\vec{s} = \mu_0 I_S, \quad (12.9)$$

where

$$I_S \equiv \int_S \vec{J} \cdot d\vec{A} \quad (12.10)$$

is the current through the surface  $S$ . Moreover by superposition this will continue to hold for the magnetic field from any configuration of multiple infinitely-long straight wires: the line integral of  $\vec{B}$  around the boundary of  $S$  will always detect whichever wires are puncturing  $S$ .

We now come to our first “new” assertion since the first day of class. So far everything we have discussed was ultimately a consequence of the Coulomb force law, superposition, and special relativity. The assertion is that (12.9) holds *without modification* for any steady current density  $\vec{J}$ ! This is the statement of **Ampere's law**:

$$\int_{\partial S} \vec{B} \cdot d\vec{s} = \mu_0 \int_S \vec{J} \cdot d\vec{A} = \mu_0 I_S. \quad (12.11)$$

You might wonder where this assertion comes from. Ultimately the answer has to be that we test it experimentally and find that it is true, which fortunately we do. It is however a quite natural guess based on our experience with Gauss's law and our understanding of infinitely-long straight wires, so we are lucky that nature was kind enough to make use of it.

### 12.3 Using Ampere's law to find the magnetic field of a thick wire

You may recall that we made considerable use of Gauss's law to find the electric field for various symmetric charge distributions. It is also sometimes possible to use Ampere's law to find the magnetic field for a symmetric current distribution. As in the electric case, the hard part is using symmetry to sufficiently constrain the form of the field so that the loop integral in Ampere's law becomes simple to evaluate. It will be easier to do this once we discuss the Biot-Savart law in the following lecture, but there is one example we can already do now: the magnetic field of a thick cylindrical wire of radius  $R$  carrying constant current density  $\vec{J}$  which is parallel to the wire.

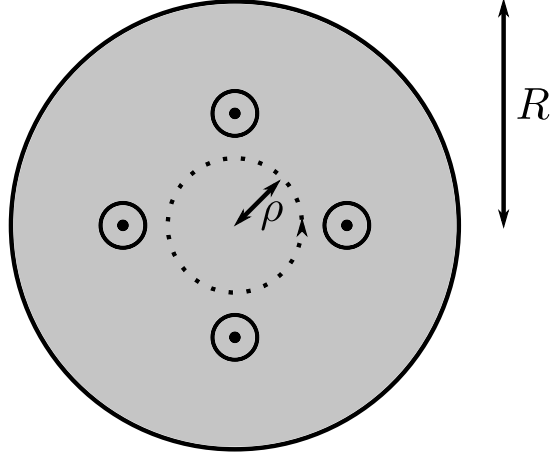


Figure 47: Applying Ampere's law to a thick wire of radius  $R$ : inside the wire there is a uniform current density pointing out of the page (indicated by the circles with dots), and we consider the line integral of  $\vec{B}$  around a circular loop  $C_r$  of radius  $r$  (shown as a dotted line).

The setup is shown in figure 47. Let's adopt cylindrical coordinates  $(z, r, \phi)$ , with the wire running along the  $z$ -axis and the current flowing towards positive  $z$ . The current density is then given by

$$\vec{J} = \begin{cases} J\hat{z} & r < R \\ 0 & r > R \end{cases}. \quad (12.12)$$

We can think of the wire as a superposition of infinitely-thin wires, so one strategy for finding the magnetic field would be just to add up the magnetic fields produced by each of them. We can do better however by using symmetry. First of all we know by superposition that there will be no magnetic field in the  $z$  direction, since none of the thin wires produce one. It is trickier to argue that there is no radial component of the magnetic field pointing in the  $r$  direction, but this also follows from symmetry. For example the radial contribution to the magnetic field at any of the four dots in figure 47 from the currents at the two closest other dots cancels, and more generally if we add up contributions in pairs of this sort the radial component always cancels. Therefore the magnetic field will only have a  $\hat{\phi}$  component, and its strength will depend only on the radial coordinate  $r$

$$\vec{B}(\vec{r}) = B_\phi(r)\hat{\phi}. \quad (12.13)$$

From Ampere's law we then have

$$\int_{C_r} \vec{B} \cdot d\vec{s} = 2\pi r B_\phi(r) = \begin{cases} \mu_0 \pi r^2 J & r < R \\ \mu_0 \pi R^2 J & r > R \end{cases}. \quad (12.14)$$

Therefore apparently the magnetic field is

$$\vec{B}(\vec{r}) = \begin{cases} \frac{\mu_0 J r}{2} \hat{\phi} = \frac{\mu_0 I r}{2\pi R^2} \hat{\phi} & (r < R) \\ \frac{\mu_0 J R^2}{2r} \hat{\phi} = \frac{\mu_0 I}{2\pi r} \hat{\phi} & (r > R) \end{cases}. \quad (12.15)$$

In other words the magnetic field grows linearly from the center of the wire, and then outside of the wire agrees with the result for an infinitely-thin wire carrying the same total current  $I = \pi R^2 J$ .

## 12.4 Differential version of Ampere's law

For Gauss's law we had both an integrated version and a differential version. The same is true for Ampere's law, and we can easily relate them using our geometric interpretation of the curl. Indeed recall that we had

$$\hat{n} \cdot (\vec{\nabla} \times \vec{V})(\vec{r}) = \lim_{\epsilon \rightarrow 0} \frac{\int_{\partial S_{\epsilon,n}(\vec{r})} \vec{V} \cdot d\vec{s}}{\int_{S_{\epsilon,n}(\vec{r})} dA}, \quad (12.16)$$

where  $S_{\epsilon,n}(\vec{r})$  is a square of size  $\epsilon$  centered at point  $\vec{r}$  and with normal vector  $\hat{n}$ . In other words to compute the  $\hat{n}$  component of the curl of  $\vec{V}$  at point  $\vec{r}$ , we take the line integral of  $\vec{V}$  around a small square centered at  $\vec{r}$  and with normal  $\hat{n}$ , divide by the area of the square, and then take the limit of zero square size. For the magnetic field we can use this to evaluate the curl by way of Ampere's law:

$$\hat{n} \cdot \vec{\nabla} \times \vec{B} = \lim_{\epsilon \rightarrow 0} \frac{\int_{\partial S_{\epsilon,n}(\vec{r})} \vec{B} \cdot d\vec{s}}{\epsilon^2} = \mu_0 \lim_{\epsilon \rightarrow 0} \frac{\int_{S_{\epsilon,n}(\vec{r})} \vec{J} \cdot d\vec{A}}{\epsilon^2} = \mu_0 \hat{n} \cdot \vec{J}. \quad (12.17)$$

Since this equation must be true for any direction  $\hat{n}$ , we have thus found the **differential version of Ampere's law**:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}. \quad (12.18)$$

This equation is quite intuitive: just as the differential Gauss's law  $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$  says that charge density sources an electric field which "points out" from the it, the differential Ampere's law says that current density sources a magnetic field which "wraps around" it. There are two important points to make about the differential version of Ampere's law:

- (1) If we instead start with the differential form of Ampere's law as our basic assumption, we can immediately use Stokes's theorem to derive the integral version:

$$\int_{\partial S} \vec{B} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \mu_0 \int_S \vec{J} \cdot d\vec{A} = \mu_0 I_S. \quad (12.19)$$

It is thus up to us which we take to be "fundamental". For my money, it is nicer to use the differential version.

- (2) By taking the divergence of both sides we find

$$\mu_0 \vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0, \quad (12.20)$$

where we have used that the divergence of a curl is always zero. Thus the differential Ampere's law *implies* that the current density is steady. Since current densities which are not steady certainly exist, we will eventually need to modify Ampere's law to allow this.

## 12.5 The divergence of the magnetic field

We have now found a mathematical expression for the curl of the magnetic field. For the electric field we found expressions for both its divergence and curl, so it is now natural to ask what we can say about the divergence of the magnetic field.

Let's first think what it would mean if we had  $\vec{\nabla} \cdot \vec{B} \neq 0$ . By analogy to Gauss's law, we would then probably want to say that there was a nonzero density for magnetic charge. By superposition we could break this charge density into small pieces, so really a statement that  $\vec{\nabla} \cdot \vec{B} \neq 0$  would be an assertion that there existed pointlike sources for the magnetic field. At this point we again have to see what experiment tells us, and so far it has been quite clear: **no significant evidence for magnetic point charges has ever been**

seen.<sup>18</sup> We thus typically take it as a basic law of nature that none exist, which we express mathematically as a requirement that

$$\vec{\nabla} \cdot \vec{B} = 0. \quad (12.21)$$

Despite the lack of experimental evidence, magnetic point charges, which are typically called **magnetic monopoles**, have been a source of fascination to theoretical physicists for more than one hundred years. For example there is a famous argument due to Paul Dirac that once quantum mechanical effects are taken into account, magnetic monopoles can only exist in a theory where electric charge is quantized in integers. As far as we can tell electric charge is indeed quantized in integers, so this seems to suggest that monopoles may well exist. Moreover many proposed generalizations of the standard model of physics attempt to “unify” electromagnetism with the strong and weak forces, and in these generalizations one always seems to find magnetic monopoles. Finally, there are quite general arguments, including some recent ones by yours truly, that successfully combining gravity and quantum mechanics *requires* the existence of magnetic monopoles! So if you asked me to bet, I would bet they exist. In science however experiments are the final arbiter, so in this class we will nonetheless conservatively assume that indeed  $\vec{\nabla} \cdot \vec{B} = 0$ .

## 12.6 The Helmholtz theorem

We now have equations which determine the divergence and curl for both the electric and magnetic fields:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \times \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J}. \end{aligned} \quad (12.22)$$

These four equations are our first iteration of **Maxwell’s equations**, which are the fundamental equations of electromagnetism. More carefully, these are Maxwell’s equations in the special case where the electric and magnetic fields do not depend on time (as you might expect when only steady currents are present). Over the next month we will see that generalizing to arbitrary currents and time-dependent fields introduces new terms on the right-hand side of the second and fourth of these equations, but in the meantime you might wonder whether or not these equations are really enough to determine the electric and magnetic fields. This question is answered by a famous theorem in vector calculus, the Helmholtz theorem. It says that two vector fields  $\vec{V}$  and  $\vec{V}'$  which have the same divergence and curl and both vanish at infinity must be equal.

Using results from our previous lectures, the proof of the Helmholtz theorem is not difficult. Say that  $\vec{V}$  and  $\vec{V}'$  are two vector fields with the same divergence and curl. As usual for our uniqueness arguments, we define a new vector field

$$\Delta \vec{V} \equiv \vec{V} - \vec{V}', \quad (12.23)$$

and our goal is to show that  $\Delta \vec{V} = 0$ . Since  $\vec{V}$  and  $\vec{V}'$  have the same curl and divergence, we must have

$$\begin{aligned} \vec{\nabla} \times \Delta \vec{V} &= 0 \\ \vec{\nabla} \cdot \Delta \vec{V} &= 0. \end{aligned} \quad (12.24)$$

We’ve already shown that any vector field with vanishing curl is the gradient of some scalar, so apparently we have

$$\Delta \vec{V} = \vec{\nabla} f \quad (12.25)$$

for  $f$  some scalar function. We then also apparently have

$$\nabla^2 f = \vec{\nabla} \cdot \Delta \vec{V} = 0, \quad (12.26)$$

---

<sup>18</sup>There was actually one apparent detection, which unfortunately has never been successfully repeated. You can google “valentine’s day monopole” to find out more about it.

so  $f$  obeys Laplace's equation. Moreover since  $\vec{V}$  and  $\vec{V}'$  both approach zero at infinity,  $f$  must approach a constant there. Finally we showed earlier that any solution  $f$  of Laplace's equation has the funny property that the average of  $f$  over the surface of any ball is equal to the value of  $f$  at the center. This includes very large balls, but then since  $f$  must approach a constant at infinity this means that it is constant everywhere. Therefore  $\Delta\vec{V} = \vec{\nabla}f = 0$ .

## 13 Vector potential, Biot-Savart law, and solenoids

We've now met the fundamental equations for determining the magnetic field produced by steady currents:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (13.1)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}. \quad (13.2)$$

Today we will use these equations to produce an "explicit" expression for the magnetic field produced by an arbitrary wire configuration, analogous to our formula

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int dV' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') \quad (13.3)$$

determining the electric field for an arbitrary charge density  $\rho$ . To do this we first need to introduce a new tool.

### 13.1 The magnetic vector potential

You may recall that the "vector antiderivatives" we discussed in the fourth lecture: a scalar whose gradient vanishes is a constant, a vector field whose curl vanishes is a gradient, and a vector field whose divergence vanishes is a curl. In particular since in electrostatics the curl of the electric field is zero, we introduced the electric potential  $\phi$ , obeying

$$\vec{E} = -\vec{\nabla}\phi. \quad (13.4)$$

Similarly since the divergence of the magnetic field is zero, there must exist some vector field  $\vec{A}(\vec{r})$ , called the **magnetic vector potential**, such that

$$\vec{B} = \vec{\nabla} \times \vec{A}. \quad (13.5)$$

Just as we usually shorten "electric potential" to "potential", we will usually just call  $\vec{A}(\vec{r})$  the "vector potential".

You probably remember that the potential  $\phi$  is not really well-defined: we can add to it any constant and still get the same electric field. There is a somewhat analogous ambiguity for the vector potential. Indeed say that  $\vec{A}$  and  $\vec{A}'$  are two vector potentials that give rise to the same magnetic field configuration  $\vec{B}$ . We therefore have

$$\vec{\nabla} \times (\vec{A}' - \vec{A}) = \vec{B} - \vec{B} = 0. \quad (13.6)$$

This then means that we can find some scalar field  $\Lambda$  such that

$$\vec{A}' - \vec{A} = \vec{\nabla}\Lambda. \quad (13.7)$$

Conversely, since the curl of a gradient always vanishes, any two vector potentials which differ only by the gradient of some scalar will produce the same magnetic field. Therefore the magnetic vector potential has a considerably larger ambiguity than the electric potential does: we can freely add to it the gradient of any scalar function without changing any of the physics! You can think of this ambiguity as a funny kind of symmetry, where the vector potential transforms as

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda \quad (13.8)$$

for whichever scalar function  $\Lambda$  we like. Such transformations are called **gauge transformations**, and although we won't use them much in this class they will be very important if you go on in physics.

## 13.2 The Biot-Savart law

Our strategy to find a magnetic field  $\vec{B}$  which satisfies (13.1) and (13.2) for some given steady current density  $\vec{J}$  will be to find a vector potential  $\vec{A}$  obeying

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}. \quad (13.9)$$

Since the divergence of a curl is zero,  $\vec{B} \equiv \vec{\nabla} \times \vec{A}$  will then satisfy both (13.1) and (13.2). We can simplify equation (13.9) using the vector calculus identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}, \quad (13.10)$$

which is true for any vector field  $\vec{A}$ . For example we can check the  $z$  component:

$$\begin{aligned} (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}))_z &= \partial_x (\vec{\nabla} \times \vec{A})_y - \partial_y (\vec{\nabla} \times \vec{A})_x \\ &= \partial_x (\partial_z A_x - \partial_x A_z) - \partial_y (\partial_y A_z - \partial_z A_y) \\ &= \partial_z (\partial_x A_x + \partial_y A_y + \partial_z A_z) - (\partial_x^2 + \partial_y^2 + \partial_z^2) A_z. \end{aligned} \quad (13.11)$$

Therefore we have

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}. \quad (13.12)$$

We can further simplify this by making use of the ambiguity in  $\vec{A}$ . Indeed I claim that by a clever gauge transformation, we can always arrange for

$$\vec{\nabla} \cdot \vec{A} = 0. \quad (13.13)$$

Here is the proof: let  $\vec{A}$  be any vector field. It will not necessarily have vanishing divergence, but let's give its divergence a name:

$$f \equiv \vec{\nabla} \cdot \vec{A}. \quad (13.14)$$

My claim is that we can find a  $\Lambda$  such that

$$\vec{\nabla} \cdot (\vec{A} + \vec{\nabla} \Lambda) = f + \vec{\nabla}^2 \Lambda = 0. \quad (13.15)$$

In fact we've already done it:

$$\vec{\nabla}^2 \Lambda = -f \quad (13.16)$$

is just Poisson's equation, where you can think of  $\Lambda$  as the electric potential  $\phi$  and  $f$  as  $\rho/\epsilon_0$ , so the solution is

$$\Lambda(\vec{r}) = \int dV' \frac{f(\vec{r}')}{4\pi|r - r'|}. \quad (13.17)$$

Therefore if we relabel

$$\vec{A} + \vec{\nabla} \Lambda \rightarrow \vec{A}, \quad (13.18)$$

our new  $\vec{A}$  will lead to the same magnetic field as before but now obey  $\vec{\nabla} \cdot \vec{A} = 0$ . Therefore we just need to find a vector field  $\vec{A}$  such that

$$\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}. \quad (13.19)$$

Here we are again in luck: each component of this vector equation is once again just equivalent to Poisson's equation! We can therefore just write down the solution:<sup>19</sup>

$$\vec{A}(\vec{r}) = \mu_0 \int dV' \frac{\vec{J}(\vec{r}')}{4\pi|r - r'|}. \quad (13.20)$$

<sup>19</sup>It is not obvious that this solution actually obeys  $\vec{\nabla} \cdot \vec{A} = 0$ . It is a fun little exercise to show that it does, see problem 6.6 in Purcell if you need help.

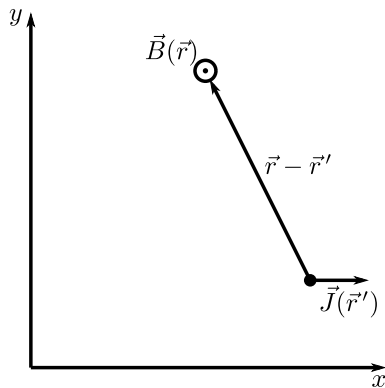


Figure 48: The contribution to the magnetic field produced by a piece of current density. By an appropriate rotation we have chosen to put both  $\vec{r}$  and  $\vec{r}'$  in the  $xy$  plane, and also arranged for  $\vec{J}(\vec{r}')$  to point in the  $x$  direction. The contribution to  $\vec{B}(\vec{r})$  points in the  $z$  direction. In general the direction of the contribution to  $\vec{B}(\vec{r})$  from  $\vec{J}(\vec{r}')$  is the same as it would be if we extended  $\vec{J}(\vec{r}')$  to an infinitely long straight wire.

The next step is to compute the curl of this  $\vec{A}$  to find  $\vec{B}$ , but before doing so I want to mention that this argument actually fills a gap in our discussion of vector antiderivatives in lecture four. There we asserted that a vector field with zero divergence is always the curl of another vector field, but we didn't prove it. Now we have! Indeed let  $\vec{B}$  be some vector field with zero divergence. We can compute its curl, and then give it the name " $\mu_0 \vec{J}$ ". We can then substitute this into (13.20) to find a vector field  $\vec{A}$  with the property that its curl has the same curl and divergence as  $\vec{B}$  does. Therefore if both  $\vec{\nabla} \times \vec{A}$  and  $\vec{B}$  go to zero at infinity (as they will if  $\vec{J}$  is nonzero only in a finite region), then by the Helmholtz theorem from last lecture they must be equal.

Now we can proceed with computing the curl. In doing so it is convenient to make use of another vector calculus identity, which tells us how the curl distributes over the product of a scalar field and a vector field:

$$\vec{\nabla} \times (f\vec{V}) = \vec{\nabla}f \times \vec{V} + f\vec{\nabla} \times \vec{V}. \quad (13.21)$$

We can again check the  $z$  component of this:

$$\begin{aligned} (\vec{\nabla} \times (f\vec{V}))_z &= \partial_x(fV_y) - \partial_y(fV_x) \\ &= (\partial_x fV_y - \partial_y fV_x) + f(\partial_x V_y - \partial_y V_x). \end{aligned} \quad (13.22)$$

In computing the curl of  $\vec{A}(\vec{r})$ , we have to be careful to remember that we are computing the curl with respect to the  $\vec{r}$  variable, *not* the  $\vec{r}'$  variable. Therefore we can bring the curl inside of the integral, with the only  $\vec{r}$ -dependence coming from  $|r - r'|$ :

$$\vec{B}(\vec{r}) = (\vec{\nabla} \times \vec{A})(\vec{r}) = \frac{\mu_0}{4\pi} \int dV' \vec{\nabla} \left( \frac{1}{|r - r'|} \right) \times \vec{J}(\vec{r}'). \quad (13.23)$$

Moreover we know how to compute the gradient of  $1/|r - r'|$ , since this is just going from the potential of a point charge to its electric field:

$$\vec{\nabla} \left( \frac{1}{|r - r'|} \right) = -\frac{\vec{r} - \vec{r}'}{|r - r'|^3}. \quad (13.24)$$

Therefore we have a magnetic field

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int dV' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|r - r'|^3}. \quad (13.25)$$

This at last is our formula for the magnetic field produced by a general steady current distribution  $\vec{J}$ , analogous to equation (13.3) for the electric field. It may look a bit scary, but if you think a little it is quite intuitive: a current density  $\vec{J}$  at location  $\vec{r}'$  contributes a magnetic field at  $\vec{r}$  which is orthogonal to both  $\vec{J}$  and  $\vec{r} - \vec{r}'$ , with the orientation given by the right-hand rule, and which has magnitude proportional to  $|J|$  and falls off with distance (see figure 48 for an illustration).

In practice, we are often especially interested in the situation where all current is being carried by thin wires. We can model a thin wire as having a current density  $\vec{J}$  which inside the wire points in the direction of the wire and outside of the wire vanishes. In (13.25) the volume integral then only needs to be done over the interior of the wire, and we can split it up as

$$dV' = ds' dA', \quad (13.26)$$

where  $s'$  is the length along the wire and  $dA'$  is the cross sectional area element. Moreover for fixed  $s'$  the vector  $\vec{r} - \vec{r}'$  will basically be uniform as we move around in the cross sectional area, so we can pull it out of the area integral. Since we are considering only steady currents, the integral of  $|J|$  over a cross sectional area will be independent of  $s'$  and just gives the current  $I$  through the wire. Therefore we have

$$\begin{aligned} \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} \int_L ds' \frac{\hat{s}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \int dA' |J| \\ &= \frac{\mu_0 I}{4\pi} \int_L ds' \frac{\hat{s}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}, \end{aligned} \quad (13.27)$$

where  $\hat{s}'$  is a unit vector that points along the wire and  $L$  denotes the path of the wire. We can simplify this a bit using  $d\vec{s}' = \hat{s}' ds'$ , leading to what is called the **Biot-Savart law**:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{s}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}. \quad (13.28)$$

The Biot-Savart law tells us how to compute the magnetic field for any current loop  $L$ , we will spend the rest of the lecture using it to work out a few examples.

### 13.3 Magnetic field of a circular current loop

We'll first consider the magnetic field of a circular loop of radius  $R$  carrying a current  $I$ . It is convenient to use cylindrical coordinates  $(z, \rho, \phi)$ , where we will take the loop to lie in the  $z = 0$  plane and be centered at  $\rho = 0$ . The setup is shown in the left diagram of figure 49, and we sketch the expected magnetic field lines in the  $yz$  plane in the right diagram. Evaluating the Biot-Savart integral is difficult for a general point  $\vec{r}$ , but there are two special cases where we can do it. The first is where we take  $\vec{r}$  to be on the  $z$ -axis:

$$\vec{r} = z\hat{z}. \quad (13.29)$$

We can parametrize the circle using  $\phi'$

$$\vec{r}' = R\hat{\rho}' = R(\cos\phi'\hat{x} + \sin\phi'\hat{y}), \quad (13.30)$$

with the infinitesimal length vector being

$$d\vec{s}' = R d\phi' \hat{\phi}' = R d\phi' (-\sin\phi'\hat{x} + \cos\phi'\hat{y}). \quad (13.31)$$

To evaluate Biot-Savart we need

$$\begin{aligned} \vec{r} - \vec{r}' &= -R \cos\phi'\hat{x} - R \sin\phi'\hat{y} + z\hat{z} \\ |\vec{r} - \vec{r}'| &= \sqrt{R^2 + z^2} \\ d\vec{s}' \times (\vec{r} - \vec{r}') &= R d\phi' \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi' & \cos\phi' & 0 \\ -R \cos\phi' & -R \sin\phi' & z \end{vmatrix} \\ &= R d\phi' (z \cos\phi'\hat{x} + z \sin\phi'\hat{y} + R\hat{z}). \end{aligned} \quad (13.32)$$



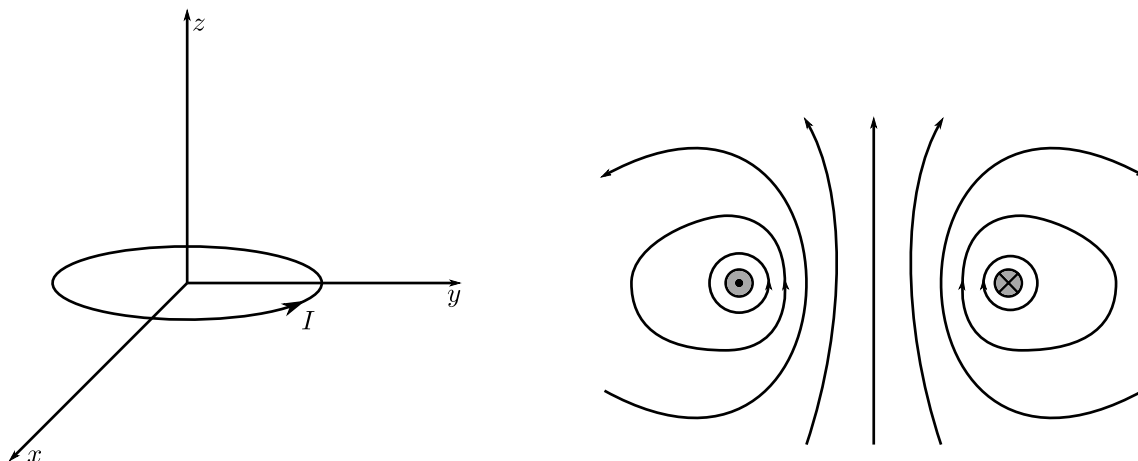


Figure 49: Magnetic field lines for a single loop of current. On the left we show the orientation of the loop, while on the right we show the magnetic field lines in the  $yz$  plane. Note that near the wires the magnetic field approaches that of an infinitely-long wire, inside the circle they combine to produce a stronger magnetic field, and outside the circle they oppose each other to produce a magnetic field that can be shown to fall off like distance cubed.

The  $x$  and  $y$  components will integrate to zero (as you could have guessed by symmetry), so we are left with

$$\begin{aligned}\vec{B}(z) &= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R^2 d\phi'}{(R^2 + z^2)^{3/2}} \hat{z} \\ &= \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}} \hat{z}.\end{aligned}\tag{13.33}$$

This always points in the positive  $z$  direction, is nonzero at  $z = 0$ , and falls off like  $1/z^3$  at large  $z$ , all of which is consistent with our sketch of the field lines.

The other case in which the integral can be done is the “asymptotic” limit of large  $\rho$  and/or  $z$ . We won’t go through the analysis here (see section 11.3 of Purcell), but the result is that in spherical coordinates  $(r, \theta, \phi)$  we have

$$\vec{B}(\vec{r}) = \frac{\mu_0 I R^2}{4r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}).\tag{13.34}$$

This is called the **magnetic dipole field**, and it has many interesting applications. Note that if  $\theta = 0$  or  $\theta = \pi$  it agrees with the large  $|z|$  limit of our formula on the  $z$ -axis.

### 13.4 Magnetic field of a solenoid

Our next example will be a wire which is wrapped into a long tightly-packed cylindrical helix, similar to a strand of DNA. A wire wrapped in this way is called a **solenoid**, we show an example in figure 50. We sketch some magnetic field lines for a solenoid of finite length in figure 51.

Since the spacing is very tight, we can think of the solenoid as a stack of circular loops, with a linear density  $n$  telling us the number of loops per meter along the axis of the solenoid. There are two strategies for computing the magnetic field. The first is to use superposition plus our results for a single circular current loop. The second is to use symmetry and Ampere’s law. The second strategy is more powerful, but we will begin with the first since it is conceptually simpler. Indeed say we want to determine the magnetic field on the  $z$  axis due to a solenoid of infinite length which is centered on the  $z$  axis. The field will be independent of  $z$ , so we might as well compute it at  $z = 0$ . We can split up the solenoid into cylinders of length  $dz'$ , each

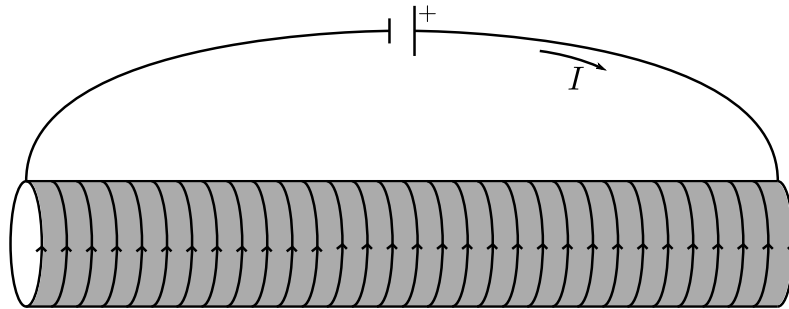


Figure 50: A long solenoid plugged into a battery. Inside the solenoid there is a nearly-uniform magnetic field pointing to the left, while outside the magnetic field is small except right near the ends.

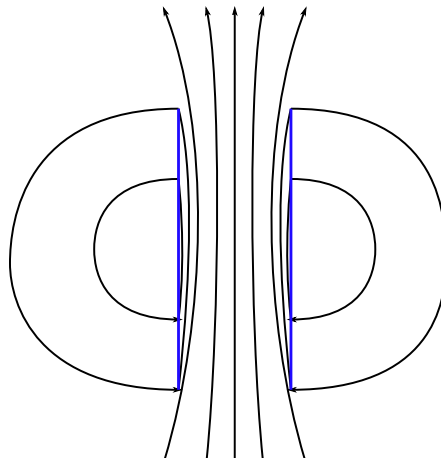


Figure 51: Magnetic field lines for a cross-section of a solenoid of finite length (the slices of the solenoid are shown in blue).

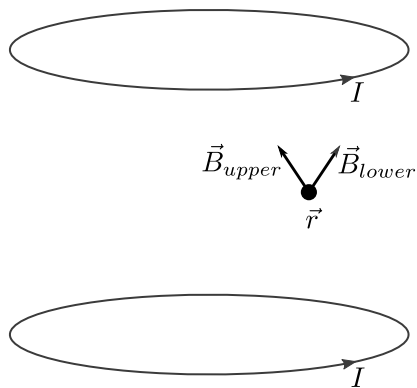


Figure 52: The magnetic field halfway between two loops of current. The radial components cancel, so the magnetic field points in the  $\hat{z}$  direction.

of which carries a current  $dI = nI dz'$ . Therefore we should substitute  $I \rightarrow nI dz'$  into our formula (13.33), replace  $z \rightarrow z'$ , and then integrate over  $z'$ :

$$\vec{B}(z) = \frac{\mu_0 n I}{2} \int_{-\infty}^{\infty} \frac{R^2 dz'}{(R^2 + z'^2)^{3/2}} \hat{z}. \quad (13.35)$$

Evaluating the integral,

$$\vec{B}(z) = \frac{\mu_0 n I}{2} \left( \frac{z'}{\sqrt{R^2 + z'^2}} \right) \Big|_{-\infty}^{\infty} \hat{z} = \mu_0 n I \hat{z}. \quad (13.36)$$

In fact you can also do this integral for a solenoid of finite length, finding the magnetic field everywhere on the  $z$ -axis, but I'll leave it to you.

It isn't obvious, but in fact the formula  $\vec{B} = \mu_0 n I \hat{z}$  is true *everywhere* inside of an infinite solenoid, and moreover outside the field is exactly zero. We will show this using a combination of Ampere's law and symmetry. Let's say  $\vec{r}$  is some point either inside or outside of the solenoid. We want to determine the direction of  $\vec{B}(\vec{r})$ . We can split up the contribution of the solenoid to the magnetic field into the contributions from pairs of rings which are at distance  $d$  above and below the point  $\vec{r}$ . Each ring gives a field which has only  $\hat{\rho}$  and  $\hat{z}$  components, and moreover the  $\hat{\rho}$  components cancel between the two rings (see figure 52). Therefore  $\vec{B}(\vec{r})$  points in the  $z$  direction everywhere inside and outside of the solenoid:

$$\vec{B}(\vec{r}) = B_z(\rho) \hat{z}. \quad (13.37)$$

Next we can use Ampere's law to show that  $B_z$  is constant inside and outside of the solenoid, but with a jump as we move from inside to outside. Indeed consider the loops shown in figure 53. Since  $\vec{B}$  points only in the  $z$  direction, only the vertical contributions to the loops contribute to the line integral of  $\vec{B}$  around them. For example let's say that the inner and outer vertical edges of loop one are at  $\rho_1$  and  $\rho_2$ , and that the vertical length is  $L$ . We then have

$$\int_1 \vec{B} \cdot d\vec{s} = L(B_z(\rho_2) - B_z(\rho_1)) = 0, \quad (13.38)$$

with the last equality being because no current is enclosed by this loop. Therefore  $B_z(\rho_2) = B_z(\rho_1)$  for all  $\rho_1, \rho_2 < R$ , so throughout the inside we just have

$$B_z(\rho) = B_z^{inside} \quad (\rho < R) \quad (13.39)$$

Similarly looking at loop three we have

$$B_z(\rho) = B_z^{outside} \quad (\rho > R). \quad (13.40)$$

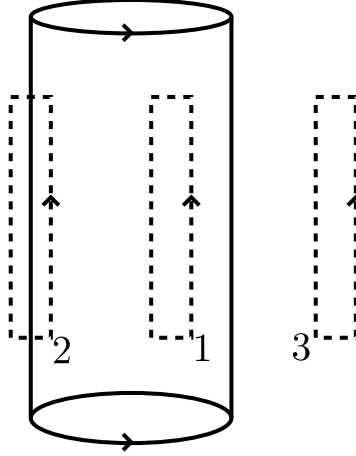


Figure 53: Amperian loops for determining the magnetic field of a solenoid.

If we then consider loop two, now there *is* current enclosed so we have

$$\int_3 \vec{B} \cdot d\vec{s} = L(B_z^{inside} - B_z^{outside}) = \mu_0 nLI, \quad (13.41)$$

or in other words

$$B_z^{inside} = B_z^{outside} + \mu_0 nI. \quad (13.42)$$

From here there are two ways to proceed. The first is to use our previous result (13.36) for the magnetic field on the  $z$  axis to determine  $B_z^{inside}$ , and then use (13.42) to show that  $B_z^{outside} = 0$ . It is perhaps more instructive however to instead argue that  $B_z^{outside}$  must be zero, and then use (13.42) to confirm that  $B_z^{inside}$  indeed agrees with (13.36). We will do this by showing that  $\vec{B}$  must go to zero far away from the solenoid. Since the field is constant outside, this then implies that  $B_z^{outside} = 0$ . To see that  $\vec{B}$  goes to zero, we can use the dipole field (13.34). At large  $\rho$ , this field falls off like

$$\vec{B} \sim \frac{1}{r^3} = \frac{1}{(\rho^2 + z^2)^{3/2}}. \quad (13.43)$$

That is the large  $\rho$  field for a single current loop, the field from the solenoid can be obtained from it by superposition:

$$\vec{B} \sim \int_{-\infty}^{\infty} \frac{dz}{(\rho^2 + z^2)^{3/2}}. \quad (13.44)$$

This is actually the same integral we just did in computing the field on the  $z$ -axis of the solenoid, we have

$$\int_{-\infty}^{\infty} \frac{dz}{(\rho^2 + z^2)^{3/2}} = 2/\rho^2, \quad (13.45)$$

and this indeed goes to zero at large  $\rho$ ! Thus we at last find

$$\vec{B}(\vec{r}) = \begin{cases} \mu_0 nI \hat{z} & \rho < R \\ 0 & \rho > R \end{cases}. \quad (13.46)$$

In fact this result is still correct even if the shape of the loop is not circular (provided you replace  $\rho < R$  with “inside” and  $\rho > R$  with “outside”), you might amuse yourself trying to show this. In practice solenoids are very useful for establishing uniform magnetic fields, and they also are important circuit elements as we will learn soon.

## 14 Induction and Faraday's law

So far we have discussed only situations where the electric and magnetic fields do not vary with time. In this lecture we will see that a time-varying magnetic field can actually act as a new kind of source for the electric field, taking us beyond Coulomb's law.

### 14.1 A second look at electromotive force

One of our basic equations for electrostatics is

$$\int_L \vec{E} \cdot d\vec{s} = 0, \quad (14.1)$$

which among other things tells us that Coulomb forces alone cannot produce a situation where moving a charge around a closed loop requires or produces a nonzero amount of work. In our discussion of circuits we certainly considered situations where circulating charges around a loop required us to do work, but this work was done for us by the non-Coulombic chemical forces inside of a battery. We can quantify this “non-Coulombic force” by defining the **loop EMF**,  $\mathcal{E}_L$ , given by the integral around a loop of the average *total* force per unit charge on the charge carriers:

$$\mathcal{E}_L \equiv \int_L \left( \frac{\vec{F}^{tot}}{q} \right)_{ave} \cdot d\vec{s}. \quad (14.2)$$

Here  $L$  is the loop under discussion, and if there are  $N$  mobile charged particles labeled by  $i$  in a small piece of the wire centered at  $\vec{r}$  then at  $\vec{r}$  we have

$$\left( \frac{\vec{F}^{tot}}{q} \right)_{ave} = \frac{1}{N} \sum_{i=1}^N \frac{\vec{F}_i^{tot}}{q_i}. \quad (14.3)$$

We can split  $\left( \frac{\vec{F}^{tot}}{q} \right)_{ave}$  into “Coulombic” and “non-Coulombic” pieces:

$$\left( \frac{\vec{F}^{tot}}{q} \right)_{ave} = \vec{E}^C + \left( \frac{\vec{F}^{NC}}{q} \right)_{ave}, \quad (14.4)$$

where  $\vec{E}^C$  is electric field produced by any stationary charges which are around, in terms of which we have

$$\mathcal{E}_L = \int_L \left( \vec{E}^C + \left( \frac{\vec{F}^{NC}}{q} \right)_{ave} \right) \cdot d\vec{s} = \int_L \left( \frac{\vec{F}^{NC}}{q} \right)_{ave} \cdot d\vec{s}. \quad (14.5)$$

Here we have used that the line integral of  $\vec{E}^C$  around any loop is zero.

The reason for defining the loop EMF  $\mathcal{E}_L$  is that it is simply related to the current which flows in the loop. For example consider the simple circuit shown in figure 54: the only non-Coulombic force is that inside of the battery, which is precisely tuned to cancel the Coulombic force inside of the battery (remember that the chemical reactions will happen at precisely the rate where this is achieved). Let's now compute the loop EMF for this circuit in two different ways. The first way is as in (14.5), which gives us the line integral of  $\left( \frac{\vec{F}^{NC}}{q} \right)_{ave}$  from the negative to the positive terminal of the battery. Since inside the battery this force is the opposite of the electric force,  $\mathcal{E}_L$  is (fortunately) equal to what we called the EMF of the battery a few lectures ago (which was the potential increase from the minus to the plus terminal). The second way to compute the loop EMF is to note that since the contributions from  $\vec{E}^C$  and  $\left( \frac{\vec{F}^{NC}}{q} \right)_{ave}$  cancel inside of the battery, we can also obtain the integral from the integral of the electric field around the rest of the circuit:

$$\mathcal{E}_L = \int_{wire} \vec{E} \cdot d\vec{s} = IR, \quad (14.6)$$

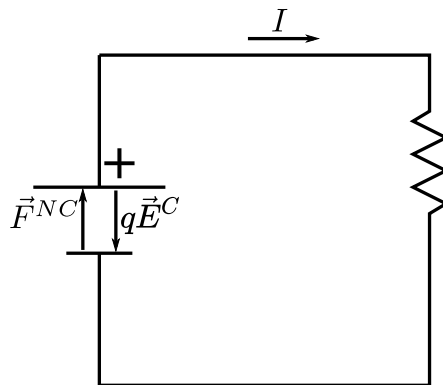


Figure 54: A simple circuit with a battery and a resistor.

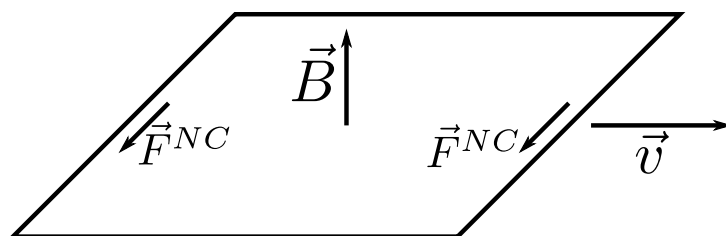


Figure 55: A square loop moving in a uniform field. The magnetic forces on the charge carriers give no net contribution to the loop EMF  $\mathcal{E}_L$ , since they cancel between the “leading” and “trailing” edges.

where in the last step we have used Ohm’s law. Thus we indeed see that if there are no non-Coulombic forces, there will be no net current flow.

## 14.2 Electromotive force on a conducting loop moving in a magnetic field

We have recently met a new force, the magnetic force, and it is now natural to ask if we can use it to drive a current around a circuit. The magnetic force only acts on charges which are moving, so we now study the loop EMF of a conducting loop moving in a magnetic field. We first consider the case of a square loop moving in a uniform field, shown in figure 55. In this case there is no loop EMF, so no current flows. It is easy to guess how we can fix this example to obtain a net EMF however: we consider a *nonuniform* magnetic field. Our main example will be a square loop of current moving with velocity  $v$  towards an infinitely-long wire carrying a current  $I$ . Since the leading edge of the loop now feels a stronger magnetic field than the trailing edge does, there is a net EMF (see figure 56 for the setup). We can compute the EMF explicitly: if the leading edge is at a distance  $x$  from the wire and the loop has length  $\ell$  on each side, we have

$$\mathcal{E}_L = \frac{\mu_0 I v \ell}{2\pi x} - \frac{\mu_0 I v \ell}{2\pi(x + \ell)}. \quad (14.7)$$

Here the orientation of the loop is as in figure 56.

It turns out that there is a nice relationship between  $\mathcal{E}_L$  and the magnetic flux through the interior of  $L$ , which we define as

$$\Phi_L \equiv \int_S \vec{B} \cdot d\vec{A}, \quad (14.8)$$

where  $S$  is any surface whose boundary is  $L$  and whose normal vector is compatible with the orientation of  $L$  (out of the page in figure 56). It does not matter which such surface  $S$  we use: if  $S$  and  $S'$  are two surfaces

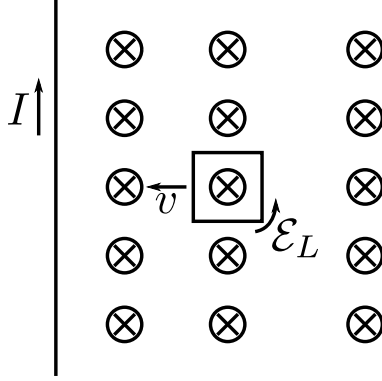


Figure 56: The induced EMF on a conducting loop moving towards a conducting wire. Note that the current produced by this EMF creates a magnetic field which *opposes* the one from the wire.

which both have boundary  $L$ , then

$$\int_S \vec{B} \cdot d\vec{A} - \int_{S'} \vec{B} \cdot d\vec{A} = \int_{S-S'} \vec{B} \cdot d\vec{A} = \int_R dV \vec{\nabla} \cdot \vec{B} = 0, \quad (14.9)$$

where  $R$  is the spatial volume between  $S$  and  $S'$  and we have used the divergence theorem and also that  $\vec{\nabla} \cdot \vec{B} = 0$ . So in particular for the loop shown in figure 56, we have

$$\Phi_L = \int_S \vec{B} \cdot d\vec{A} = -\frac{\mu_0 I \ell}{2\pi} \int_x^{x+\ell} \frac{dx'}{x'} = -\frac{\mu_0 I \ell}{2\pi} \log \frac{x+\ell}{x}, \quad (14.10)$$

where  $S$  is the square whose boundary is the loop. The minus sign arises from the normal vector being opposite to the magnetic field. The relationship between  $\mathcal{E}_L$  and  $\Phi_L$  is given by what will be our first iteration of **Faraday's law**:

$$\mathcal{E}_L = -\frac{d\Phi_L}{dt}. \quad (14.11)$$

We can check it by computing the time derivative of the flux:

$$-\frac{d\Phi_L}{dt} = \frac{\mu_0 I \ell}{2\pi} \left( \frac{\dot{x}}{x+\ell} - \frac{\dot{x}}{x} \right) = \frac{\mu_0 I v \ell}{2\pi} \left( \frac{1}{x} - \frac{1}{x+\ell} \right), \quad (14.12)$$

where we have used that  $\dot{x} = -v$ .

### 14.3 Faraday's law for a general loop moving in a time-independent magnetic field

The relationship we just derived between loop EMF and magnetic flux may seem like it might be a consequence of special properties of our example, but actually it is true for a loop of any shape and size moving in an arbitrary time-independent magnetic field. To prove this, let's consider the change in magnetic flux caused by the motion of a piece of the loop of length  $ds'$ . In a time  $dt$  the area swept out by this piece of the loop is  $v dt \sin \theta ds'$ , as shown in figure 57. The change in flux is therefore  $v dt \sin \theta ds' B_{\perp}$ , where  $B_{\perp}$  is the component of  $B$  pointing out of the page. We can rewrite this as  $(\vec{v} \times d\vec{s}') \cdot \vec{B} dt$ , so apparently the change in flux is

$$d\Phi = dt \int_L (\vec{v} \times d\vec{s}') \cdot \vec{B}. \quad (14.13)$$

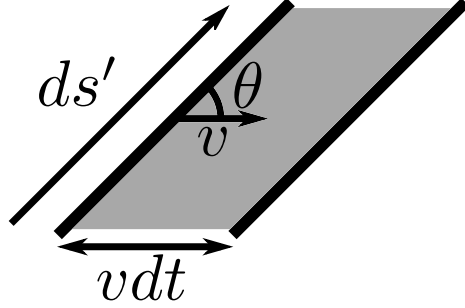


Figure 57: The area swept out by the motion of a loop segment of length  $ds'$  with velocity  $v$  in a time  $dt$ .

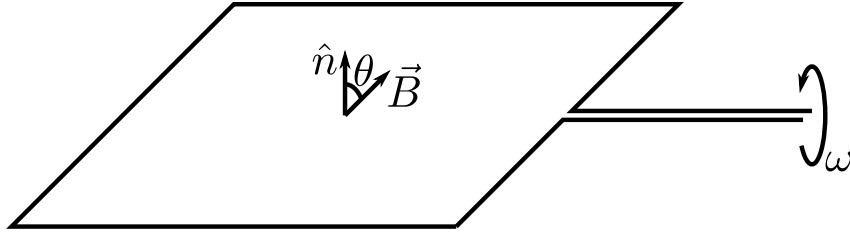


Figure 58: An electric generator.

Moreover for any three vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  it is always the case that

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{w} \times \vec{u}) \cdot \vec{v}, \quad (14.14)$$

so apparently we have

$$\frac{d\Phi}{dt} = \int_L (\vec{B} \times \vec{v}) \cdot d\vec{s} = - \int_L (\vec{v} \times \vec{B}) \cdot d\vec{s} = -\mathcal{E}_L. \quad (14.15)$$

Note that this argument did not require  $\vec{v}$  to be the same everywhere on the loop, so apparently Faraday's law holds also for situations where the loop is rotating and/or deforming.

In situations where Faraday's law predicts an EMF around a loop due to a changing magnetic flux through the loop, it is customary to say that this EMF has been **induced** by the changing magnetic flux. Faraday's law is therefore sometimes called Faraday's law of induction.

#### 14.4 An electric generator

Faraday induction has many useful applications, one simple one is that we can use it to convert mechanical work to an electric current. For example consider the setup shown in figure 58: a loop of conducting wire with area  $A$  is rotated at constant angular velocity  $\omega$  in a uniform magnetic field  $\vec{B}$ . The magnetic flux is

$$\Phi = A\vec{B} \cdot \hat{n} = A|B| \cos\theta(t) = A|B| \cos(\omega t). \quad (14.16)$$

Since this flux is time dependent, there is an EMF

$$\mathcal{E}_L = -\dot{\Phi} = A|B|\omega \sin(\omega t) \quad (14.17)$$

around the loop. If we now make a small cut in the loop and run two wires out of it (as in figure 58), we get a circuit element which acts as a "battery of oscillating voltage", or in other words a source of alternating current (AC). This mechanism is realized in some way or other in almost all power plants.



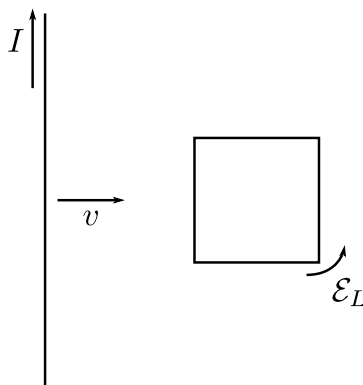


Figure 59: A current-carrying wire moving towards a stationary loop induces an EMF around the loop.

### 14.5 The case of the missing EMF

We now consider the situation from figure 56 again, but this time from the point of view of the rest frame of the loop (see figure 59). We are immediately confronted with a puzzle: the two frames need to agree on whether or not current flows around the loop, but in this new frame the loop is not moving so its charge carriers do not feel any magnetic force. How then are we to explain the EMF in this frame?

The only possibility is that somehow in this new frame there is an electric field which contributes a nonzero EMF. The appearance of this electric field may not be such a surprise, after all in our original motivation for the magnetic field we showed that it arose from an electric field in a different reference frame. The real surprise here though is that this electric field gives a nonzero loop EMF: so far in this class we have always assumed that

$$\int_L \vec{E} \cdot d\vec{s} = 0, \quad (14.18)$$

but apparently in the presence of a moving current-carrying wire this is not true!

### 14.6 Faraday's law in general

As with our introduction of Ampere's law, it is not possible to give a general derivation of the electric field produced by a moving current-carrying wire. We can however come up with a guess, and then test it against experiment. Based on our above experience with Faraday induction, there is a somewhat natural proposal:

$$\mathcal{E}_L = \int_L \vec{E} \cdot d\vec{s} = -\dot{\Phi}. \quad (14.19)$$

In our previous frame the time-dependence of the magnetic flux through  $L$  arose from the motion of the loop, but now we are proposing that even when the loop is stationary Faraday's law continues to accurately compute the EMF. By carefully studying the Lorentz transformations of the electric magnetic fields in the situation of figure 59 this proposal can actually be justified for the specific case of an infinitely-long perfectly-straight wire, but in general it has to be taken as a new axiom. We can rewrite this axiom in a nice way by shrinking the loop to zero size and using our geometric interpretation of the curl:

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}. \quad (14.20)$$

This is the differential version of Faraday's law, and it is one of Maxwell's equations. In situations where the magnetic field is time-independent, it reduces to our old electrostatic equation  $\vec{\nabla} \times \vec{E} = 0$ .

It is worth emphasizing that we have now discussed two seemingly-distinct versions of Faraday's law: one for a time-dependent loop in a time-independent magnetic field, and one for a time-independent loop in

a time-dependent magnetic field. In either case we saw that the EMF around the loop was given by minus the time derivative of the magnetic flux, but the explanation for the EMF was different in the two cases. In the former it arose from magnetic forces, while in the latter it arose from electric forces. You might wonder about the more general situation of a time-dependent loop moving in a time-dependent magnetic field: we then need to take *both* effects into account. In fact this is easy to do, since both the forces and the contributions to the time derivative of the magnetic flux simply add. Therefore we always have

$$\mathcal{E}_L = \int_L (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{s} = -\dot{\Phi}, \quad (14.21)$$

even when both the loop and the magnetic field are changing with time.

You may worry that now we have allowed for the curl of  $\vec{E}$  to be nonvanishing, all of our discussion of the electric potential  $\phi$  needs to be thrown out. In fact the situation is not so bad as that: we can rewrite Faraday's law as

$$\vec{\nabla} \times (\vec{E} + \dot{\vec{A}}) = 0, \quad (14.22)$$

so we are still allowed to introduce a scalar potential provided that we now take it to obey

$$\vec{E} + \dot{\vec{A}} = -\vec{\nabla}\phi. \quad (14.23)$$

We can rewrite this as

$$\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}, \quad (14.24)$$

so now to determine  $\vec{E}$  we need to know both the scalar and vector potentials.

## 14.7 Lenz's law

The minus sign in Faraday's law has a simple interpretation which is formalized by the following principle, known as **Lenz's law**: a current induced by a changing magnetic flux through a loop will always flow in the direction such that the magnetic field it produces *opposes* the change in magnetic flux which did the inducing. For example in figure 56, as the loop moves towards the wire the magnetic flux into the page is increasing, so the current flows to produce a magnetic field which is out of the page. More generally, you can think of the inductance of a loop as a kind of "electromagnetic inertia": it resists changes in the magnetic field. We will see this analogy between inductance and inertia is quite precise in the next few lectures.

# 15 Inductors and magnetic energy

We learned last lecture that a time-varying magnetic flux through a loop induces an EMF around that loop. In thinking about how to harness this effect for practical purposes, it is useful to introduce a few new concepts.

## 15.1 Mutual inductance

Consider the situation where we have a wire lying on a loop  $L_1$  carrying a current  $I_1$ . This current creates a magnetic field  $\vec{B}_1$ , and one natural question we can ask is what is the magnetic flux  $\Phi_{L_2}$  of this  $\vec{B}_1$  through some other loop  $L_2$ ? In general this requires a complicated calculation, but one thing we can say already is that this magnetic flux must be linear in the current  $I_1$ :

$$\Phi_{L_2} = M_{21}I_1. \quad (15.1)$$

This linearity is an immediate consequence of the Biot-Savort law. The quantity  $M_{21}$  is called the **mutual inductance** of loop one on loop two. The units of mutual inductance are called **Henry's**:

$$[M_{21}] = \frac{\text{kg m}^2}{\text{C}^2} \equiv 1 \text{ Henry} = 1 \text{ H}. \quad (15.2)$$

As usual a reasonable person would have defined the mutual inductance to be  $M_{21}/\mu_0$ , in which case it would have just had units of meters, but we are stuck with the poor decisions of our predecessors.

There are a few cases where  $M_{21}$  can be computed explicitly, one example where the loops are concentric circles lying in the same plane *and* the radius  $R_1$  of  $L_1$  is much larger than the radius  $R_2$  of  $L_2$ . We may then approximate the magnetic field everywhere within the disk bounded by  $L_2$  using our expression for the magnetic field on the axis of a loop of current:

$$\Phi_{L_2} = \pi R_2^2 \times \frac{\mu_0 I}{2R_1}, \quad (15.3)$$

so

$$M_{21} = \frac{\mu_0 \pi R_2^2}{2R_1}. \quad (15.4)$$

Perhaps the most interesting property of mutual inductance is that for any pair of loops  $L_1$  and  $L_2$ , the mutual inductance of  $L_1$  on  $L_2$  is always equal to the mutual inductance of  $L_2$  on  $L_1$ :

$$M_{12} = M_{21}. \quad (15.5)$$

This is not at all obvious from the definition, for example in the above example with the concentric circles to compute the mutual inductance of  $L_2$  on  $L_1$  you would need to integrate our expression for the magnetic dipole field created by the small loop to find the flux through the big loop and that integral has no apparent connection to the (trivial) integral we did to find  $M_{21}$ . Here is the proof. Let  $S_2$  be a surface whose boundary is  $L_2$ . By Stokes's theorem we then have

$$\Phi_{L_2} = \int_{S_2} \vec{B}_1 \cdot d\vec{A} = \int_{S_2} (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{A} = \int_{L_2} \vec{A}_1 \cdot d\vec{s}. \quad (15.6)$$

Next recall our general expression for the vector potential produced by a current density:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int dV' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (15.7)$$

For the case where the only current is in a thin wire going around a loop  $L$  and carrying a current  $I$ , we can rewrite this (as we did in deriving Biot-Savort) as

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{s}'}{|\vec{r} - \vec{r}'|}. \quad (15.8)$$

We may then use this in our expression for  $\Phi_{L_2}$ :

$$\Phi_{L_2} = \frac{\mu_0 I_1}{4\pi} \int_{L_2} d\vec{s} \cdot \int_{L_1} d\vec{s}' \frac{1}{|\vec{r} - \vec{r}'|}. \quad (15.9)$$

Therefore we have

$$M_{21} = \frac{\mu_0}{4\pi} \int_{L_2} d\vec{s} \cdot \int_{L_1} d\vec{s}' \frac{1}{|\vec{r} - \vec{r}'|}. \quad (15.10)$$

But actually this expression is symmetric under exchanging 1 and 2, since we can undo the exchange merely by also exchanging the names of the integration variables.

## 15.2 Self-inductance

It is also interesting to consider the magnetic flux a current loop produces through itself! The magnetic flux  $\Phi$  through the loop will again be linear in the current  $I$  which flows around the loop, so we can define a **self-inductance**  $L$  for the loop as

$$\Phi = LI. \quad (15.11)$$

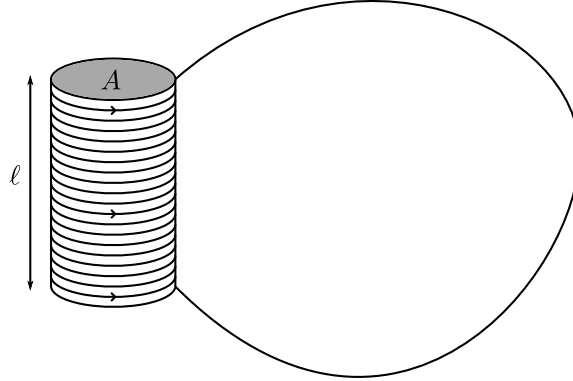


Figure 60: A loop of wire containing a solenoid closed by a single wire. The cross section of the solenoid does not to be a circle, although I've drawn one where it is.

For example let's consider a loop consisting of a long solenoid of length  $\ell$ , cross-sectional area  $A$ , and with  $N$  turns, with the two endpoints connected via single long wire which extends far out from the solenoid and then returns (see figure 60). Since the solenoid is long, to a good approximation magnetic field inside is

$$\vec{B} = \frac{\mu_0 N I}{\ell} \hat{z}, \quad (15.12)$$

where  $\hat{z}$  points along the axis of the solenoid. Therefore the magnetic flux due to this field is

$$\Phi = \frac{\mu_0 N^2 I A}{\ell}. \quad (15.13)$$

At least naively we expect the magnetic field outside to be quite small, in which case we can conclude we may neglect the contribution to the flux from the magnetic flux through the “single-wire” part of the loop. We then conclude that the self-inductance of this loop is

$$L = \frac{\mu_0 N^2 A}{\ell} \quad (15.14)$$

Unfortunately our neglect of the magnetic field outside of the solenoid was not quite correct. We can understand the source of the problem by instead trying to compute the self-inductance of what you might think should be a simpler setup: a single circular loop of radius  $R$ . One immediate problem with this is that we did not actually compute the magnetic field everywhere on the disk bounded by a circular current loop, so we don't know what function to integrate. There is however a more serious problem: even if we did know that, the magnetic flux is actually infinite! This happens because the magnetic field very close to the wire diverges like  $1/\rho$ , where  $\rho$  is the distance to the wire, and the integral of  $1/\rho$  diverges logarithmically at  $\rho = 0$ . This is not a problem for real wires of course, which have finite thickness, but for a wire of finite thickness there is a new problem: we don't know precisely which loop to use to compute the flux, and we also don't know how to distribute the current  $I$  through cross-section of the wire. Therefore it seems that we have a choice between a self-inductance which is infinite and a self-inductance which is ambiguous! Why then are we talking about such a confusing quantity? The answer to this question is quite interesting, but I do not have time to get into it in detail here. The answer however is that if we replace our circular loop of radius  $R$  by a finite tube of thickness  $\epsilon$ , the self-inductance is

$$L = \mu_0 R \left( \log \left( \frac{R}{\epsilon} \right) + C \right), \quad (15.15)$$

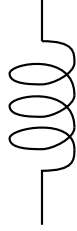


Figure 61: The standard symbol for an inductor in a circuit diagram.

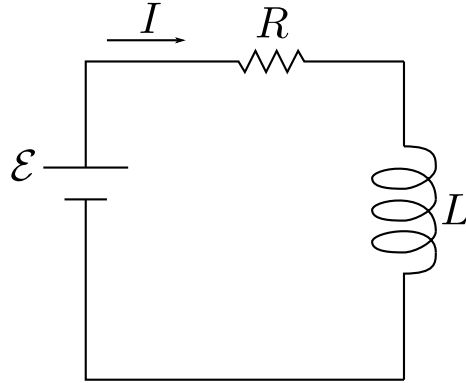


Figure 62: A simple circuit containing an inductor. This kind of circuit is usually called an “RL circuit”.

where  $C$  is a constant whose value depends on exactly which loop inside the tube we pick to define the flux and also how we distribute the current density. For a fixed choice of these things,  $C$  is independent of  $R$  and  $\epsilon$ . What this means is that in the limit  $R \gg \epsilon$  the ambiguities in  $L$  are small compared to its overall value. On the other hand the logarithm function does not grow very fast, so for example if  $R$  is a centimeter and  $\epsilon$  is the size of an atom,  $\log\left(\frac{R}{\epsilon}\right)$  will only be about eighteen! So in practice we have to live with an  $L$  given by  $\mu_0 R$  times an order one number of dubious nature.

You presumably are still not particularly happy with this state of affairs, but now let’s return to our loop shown in figure 60. We’ve now learned that our computation of the magnetic flux was off by some contribution which is in detail ambiguous but is roughly  $\mu_0$  times the size of the return loop. But actually for sufficiently large  $N^2 A/\ell$  this ambiguous contribution will indeed be swamped by the contribution from the solenoid so  $L$  will indeed be given by (15.14) to an excellent approximation! This is one of those special cases where to get the story straight it is essential to remember that wires have finite thickness, even though the actual thickness ends up not mattering.

### 15.3 Inductors

The solenoid in the loop of figure 60 is the canonical example of a new kind of circuit element, an **inductor**. In the idealized circuit model, an inductor as an object we can insert into a circuit which gives contributes to whatever loop it is inserted into a self-inductance  $L$  which is much larger than any self-inductance associated to the same loop prior to inserting any inductors. This is precisely what we argued was the case for the solenoid (for sufficiently large  $N^2 A/\ell$ ). Moreover in the circuit model we assume that this inductor has zero resistance, although of course that won’t be true in practice (a real inductor can be modeled by an inductor and a resistor in series). We show the standard symbol for an inductor in figure 61.

The role of inductors in circuits is that they suppress changes in the current. To see this more concretely,

consider the circuit shown in figure 62. Previously we analyzed such circuits by requiring that the line integral of  $\vec{E}$  around the loop vanished, but now we instead have the integral version of Faraday's law:

$$\int \vec{E} \cdot d\vec{s} = -\dot{\Phi}. \quad (15.16)$$

We can rewrite this using the self-inductance of the loop,

$$\int \vec{E} \cdot d\vec{s} + L\dot{I} = 0, \quad (15.17)$$

so what we've learned is that actually we can still use Kirchoff's rules for circuits containing inductors provided that we "pretend" that we can think of them as just contribute a voltage drop  $L\dot{I}$ . So in particular for the circuit shown in figure 62, we have

$$L\dot{I} + IR - \mathcal{E} = 0. \quad (15.18)$$

If  $L$  is large, it means that any time derivative for  $I$  leads to the first term in this equation being quite large. In particular it is basically only possible if the current itself is also quite large. You might wonder what happens if we take this circuit while it is running and then open a switch, making the current flow impossible. Doing this makes the inductor *very* unhappy, and typically results in large arcing sparks as the air is ionized to allow the current to continue to keep flowing!<sup>20</sup>

We can solve the differential equation (15.18) to determine the current as a function of time. Let's say we start with zero current at time  $t = 0$ . This equation is similar to the one we solved for the RC circuit a few lectures ago, the solution is

$$I = \frac{\mathcal{E}}{R} \left(1 - e^{-\frac{R}{L}t}\right), \quad (15.19)$$

as you can check by substituting into (15.18). If  $L$  is large, this current builds up very slowly, over a time of order  $L/R$ , and eventually saturates at the value  $\mathcal{E}/R$  predicted by Ohm's law in the absence of the inductor. This makes good sense, since once the current is not changing we are indeed allow to ignore the inductor. We can also consider what happens if at  $t = 0$  there is some initial current  $I_0$  and we remove the battery. The solution is then

$$I = I_0 e^{-\frac{R}{L}t}, \quad (15.20)$$

so the current gradually decays back to zero.

## 15.4 Energy in the magnetic field

In the decay of the RL circuit we considered in the previous section, the energy burned off by the resistor had to come from somewhere. A natural proposal is that it was stored in the magnetic field inside the inductor! Let's first compute how much energy was dissipated:

$$U = \int_0^\infty dt RI^2(t) = I_0^2 R \int_0^\infty e^{-2tR/L} dt = \frac{LI_0^2}{2}. \quad (15.21)$$

Therefore we can say that in general an inductor through which a current  $I$  is passing carries energy

$$U = \frac{LI^2}{2}. \quad (15.22)$$

Let's study this for the particular case of a solenoid. We then have

$$U = \frac{\mu_0 N^2 A I^2}{2\ell} = \frac{1}{2\mu_0} (\mu_0 N I / \ell)^2 \ell A = \frac{1}{2\mu_0} \int dV |B|^2. \quad (15.23)$$

<sup>20</sup>Strongly inductive circuits are thus quite dangerous, and people have been killed trying to open them.

Therefore we are led to propose that in general the magnetic field stores an energy

$$U = \frac{1}{2\mu_0} \int dV |B|^2. \quad (15.24)$$

This is indeed true, and we can show it using (15.22) for the field produced by any wire. Indeed we first observe that

$$\frac{1}{2\mu_0} \int_{B_R} dV |B|^2 = \frac{1}{2\mu_0} \int_{B_R} dV \vec{B} \cdot (\vec{\nabla} \times \vec{A}), \quad (15.25)$$

where  $B_R$  is a large solid ball of radius  $R$ , with  $R$  large enough that all currents lie within  $B_R$ . We then make use of the vector calculus identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}), \quad (15.26)$$

which is true for any vector fields  $\vec{A}$  and  $\vec{B}$ , together with the divergence theorem and Ampere's law, to see that

$$\begin{aligned} \frac{1}{2\mu_0} \int_{B_R} \vec{B} \cdot (\vec{\nabla} \times \vec{A}) &= \frac{1}{2\mu_0} \int_{B_R} dV (\vec{A} \cdot (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \cdot (\vec{A} \cdot \vec{B})) \\ &= \frac{1}{2} \int_{B_R} dV \vec{A} \cdot \vec{J} + \frac{1}{2\mu_0} \int_{\partial B_R} (\vec{A} \times \vec{B}) \cdot d\vec{A}. \end{aligned} \quad (15.27)$$

In the limit that  $R \rightarrow \infty$ , the boundary term on the right-hand side goes to zero since  $\vec{A} \sim 1/R$  and  $\vec{B} \sim 1/R^2$  (see our expressions for  $\vec{A}$  and  $\vec{B}$  in terms of  $I$ ), which together beat the  $R^2$  growth from the area of  $\partial B_R$ . Therefore we have that

$$\frac{1}{2\mu_0} \int dV |B|^2 = \frac{1}{2} \int dV \vec{A} \cdot \vec{J} = \frac{I}{2} \int_C \vec{A} \cdot d\vec{s}, \quad (15.28)$$

where the volume integrals are taken over all of space and we have used (via the same manipulation that went into deriving Biot-Savort) that all of the current lies in a wire carrying a current  $I$  and whose path we call  $C$ . Finally introducing a surface  $S$  such that  $C = \partial S$ , we can use Stokes's theorem to see that:

$$\begin{aligned} \frac{I}{2} \int_C \vec{A} \cdot d\vec{s} &= \frac{I}{2} \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{A} \\ &= \frac{I}{2} \int_S \vec{B} \cdot d\vec{A} \\ &= \frac{I}{2} \Phi_C \\ &= \frac{LI^2}{2}. \end{aligned} \quad (15.29)$$

Therefore we indeed have that

$$U = \frac{LI^2}{2} = \frac{1}{2\mu_0} \int dV |B|^2 \quad (15.30)$$

## 16 RLC circuits

In this lecture we will study the behavior of "RLC circuits", meaning circuits which have resistors, inductors, and capacitors. In particular we will consider the circuit shown in figure 63. The loop rule (including the back EMF from the inductor) gives us the equation

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} - \mathcal{E} = 0, \quad (16.1)$$

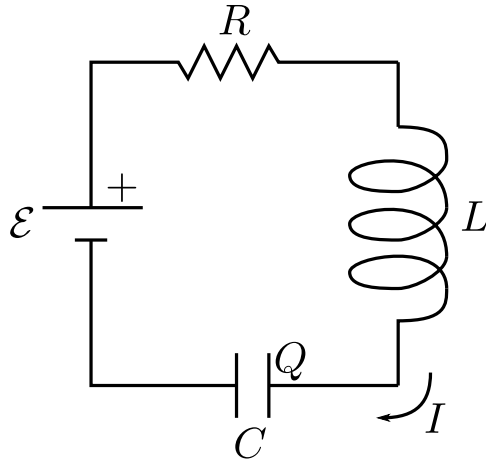


Figure 63: The circuit diagram for an RLC circuit with a DC source. The direction of current flow is as indicated, and  $Q$  denotes the accumulated charge on the right plate of the capacitor.

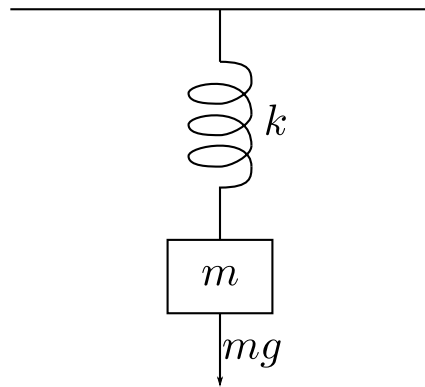


Figure 64: A mechanical analogy to an RLC circuit: a mass bouncing on a spring that is hanging from the ceiling in the presence of gravity and air resistance.

where we have used the relationship

$$I = \dot{Q} \tag{16.2}$$

between current and charge. Equation (16.1) is a bit more complicated than the circuit equations we have studied so far, so it is worthwhile making a few general observations about it before working out its solutions. The first observation is that equation (16.1) is an example of what is called an **ordinary second-order differential equation**, which means that it is an equation setting some combination of a function and its first and second time derivatives to zero. Here “ordinary” is the term which tells us that the unknown function depends on only a single variable; otherwise it would be a “partial” differential equation. “Second-order” means that the equation involves only first and second derivatives. Moreover in this case we are fortunate to have an equation which is **linear**, meaning that the quantity on the left-hand side of (16.1) is a linear function of  $Q$  and its derivatives. In general it is quite difficult to find the solutions of differential equations, but for linear equations quite a lot is known, and in particular equation (16.1) is one of the ones we know how to solve.

Before solving this equation, it is useful to first notice that it is very closely related to the mechanical problem of a mass  $m$  hanging from a spring of spring-constant  $k$  attached to the ceiling in the presence of



air resistance (see figure 64). Newton's law for this mass is

$$m\ddot{z} = mg - kz - D\dot{z}, \quad (16.3)$$

where  $g$  is  $9.8m/s^2$ ,  $D$  is a drag coefficient, and  $z$  is the distance of the mass below the point where the spring is "unstretched" (increasing  $z$  goes down). Up to relabelings, this is precisely the same equation as (16.1)! Indeed the map between them is

$$\begin{aligned} Q &\rightarrow z \\ L &\rightarrow m \\ R &\rightarrow D \\ \frac{1}{C} &\rightarrow k \\ \mathcal{E} &\rightarrow mg. \end{aligned}$$

This mathematical analogy is quite useful in anticipating what the circuit will do: resistance is like a drag force, inductance is like mass, a battery is like the gravitational force, and so on. For example we know from experience that there will be an "equilibrium" position where the spring tension balances the gravitational force and the mass does not move. Moreover if the drag force is weak and we disturb the mass from this equilibrium, then it will bounce repeatedly on the end of the spring, with the amplitude of the oscillations gradually decaying with time due to air resistance. On the other hand if the drag force is strong (say we put the mass and spring in molasses), then the mass will slowly relax to its equilibrium without any oscillation. We will now see quantitatively that analogous statements are for the RLC circuit.

## 16.1 Solving the RLC circuit equation

The first thing to note is that we can get rid of the battery EMF in equation 16.1 by an additive shift in  $Q$ : if we define

$$Q = \hat{Q} + \mathcal{E}C, \quad (16.4)$$

then  $\hat{Q}$  obeys (16.1) with  $\mathcal{E} = 0$ :

$$L\ddot{\hat{Q}} + R\dot{\hat{Q}} + \frac{\hat{Q}}{C} = 0. \quad (16.5)$$

In our mechanical analogy, this is the observation that the only effect of gravity is to shift the equilibrium location of the mass from  $z = 0$  to  $z = \frac{mg}{k}$ .

Now we come to the key trick: since this is a linear differential equation, it is natural to guess that the solution is exponential:

$$\hat{Q} = Ae^{\alpha t}. \quad (16.6)$$

Since time derivatives of  $\hat{Q}$  just bring down powers of  $\alpha$ , inserting this into equation (16.5) gives us

$$\left(L\alpha^2 + R\alpha + \frac{1}{C}\right) Ae^{\alpha t} = 0. \quad (16.7)$$

This equation has a boring solution where  $A = 0$ , which describes the situation where the capacitor is fully charged and no current is flowing (in the mechanical analogy the mass is just sitting at its equilibrium position with the restoring force balances gravity). To get a more interesting solution we should take  $A \neq 0$ , in which case we must choose  $\alpha$  so that

$$L\alpha^2 + R\alpha + \frac{1}{C} = 0. \quad (16.8)$$

This is a quadratic equation, so it has two solutions:

$$\begin{aligned}\alpha_{\pm} &= -\frac{R}{2L} \pm \frac{1}{2L} \sqrt{R^2 - \frac{4L}{C}} \\ &= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}.\end{aligned}\tag{16.9}$$

Since both solutions are equally good, and moreover since the sum of two solutions of the  $\hat{Q}$  equation is also a solution, the general solution of our RLC equation is

$$Q(t) = \mathcal{E}C + A_+e^{\alpha_+t} + A_-e^{\alpha_-t}.\tag{16.10}$$

The constants  $A_{\pm}$  are determined from the initial conditions, for example the charge and current at  $t = 0$ . This is consistent with the general mathematical fact that an ordinary second-order differential equation requires two initial conditions to specify a unique solution; in our mechanical analogy these are the initial position and velocity.

There is something about our expressions for  $\alpha_{\pm}$  which is perhaps a bit puzzling. The resistance  $R$  is analogous to the drag constant  $D$  in the spring system, so we expect that at small  $R$  we get an oscillating solution. What seems to happen instead is that the quantity inside the square root becomes negative in this limit, or in other words  $\alpha_{\pm}$  become complex. We can emphasize this by writing

$$\alpha_{\pm} = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}},\tag{16.11}$$

where now the square root is real for small  $R$ . This may seem absurd: what could it possibly mean for the charge to be complex? In fact this is actually the right physics! To see why, we need to introduce one of the most remarkable formulas in all of mathematics.

## 16.2 Complex exponentials

You presumably are familiar with complex numbers: any complex number can be written as a sum of real and imaginary parts as

$$z = a + bi,\tag{16.12}$$

where  $a$  and  $b$  are real numbers. Complex numbers are added and multiplied using the definition that  $i^2 = -1$ . At first this may seem bizarre, but you can simply think of the complex numbers as introducing a way to add and multiply points in the two-dimensional plane  $\mathbb{R}^2$ . These points are just as “real” as the integers are, in particular  $i$  is just the point  $(0, 1)$ , so there really isn’t anything mysterious about the complex numbers. Indeed geometrically the addition of complex numbers is just the addition of vectors in  $\mathbb{R}^2$ , while their multiplication is defined by multiplying the lengths of the two position vectors and adding their polar angles. So in particular since  $\pi/2 + \pi/2 = \pi$ ,  $(0, 1)^2 = (-1, 0)$ .

In this class we will only begin to scratch the surface, but the real fun of complex numbers is that you can do calculus with them. This leads to the wonderful subject of **complex analysis**, and I encourage any of you who wish to be physics majors to take a course in complex analysis! The main result we will need is Euler’s beautiful formula for the exponential of a complex number:

$$e^{a+bi} = e^a(\cos b + i \sin b).\tag{16.13}$$

One way to derive this is to multiply through by  $e^{-a}$  and then use the Taylor series definition of the exponential function:

$$\begin{aligned}e^{bi} &= 1 + bi + \frac{(bi)^2}{2} + \frac{(bi)^3}{3!} + \dots \\ &= \left(1 - \frac{b^2}{2} + \dots\right) + i\left(b - \frac{b^3}{3!} + \dots\right) \\ &= \cos b + i \sin b.\end{aligned}\tag{16.14}$$

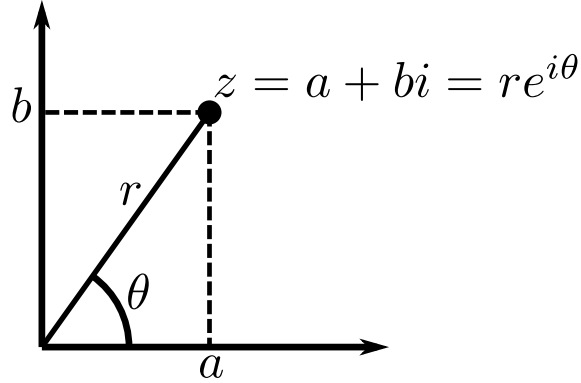


Figure 65: Representing a complex number in the two-dimensional plane. We can use either its real and imaginary parts  $a$  and  $b$  or its magnitude and phase  $r$  and  $\theta$ .

Using this expression, any complex number can be rewritten in terms of a magnitude  $r$  and phase  $\theta$ :

$$z = re^{i\theta}. \quad (16.15)$$

See figure 65 for an illustration.

### 16.3 Back to the RLC circuit

Using the complex exponential, even if  $\alpha_{\pm}$  are complex we can re-interpret equation (16.10) as giving the general *complex* solution of the differential equation (16.1) (provided that we also allow the coefficients  $A_{\pm}$  to be complex). In the end however we are of course only interested in solutions where  $Q(t)$  is real. To find the general real solution, we now need to make up our minds about the sign of the quantity in the square root.

We will first consider the case

$$\frac{R^2}{4L^2} > \frac{1}{LC}, \quad (16.16)$$

which is called the **overdamped regime**. In this situation  $\alpha_{\pm}$  are both real, so the general real solution is just given by

$$Q(t) = \mathcal{E}C + A_+e^{\alpha_+t} + A_-e^{\alpha_-t} \quad (16.17)$$

with  $A_{\pm}$  both real. Moreover both  $\alpha_+$  and  $\alpha_-$  are negative, so  $Q(t)$  rapidly approaches its limiting equilibrium value of  $\mathcal{E}C$  without any oscillation. In the spring analogy, this is the regime where the drag force is strong.

More interesting is the case

$$\frac{R^2}{4L^2} < \frac{1}{LC}, \quad (16.18)$$

which is called the **underdamped regime**. In this situation  $\alpha_{\pm}$  are complex, so we are looking at a solution of the form

$$\begin{aligned} Q(t) &= \mathcal{E}C + A_+e^{\alpha_+t} + A_-e^{\alpha_-t} \\ &= \mathcal{E}C + e^{-t/\tau} \left[ (A_+ + A_-) \cos \omega t + i(A_+ - A_-) \sin \omega t \right], \end{aligned} \quad (16.19)$$

with

$$\omega \equiv \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad (16.20)$$

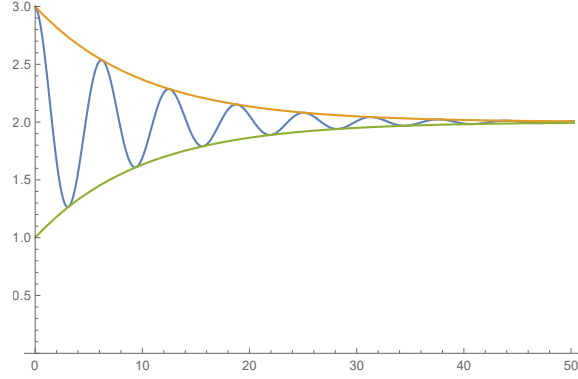


Figure 66: Damped oscillation: the blue curve shows  $Q(t)$  in the underdamped regime for some particular choice of  $R$ ,  $L$ ,  $C$ , and  $\mathcal{E}$ . The orange and green curves show the decaying exponential envelope of the oscillation. The “quality factor” for this particular oscillator is five.

and

$$\tau \equiv \frac{2L}{R}. \quad (16.21)$$

In order for this solution to be real at all times, the imaginary part of  $A_+ + A_-$  and the real part of  $A_+ - A_-$  must both vanish. In other words  $A_+$  and  $A_-$  must be complex conjugates. We can therefore rewrite the solution as

$$Q(t) = \mathcal{E}C + e^{-t/\tau} [A_c \cos \omega t + A_s \sin \omega t], \quad (16.22)$$

with  $A_c$  and  $A_s$  both real. This solution describes an oscillation about the equilibrium value of the charge, with an amplitude that decays exponentially in time (see figure 66). In the spring analogy, this is the situation where the drag force is weak.

In many applications, an oscillator which is “good” is one which goes through many oscillations before the amplitude decays appreciably. We can quantify the quality of a damped oscillator using something called the **quality factor**:

$$Q \equiv \frac{\omega\tau}{2}. \quad (16.23)$$

Tragically the same symbol is used for the quality factor and the electric charge, I can’t be held responsible. The factor of two in the definition is an arbitrary convention. An oscillator with large quality factor is one where the period of oscillation is small compared to the decay time  $\tau$ , roughly speaking the quality factor tells us how many “good” oscillations we get before the decay kicks in (see figure 66 for an example).

It is interesting to consider the limit  $R \rightarrow 0$ , in which case our RLC circuit just becomes an LC circuit. This sends  $\tau \rightarrow \infty$ , so the damping is removed and we get pure oscillation at frequency

$$\omega = \sqrt{\frac{1}{LC}}. \quad (16.24)$$

There is an elegant way to think about this oscillation. For simplicity let’s also set  $\mathcal{E} = 0$ , and consider the solution

$$Q(t) = Q_0 \cos \omega t. \quad (16.25)$$

The current is given by

$$I(t) = \dot{Q} = -Q_0\omega \sin \omega t. \quad (16.26)$$

Therefore the charge and the current are out of phase by 90 degrees, so one is big when the other is small. This is quite natural if you think about energy: when  $Q(t)$  is large most of the energy in the system is stored

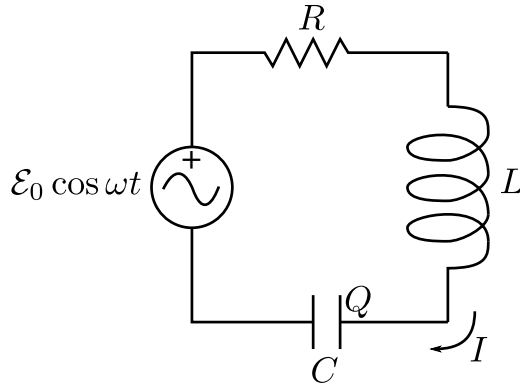


Figure 67: An RLC circuit with an AC power source. I've used a "+" to indicate which side of the source has higher potential at  $t = 0$ ; people often don't do this, but then the meaning of the symbol is ambiguous.

in the electric field inside the capacitor, while when  $I(t)$  is large most of the energy is stored in the magnetic field in the inductor. Once we turn the resistance back on we can still think of the oscillation arising from energy sloshing back and forth between the capacitor and the inductor, but it is now also gradually leaking out of the system through the resistor.

Finally for those of you who are completionists, let's briefly consider the case

$$\frac{R^2}{4L^2} = \frac{1}{LC}, \quad (16.27)$$

which is called **critical damping**. In this case  $\alpha_+ = \alpha_-$ , so we do not have two independent solutions and a new one must be found to account for all possible initial data. I will not go through the details, but you can check on your own that in this case the general real solution is

$$Q(t) = \mathcal{E}C + (A + Bt)e^{-\frac{R}{2L}t}, \quad (16.28)$$

where  $A$  and  $B$  are arbitrary real numbers.

## 17 AC sources and resonance

So far the current in all circuits we've considered has been driven by DC sources of EMF, with a battery being the standard example of such a source. We now turn to current sources that oscillate with time: current driven by such a source is called **alternating current** (AC), and the source is called an AC source. By far the most common kind of AC source to consider is a pure sinusoid, which produces an EMF

$$\mathcal{E} = \mathcal{E}_0 \cos \omega t. \quad (17.1)$$

An example of such a source is the electric generator we described a few lectures ago, which used a conducting loop rotating in a constant magnetic field to generate a sinusoidal EMF. We show an example of an RLC circuit with an AC power source in figure 67. We will spend this whole lecture studying the behavior of the circuit shown in figure 67.

### 17.1 General properties of the AC RLC circuit.

Let's first write out the loop equation for the circuit in figure 67:

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \mathcal{E}_0 \cos \omega t. \quad (17.2)$$

In the previous lecture we found the general solution of this equation in the special case of  $\mathcal{E}_0 = 0$ ,

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = 0, \quad (17.3)$$

which we saw was labeled by two real parameters corresponding to the initial charge and current. Moreover we saw that the solution decayed exponentially with time (both in the under- and over-damped regimes). Our analysis of this equation will be a little complicated, so it is important to first think about what we expect. One immediate observation is that if  $Q(t)$  is a solution of (17.2) and  $\hat{Q}(t)$  is a solution of (17.3), then  $Q(t) + \hat{Q}(t)$  is *also* a solution of (17.2). Let's check:

$$\begin{aligned} L(\ddot{Q} + \ddot{\hat{Q}}) + R(\dot{Q} + \dot{\hat{Q}}) + \frac{1}{C}(Q + \hat{Q}) &= \left( L\ddot{Q} + R\dot{Q} + \frac{Q}{C} \right) + \left( L\ddot{\hat{Q}} + R\dot{\hat{Q}} + \frac{\hat{Q}}{C} \right) \\ &= \mathcal{E}_0 \cos \omega t + 0 \\ &= \mathcal{E}_0 \cos \omega t. \end{aligned} \quad (17.4)$$

This tells us something very interesting: whatever initial conditions our solution of (17.2) needs to obey, we can always satisfy them just by adjusting the two real parameters in  $\hat{Q}$ . In other words any solution of (17.2) will consist of a unique **steady-state** solution plus a **transient** solution of (17.3), with the latter decaying exponentially in time. An RLC circuit driven by an AC source rapidly “forgets” its initial configuration, and settles down into a unique long-time behavior. Our goal in this lecture is to understand this long-time behavior.

## 17.2 Solving the AC RLC circuit

As is usual in this class, we will solve the equation (17.2) by guess and check. We are interested in its behavior after all transients have died away, so it is natural to guess that the steady-state solution is oscillating with frequency  $\omega$  and constant amplitude:

$$Q(t) = \frac{I_0}{\omega} \sin(\omega t + \delta). \quad (17.5)$$

The factor of  $\omega$  is conventional, and is chosen so that the current is

$$I(t) = \dot{Q}(t) = I_0 \cos(\omega t + \delta). \quad (17.6)$$

The quantity  $\delta$  is called the **phase shift**, and it tells us to what extent the phase of the current is ahead or behind the phase of the oscillating voltage source.

One way we could proceed from here is to substitute our guess (17.5) into the loop equation (17.2) and see if we can choose  $I_0$  and  $\delta$  to get a solution. This indeed works, but the algebra is unpleasant and generalizing to more complicated AC circuits is very painful. A much nicer approach is possible using complex exponentials, so we will instead do that. The idea is to begin by looking for *complex* solutions of the equation

$$L\ddot{\tilde{Q}} + R\dot{\tilde{Q}} + \frac{\tilde{Q}}{C} = \mathcal{E}_0 e^{i\omega t}. \quad (17.7)$$

This may seem like a silly thing to do, but given such a solution we may then recover a solution of (17.2) by taking the real part. Indeed note that by taking the complex conjugate, we have

$$L\ddot{\tilde{Q}^*} + R\dot{\tilde{Q}^*} + \frac{\tilde{Q}^*}{C} = \mathcal{E}_0 e^{-i\omega t} \quad (17.8)$$

Therefore

$$\Re \tilde{Q} \equiv \frac{\tilde{Q} + \tilde{Q}^*}{2} \quad (17.9)$$

obeys

$$L(\ddot{\tilde{Q}}) + R(\dot{\tilde{Q}}) + \frac{\mathfrak{R}\tilde{Q}}{C} = \mathcal{E}_0 \left( \frac{e^{i\omega t} + e^{-i\omega t}}{2} \right) = \mathcal{E}_0 \cos \omega t. \quad (17.10)$$

Our candidate solution for (17.7) will be

$$\tilde{Q}(t) = \frac{\tilde{I}_0}{i\omega} e^{i\omega t}, \quad (17.11)$$

with  $\tilde{I}_0$  now allowed to be complex. This leads to a complex current

$$\tilde{I}(t) = \tilde{I}_0 e^{i\omega t}. \quad (17.12)$$

In taking the real part, it is convenient to first do a polar decomposition of  $\tilde{I}_0$ :

$$\tilde{I}_0 = I_0 e^{i\delta}, \quad (17.13)$$

with  $I_0 > 0$  and  $\delta$  real. We then have

$$\tilde{I}(t) = I_0 e^{i(\omega t + \delta)}, \quad (17.14)$$

so the real part is

$$\Re \tilde{I}(t) = I_0 \cos(\omega t + \delta). \quad (17.15)$$

Therefore our guess (17.11) for the solution of (17.7) is just a repackaging of our guess (17.5) for (17.2). The amplitude and phase shift are now combined into a single complex amplitude  $\tilde{I}_0$ .

Now at last let's solve (17.7). Substituting in our guess (17.11), we find

$$\left( i\omega L + R + \frac{1}{i\omega C} \right) \tilde{I}_0 e^{i\omega t} = \mathcal{E}_0 e^{i\omega t}. \quad (17.16)$$

Therefore we will have a solution provided that we take

$$\tilde{I}_0 = \frac{\mathcal{E}_0}{i\omega L + R + \frac{1}{i\omega C}} = \frac{\mathcal{E}_0}{R + i\left(\omega L - \frac{1}{\omega C}\right)}. \quad (17.17)$$

Note that if we had taken the frequency of our guess to be anything other than  $\omega$ , the time-dependence on the two sides of the equation would not have canceled: at long times an AC circuit has no choice but to oscillate at the driving frequency, even if this is *not* the frequency that the RLC circuit would have liked to oscillate at on its own.

To extract the real amplitude  $I_0$  and phase shift  $\delta$ , we need to work out the polar decomposition of  $\tilde{I}_0$ . The amplitude is easy:

$$I_0 = \sqrt{|\tilde{I}_0|^2} = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \quad (17.18)$$

To get the phase, we can multiply top and bottom by  $R - i\left(\omega L - \frac{1}{\omega C}\right)$  to find

$$\tilde{I}_0 = \frac{\mathcal{E}_0(R - i\left(\omega L - \frac{1}{\omega C}\right))}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = I_0 (\cos \delta + i \sin \delta). \quad (17.19)$$

Therefore the tangent of  $\delta$  is equal to the ratio of the imaginary part to the real part of  $\tilde{I}_0$ :

$$\tan \delta = \frac{1}{\omega RC} - \frac{\omega L}{R}. \quad (17.20)$$

Thus at last our real solution for the current is given by

$$I(t) = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t + \delta), \quad (17.21)$$

with  $\delta$  determined by (17.20).

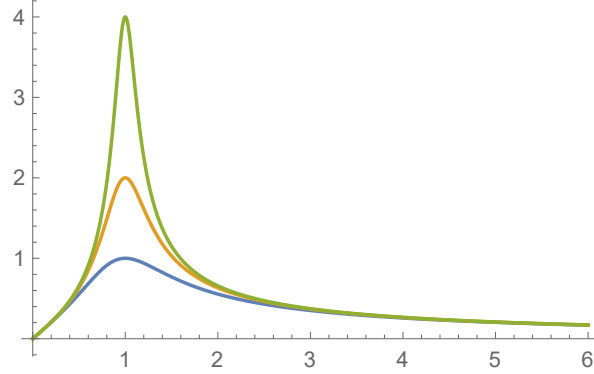


Figure 68: Current amplitude as a function of frequency as we vary the resistance  $R$  keeping  $L$  and  $C$  fixed: the resistance for the orange curve is half that of the blue curve, and the resistance for the green curve is half that of the orange curve.

### 17.3 Resonance

We now consider a few properties of our solution (17.21) for the current in an AC RLC circuit. Perhaps the most important is that the amplitude of the current depends on the frequency: if  $\omega$  is either very large or very small, then  $I_0$  goes to zero. In between there is a maximum when

$$\omega L - \frac{1}{\omega C} = 0, \quad (17.22)$$

or in other words when

$$\omega = \omega_{res} \equiv \frac{1}{\sqrt{LC}}. \quad (17.23)$$

This frequency may look familiar: it is the frequency at which the circuit would have oscillated in the absence of both the AC source and the resistance! In other words, although we can drive the circuit at any frequency  $\omega$  that we like, usually it doesn't enjoy it very much. On the other hand it really likes to be driven at the frequency  $\omega_{res}$ , and if we oblige it then we get a big response! This is the phenomenon of **resonance**, and it occurs in some form or other in many interesting systems. For example resonance is the mechanism by which you choose which radio station to listen to: the radio waves in the air interact with the antenna of your radio to create an oscillating voltage. By tuning the dial you are changing the parameters of a resonant circuit inside of your radio, and thus its resonant frequency. Your antenna is picking up all radio frequencies simultaneously, but the current in your radio is only responding substantially to the signal for the channel which happens to be broadcasting at the resonant frequency you've chosen. There are also mechanical versions of resonance, for example on a playground swing if you move your feet at the same frequency as the swing oscillation then the amplitude of oscillation increases.

In understanding resonance it is important to work out not just the resonant frequency  $\omega_{res}$ , but also the "width" of the resonant peak. Conventionally this is defined as the amount  $\Delta\omega$  we have to increase or decrease  $\omega$  away from  $\omega_{res}$  such that the amplitude of the current is smaller by a factor of  $\sqrt{2}$ . Since the peak amplitude is  $\frac{\mathcal{E}_0}{R}$ , we get an amplitude of  $\frac{\mathcal{E}_0}{\sqrt{2}R}$  when

$$\omega L - \frac{1}{\omega C} = \pm R. \quad (17.24)$$

We can rewrite this as a quadratic equation

$$L\omega^2 \mp R\omega - \frac{1}{C} = 0, \quad (17.25)$$



which has positive solutions

$$\omega_{\pm} = \pm \frac{R}{2L} + \frac{1}{2L} \sqrt{R^2 + \frac{4L}{C}} = \pm \frac{R}{2L} + \sqrt{\frac{1}{LC} + \frac{R^2}{4L^2}}. \quad (17.26)$$

Typically we are interested in the situation where the “quality factor”

$$Q = \frac{L}{R} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad (17.27)$$

of the RLC circuit is large, meaning that

$$\frac{1}{\sqrt{LC}} \gg \frac{R}{L} \quad (17.28)$$

and

$$Q \approx \frac{L}{R} \sqrt{\frac{1}{LC}} = \frac{L\omega_{res}}{R}, \quad (17.29)$$

in which case we can write the frequencies  $\omega_{\pm}$  where the amplitude is down by  $1/\sqrt{2}$  as

$$\omega_{\pm} \approx \omega_{res} \pm \frac{R}{2L}. \quad (17.30)$$

Thus in this limit the width is given by

$$\frac{\Delta\omega}{\omega_{res}} \approx \frac{R}{2L\omega_{res}} \approx \frac{1}{2Q}. \quad (17.31)$$

Thus for a resonant circuit of high quality factor, the resonant peak is quite narrow. We illustrate this for some particular choices of  $L$ ,  $R$ , and  $C$  in figure 68.

## 17.4 Phase shift

Finally let’s discuss a bit the interpretation of the phase shift  $\delta$ . We can rewrite equation (17.20) as

$$\tan \delta = \frac{L}{\omega R} \left( \frac{1}{LC} - \omega^2 \right) = \frac{L}{\omega R} (\omega_{res}^2 - \omega^2), \quad (17.32)$$

so right at the resonant frequency the current and the voltage are perfectly in phase. Away from the resonance it is useful to consider the limit of large quality factor, in which case we have

$$\tan \delta \approx Q \frac{\omega_{res}^2 - \omega^2}{\omega\omega_{res}}. \quad (17.33)$$

Thus in this limit the phase shift is given by

$$\delta \approx \begin{cases} \frac{\pi}{2} & \omega < \omega_{res} \\ -\frac{\pi}{2} & \omega > \omega_{res} \end{cases}. \quad (17.34)$$

Therefore at large quality factor the current “leads the voltage by 90 degrees” at low frequencies and “lags the voltage by 90 degrees” at high frequencies.

## 18 Impedance and Power in AC circuits

So far we have considered only AC circuits with a single loop. In most real AC circuits there is more than one loop, so we now turn to discussing how to deal with more complicated AC circuits. As with DC circuits, in studying multi-loop circuits it is useful to adopt an algorithmic approach. The principles are the same as for Kirchoff's laws:

- **Charge conservation:** At each wire junction, the current in must equal the current out.
- **Faraday's law:** Around each loop  $L$ , we must have

$$\int_L \vec{E} \cdot d\vec{s} = -\dot{\Phi}_L, \quad (18.1)$$

where  $\Phi_L$  is the magnetic flux through the loop.

As in our earlier discussion of the circuit model, a key point is that both of these conditions are *linear* in the currents through the various wires. In particular this means that if we can find a set of complex currents which satisfy these two requirements, we can find a real solution just by taking the real part. In formulating the rules we will allow for more than one AC voltage source, but we will insist that all AC sources are oscillating at the same frequency  $\omega$ .<sup>21</sup> We will allow these different sources to oscillate with different phases, which we can implement by allowing them to have complex amplitudes  $\tilde{\mathcal{E}}_0$ : each AC source then contributes  $\tilde{\mathcal{E}}_0 e^{i\omega t}$  to the loop equation, with the amplitude  $\tilde{\mathcal{E}}_0$  typically differing from source to source. We thus have the following algorithm:

- (1) Assign a direction to each wire in the circuit, and label the complex current through each wire in that direction as  $\tilde{I}_i = \tilde{I}_{0,i} e^{i\omega t}$  (here  $i$  labels the wires).  $\tilde{I}_{0,i}$  is the complex amplitude of the current through the  $i$ th wire.
- (2) For each  $Y$ -junction in the circuit, write the equation that the sum of the complex current amplitudes  $\tilde{I}_{0,i}$  going in equals the sum of complex current amplitudes going out.
- (3) For each loop, write the equation that the line integral of  $\vec{E}$  around the loop equals minus the time derivative of the flux. More concretely, require that the sum of  $R\tilde{I}_0$  for each resistor,  $i\omega L\tilde{I}_0$  for each inductor,  $\frac{1}{i\omega C}\tilde{I}_0$  for each capacitor, and  $-\tilde{\mathcal{E}}_0$  for each AC source vanishes.
- (4) Solve the equations from (2)-(3) to determine the current amplitudes  $\tilde{I}_{0,i}$ , and then take the real parts of the complex currents  $\tilde{I}_i(t) = \tilde{I}_{0,i} e^{i\omega t}$  to find the actual currents  $I_i(t)$ .

To illustrate this, let's consider the circuit shown in figure 69. Either of the  $Y$  junctions gives us the relation

$$\tilde{I}_{0,1} = \tilde{I}_{0,2} + \tilde{I}_{0,3}, \quad (18.2)$$

the loop on the left gives us

$$R_1 \tilde{I}_{0,1} + \frac{1}{i\omega C} \tilde{I}_{0,3} - \tilde{\mathcal{E}}_{0,1} = 0, \quad (18.3)$$

and the loop on the right gives us

$$R_2 \tilde{I}_{0,2} + i\omega L \tilde{I}_{0,2} - \frac{1}{i\omega C} \tilde{I}_{0,3} - \tilde{\mathcal{E}}_{0,2} = 0. \quad (18.4)$$

We can view these equations as three linear equations for three unknown complex current amplitudes, so we can solve them to find the complex current amplitudes. Already in writing these equations you can see

<sup>21</sup>This is more a convenience than a necessity: the rules determining the currents are linear, so if there are sources with more than one frequency then the current will just be a superposition of the steady state solutions produced by considering the different frequencies separately. This will *not* be true for the power however, as this involves the square of the current.

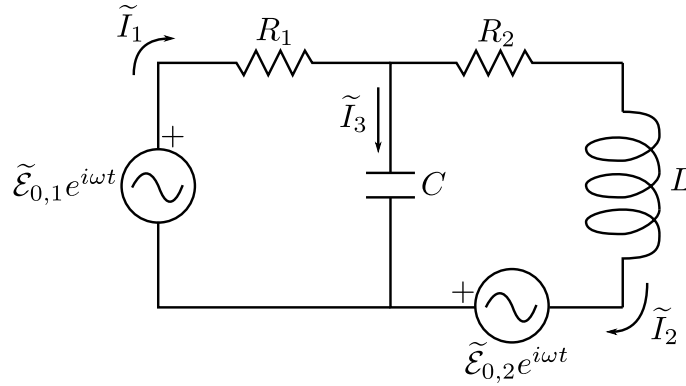


Figure 69: A two-loop AC circuit. Note that we have used a + to indicate which of the terminals of each AC source is positive when the phase of  $\tilde{\mathcal{E}}_0 e^{i\omega t}$  vanishes: this is not part of standard circuit notation, but it should be (otherwise you can get the wrong answer for circuits with multiple AC sources).

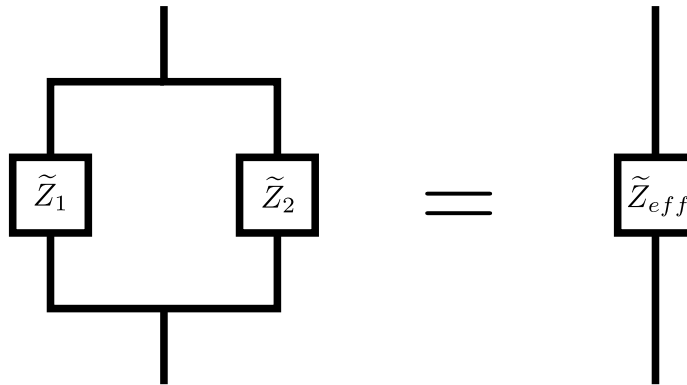


Figure 70: Combining impedance in parallel.

that there is an irresistible temptation to drop the 0 everywhere and conflate the symbols  $\tilde{I}_{0,i}$  and  $\tilde{\mathcal{E}}_{0,i}$  for the current/source amplitudes with the symbols  $\tilde{I}_i$   $\tilde{\mathcal{E}}_i$  for the time-dependent complex currents/sources. You have my permission to do this, but only if you remember that to recover the actual currents and sources you need to restore the  $e^{i\omega t}$  before taking the real part!

As with DC circuits, there is a convenient relabeling of the current variables which automatically implements the  $Y$ -junction equations. This is to assign currents to loops rather than wires, and then at each circuit element add the contributions to the current from each loop. For the circuit in figure 69, we can redesignate the symbol  $\tilde{I}_1$  to be the (clockwise) complex current through the loop on the left and  $\tilde{I}_2$  to be the (clockwise) complex current through the loop on the right. Adopting our new convention of dropping the 0, we then have the two loop equations:

$$\begin{aligned}
 R_1 \tilde{I}_1 + \frac{1}{i\omega C} (\tilde{I}_1 - \tilde{I}_2) - \tilde{\mathcal{E}}_1 &= \\
 R_2 \tilde{I}_2 + i\omega L \tilde{I}_2 + \frac{1}{i\omega C} (\tilde{I}_2 - \tilde{I}_1) - \tilde{\mathcal{E}}_2 &= 0
 \end{aligned}
 \tag{18.5}$$

## 18.1 Impedance/Admittance

You may have noticed that in these equations it seems like the distinction between resistors, capacitors, and inductors has almost disappeared. Each just contributes a term consisting of some complex number times the current amplitude to the loop equation, with this number being  $R$  for resistor,  $i\omega L$  for an inductor, and  $\frac{1}{i\omega C}$  for a capacitor. It is conventional to give this number a name: we call it the **impedance**,  $\tilde{Z}$ , of the circuit element. The usefulness of impedance is that it allows us to apply intuition from ordinary resistors in DC circuits to AC circuits. For example the impedance of circuit elements in series combines additively, just as resistance does in DC circuits. So in particular for our old single-loop AC circuit, we can now write the loop equation quite concisely:

$$\mathcal{E}_0 = \left( R + i\omega L + \frac{1}{i\omega C} \right) \tilde{I}_0 = \tilde{Z}\tilde{I}_0. \quad (18.6)$$

You can think of this as an AC version of Ohm's law, where impedance plays the role of resistance, the AC amplitude  $\mathcal{E}_0$  plays the role of the voltage, and the complex current amplitude  $\tilde{I}_0$  plays the role of the current. Impedance also combines in parallel in the same way as resistance does (see figure 70),

$$\tilde{Z}_{eff} = \frac{\tilde{Z}_1\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}, \quad (18.7)$$

as you are welcome to work out for yourself using the loop equation.

In the general spirit of coming up with useless additional definitions, sometimes people like to call the inverse of the impedance the **admittance**:

$$\tilde{Y} \equiv \frac{1}{\tilde{Z}}. \quad (18.8)$$

## 18.2 Power in AC circuits

Let's now consider the role of energy in AC circuits. There are two basic effects: energy is delivered to the circuit by the AC sources, and it is dissipated at the resistors. In the steady-state behavior of the circuit these effects must match each other when averaged over an oscillation period, since otherwise the energy in the circuit would be increasing or decreasing with time. The most important thing to remember about power in AC circuits is that it is *quadratic* in the current (and/or voltage). This means that we must take the real part of the current (and/or voltage) *before* we compute the power, or we may get the wrong answer. For example let's consider the energy being delivered by an AC EMF source

$$\mathcal{E}(t) = \mathcal{E}_0 \cos \omega t \quad (18.9)$$

to a circuit such that the current through the EMF source is

$$I(t) = I_0 \cos(\omega t + \delta). \quad (18.10)$$

The power delivery as a function of time is thus

$$P(t) = \mathcal{E}(t)I(t) = \mathcal{E}_0 I_0 \cos(\omega t + \delta) \cos \omega t. \quad (18.11)$$

Had we tried first multiplying  $\tilde{\mathcal{E}}(t)$  and  $\tilde{I}(t)$  and then taking the real part, we'd have ended up with

$$P(t) \text{ " = " } \Re \left( \mathcal{E}_0 I_0 e^{i(2\omega t + \delta)} \right) = \mathcal{E}_0 I_0 \cos(2\omega t + \delta), \quad (18.12)$$

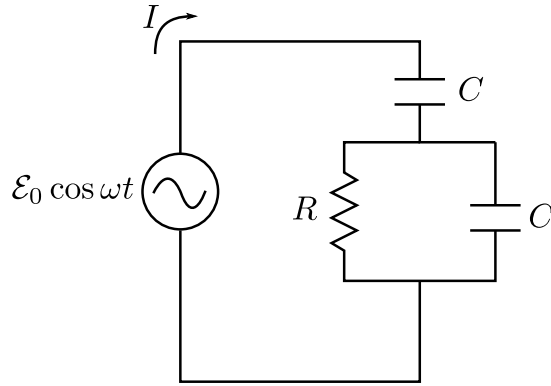


Figure 71: An example AC circuit.

which is not the same as (18.11). The distinction is especially clear if we look at the average power delivered over a period of oscillation:

$$\begin{aligned}
 P_{ave} &\equiv \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} dt P(t) \\
 &= \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} dt \mathcal{E}_0 I_0 (\cos(\omega t) \cos \delta - \sin(\omega t) \sin \delta) \cos \omega t \\
 &= \frac{1}{2} \mathcal{E}_0 I_0 \cos \delta.
 \end{aligned} \tag{18.13}$$

This is typically not zero, while the time average of (18.12) is always zero. Equation (18.13) tells us that the power delivery depends not just on the real amplitudes of the current and EMF, but also on the phase shift.

Similarly we can also look at the energy dissipated by a resistor through which an alternating current

$$I(t) = I_0 \cos(\omega t + \delta) \tag{18.14}$$

is passing. The power dissipation as a function of time is

$$P(t) = I(t)^2 R = I_0^2 \cos^2(\omega t + \delta), \tag{18.15}$$

and the average power dissipated over a period of oscillation is

$$P_{ave} \equiv \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} dt P(t) = \frac{I_0^2 R}{2}. \tag{18.16}$$

If we add up the average power deliveries (18.13) from each AC source we should get the same thing as if we add up the average power dissipations (18.16) from each resistor.

As an illustration, let's consider the circuit shown in figure 71. We can combine the capacitors and the

resistor into a single effective impedance:

$$\begin{aligned}
\tilde{Z} &= \frac{1}{i\omega C} + \frac{\frac{R}{i\omega C}}{R + \frac{1}{i\omega C}} \\
&= \frac{1}{i\omega C} + \frac{R}{1 + i\omega CR} \\
&= \frac{2\omega CR - i}{\omega C(1 + i\omega CR)} \\
&= \frac{\omega CR - i(1 + 2R^2\omega^2 C^2)}{\omega C(1 + \omega^2 C^2 R^2)}.
\end{aligned} \tag{18.17}$$

We can parametrize this as

$$\tilde{Z} = Z_0 e^{-i\delta}, \tag{18.18}$$

with

$$Z_0 = \sqrt{\frac{1 + 4\omega^2 C^2 R^2}{\omega^2 C^2 (1 + \omega^2 C^2 R^2)}} \tag{18.19}$$

and

$$\tan \delta = \frac{1 + 2\omega^2 C^2 R^2}{\omega CR}, \tag{18.20}$$

so using the loop equation

$$\mathcal{E}_0 = \tilde{Z}\tilde{I} \tag{18.21}$$

we see that the current across the AC source is

$$I(t) = \frac{\mathcal{E}_0}{Z_0} \cos(\omega t + \delta). \tag{18.22}$$

Therefore the average power delivery is

$$P_{ave} = \frac{\mathcal{E}_0^2}{2Z_0} \cos \delta, \tag{18.23}$$

with  $Z_0$  and  $\delta$  taken from (18.19) and (18.20).

We can also think about the power dissipated by the resistor. If we denote the complex current amplitude through the resistor as  $\tilde{I}_R$ , then the loop equation around the loop containing the resistor and the capacitor gives

$$\tilde{I}_R R = \frac{1}{i\omega C} (\tilde{I} - \tilde{I}_R). \tag{18.24}$$

Solving for  $\tilde{I}_R$ , we find

$$\tilde{I}_R = \frac{\tilde{I}}{(1 + i\omega CR)}. \tag{18.25}$$

so the average power dissipation is

$$P_{ave} = \frac{R|\tilde{I}_R|^2}{2} = \frac{R\mathcal{E}_0^2}{2(1 + \omega^2 C^2 R^2)Z_0^2}. \tag{18.26}$$

This will agree with (18.23) if we can show that

$$\frac{R}{Z_0(1 + \omega^2 C^2 R^2)} = \cos \delta. \tag{18.27}$$

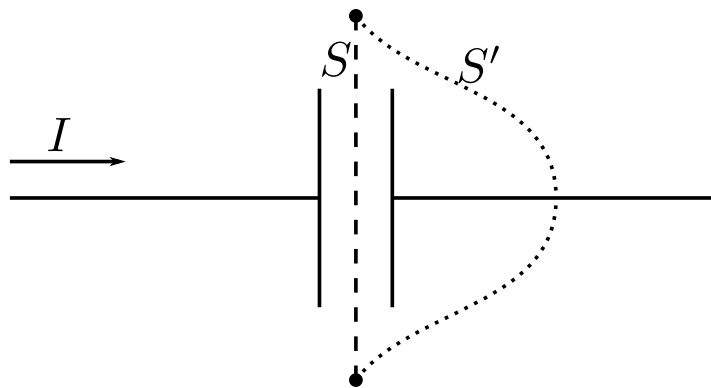


Figure 72: Ampere's law near a charging capacitor. The two dots represent a circle  $C$  surrounding the capacitor, which we can view either as the boundary of a disk  $S$  which passes between the plates of the capacitor or a disk  $S'$  which is punctured by the outgoing wire. We choose the orientation of  $C$  so that the normal vectors of  $S$  and  $S'$  point to the right.

This is indeed true:

$$\cos \delta = \frac{1}{\sqrt{1 + \tan^2 \delta}} \quad (18.28)$$

$$= \sqrt{\frac{1}{1 + \frac{(1+2\omega^2 C^2 R^2)^2}{\omega^2 C^2 R^2}}} \quad (18.29)$$

$$= \omega C R \sqrt{\frac{1}{(1 + 4\omega^2 C^2 R^2)(1 + \omega^2 C^2 R^2)}} \quad (18.30)$$

$$= \frac{R}{(1 + \omega^2 C^2 R^2)} \sqrt{\frac{\omega^2 C^2 (1 + \omega^2 C^2 R^2)}{1 + 4\omega^2 C^2 R^2}} \quad (18.31)$$

$$= \frac{R}{Z_0(1 + \omega^2 C^2 R^2)}. \quad (18.32)$$

Thus as expected the average power delivery equals the average power dissipation!

## 19 Displacement current and Maxwell's equations

So far our discussion of the magnetic field has been restricted to steady currents, obeying  $\vec{\nabla} \cdot \vec{J} = 0$ . Indeed given Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \quad (19.1)$$

by taking the divergence of both sides we immediately see that  $\vec{\nabla} \cdot \vec{J} = 0$ . The continuity equation then tells us we must have

$$\dot{\rho} = -\vec{\nabla} \cdot \vec{J} = 0, \quad (19.2)$$

so we've been restricted to situations where the charge density is independent of time.

It is interesting to see how Ampere's law can fail when we relax the assumption of steady currents. For example consider the charging of a capacitor by a constant current  $I$ , as shown in figure 72. Let's try to compute the line integral of the magnetic field around the circle  $C$  using Ampere's law. We need to pick a surface whose boundary is  $C$ , two options are shown in figure 72. If we use surface  $S$ , we find

$$\int_C \vec{B} \cdot d\vec{s} = \mu_0 \int_S \vec{J} \cdot d\vec{A} = 0. \quad (19.3)$$

On the other hand if we use the surface  $S'$ , we find

$$\int_C \vec{B} \cdot d\vec{s} = \mu_0 \int_{S'} \vec{J} \cdot d\vec{A} = \mu_0 I. \quad (19.4)$$

Clearly at least one of these results must be wrong, and therefore Ampere's law must be wrong as well!

Let's see if we can guess how to modify Ampere's law when currents are not steady. We'll propose a modification of the form

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \vec{C}, \quad (19.5)$$

where  $\vec{C}$  is a quantity we'll try to determine. Let's try taking the divergence:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \vec{C} = 0, \quad (19.6)$$

so applying the continuity equation we apparently need

$$\vec{\nabla} \cdot \vec{C} = -\mu_0 \vec{\nabla} \cdot \vec{J} = \mu_0 \dot{\rho}. \quad (19.7)$$

We can rewrite this using Gauss's law to find

$$\vec{\nabla} \cdot \vec{C} = \mu_0 \epsilon_0 \vec{\nabla} \cdot \dot{\vec{E}}, \quad (19.8)$$

which we see will be satisfied if we take

$$\vec{C} = \mu_0 \epsilon_0 \dot{\vec{E}}. \quad (19.9)$$

Therefore we are led to guess, as Maxwell did in 1861, that Ampere's law should be modified to

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \dot{\vec{E}}). \quad (19.10)$$

The quantity

$$\vec{J}_d \equiv \epsilon_0 \dot{\vec{E}} \quad (19.11)$$

is called **Maxwell's displacement current**, since it sources the magnetic field in the same way that  $\vec{J}$  does. It is important to emphasize that this argument for (19.10) is *not* a logical derivation. For one thing we assumed the continuity equation, and for another we could have added to  $\vec{C}$  any vector field with vanishing divergence without affecting things. As with the  $-\vec{B}$  term in Faraday's law, Maxwell's modification of Ampere's law is ultimately a theoretical hypothesis that needed to be tested experimentally to confirm its validity.

Let's now see how Maxwell's displacement current resolves our paradox with the charging capacitor. Integrating (19.10), we find that the new integrated version of Ampere's law is

$$\begin{aligned} \int_{\partial S} \vec{B} \cdot d\vec{s} &= \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} \\ &= \int_S (\mu_0 \vec{J} + \mu_0 \epsilon_0 \dot{\vec{E}}) \cdot d\vec{A} \\ &= \mu_0 I[S] + \mu_0 \epsilon_0 \dot{\Phi}_E[S], \end{aligned} \quad (19.12)$$

where  $I[S]$  and  $\Phi_E[S]$  are the current and electric flux through any surface  $S$ . In other words in addition to a contribution from the current, there is now also a Faraday-like contribution from the time derivative of the *electric* flux through  $S$ . Applying this now to the surfaces  $S$  and  $S'$  in figure 72, we first note that as the current flows, positive charge accumulates on the left plate and negative charge accumulates on the right plate. This leads to an electric field between the plates which is pointing to the right and growing in time. On  $S'$  this electric field is small however (zero in the limit of zero separation between the plates), so the line integral of  $\vec{B}$  around the circle  $\partial S$  must still be  $\mu_0 I$ . On the other hand on  $S$  there is no  $\vec{J}$ , but we



now have a time-dependent electric field. Let's assume that the plates are very close together, so that the field inside is

$$\vec{E} = \frac{\sigma(t)}{\epsilon_0} \hat{x} = \frac{Q(t)}{A\epsilon_0} \hat{x}, \quad (19.13)$$

where  $\sigma$  is the charge density on the left plate,  $A$  is the area of the plates,  $Q$  is the total charge on the left plate, and we've chosen the current to flow in the  $x$  direction. Apparently the time derivative of the electric field is

$$\dot{\vec{E}} = \frac{\dot{Q}}{A\epsilon_0} \hat{x} = \frac{I}{A\epsilon_0} \hat{x}, \quad (19.14)$$

so we have

$$\int_C \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \int_S \dot{\vec{E}} \cdot d\vec{A} = \mu_0 I, \quad (19.15)$$

consistent with the answer we got integrating on  $S'$ .

## 19.1 Maxwell's equations

We've now reached a milestone in this class: we can write down **Maxwell's equations**:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad (19.16)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (19.17)$$

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}} \quad (19.18)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \dot{\vec{E}}). \quad (19.19)$$

These are the fundamental equations which, together with the Lorentz force law

$$\vec{F} = q (\vec{E} + v \times \vec{B}), \quad (19.20)$$

describe all electromagnetic phenomena in our world. This is true even after we introduce quantum mechanics, although the meanings of the symbols change a bit. As with any laws of physics, future experiments may eventually reveal deviations from Maxwell's equations. For example one day we might find magnetic monopoles. So far however no deviation has been seen, despite a few hundred years of looking!

A very interesting special case of Maxwell's equations is obtained by setting  $\rho = 0$  and  $\vec{J} = 0$ ,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad (19.21)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (19.22)$$

$$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}} \quad (19.23)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \dot{\vec{E}}, \quad (19.24)$$

which are sometimes called **Maxwell's equations in vacuum**. With the inclusion of the displacement current term in Ampere's law, these equations have a nice symmetry: they are invariant under the replacements<sup>22</sup>

$$\begin{aligned} \vec{E} &\rightarrow \frac{1}{\sqrt{\epsilon_0 \mu_0}} \vec{B} \\ \vec{B} &\rightarrow -\sqrt{\epsilon_0 \mu_0} \vec{E}. \end{aligned} \quad (19.25)$$

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<sup>22</sup>The annoying factors of  $\sqrt{\epsilon_0 \mu_0}$  in these replacements arise from the tragic fact that in SI electric and magnetic fields do not have the same units.

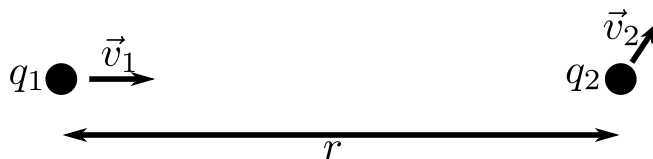


Figure 73: Two moving charges, at separation  $r$ .

This invariance is called **electric-magnetic duality**, and it says that in the absence of electric charges there isn't really any difference between electric and magnetic fields. Electric-magnetic duality has many interesting applications in high-energy and condensed-matter physics.

Perhaps the most interesting feature of Maxwell's equations in vacuum is that the displacement current allows for them to have nontrivial solutions. Let's return for a moment to our old-fashioned Ampere's law (19.1). If we set  $\vec{J} = 0$ , the Biot-Savart law then immediately tells us that we must also have  $\vec{B} = 0$ . If we also set  $\rho = 0$ , then Gauss's law and Faraday's law, together with the Helmholtz theorem, tell us that we must also have  $\vec{E} = 0$ . Therefore without the displacement current, the vacuum Maxwell equations have only boring solutions. This is why our discussion of electric and magnetic fields so far has always assumed the existence of some kind of charge! We will soon see however that the vacuum Maxwell equations *including* the displacement current have very interesting solutions: electromagnetic waves. The displacement current thus allows electric and magnetic fields to develop lives of their own, liberating them from their attachments to charged particles.

## 19.2 When is the displacement current important?

You may be wondering how we have been able to get away with neglecting the displacement current so far in this class. For example inside of the resistors in our time-dependent circuits, by Ohm's law  $\vec{J} = \sigma \vec{E}$  there are certainly time-dependent electric fields; why didn't we need to include the magnetic field induced by this in our consideration of the behavior of the circuit? And further back, in our experiments with Van der Graaf generators we were moving various charges around; this also produces a time-dependent electric field. Why didn't we need to think about the magnetic field which was produced? The answers to these questions are quite interesting, we'll now say a bit about them.<sup>23</sup>

Let's first consider the effect of the displacement current term in Maxwell's equations on the force between two charge particles. We will compute the force on a charge  $q_2$  moving with velocity  $\vec{v}_2$  due to a charge  $q_1$  moving with velocity  $\vec{v}_1$ . To maximize the change in electric field at the location of  $q_2$ , we'll take  $\vec{v}_1$  to be pointing direction at  $q_2$  (see figure 73). The coulomb electric field from charge  $q_1$  at the location of  $q_2$  is

$$\vec{E} = \frac{q_1}{4\pi\epsilon_0 r^2} \hat{x}, \quad (19.26)$$

so the time derivative is

$$\dot{\vec{E}} = -\frac{q_1 \dot{r}}{4\pi\epsilon_0 r^3} \hat{x} = \frac{q_1 |v_1|}{4\pi\epsilon_0 r^3} \hat{x}. \quad (19.27)$$

There is no current density at the location of  $q_2$ , so apparently we must have

$$\vec{\nabla} \times \vec{B} = \frac{q_1 |v_1| \mu_0}{4\pi r^3} \hat{x} \quad (19.28)$$

there. We will work out the explicit form of  $\vec{B}$  for a moving point charge in the last lecture of the semester, but here it is enough to do a crude estimate: since the magnetic field near  $q_2$  should only change appreciably

<sup>23</sup>The arguments here are qualitative, they are typical of the kinds of arguments physicists use in deciding when effects are important to consider but you may be uncomfortable with the relatively low level of rigor. I won't ask you to produce such arguments on an exam, but if you want to be a physicist you need to learn how to make them eventually!

if we change location by a distance of order  $r$ , we can estimate that near  $q_2$  we have

$$|\vec{\nabla} \times \vec{B}| \sim |B|/r. \quad (19.29)$$

Therefore we have

$$|B| \sim \frac{q_1 |v_1| \mu_0}{r^2}. \quad (19.30)$$

Apparently the magnitude of the magnetic force on  $q_2$  is then of order

$$|F_{mag}| \sim \frac{q_1 q_2 |v_1| |v_2| \mu_0}{r^2}. \quad (19.31)$$

We can compare this to the Coulomb force between the charges,

$$|F_{coulomb}| = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}, \quad (19.32)$$

apparently

$$|F_{mag}| \sim \mu_0 \epsilon_0 |v_1| |v_2| |F_{coulomb}| = \frac{|v_1| |v_2|}{c^2} |F_{coulomb}|. \quad (19.33)$$

Thus as long as the charges are not moving close to the speed of light, the magnetic force between the two charges will be much smaller than the Coulomb force.

From the analysis of the previous paragraph, you might wonder why we *ever* consider the magnetic force. The reason is that we can arrange for situations (such as neutral steady currents) where the Coulomb force is canceled but the magnetic force is not. In such situations (which we often encountered in circuits), why then were we able to get away with ignoring the displacement current? Or in other words, why didn't Faraday discover the displacement current while doing his experiments with circuits? In fact as long we restrict to circuits where the time it takes for the current to change appreciably is large compared to the time it takes for light to move from one side of the circuit to the other, ignoring the displacement current is a good approximation. We'll now sketch how to justify this. We first note that our derivation of the vector potential and magnetic field due to a current density  $\vec{J}$  goes through also in the presence of displacement current:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int dV' \frac{\vec{J}(\vec{r}') + \vec{J}_d(\vec{r}')}{|\vec{r} - \vec{r}'|}. \quad (19.34)$$

We'd like to argue that the contribution to this vector potential from the displacement current  $\vec{J}_d$  is small unless the current is changing relativistically. We can always write the displacement current as

$$\vec{J}_d = \vec{\nabla} f + \vec{V}, \quad (19.35)$$

with  $\vec{\nabla} \cdot \vec{V} = 0$ . Indeed we can take  $f$  to be the solution of

$$\vec{\nabla}^2 f = \vec{\nabla} \cdot \vec{J}_d \quad (19.36)$$

which falls off at infinity, and then define  $\vec{V} = \vec{J}_d - \vec{\nabla} f$ . Now let's compute the curl of the displacement current:

$$\vec{\nabla} \times \vec{J}_d = \vec{\nabla} \times \vec{V} = \epsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E} = -\epsilon_0 \ddot{\vec{B}}. \quad (19.37)$$

If the magnetic field is only varying slowly in time, the right-hand side will be small so we have  $\vec{\nabla} \times \vec{V} \approx 0$ . Since  $\vec{\nabla} \cdot \vec{V} = 0$  as well, by the Helmholtz theorem we have  $\vec{V} \approx 0$ . More carefully, if  $T$  is the time over which the current is changing and  $R$  is the size of the circuit, we expect

$$|V| \approx \frac{\epsilon_0 R}{T^2} |B|. \quad (19.38)$$

Now let's compute the contribution of  $\vec{J}_d$  to  $\vec{A}$ . Since  $\vec{V}$  is small, we expect the leading contribution of the displacement current to  $\vec{A}$  to be

$$\vec{A} \supset \frac{\mu_0}{4\pi} \int dV' \frac{\vec{\nabla}' f(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{\mu_0}{4\pi} \int dV' \left[ \vec{\nabla}' \left( \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - f(\vec{r}') \vec{\nabla}' \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \right]. \quad (19.39)$$

The first term is a total derivative which vanishes since  $f \rightarrow 0$  at  $\infty$ , and we can simplify the second using that

$$\vec{\nabla}' \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -\vec{\nabla} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right), \quad (19.40)$$

where note that we have changed which coordinate the gradient acts on. We then have

$$\vec{A} \supset \frac{\mu_0}{4\pi} \int dV' f(\vec{r}') \vec{\nabla} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = \vec{\nabla} \left( \frac{\mu_0}{4\pi} \int dV' \frac{f(\vec{r}')}{|\vec{r} - \vec{r}'|} \right). \quad (19.41)$$

Note however that since this is a gradient with respect to  $\vec{r}$ , it will vanish when we take the curl of  $\vec{A}(\vec{r})$  to find the magnetic field  $\vec{B}$ . Thus the  $\vec{\nabla}' f$  term in  $\vec{J}_d$  makes no net contribution to the magnetic field! Thus we are left with the smaller  $\vec{V}$  term. This would contribute

$$\vec{B} \supset \frac{\mu_0}{4\pi} \int dV' \frac{\vec{V}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}, \quad (19.42)$$

which we can estimate using (19.38) as

$$\frac{\mu_0 \epsilon_0 R^2}{T^2} |B| = \frac{R^2}{T^2 c^2} |B|, \quad (19.43)$$

where  $|B|$  is the magnitude of the magnetic field produced by the ordinary current density.  $R/c$  is precisely the time it takes for light to cross the circuit, so we therefore want to assume that this time is small compared to the time  $T$  over which the current is changing:

$$\frac{R}{Tc} \ll 1. \quad (19.44)$$

Thus the contribution to the magnetic field from the displacement current is quite small compared to the contribution from the ordinary current density! It is only in relativistic situations (such as electromagnetic waves) where the displacement current gives a leading order contribution to the magnetic field.

## 20 Waves

We've now written down the final version of Maxwell's equations. In the next lecture we will show that these equations have very interesting solutions: electromagnetic waves. These solutions are a bit complicated, so in this lecture we will get ready for them by studying waves in simpler systems. Our discussion here will only be a taste, to get the full treatment you need to take 8.03!

### 20.1 Displacement of a string under tension

We'll begin with the problem of the motion of a stretched string of tension  $T$  and linear mass density  $\mu$ . We'll take the string to be infinitely long, and to be stretched along the  $x$  axis from  $x = -\infty$  to  $x = \infty$ . We will consider only deformations of the string which stay in the  $xy$  plane, and we will further assume that the deviation of the string away from the  $x$  axis is small enough that the angle between each piece of the string and the  $x$  direction is small. We can then parametrize the configuration of the string by a function  $y(x)$ .

We can understand the motion of this string by writing out Newton's second law for each infinitesimal piece of it. This is easiest if we discretize the string as a bunch of point masses of mass  $\mu\epsilon$  equally spaced

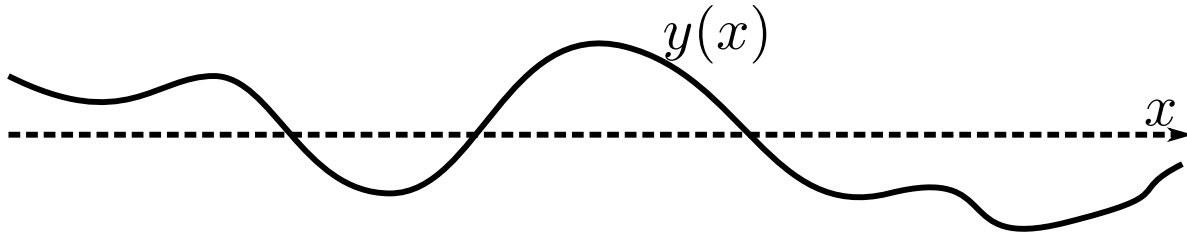


Figure 74: The displacement of a string about the  $x$ -axis, parameterized by a function  $y(x)$ .

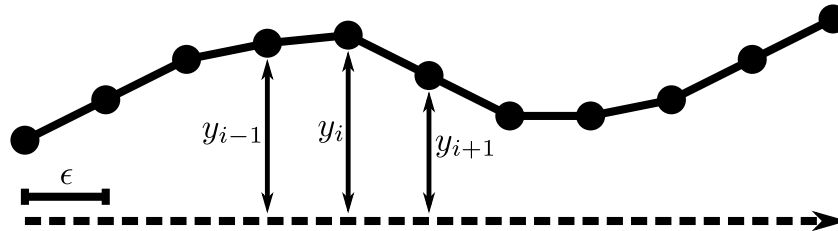


Figure 75: Discretizing a string into a bunch of point masses connected by massless ropes.

in the  $x$  direction with separation  $\epsilon$  and connected by massless ropes of tension  $T$ . We then take  $\epsilon \rightarrow 0$  to recover the continuum string. We can label the masses by an index  $i$ . Let's compute the force on the  $i$ th mass. This arises from the tension in the ropes attaching it to the  $i - 1$ st and  $i + 1$ st masses, as in figure 75. The net force (in the  $y$  direction) on the  $i$ th mass is

$$F_y = T \left( \frac{y_{i+1} - y_i}{\epsilon} + \frac{y_{i-1} - y_i}{\epsilon} \right) = \epsilon T \left( \frac{y_{i+1} - y_i}{\epsilon^2} - \frac{y_i - y_{i-1}}{\epsilon^2} \right), \quad (20.1)$$

where we have used that for small displacements the sine of the angle  $\theta_j$  off of horizontal of the string segment from mass  $j$  to mass  $j + 1$  is

$$\sin \theta_j \approx \tan \theta_j \approx \frac{y_{j+1} - y_j}{\epsilon}. \quad (20.2)$$

We may then write Newton's second law:

$$\mu \epsilon \frac{\partial^2 y_i}{\partial t^2} = \epsilon T \left( \frac{y_{i+1} - y_i}{\epsilon^2} - \frac{y_i - y_{i-1}}{\epsilon^2} \right). \quad (20.3)$$

In the limit that  $\epsilon \rightarrow 0$ , this becomes

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}. \quad (20.4)$$

We can rewrite this using our abbreviated notation for partial derivatives as

$$\partial_x^2 y = \frac{1}{v^2} \ddot{y}, \quad (20.5)$$

where

$$v \equiv \sqrt{\frac{T}{\mu}}. \quad (20.6)$$

Equation (20.5) is the one-dimensional version of one the best equations in physics: the **wave equation**. This is a linear second-order partial differential equation, and understanding its solutions is our goal for the rest of this lecture.

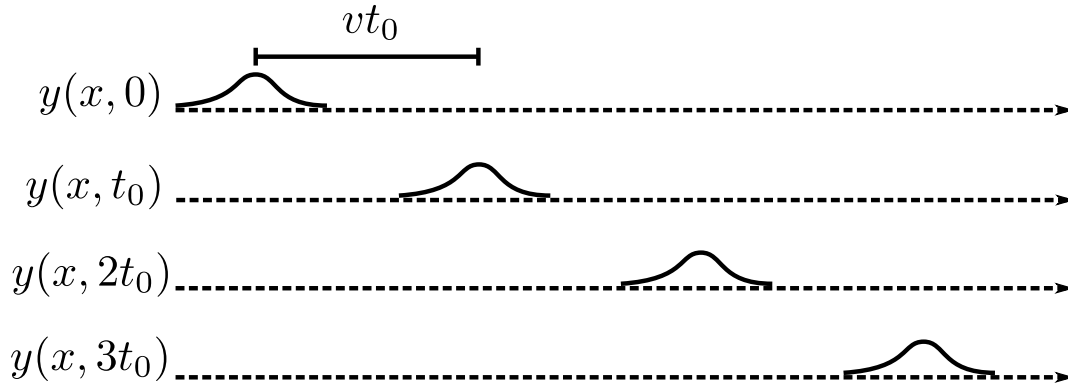


Figure 76: A solution of the wave equation, of the form  $y(x, t) = f(x - vt)$ .

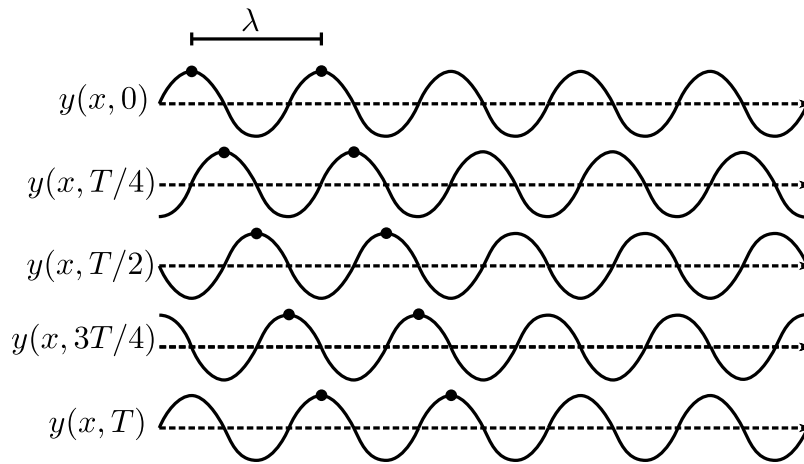


Figure 77: One period of oscillation for a right-moving sinusoid.

## 20.2 Solutions of the wave equation in one spatial dimension

The first thing to notice about one-dimensional wave equation (20.5) is that any function of the form

$$y(x, t) = f(x \mp vt) \quad (20.7)$$

is a solution! Indeed we have

$$\frac{1}{v^2} \ddot{y}(x, t) = \frac{(\mp v)^2}{v^2} f''(x \mp vt) = f''(x \mp vt) = \partial_x^2 y(x, t). \quad (20.8)$$

These solutions have a very simple physical interpretation: at  $t = 0$  the configuration of the string is described by a function  $f(x)$  of our choice. As time advances, this configuration simply moves with velocity  $v$  to the right or to the left (right if we choose the minus sign, left if we choose the plus sign), maintaining its shape (see figure 76 for an example). This is the basic physics of the wave equation: its solutions describe local disturbances which move in any direction at velocity  $v$ .

There is an especially useful set of wave equation solutions of the form (20.7) called **pure sinusoids**, of the form

$$y(x, t) = y_0 \cos(kx - \omega t + \delta). \quad (20.9)$$

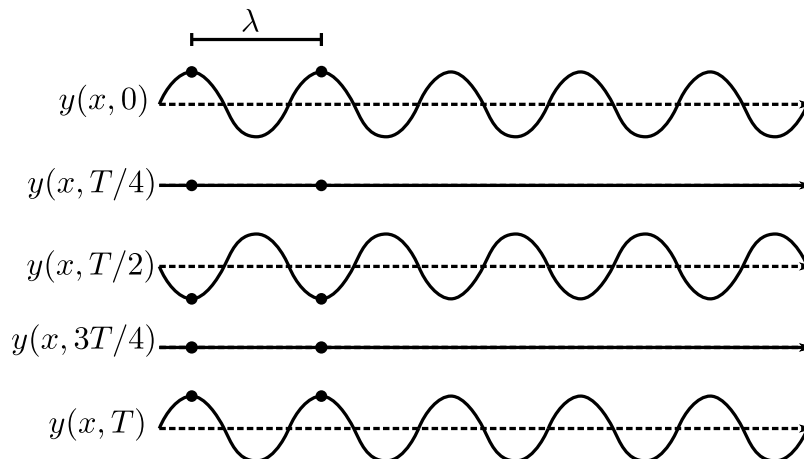


Figure 78: One period of oscillation for a standing wave.

By convention we will take  $\omega > 0$ , so this wave is right-moving if  $k > 0$  and left-moving if  $k < 0$  (see figure 77).  $k$  is called the **wave number** of this wave, and  $\omega$  is called the **angular frequency**. We note for future reference that this solution is the real part of a *complex* solution of the wave equation:

$$\tilde{y}(x, t) = \tilde{y}_0 e^{i(kx - \omega t)} \equiv y_0 e^{i\delta} e^{i(kx - \omega t)}. \quad (20.10)$$

At any fixed time the solution (20.9) describes a string configuration which is a cosine function with amplitude  $y_0$ , and as time advances this cosine picks up a phase shift that moves it to the right or left depending on the sign of  $k$ . In order for this solution to indeed be of the form (20.7), the wave number and angular frequency must be related by

$$\omega = v|k|. \quad (20.11)$$

The **period** of a pure sinusoid,  $T$ , is defined as the amount of time it takes for the piece of string at any particular location to complete one full oscillation: it is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{v|k|}. \quad (20.12)$$

The **frequency** of a pure sinusoid,  $f$ , is defined as the number of oscillations each piece of string undergoes per second. It is the inverse of the period:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{v|k|}{2\pi}. \quad (20.13)$$

The **wavelength** of a pure sinusoid,  $\lambda$ , is defined as the distance from oscillation peak to oscillation peak at a fixed time: it is given by

$$\lambda = \frac{2\pi}{|k|} = \frac{2\pi v}{\omega}. \quad (20.14)$$

For a fixed wave velocity  $v$ , the quantities  $|k|$ ,  $\omega$ ,  $T$ ,  $f$ , and  $\lambda$  all contain exactly the same information about the wave ( $k$  contains a bit more since its sign tells you the direction of propagation). Each one expresses it in a slightly different way however, and it is best to be familiar with all of them.

Sinusoids are interesting for many reasons. One reason is that many traveling waves one encounters in nature can be well-approximated by sinusoids, for example the ocean waves you see at the beach or the ripples which move down a string if you shake one end with a steady frequency. A deeper reason is that actually *any* solution of the wave equation that vanishes at spatial infinity can always be written as a sum

of sinusoids. This somewhat surprising statement is one of the main consequences of a beautiful branch of mathematics called **Fourier analysis**, and it makes use of the fact that the wave equation is linear and thus the sum of two solutions is also a solution. A simple example is a **standing wave**<sup>24</sup>

$$y(x, y) = y_0 \cos(kx + \delta) \cos(\omega t) = \frac{y_0}{2} \left( \cos(kx - \omega t + \delta) + \cos(kx + \omega t + \delta) \right), \quad (20.16)$$

which describe a sequence of string configurations whose peaks and troughs are always at the same locations but whose amplitudes oscillates with time (see figure 78). The strings on pianos and violins vibrate in standing waves. The proof that any solution of the wave equation is a sum of sinusoids goes beyond the scope of this class, but just for fun you can try showing (or using Mathematica/Maple/etc to check) that the rectangle function

$$f(x) = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1/2 \end{cases}, \quad (20.17)$$

which certainly doesn't look periodic at all, can be represented as a sum (integral really) of cosines:<sup>25</sup>

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\pi k} \sin\left(\frac{k}{2}\right) \cos(kx). \quad (20.19)$$

### 20.3 Solutions of the wave equation in two and three spatial dimensions

Having understood the wave equation in one spatial dimension, we now consider its two-dimensional and three-dimensional versions. Two-dimensional waves can propagate for example on a stretched rubber sheet, where we can use an argument similar to the one we gave for the string to show that the vertical displacement  $z(x, y, t)$  of the sheet obeys the equation

$$\partial_x^2 z + \partial_y^2 z = \frac{1}{v^2} \ddot{z}, \quad (20.20)$$

where  $v$  is given by

$$v = \sqrt{\frac{\tau}{\sigma}}, \quad (20.21)$$

with  $\tau$  the surface tension (with units of Newtons per meter) and  $\sigma$  the surface mass density. The mathematical properties of the two-dimensional wave equation (20.20) are quite similar to those of the three-dimensional wave equation, so we will just proceed directly to three spatial dimensions.

In three spatial dimensions perhaps the most familiar example of solutions of the wave equation are sound waves: these are waves carried by variations of the mass density  $\rho(\vec{r}, t)$  of air.<sup>26</sup> These obey the three-dimensional wave equation

$$\nabla^2 \rho = \frac{1}{v^2} \ddot{\rho}, \quad (20.22)$$

---

<sup>24</sup>If (like me) you are not good at remembering trig identities, the complex exponential is a big help. Observe:

$$\begin{aligned} \cos(a + b) + i \sin(a + b) &= e^{i(a+b)} = e^{ia} e^{ib} \\ &= (\cos a + i \sin a)(\cos b + i \sin b) \\ &= (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \cos b \sin a). \end{aligned} \quad (20.15)$$

<sup>25</sup>Here is one trick you can try: differentiating with respect to  $x$ , we get

$$f'(x) = - \int_{-\infty}^{\infty} \frac{dk}{\pi} \sin\left(\frac{k}{2}\right) \sin(kx). \quad (20.18)$$

This integrates to zero unless  $x = \pm \frac{1}{2}$ , so  $f$  is constant both inside and outside of the region  $(-1/2, 1/2)$ . Now you just have to show that inside this region the constant is one, while outside it is zero. It helps to know that  $\int_{-\infty}^{\infty} dy \frac{\sin y}{y} = \pi$ .

<sup>26</sup>Not the charge density, sorry for the doubled notation.



where  $v$  is about  $340\text{m/s}$  at room temperature and one atmosphere of pressure. If you want a formula, we have

$$v = \sqrt{\frac{\gamma P}{\rho}}, \quad (20.23)$$

where  $\rho$  is the average mass density (the oscillations aren't around  $\rho = 0$ ),  $P$  is the pressure, and  $\gamma$  is called the adiabatic index and depends on what kind of molecules make up the air.  $\gamma$  is about  $7/5$  for the Earth's atmosphere, more generally it is  $1 + 2/n$  where  $n$  is the number of degrees of freedom per molecule (most air molecules are diatomic, so they have three translation degrees of freedom and two rotational degrees of freedom and  $n = 5$ ).<sup>27</sup>

As with the one-dimensional wave equation, there is a simple very general family of solutions of the three-dimensional wave equation:

$$\rho(\vec{r}, t) = f(\vec{k} \cdot \vec{r} - \omega t), \quad (20.24)$$

where  $\omega > 0$  and  $\vec{k}$  is any vector obeying

$$\omega = v|k|. \quad (20.25)$$

Let's check:

$$\frac{1}{v^2} \ddot{\rho} = \frac{\omega^2}{v^2} f''(\vec{k} \cdot \vec{r} - \omega t) = |k|^2 f''(\vec{k} \cdot \vec{r} - \omega t) = \nabla^2 \rho. \quad (20.26)$$

These solutions describe a mass density configuration  $f(\vec{k} \cdot \vec{r})$  which moves in the  $\vec{k}$  direction with velocity  $v$  and at each time is uniform in the directions orthogonal to  $\vec{k}$ . One simple way to see this is to choose our coordinates so that  $\vec{k}$  points in the  $x$  direction: we then simply have

$$\rho(\vec{r}, t) = f(|k|(x - vt)), \quad (20.27)$$

which is just the kind of right-moving one-dimensional configuration we were considering before and has no dependence on  $y$  and  $z$ .

As before we can consider pure sinusoid solutions, which here are also called **plane waves**:

$$\rho(\vec{r}, t) = \rho_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta). \quad (20.28)$$

This solution is a one-dimensional pure sinusoid in the  $\vec{k}$  direction, also called the direction of propagation or the longitudinal direction, and it is constant in the two directions which are orthogonal to  $\vec{k}$ , also called the transverse directions (hence "plane wave"). The vector  $\vec{k}$  is called the **wave vector**, it simultaneously tells us the direction of propagation and the wave number  $|k|$  of the one-dimensional sinusoid propagating in that direction. The period, frequency, and wavelength are all defined as for the one-dimensional sinusoid:

$$\begin{aligned} T &= \frac{2\pi}{\omega} = \frac{2\pi}{v|k|} \\ f &= \frac{1}{T} = \frac{\omega}{2\pi} = \frac{v|k|}{2\pi} \\ \lambda &= \frac{2\pi}{|k|} = \frac{2\pi v}{\omega}. \end{aligned} \quad (20.29)$$

The wave equation in three dimensions is still a linear equation, so the sum of solutions is still a solution. We may thus still prepare standing waves as in figure 78, and it is still true that an arbitrary solution (which vanishes at spatial infinity) can be written as a sum (or integral) of plane waves.

<sup>27</sup>Isaac Newton tried to compute the speed of sound in air, but he missed the factor of  $\gamma$  and got an answer which was too slow. The factor of  $\gamma$  was first included by Pierre-Simon Laplace.

## 20.4 Energy density in scalar waves

We'll close by saying a little about the energy density of the waves in the one-dimensional string. Each mass contributes a kinetic energy

$$K_i = \frac{1}{2}\mu\epsilon\dot{y}_i^2, \quad (20.30)$$

and each rope contributes a potential energy

$$V_i = T\delta L_i, \quad (20.31)$$

where  $\delta L_i$  is the increase in length of the rope stretching from mass  $i$  to mass  $i + 1$  from its equilibrium value  $\epsilon$ . For small displacements we have

$$\delta L_i = \sqrt{\epsilon^2 + (y_{i+1} - y_i)^2} - \epsilon = \epsilon \left( \sqrt{1 + \frac{(y_{i+1} - y_i)^2}{\epsilon^2}} - 1 \right) \approx \frac{\epsilon}{2} \left( \frac{y_{i+1} - y_i}{\epsilon} \right)^2. \quad (20.32)$$

Thus we can write the total energy as

$$U = \sum_i \epsilon \left( \frac{\mu}{2}\dot{y}_i^2 + \frac{T}{2} \left( \frac{y_{i+1} - y_i}{\epsilon} \right)^2 \right), \quad (20.33)$$

which in the limit  $\epsilon \rightarrow 0$  becomes

$$U = \int_{-\infty}^{\infty} dx \left( \frac{\mu}{2}\dot{y}^2 + \frac{T}{2}(\partial_x y)^2 \right). \quad (20.34)$$

Therefore the energy density in the string is

$$u = \frac{\mu}{2}\dot{y}^2 + \frac{T}{2}(\partial_x y)^2. \quad (20.35)$$

We can evaluate this on the pure sinusoid (20.9), finding

$$\begin{aligned} u &= \frac{\mu}{2}\omega^2 y_0^2 \sin^2(kx - \omega t + \delta) + \frac{T}{2}|k|^2 y_0^2 \sin^2(kx - \omega t + \delta) \\ &= \mu\omega^2 y_0^2 \sin^2(kx - \omega t + \delta). \end{aligned} \quad (20.36)$$

We can average this over a period to find

$$u_{ave} = \frac{\mu}{2}\omega^2 y_0^2 : \quad (20.37)$$

the average energy density of a pure sinusoid increases quadratically with both the amplitude and the frequency of the wave. This is hopefully quite intuitive, waves of large amplitude and high frequency are usually more exciting than those of small amplitude and low frequency! For example you can think about the work you would do producing such a wave by shaking an end of the string.

## 21 Electromagnetic waves

In the previous lecture we reviewed some of the physics of waves propagating in scalar fields such as the displacement of a string or the mass density of the air. The electric and magnetic fields are vector fields, so any waves which propagate in them will be a little more complicated. In this lecture we will begin our discussion of electromagnetic waves, understanding how they arise from Maxwell's equations.

## 21.1 The speed of light

In the lecture before last we met the vacuum version of Maxwell's equations:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\dot{\vec{B}} \\ \vec{\nabla} \times \vec{B} &= \mu_0 \epsilon_0 \dot{\vec{E}}.\end{aligned}\tag{21.1}$$

We can learn something very useful by taking the curl of the third of these and then using the first and fourth:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E} = -\vec{\nabla} \times \dot{\vec{B}} = -\mu_0 \epsilon_0 \ddot{\vec{E}}.\tag{21.2}$$

Similarly by taking the curl of the fourth and then using the second and third, we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -\nabla^2 \vec{B} = \mu_0 \epsilon_0 \vec{\nabla} \times \dot{\vec{E}} = -\mu_0 \epsilon_0 \ddot{\vec{B}}.\tag{21.3}$$

In other words, the components of the electric and magnetic fields obey the wave equations

$$\begin{aligned}\nabla^2 \vec{E} &= \frac{1}{c^2} \ddot{\vec{E}} \\ \nabla^2 \vec{B} &= \frac{1}{c^2} \ddot{\vec{B}},\end{aligned}\tag{21.4}$$

with velocity

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.\tag{21.5}$$

Using our values for  $\epsilon_0$  and  $\mu_0$ , we can evaluate this to find

$$c \approx 2.998 \times 10^8 \text{m/s}.\tag{21.6}$$

These days it is second nature to most physicists and engineers that light is indeed really just waves rippling through the electric and magnetic fields, and that the quantity  $c$  is indeed the speed of light. To Maxwell and his contemporaries however this was an incredible revelation:  $\epsilon_0$  and  $\mu_0$  were for them defined by the Coulomb force law and Ampere's law, and these phenomena *a priori* need not have anything to do with the light we use to perceive the world around us. You can imagine how excited Maxwell must have felt the first time he realized that the value of  $c$  given by equation (21.5) was indeed quite close to the measured value of the speed of light!

You may wonder how the scientists of Maxwell's day were able to measure the speed of light without the benefit of any lasers or high-speed electronics. In fact the first reasonably accurate measurement of the speed of light came much earlier, in 1676. This was done by the Danish astronomer Ole Rømer, who realized that the apparent orbital period of Jupiter's moon Io has a small dependence on whether Earth is moving towards or away from Jupiter. Today we would call this the Doppler effect, although Rømer lived long before Doppler. If  $T$  is the true period of Io and  $v$  is the relative velocity of the Earth towards Jupiter, you can try to convince yourself that the observed period is

$$T_{obs} = \frac{T}{1 + v/c}.\tag{21.7}$$

Thus if we know  $T$ ,  $T_{obs}$ , and  $v$  then we can solve to find  $c$ . The difference between  $T$  and  $T_{obs}$  was not an easy effect to measure in 1676, the orbital period  $T$  is about 42.5 hours and  $|T - T_{obs}|$  is at most only about 15 seconds. Fortunately the deviations accumulate over multiple periods, so by measuring the duration of around 30 periods Rømer was able to get a decent estimate of  $c$  (he was off by about 25%).<sup>28</sup>

<sup>28</sup>Rømer didn't actually know the velocity  $v$  of Earth relative to Jupiter in conventional human-sized units, so he presented

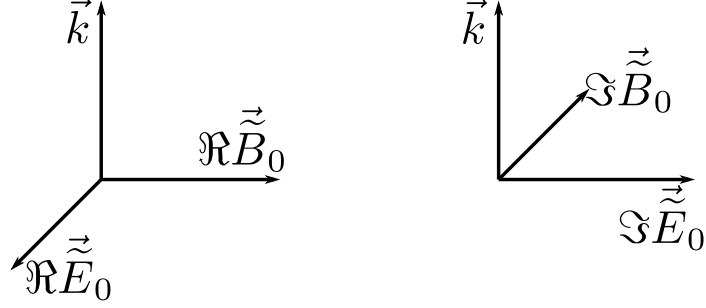


Figure 79: The relative orientations of the wave vector  $\vec{k}$  and the real/imaginary parts of the complex vector amplitudes  $\vec{E}_0$ ,  $\vec{B}_0$ . Note that the real and imaginary parts do not need to be aligned, here we have taken  $\vec{k}$  to point in the  $\hat{z}$  direction,  $\Re\vec{E}_0$  to point in the  $\hat{x}$  direction, and  $\Im\vec{E}_0$  to point in the  $\hat{y}$  direction.  $\Re\vec{B}_0$  then must point in the  $\hat{y}$  direction and  $\Im\vec{B}_0$  in the  $-\hat{x}$  direction. The directions of the electric and magnetic fields are *always* orthogonal to the direction of wave propagation.

## 21.2 Solutions of the vacuum Maxwell equations

The wave equations (21.4) do not express the full power of Maxwell's equations. In other words obeying (21.4) is necessary but not sufficient for a pair of  $\vec{E}$  and  $\vec{B}$  fields to obey Maxwell equations. We can however use (21.4) to motivate a guess for a full solution, and then see how full equations constrain our guess. As with our discussion of AC circuits, it is convenient to first consider *complex* solutions of Maxwell's equations. We will take these to have the “plane wave” form

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{B}(\vec{r}, t) &= \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)},\end{aligned}\tag{21.8}$$

where  $\vec{E}_0$  and  $\vec{B}_0$  are complex vector amplitudes. Real solutions may then be obtained by taking the real parts of  $\vec{E}$  and  $\vec{B}$ . Substituting our complex plane waves (21.8) into the vacuum Maxwell equations (21.1), we find

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= i\vec{k} \cdot \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0 \\ \vec{\nabla} \cdot \vec{B} &= i\vec{k} \cdot \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0 \\ \vec{\nabla} \times \vec{E} &= i(\vec{k} \times \vec{E}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -\dot{\vec{B}} = i\omega\vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \vec{\nabla} \times \vec{B} &= i(\vec{k} \times \vec{B}_0) e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \mu_0\epsilon_0\dot{\vec{E}} = -i\omega\mu_0\epsilon_0\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}.\end{aligned}\tag{21.9}$$

his result as an estimate of the time it takes light to cross the diameter of the orbit of Earth. The first somewhat accurate measurements of this diameter in conventional units happened at around the same time as Rømer's analysis, and they were combined with Rømer's results by Huygens to get the first value for  $c$  in conventional units.

Thus we learn that to satisfy Maxwell's equations the complex vector amplitudes  $\vec{\tilde{E}}_0$  and  $\vec{\tilde{B}}_0$  must obey

$$\begin{aligned}\vec{k} \cdot \vec{\tilde{E}}_0 &= 0 \\ \vec{k} \cdot \vec{\tilde{B}}_0 &= 0 \\ \vec{k} \times \vec{\tilde{E}}_0 &= \omega \vec{\tilde{B}}_0 \\ \vec{k} \times \vec{\tilde{B}}_0 &= -\frac{\omega}{c^2} \vec{\tilde{E}}_0.\end{aligned}\tag{21.10}$$

The first two of these equations say that both the real and imaginary parts of  $\vec{\tilde{E}}_0$  and  $\vec{\tilde{B}}_0$  must be vectors which are orthogonal to  $\vec{k}$ , which is often described as saying that electromagnetic waves are **transverse**. The third (or fourth) of these equations says that the real/imaginary part of  $\vec{\tilde{B}}_0$  of the magnetic complex vector amplitude points in the direction of the cross product of  $\vec{k}$  with the real/imaginary part of  $\vec{\tilde{E}}_0$  (see figure 79 for an illustration). The easiest way to remember this orientation is that  $\Re \vec{\tilde{E}}_0 \times \Re \vec{\tilde{B}}_0$  and  $\Im \vec{\tilde{E}}_0 \times \Im \vec{\tilde{B}}_0$  must both point in the direction of propagation. Having determined the relative orientations of  $\vec{k}$ ,  $\vec{\tilde{E}}_0$ , and  $\vec{\tilde{B}}_0$ , we now want to see what we can learn about their magnitudes. Taking the magnitudes of the real and imaginary parts of the third and fourth equations give us

$$\begin{aligned}|k| |\Re \vec{\tilde{E}}_0| &= \omega |\Re \vec{\tilde{B}}_0| \\ |k| |\Im \vec{\tilde{E}}_0| &= \omega |\Im \vec{\tilde{B}}_0| \\ |k| |\Re \vec{\tilde{B}}_0| &= \frac{\omega}{c^2} |\Re \vec{\tilde{E}}_0| \\ |k| |\Im \vec{\tilde{B}}_0| &= \frac{\omega}{c^2} |\Im \vec{\tilde{E}}_0|\end{aligned}\tag{21.11}$$

Combining the first and third of these equations gives

$$\omega^2 = |k|^2 c^2,\tag{21.12}$$

which we already knew from the wave equations (21.4), together with

$$|\Re \vec{\tilde{B}}_0| = \frac{1}{c} |\Re \vec{\tilde{E}}_0|.\tag{21.13}$$

Similarly combining the second and fourth equations again confirms the relationship between  $\omega$  and  $\vec{k}$ , and also tells us that

$$|\Im \vec{\tilde{B}}_0| = \frac{1}{c} |\Im \vec{\tilde{E}}_0|.\tag{21.14}$$

Thus both the direction and the amplitude of the magnetic field in an electromagnetic wave are determined from the direction and magnitude of the electric field, so to completely describe the wave we only need specify the latter together with the wave vector  $\vec{k}$ . Let's summarize what we've learned from Maxwell's equations about  $\vec{k}$ ,  $\vec{\tilde{E}}_0$ , and  $\vec{\tilde{B}}_0$ :

- As with the scalar waves we studied last time, the frequency and wave vector are related by

$$\omega^2 = |k|^2 c^2.\tag{21.15}$$

- The magnitude of the real/imaginary part of  $\vec{\tilde{B}}_0$  is just  $1/c$  times the magnitude of the real/imaginary part of  $\vec{\tilde{E}}_0$ .
- The real/imaginary parts of  $\vec{\tilde{E}}_0$  can each point in any direction which is orthogonal to  $\vec{k}$ , but once these directions are specified the directions of the real/imaginary parts of  $\vec{\tilde{B}}_0$  are fixed as in figure 79.

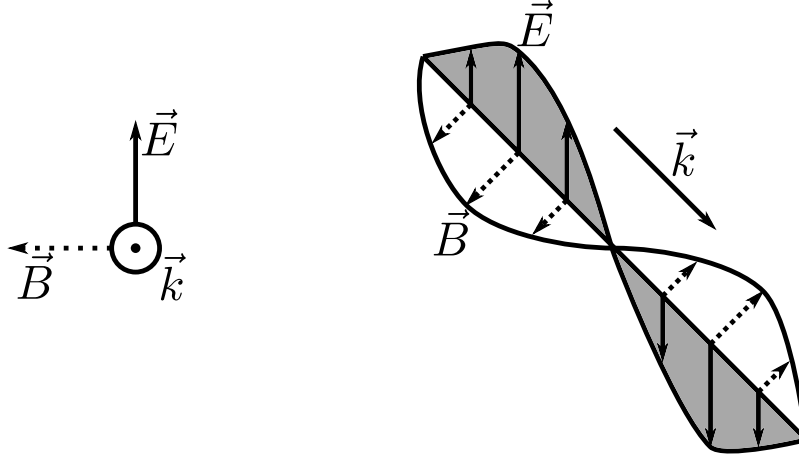


Figure 80: A linearly polarized wave. In the left diagram the wave is propagating out of the page, the electric field is oscillating up and down, and the magnetic field is oscillating left and right. The right diagram shows a piece of the same wave from a different angle.

### 21.3 Polarization

Although complex vector amplitudes are useful, they are admittedly a bit abstract. To get some intuition, let's consider the case where

$$\begin{aligned}\vec{k} &= k\hat{z} \\ \vec{\tilde{E}}_0 &= E_0\hat{x} \\ \vec{\tilde{B}}_0 &= \frac{E_0}{c}\hat{y},\end{aligned}\tag{21.16}$$

with  $k > 0$  and  $E_0 > 0$ . The real electric and magnetic fields are

$$\begin{aligned}\vec{E}(\vec{r}, t) &= E_0 \cos(k(z - ct))\hat{x} \\ \vec{B}(\vec{r}, t) &= \frac{E_0}{c} \cos(k(z - ct))\hat{y}.\end{aligned}\tag{21.17}$$

This solution describes an electromagnetic wave propagating in the  $\hat{z}$  direction, whose electric field is oscillating in the  $\hat{x}$  direction. Electromagnetic waves (like this one) with the property that at each point in space the electric field oscillates in a fixed direction are said to be **linearly polarized**. More generally any linearly polarized wave can be described via a complex vector amplitude

$$\vec{\tilde{E}}_0 = E_0 e^{i\delta} \hat{e},\tag{21.18}$$

where  $\hat{e}$  is a real unit vector which is orthogonal to  $\vec{k}$ .  $\hat{e}$  is called the **polarization vector**,  $E_0$  is called the **amplitude**, and  $\delta$  is called the **phase shift**. The real electric and magnetic fields are

$$\begin{aligned}\vec{E}(\vec{r}, t) &= E_0 \hat{e} \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) \\ \vec{B}(\vec{r}, t) &= \frac{E_0}{\omega} (\vec{k} \times \hat{e}) \cos(\vec{k} \cdot \vec{r} - \omega t + \delta).\end{aligned}\tag{21.19}$$

See figure 80 for an illustration of linear polarization.

For linearly polarized waves the real and imaginary parts of  $\vec{\tilde{E}}_0$  point in the same direction. Another interesting situation is when they are orthogonal to each other: waves of this type are said to be **circularly**

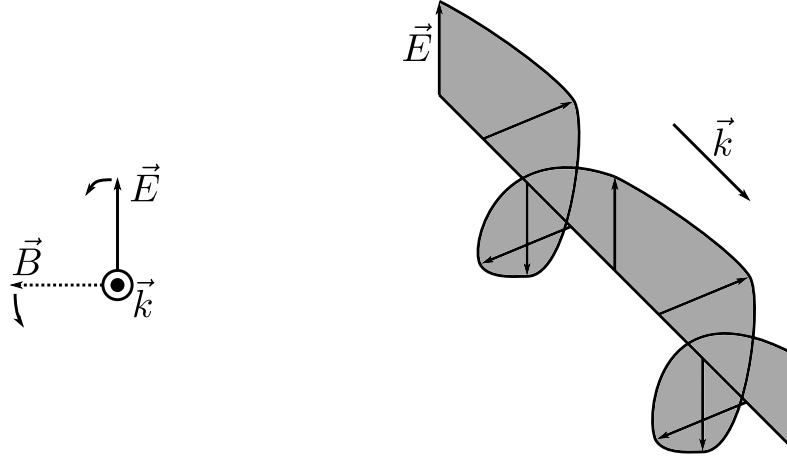


Figure 81: A right-handed circularly polarized wave. In the left diagram we show the direction of rotation of  $\vec{E}$  and  $\vec{B}$  about  $\vec{k}$  at a fixed spatial point, while in the right diagram we (attempt to) show the profile of the electric field over two wavelengths at fixed time.

**polarized** since they have the interesting property that the electric field at each spatial point rotates in a circle in the plane orthogonal to the direction of motion. As a concrete example, let's take

$$\begin{aligned}\vec{k} &= k\hat{z} \\ \vec{E}_0 &= E_0 e^{i\delta} \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}} \\ \vec{B}_0 &= \frac{E_0}{c} e^{i\delta} \frac{\hat{y} \mp i\hat{x}}{\sqrt{2}}\end{aligned}\tag{21.20}$$

with  $k > 0$  and  $E_0 > 0$ . The real electric and magnetic fields are

$$\vec{E}(\vec{r}, t) = \frac{E_0}{\sqrt{2}} \hat{x} \cos(k(z - ct) + \delta) \mp \frac{E_0}{\sqrt{2}} \hat{y} \sin(k(z - ct) + \delta)\tag{21.21}$$

$$\vec{B}(\vec{r}, t) = \frac{E_0}{c\sqrt{2}} \hat{y} \cos(k(z - ct) + \delta) \pm \frac{E_0}{c\sqrt{2}} \hat{x} \sin(k(z - ct) + \delta).\tag{21.22}$$

At a fixed time these electric and magnetic fields sweep out a helical shape in space, and as time evolves the helix moves in the  $z$  direction. The  $+$  choice of sign in  $\vec{E}_0$  is called **right-handed circular polarization**, while the  $-$  choice is called **left-handed circular polarization**. We illustrate right-handed circular polarization in figure 81.

The real and imaginary parts of  $\vec{E}_0$  will in general neither be parallel nor orthogonal, and if neither is true then we then say that the wave is **elliptically polarized**. What kind of polarization an electromagnetic wave has depends on how it was produced. For example sunlight which is reflected off of the ground (or a body of water) tends to be linearly polarized with a horizontal polarization vector  $\hat{e}$ . Polarized sunglasses are designed to only allow through linearly-polarized light whose polarization vector is *vertical*, so they block most of this reflected light and thus cut down on glare.

You may have noticed already that our circularly polarized wave (21.22) is a superposition of two linearly polarized waves. This is a special case of a far more general fact: any solution of the vacuum Maxwell equations (21.1) that vanishes sufficiently fast at spatial infinity can be written as a sum (or integral) of linearly-polarized plane waves. As mentioned last time, this is one of the main result of the mathematical subject of Fourier analysis. We do not have time to get into Fourier analysis in this class, but there is one

kind of superposition that we will need to use next time: a **standing wave**. This is a wave where the electric and magnetic fields at each point in space are all oscillating with the same phase, one example is

$$\begin{aligned}\vec{E}(\vec{r}, t) &= E_0 \hat{x} \cos kz \cos \omega t = \frac{E_0}{2} \hat{x} \left( \cos(kz - \omega t) + \cos(kz + \omega t) \right) \\ \vec{B}(\vec{r}, t) &= \frac{E_0}{c} \hat{y} \cos kz \cos \omega t = \frac{E_0}{2c} \hat{y} \left( \cos(kz - \omega t) + \cos(kz + \omega t) \right).\end{aligned}\tag{21.23}$$

## 21.4 Wavelength and frequency of electromagnetic waves

The electromagnetic waves which we perceive as visible light are only those whose wavelength  $\lambda$  is in the range from 400 to 700 nanometers. We see the different wavelengths within this band as having different colors, with the order of colors in decreasing wavelength being red, orange, yellow, green, blue, purple. In modern science and engineering however, we are interested in electromagnetic waves whose wavelengths vary over a much larger range. It is therefore traditional to classify electromagnetic waves based on wavelength/frequency, using the following categories:

- **Radio waves:** Electromagnetic waves with wavelength greater than 30 centimeters. These are produced by larger macroscopic objects such as antennae and diffuse astrophysical plasmas, and their many applications include radar and radio transmission.
- **Microwaves:** Electromagnetic waves with wavelength between one millimeter and 30 centimeters. These are produced by smaller macroscopic objects, such as cellular phones and microwave ovens. They are also by far the most common kind of electromagnetic wave in the universe, as the **cosmic microwave background** left behind by the big bang consists mostly of microwaves.
- **Infrared light:** Electromagnetic waves with wavelengths from 700 nanometers to one millimeter; their color is “beyond red”. These are produced by vibrating molecules at temperatures which aren’t too far from 300K. We tend to perceive these vibrations as heat, so we can detect infrared light using our skin instead of our eyes.
- **Visible light:** Electromagnetic waves with wavelengths from 400 to 700 nanometers. These are what we see! They are produced by relatively hot molecules, for example at the surface of the sun where the temperature is around 5000K.
- **Ultraviolet light:** Electromagnetic waves with wavelengths from 10 to 400 nanometers; their color is “beyond violet”. These are produced and absorbed in chemical reactions, and they are thus the first entry in this list which is unhealthy for you to absorb even in fairly moderate doses.
- **X-rays:** Electromagnetic waves with wavelengths from .01 to 10 nanometers. These are produced and absorbed in the most extreme chemical reactions, and also by charged particles being accelerated in particle accelerators. The most familiar application of these is in medical imaging, and they are also important in astronomy as probes of extreme environments such as the accretion disks around black holes.
- **Gamma rays:** Electromagnetic waves with wavelengths less than .01 nanometers. On Earth these are primarily produced in nuclear reactions, and also in particle accelerators. Astronomically they are produced in significant quantities only in extraordinarily violent events, such as supernovas or collisions of black holes with neutron stars. Gamma rays are sometimes also used for medical imaging, for example in PET scans.



## 22 Energy, momentum, and transmission of electromagnetic waves

We've now found that Maxwell's equations allow propagating wave solutions, for example the linearly polarized wave

$$\begin{aligned}\vec{E}(\vec{r}, t) &= E_0 \hat{x} \cos(kz - \omega t) \\ \vec{B}(\vec{r}, t) &= \frac{E_0}{c} \hat{y} \cos(kz - \omega t).\end{aligned}\tag{22.1}$$

From our daily experience it seems clear that these waves carry momentum and energy, we'll now make this more precise.

### 22.1 Energy density in electromagnetic waves

We've already found an expression for the **energy density in the electric and magnetic fields**:

$$u = \frac{\epsilon_0}{2} |E|^2 + \frac{1}{2\mu_0} |B|^2.\tag{22.2}$$

Evaluating this on our plane wave (22.1), we find

$$\begin{aligned}u &= \frac{\epsilon_0}{2} E_0^2 \cos^2(k(z - ct)) + \frac{1}{2\mu_0 c^2} E_0^2 \cos^2(k(z - ct)) \\ &= \epsilon_0 E_0^2 \cos^2(k(z - ct)).\end{aligned}\tag{22.3}$$

Averaging over a period, we find an average energy density

$$u_{ave} = \frac{\epsilon_0 E_0^2}{2}.\tag{22.4}$$

Thus the average energy density in an electromagnetic wave is proportional to its amplitude squared, just as we found for waves on a string. Sometimes people like to rewrite this in terms of a “root-mean-square” amplitude

$$E_{rms} \equiv \frac{E_0}{\sqrt{2}},\tag{22.5}$$

which you can think of as the square root of the average over a period of the amplitude squared. We then have

$$u = \epsilon_0 E_{rms}^2.\tag{22.6}$$

You may be surprised that we did not also find a factor of the frequency squared in the average energy density: this seems to contradict our expectation that waves of higher frequency carry more energy. On the other hand, without discussing how electromagnetic waves are produced it is not so clear how hard we have to work to produce waves of a given amplitude. In fact it is basically always the case that if we keep the size of the transmitting system fixed, the amplitude (and thus the energy) increases with the frequency. One nice way to think about this, although it goes beyond the scope of this class, is to realize that in quantum mechanics electromagnetic waves are built out of particles called **photons**. A classical plane wave such as the one in equation (22.1) can be thought of as being made out of huge numbers of photons all moving in the same direction, each carrying an energy

$$E = \hbar\omega.\tag{22.7}$$

Here  $\hbar$  is a fundamental constant called the reduced Planck's constant; in SI units it is given by

$$\hbar = 1.054571817 \times 10^{-34} \text{J} \cdot \text{s}.\tag{22.8}$$

Thus the energy per photon indeed increases with frequency. The smallness of  $\hbar$  makes it hard to detect individual photons, which is part of why we don't have to worry about quantum mechanics in our daily lives.

## 22.2 Energy transport by electromagnetic waves

In our plane wave (22.1), we have an average energy density  $u_{ave}$  which is moving at speed  $c$ . We therefore expect that the average power density carried by this wave (per square meter), a quantity referred to as its **intensity**, is

$$I = u_{ave}c = \frac{\epsilon_0 c E_0^2}{2} = \epsilon_0 c E_{rms}^2. \quad (22.9)$$

For example the intensity of solar radiation arriving at noon on the Earth's surface is usually around

$$I \approx 1300 \frac{\text{W}}{\text{m}^2}, \quad (22.10)$$

which converts to an amplitude of

$$E_0 \approx 990 \frac{\text{N}}{\text{C}}. \quad (22.11)$$

We can think about energy transport by the electromagnetic field in a more general way as follows. Using Maxwell's equations in vacuum, and also the vector calculus identity (check it if you don't believe me!)

$$\vec{\nabla} \cdot (\vec{V} \times \vec{W}) = \vec{W} \cdot (\vec{\nabla} \times \vec{V}) - \vec{V} \cdot (\vec{\nabla} \times \vec{W}), \quad (22.12)$$

we have that

$$\begin{aligned} -\dot{u} &= -\epsilon_0 \vec{E} \cdot \dot{\vec{E}} - \frac{1}{\mu_0} \vec{B} \cdot \dot{\vec{B}} \\ &= \frac{1}{\mu_0} \left( \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \right) \\ &= \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}). \end{aligned} \quad (22.13)$$

Therefore if we define a vector field

$$\vec{S} \equiv \frac{\vec{E} \times \vec{B}}{\mu_0}, \quad (22.14)$$

called the **Poynting vector**, we see that we have a new version of the continuity equation:

$$-\dot{u} = \vec{\nabla} \cdot \vec{S}. \quad (22.15)$$

Therefore if we think of the Poynting vector  $\vec{S}$  as the “current density” for the energy in the electromagnetic field, telling us the rate of energy transport per unit area, we see that energy is locally conserved in the same way that charge is! Let's evaluate the Poynting vector for our plane wave (22.1):<sup>29</sup>

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{E_0^2}{\mu_0 c} \hat{z} \cos^2(kz - \omega t) = \epsilon_0 c E_0^2 \hat{z} \cos^2(kz - \omega t). \quad (22.16)$$

We can average this over a period of oscillation to find the average Poynting vector

$$\vec{S}_{ave} = \frac{\epsilon_0 c E_0^2}{2} \hat{z}, \quad (22.17)$$

whose magnitude indeed agrees with our expression (22.9) for the intensity of the wave. More generally the flux of electromagnetic energy through any surface  $\Sigma$  is given by

$$P_\Sigma = \int_\Sigma \vec{S} \cdot d\vec{A}. \quad (22.18)$$

<sup>29</sup>Note that the Poynting vector “poynts” in the direction of wave propagation, how convenient!

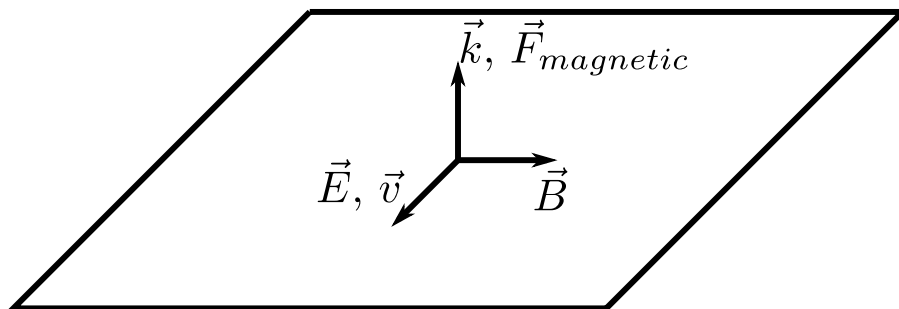


Figure 82: The force of a plane wave on an absorber. The electric field causes charges in the absorber to oscillate (think  $\vec{J} = \sigma \vec{E}$ ), and they then feel a magnetic force in the direction of propagation.

You may wonder what would have happened if we had used the full Maxwell equations in this derivation instead of the vacuum ones. You can check that we instead would have derived the equation

$$-\dot{u} = \vec{\nabla} \cdot \vec{S} + \vec{E} \cdot \vec{J}. \quad (22.19)$$

This may not look like a conservation equation anymore, but actually it is:  $\vec{E} \cdot \vec{J}$  is the rate per volume at which work is being done by the electric field to move charged particles around. This energy must be coming from the energy stored in the electric and magnetic field, so equation (22.19) says that the energy density in the electric and magnetic fields can only decrease by flowing away via the Poynting vector *or* doing work on charges.

There is a potential weak point in our discussion so far: strictly speaking, our formula (22.2) for the energy density in the electric and magnetic fields was only derived for fields produced by stationary charges and steady currents. It is not obvious that it should continue to be correct for arbitrary moving charges and electromagnetic waves. On the other hand, why should the energy in  $\vec{E}$  and  $\vec{B}$  at some point depend on how they were produced? And moreover we see that the energy density we define this way is conserved in the sense of equation (22.15) (or equation (22.19)), so it is natural to assume that equation (22.2) is true in general. To really be sure, however, we need to understand more deeply what energy actually is. This is a fascinating question with a fascinating answer, but it is beyond the scope of this class so you will just have to trust me that the conclusion is that (22.2) is indeed always the correct expression for the energy stored in the electric and magnetic fields.<sup>30</sup>

### 22.3 Momentum density in electromagnetic waves

We've now seen that electromagnetic waves carry energy. It is natural to guess that they must also carry momentum, and indeed this is true. Here is a simple way to see why this must be the case. Let's say that our plane wave (22.1) is propagating along in the  $z$  direction when it encounters an absorbing material filling the region  $z > 0$ . The mechanism of absorption is that the oscillating electric field in the wave moves charged particles in the material back and forth, doing work on them. These moving particles then feel a *magnetic* force due to the magnetic field in the wave, and with a little thought we can see that this force is always in the direction of propagation of the wave. Therefore the wave exerts a force on the absorbing material, and thus transfers momentum to it. The basic idea is illustrated in figure 82.

<sup>30</sup>If you want to get started thinking about it, the key concept is that “energy is the generator of time translations”, and similarly “momentum is the generator of spatial translations”. These both turn out to be related to the idea that “energy and momentum are the source of the gravitational field”. Our modern understanding of these ideas is due to the famous German mathematician Emmy Noether, who among other things proved a fundamental theorem showing that continuous symmetries of physical systems such as time and space translations always lead to conserved quantities such as energy and momentum. Noether was notoriously blocked for years from joining the faculty at the University of Göttingen on the grounds that it would be humiliating for soldiers to learn from a woman.

In fact we can make this argument quantitative, deriving an expression for the momentum density in the wave. Indeed the rate at which the electric field is doing work on each particle is given by

$$\frac{dU_{particle}}{dt} = \vec{F} \cdot \vec{v} = q\vec{v} \cdot \vec{E}. \quad (22.20)$$

The magnetic force on the particle is

$$\vec{F}_{magnetic} = q\vec{v} \times \vec{B}. \quad (22.21)$$

For the plane wave (22.1), the magnetic field can be written in terms of the electric field as

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}. \quad (22.22)$$

Therefore we have

$$\vec{F}_{magnetic} = \frac{q}{\omega} \vec{v} \times (\vec{k} \times \vec{E}) = \frac{q}{\omega} \left( (\vec{v} \cdot \vec{E})\vec{k} - (\vec{v} \cdot \vec{k})\vec{E} \right). \quad (22.23)$$

Since the absorbing material begins at rest, the velocity  $\vec{v}$  will be orthogonal to the wave vector  $\vec{k}$  and thus we have

$$\vec{F}_{magnetic} = q\vec{v} \cdot \vec{E} \frac{\vec{k}}{\omega} = \frac{dU_{particle}}{dt} \frac{\hat{z}}{c}. \quad (22.24)$$

Moreover since force is just the time derivative of momentum, the rate of change of the momentum of the particle is

$$\frac{d\vec{P}_{particle}}{dt} = \frac{dU_{particle}}{dt} \frac{\hat{z}}{c}. \quad (22.25)$$

Since this is true for each particle, the rate of momentum change for the entire absorber must be related to total energy flux in the same way:

$$\frac{d\vec{P}_{absorber}}{dt} = \frac{dU_{absorber}}{dt} \frac{\hat{z}}{c} = \frac{A}{c} \vec{S}, \quad (22.26)$$

where  $\vec{S}$  is the Poynting vector and  $A$  is the cross-sectional area of the absorber. Therefore the momentum of the absorber is increasing with time. If we want momentum to be conserved, then we must say that this momentum is arriving from the electromagnetic wave. Since this momentum is arriving at the speed of light, the momentum which arrives in a time  $dt$  must be equal to the amount of momentum stored in a volume

$$dV = Acdt. \quad (22.27)$$

Therefore the **momentum density stored in the electric and magnetic fields**, which we'll call  $\vec{p}$ , must obey

$$\vec{p}Acdt = \frac{A}{c} \vec{S}dt, \quad (22.28)$$

and must thus be given by

$$\vec{p} = \frac{\vec{S}}{c^2}. \quad (22.29)$$

Thus we have a second interpretation for the Poynting vector  $\vec{S}$ : it is equal to  $c^2$  times the momentum density of the electromagnetic field! We derived this only for the particular case of a plane wave, but in fact the momentum density in the electromagnetic field is always given by (22.29).

This derivation of the momentum density in a plane wave is a little complicated, using a bit of quantum mechanics we can actually give a simpler derivation. Namely we can think of our electromagnetic wave as being made out of photons, which are massless particles. The energy and momentum of a massless particle are related by

$$E = |p|c, \quad (22.30)$$

so in the plane wave (22.1), where all photons are propagating in the  $\hat{z}$  direction, this relation should also be true for the energy and momentum densities:

$$\vec{p} = \frac{u}{c} \hat{z} = \frac{uc\hat{z}}{c^2} = \frac{\vec{S}}{c^2}. \quad (22.31)$$

In equation (22.26) we found that an incident plane wave on an absorbing material generates a force per area, given by

$$\frac{1}{A} \frac{d\vec{P}_{absorber}}{dt} = \frac{\vec{S}}{c}. \quad (22.32)$$

Force per area is also called pressure, so the magnitude of this quantity is usually called the **radiation pressure** on the absorber. In daily situations radiation pressure is quite small, but it is important in astrophysics and it has recently been proposed as a somewhat plausible mechanism for launching miniature spacecraft to Alpha Centauri.

## 22.4 Production of electromagnetic waves

So far the electromagnetic waves we have discussed propagate in from spatial infinity. In the real world this is of course not where most of the electromagnetic waves we encounter come from. We thus should say at least a little about how real electromagnetic waves are produced. Unfortunately in this class we do not have time for any detailed discussion of this, so we will instead just discuss one very simple example and mention a few general lessons.

Our simple example is that we again consider a linearly polarized plane wave moving in the  $\hat{z}$  direction, but now in the presence of a perfect conductor filling the space with  $z > 0$ . There therefore cannot be any electric field for  $z > 0$ , and at  $z = 0$  any electric field must be normal to the surface of the conductor. The way this can be realized is for the field at  $z < 0$  to be in a standing wave configuration, which we can view as a superposition of an **incident** plane wave

$$\vec{E}_I(\vec{r}, t) = \frac{E_0}{2} \hat{x} \cos(kz - \omega t) \quad (22.33)$$

and a **reflected** plane wave

$$\vec{E}_R(\vec{r}, t) = -\frac{E_0}{2} \hat{x} \cos(-kz - \omega t). \quad (22.34)$$

Thus the full electric field configuration is

$$\vec{E}(\vec{r}, t) = \begin{cases} 0 & z > 0 \\ E_0 \hat{x} \sin(kz) \sin \omega t = \frac{E_0}{2} \hat{x} (\cos(kz - \omega t) - \cos(-kz - \omega t)) & z < 0 \end{cases}. \quad (22.35)$$

We can think about this in the following way: as in the absorber we considered in the previous section, the incident wave causes charges in the conductor to oscillate. But unlike the absorber, these oscillating charges will all be localized right at the surface of the conductor. This is because the wave must immediately be blocked, as otherwise the electric field would be able to penetrate into the conductor.<sup>31</sup> We may then think about the electric field (22.35) as a superposition of two parts: the “original” plane wave we would obtain by extending the incident wave  $\vec{E}_I$  to  $z > 0$  as if the charges were not there and a “radiated” wave

$$\vec{E}_{rad} \equiv \begin{cases} -\frac{E_0}{2} \hat{x} \cos(-kz - \omega t) & z < 0 \\ -\frac{E_0}{2} \hat{x} \cos(kz - \omega t) & z > 0 \end{cases} \quad (22.36)$$

<sup>31</sup>For a good but not perfect conductor, the oscillating charges are localized within a small distance  $\delta \sim \lambda\sqrt{\tau\omega}$ , called the **skin depth**, of the surface. Here  $\lambda$  and  $\omega$  are the wavelength and angular frequency of the wave and  $\tau = \frac{\epsilon_0}{\sigma}$  is the relaxation time for charges in the conductor. Thus the skin depth is small compared to the wavelength provided that the period of oscillation is long compared to the time it takes charge to dissipate in the conductor.

which we can think of as being produced by the oscillating charges. This is the key point: **electromagnetic waves are produced by oscillating charges**. The configuration  $\vec{E}_{rad}$  describes a radiation field which carries energy away from the oscillating charges at  $z = 0$  in both directions, and it is polarized so that the electric field points in the same direction as the charges were oscillating. The radiation which goes to the right is just what is needed to cancel the original incident wave, while the radiation which goes to the left produces the reflected wave  $\vec{E}_R$ . An example of the same phenomenon is a **radio antenna**, which is a tall metal tower connected to an AC source that causes charge to move up and down within the tower. This oscillatory motion produces electromagnetic waves, which propagate away from the tower along the surface of the earth. Typically the electric and magnetic fields produced by an antenna fall off like  $1/r$ , where  $r$  is the distance from the antenna. This is necessary for energy conservation, since the flux of the Poynting vector  $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$  through a sphere of radius  $r$  surrounding the antenna must be independent of  $r$ .

It is interesting to note that the reflection of an incident wave by a conductor produces a radiation pressure on the conductor, just as we found for the absorption of an incident wave by an absorber. But actually the pressure is larger by a factor of two, since in addition to absorbing the incident wave the conductor must also produce the reflected wave:

$$\frac{1}{A} \frac{d\vec{P}_{conductor}}{dt} = 2 \frac{\vec{S}_I}{c}, \quad (22.37)$$

where  $\vec{S}_I$  is the Poynting vector of the incident wave  $\vec{E}_I$  at  $z = 0$ . Therefore most proposed applications of radiation pressure use conductors instead of absorbers.

## 23 Relativity

One of the deepest principles in life, and more importantly in physics, is that it does not matter what you call things. Notation and language are social conventions, agreed upon for mutual convenience, but they have no absolute meaning. What really matters is how the set of names and symbols you use for things can be translated to the set of names and symbols I use for things, and vice versa: this can be called their *relative* meaning. In physics we refer to the study of relative meaning as **relativity**, and it will be the topic of our last two lectures.<sup>32</sup> In principle this includes questions such as how to translate between Chinese and German, but as physicists we usually focus on the topic of how different observers label points in space and time. We begin with this topic as it was understood in the time of Galileo.

### 23.1 Galilean Relativity

Say that you want to come to my office hours. In order to accomplish this, there are four pieces of information you need to possess. You need to know the latitude, longitude, and altitude of my office (usually conveyed imperfectly via a building and room number), and you also need to know at what time to show up. In other words you need to know the location vector  $\vec{r}$  of my office, and also the time  $t$  at which I am having office hours. The set of possible locations and times for my office hours is called **spacetime**, and we can label points in spacetime as  $(t, \vec{r})$ .

If you think for a little bit however, you will realize that this state of affairs is somewhat ambiguous. For example in giving you the latitude, longitude, altitude, and time of my office hours, I am implicitly assuming that you and I are both measuring latitude from the equator, longitude from the prime meridian passing through Greenwich, and altitude from sea level, and I am also assuming that you and I are both using Eastern Standard Time (or Eastern Daylight time during the summer). If you for some reason are using a different time zone, then you are likely to miss our meeting.<sup>33</sup> We can describe this mathematically in a simple way. Let's say you and I both use Cartesian coordinates, but we happen to have chosen different

<sup>32</sup>Don't worry if this introduction to relativity feels rushed: there is an entire class (8.033) on the subject that you will take if you go on in physics, so you will have plenty of time to fill in the details later.

<sup>33</sup>This is not a purely hypothetical concern: google calendar is notorious for scheduling events in the wrong time zone, and many people I know (including me) have missed flights because of this!

time zones and also different origins of space. The relationship between my spacetime coordinates  $(t', \vec{r}')$  and your spacetime coordinates  $(t, \vec{r})$  will be

$$\begin{aligned} t' &= t + a \\ \vec{r}' &= \vec{r} + \vec{b}, \end{aligned} \tag{23.1}$$

where  $a$  is the time at which I think your  $t = 0$  happens and  $\vec{b}$  is the vector which points from your origin of space to my origin of space. In other words an event which you say happens at  $(t, \vec{r})$  I will say happens at  $(t + a, \vec{r} + \vec{b})$ . A pair of spacetime coordinates  $(t, \vec{r})$  and  $(t', \vec{r}')$  which are related as in equation 23.1 are said to differ by a **spacetime translation**, and as far as we know the laws of physics look absolutely identical in any pair of coordinates which differ by a spacetime translation.

In addition to spacetime translations, there is another way that two observers could differ in how they set up their Cartesian coordinates. This is that their spatial coordinates could differ by a rotation:

$$\begin{aligned} t' &= t \\ \vec{r}' &= R\vec{r}, \end{aligned} \tag{23.2}$$

where  $R$  is the  $3 \times 3$  matrix which implements the rotation. For example for a rotation by an angle  $\theta$  about the  $z$  axis, we have

$$R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23.3}$$

As far as we know the laws of physics also look absolutely identical in any pair of spacetime coordinates which differ by a rotation.

What Galileo proposed in 1642 was that not only do the laws of physics look the same in Cartesian coordinates which differ by spacetime translations and rotations, they also look the same in the Cartesian coordinates used by observers who are moving at constant velocity with respect to each other.<sup>34</sup> To simplify our equations, in this lecture we will restrict to situations where the two observers agree on which spacetime point is labeled  $\vec{r} = t = 0$ , and we will also assume that their relative velocity is always in the  $x$  direction. These are not major restrictions, we can recover the general situation from what we say here by using rotations and spacetime translations.

Let's consider a situation where an observer  $O$  who is at rest in some Cartesian spacetime coordinates  $(t, \vec{r})$  sees an observer  $O'$  who is moving at constant velocity  $v$  in the  $x$  direction.  $O'$  has her own set of Cartesian spacetime coordinates  $(t', \vec{r}')$  in which she is at rest. To save space it is standard to refer to the  $(t, \vec{r})$  coordinates as the **rest frame of  $O$**  and the  $(t', \vec{r}')$  coordinates as the **rest frame of  $O'$** . Galileo argued that dictionary between these coordinates, meaning the map that tells us how to relate the labels which  $O$  and  $O'$  give to points in spacetime, is given by the **Galilean transformation**

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z. \end{aligned} \tag{23.4}$$

Each of these equations merits some explanation:

- $t' = t$ : this equation says that  $O$  and  $O'$  agree on the rate at which time is flowing. More practically, it says that two clocks which are moving relative to each other will nonetheless stay synchronized. To Newton this was a fundamental postulate, in his words “Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to anything external.”

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<sup>34</sup>This was part of Galileo's defense of the Copernican picture of the solar system: he wanted to refute the argument that the Earth could not be in motion around the sun since we would have felt this motion.

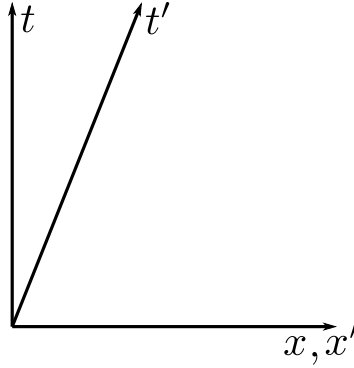


Figure 83: A Galilean transformation on spacetime. The horizontal and vertical axes label the  $x$  and  $t$  coordinates on spacetime according to observer  $O$ , and the  $t'$  axis for the observer  $O'$  (which is the set of points where  $x' = 0$ ) is also shown. In Galilean relativity all observers agree on the time coordinate, so the  $x$  and  $x'$  axes (which are the set of points where  $t = 0$  or  $t' = 0$  respectively) are identical. The slope of the  $t'$  axis in this figure is  $1/v$ .

- $x' = x - vt$ : this equation is what tells us that  $O'$  is moving with velocity  $v$  in the  $x$  direction with respect to  $O$ . Indeed since  $O'$  is at rest in the  $(t', \vec{r}')$  coordinates, say at  $x' = x'_0$ , the Galilean transformation tells us that in the  $(t, x)$  coordinates she is moving along the trajectory  $x(t) = x'_0 + vt$ . Similarly let's say that  $O$  is sitting at rest at  $x = x_0$ . Then the  $O'$  observer sees him moving along the trajectory  $x'(t') = x_0 - vt'$ , so according to her he is moving with velocity  $-v$ .
- $y' = y, z' = z$ : these equations say that  $O$  and  $O'$  agree on the coordinates which are orthogonal to their direction of relative motion.

The Galilean transformation can be visualized using a **spacetime diagram**, as shown in figure 83. A spacetime diagram is a figure which shows the trajectories of objects through spacetime, with time conventionally taken to flow upwards. We won't use them much in this class, but they are very useful in sorting out problems in relativity.

One useful application of the Galilean transformation (23.4) is that it shows us how velocities combine as we move one set of spacetime Cartesian coordinates to another. For example let's say that at time  $t = 0$ ,  $O'$  throws a tennis ball in the  $x$  direction with velocity  $v'$  from her location at  $x' = x'_0$ . She will describe the trajectory of her tennis ball as

$$x'(t') = v't' + x'_0. \quad (23.5)$$

What does this trajectory look like for  $O$  in his rest frame? The Galilean transformation tells us that we have

$$x(t) = x'(t') + vt' = x'_0 + (v' + v)t, \quad (23.6)$$

so  $O$  thinks the tennis ball is moving at velocity  $v + v'$  in the  $x$  direction. This is the **Galilean rule for the composition of velocities**: if  $O$  thinks  $O'$  is moving at velocity  $v$  and  $O'$  thinks that some object is moving at velocity  $v'$ , then  $O$  thinks that object is moving at velocity

$$v'' = v + v'. \quad (23.7)$$

This rule is surely familiar from your daily experience: if I am walking towards you and I throw you a tennis ball, it will be moving faster when you catch it than if I had first stopped and then thrown it!



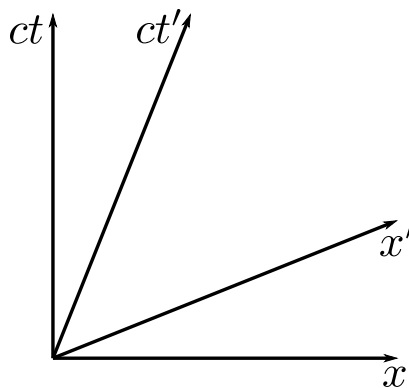


Figure 84: A Lorentz transformation on spacetime. To make the picture more symmetric between space and time we have shown  $ct$  and  $ct'$  axes instead of  $t$  and  $t'$  axes, and due to the nontrivial transformation of time the  $x$  and  $x'$  axes no longer coincide. Light propagating in the  $x$  direction moves at a 45-degree angle in this diagram.

## 23.2 Lorentz transformations and the speed of light

The correctness of Galileo's transformation (23.4) was accepted mostly without question for the first 263 years after he proposed it. In particular the rule (23.7) for composing velocities seemed so obviously consistent with daily experience that (as far as I know) there were no serious attempts to modify it. This all changed with Einstein's famous 1905 paper "On the electrodynamics of moving bodies" (or "Zur Elektrodynamik Bewegter Körper" in German), in which he introduced the **theory of special relativity** to the world.

Einstein's motivation for modifying Galileo's transformation rules (23.4) came from thinking carefully about light. We've now seen that Maxwell's equations predict the existence of electromagnetic waves propagating at the speed

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \approx 2.998 \times 10^8 \text{ m/s}. \quad (23.8)$$

There are no wave solutions of Maxwell's equations propagating at any other speed. What bothered Einstein however is that this seems to be manifestly inconsistent with the Galilean rule (23.7) for composing velocities. Namely say that instead of throwing a tennis ball while walking towards you, I instead shine a laser pointer at you. In my rest frame I think that this laser pulse propagates away from me at velocity  $c$ . Galileo would then say that if I am walking towards you with velocity  $v$ , in your rest frame the laser pulse should be moving at velocity  $v + c$ . This however is impossible, since light can only propagate at velocity  $c$ ! Thus either Galileo or Maxwell must be wrong, and rather audaciously Einstein proposed that the error lies with Galileo.

Einstein's proposal was that if  $O'$  is moving with velocity  $v$  in the  $x$  direction in the rest frame of  $O$ , the dictionary from the rest frame of  $O$  to the rest frame of  $O'$  should not be given by the Galilean transformation (23.4), but instead by the **Lorentz transformation**<sup>35</sup>

$$\begin{aligned} t' &= \gamma \left( t - \frac{vx}{c^2} \right) \\ x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z. \end{aligned} \quad (23.9)$$

<sup>35</sup>Lorentz wrote down this transformation prior to Einstein's 1905 paper, hence the name, but he did not realize the its full physical consequences. What Einstein explained was that the Lorentz transformation can be viewed as a symmetry of Maxwell's equations only if we modify Newtonian mechanics.

Here  $\gamma$  is called the **Lorentz factor**, and it stands for

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (23.10)$$

The Lorentz transformation (23.9) becomes very close to the Galilean transformation (23.4) in the limit that  $v \ll c$ , which is why we don't notice the difference between them in everyday life. The Lorentz transformation can also be visualized using a spacetime diagram, see figure 24.1.

To see how this proposal solves Einstein's problem, let's see what it says about the composition of velocities. Namely say that in the rest frame of  $O'$ , we have an object moving with velocity  $v'$  along the trajectory

$$x'(t') = v't'. \quad (23.11)$$

Substituting this into the Lorentz transformation we find

$$x' = \gamma(x - vt) = v't' = v'\gamma\left(t - \frac{vx}{c^2}\right), \quad (23.12)$$

which we can solve to find the trajectory  $x(t)$ :

$$x(t) = \frac{v + v'}{1 + \frac{vv'}{c^2}}t. \quad (23.13)$$

Thus we have a new rule for the composition of velocities,

$$v'' = \frac{v + v'}{1 + \frac{vv'}{c^2}}. \quad (23.14)$$

This rule is very close to the Galilean rule (23.7) when both  $v$  and  $v'$  are much less than  $c$ . Say however that we consider Einstein's problem, where  $v' = c$  since it is the velocity of the laser pulse in my rest frame. The velocity of the pulse in your frame is then

$$v'' = \frac{v + c}{1 + v/c} = \frac{c(1 + v/c)}{1 + v/c} = c, \quad (23.15)$$

so now you *also* think the pulse is moving with velocity  $c$ , just as predicted by Maxwell's equations! We won't discuss it here, but you can show that the Lorentz transformation is actually the only possible one which can achieve this.

### 23.3 Time dilation and length contraction

There are three especially interesting consequences of the Lorentz transformation which you should remember. We have already encountered the first: it modifies the law of composition of velocities from (23.7) to (23.14).

The second interesting consequence is **time dilation**, which means that a clock which is moving at constant velocity appears to run more slowly than one which is at rest. For example consider a clock sitting at  $x' = 0$  in the rest frame of  $O'$ . We've already seen that it follows a trajectory  $x = vt$  in the rest frame of  $O$ , which leads to a simple relationship between the clock time  $t'$  and the time  $t$  in the rest frame of  $O$ :

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \gamma\left(t - \frac{v \cdot vt}{c^2}\right) = \frac{t}{\gamma}. \quad (23.16)$$

If  $v > 0$  then  $\gamma > 1$ , so this equation says that when  $O$  thinks a time  $t$  has elapsed the moving clock only thinks a time  $t/\gamma$  has elapsed.

This time dilation effect is not noticeable in our daily experience, since when  $v$  is much less than  $c$  the Lorentz factor  $\gamma$  is very close to one. Time dilation is very noticeable however in particle physics experiments. One fun example arises in the physics of cosmic rays. These enter the Earth's atmosphere from space, and there is a particular kind of subatomic particle, called a muon, which is often produced when cosmic rays crash into air molecules in the stratosphere. Muons take about two microseconds to decay, which leads to an interesting puzzle. In two microseconds, a particle moving near the speed of light travels a distance

$$d \approx (2 \times 10^{-6}\text{s})(3 \times 10^8\text{m/s}) = 600\text{m}. \quad (23.17)$$

The typical altitude at which muons are produced is 15km, so one might expect that almost all the muons produced by cosmic ray collisions should decay before reaching the surface of the earth. Nonetheless we see a fairly regular flux of muons arriving. How do these muons have time to make it down to us before decaying? The answer is time dilation: due to its motion relative to us, from the point of view of a muon the time between its creation in the upper atmosphere and its detection by us is considerably less than the time we see between those two events. The typical velocities are such that the Lorentz factor is  $\gamma \approx 9.1$ , so in fact we should expect atmospheric muons to travel a distance  $d \approx 5.4\text{km}$  before beginning to decay: this is far enough that there is still an appreciable number for us to see at sea level.

The third interesting consequence of the Lorentz transformation is **length contraction**, which means that objects which are moving relative to us appear to be shortened along their direction of motion. For example consider a rod of length  $L$  which is at rest in the rest frame of  $O'$ . We can take its endpoints to follow the trajectories

$$\begin{aligned} x'_1(t') &= 0 \\ x'_2(t') &= L \end{aligned} \quad (23.18)$$

What we would like to understand is where these endpoints are at  $t = 0$  in the rest frame of  $O$ . For the first endpoint this is easy, since the two frames agree on the location of the point  $t = \vec{r} = 0$ . We can determine the trajectory  $x_2(t)$  of the second endpoint in the rest frame of  $O$  by using the Lorentz transformation: from

$$L = x'_2 = \gamma(x_2 - vt), \quad (23.19)$$

we have

$$x_2(t) = \frac{L}{\gamma} + vt. \quad (23.20)$$

In particular at  $t = 0$ , the location of the second endpoint is

$$x_2 = \frac{L}{\gamma}. \quad (23.21)$$

Since  $\gamma > 1$  when  $v > 0$ , we see that the rod appears shorter than its actual length by a factor of  $1/\gamma$ .

When  $v \ll c$  this length contraction effect is quite small, which is why we don't notice it when looking at moving objects. As with time dilation however, it can be quite important in particle physics applications. For example in the above discussion of the muon, you might have wondered what the story is in the rest frame of the muon: then there is no time dilation, so how does the muon account for reaching the surface of the earth? The answer is that the muon thinks that the atmosphere through which it is moving is length-contracted by a factor of  $\gamma$ , and thus it only has to go 1.6km to reach sea level.

## 23.4 Energy, momentum, and force in special relativity

What is the energy and momentum of a particle moving with velocity  $v$  in special relativity? The idea is to look for definitions which reduce to the usual ones when  $v \ll c$ , and which mix together under Lorentz transformations in the same way that space and time do. More carefully, the energy  $U'$  and momentum  $\vec{p}'$

in the rest frame of  $O'$  are related to those in the rest frame of  $O$  in such a way that  $U/c^2$  and  $\vec{p}$  transform in the same way as  $t$  and  $\vec{r}$ :<sup>36</sup>

$$\begin{aligned}\frac{U'}{c^2} &= \gamma\left(\frac{U}{c^2} - \frac{vp_x}{c^2}\right) \\ p'_x &= \gamma\left(p_x - \frac{vU}{c^2}\right) \\ p'_y &= p_y \\ p'_z &= p_z.\end{aligned}\tag{23.22}$$

In particular let's say that the particle of interest is at rest in the  $O'$  frame, and therefore its momentum  $\vec{p}'$  is zero in that frame. We will *not* assume that its energy  $U'$  is also zero, as this would imply that its momentum and energy are zero in any frame! We will instead leave its energy in the  $O'$  frame to be determined, and call it  $\alpha$ . Our proposed Lorentz transformation (23.22) then tells us that the energy and momentum in the  $O$  frame, in which the particle is moving with velocity  $v$  in the  $x$  direction, must obey

$$\begin{aligned}\alpha &= \gamma(U - vp_x) \\ 0 &= p_x - \frac{vU}{c^2}.\end{aligned}\tag{23.23}$$

Substituting the second equation into the first we have

$$\alpha = \gamma U \left(1 - \frac{v^2}{c^2}\right) = \frac{U}{\gamma},\tag{23.24}$$

so apparently

$$\begin{aligned}U &= \alpha\gamma \\ \vec{p} &= \frac{\alpha\gamma}{c^2}\vec{v}.\end{aligned}\tag{23.25}$$

Now we require that the momentum approaches Newton's answer of  $\vec{p} = m\vec{v}$  in the limit  $v \ll c$ : this tells us that  $\alpha = mc^2$ , and therefore that the energy  $U$  and momentum  $\vec{p}$  of a particle of mass  $m$  moving with velocity  $v$  are

$$\begin{aligned}U &= \gamma mc^2 \\ \vec{p} &= \gamma m\vec{v},\end{aligned}\tag{23.26}$$

where  $\gamma$  is again the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.\tag{23.27}$$

The conservation of relativistic energy and momentum when defined in this way have been confirmed to extraordinary precision at particle colliders such as the Large Hadron Collider (LHC) in Geneva. Note in particular that when  $\vec{v}$  is zero we have  $U = mc^2$ , which is Einstein's famous formula for the rest energy of a massive particle.

It is interesting to note that the relativistic formula (23.26) for the energy of a massive particle is divergent as  $v \rightarrow c$ . In other words, to accelerate a massive particle to the speed of light would require infinite energy:

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<sup>36</sup>It is not obvious why this is the right requirement to impose. Ultimately it is a consequence of something we mentioned in the previous lecture: Noether's theorem tells us that "energy is the generator of time translations" and "momentum is the generator of space translations", and so energy is related to time and momentum to space. Unfortunately you will need to wait for more a more advanced class to understand this statement in detail. The factors of  $c^2$  in (23.22) are there because our predecessors made the mistake of defining time to have units of seconds instead of meters.

$c$  is a *universal speed limit*. We can see this in another way by looking at the motion of a particle in a constant electric field. The relativistic version of the Lorentz force law is<sup>37</sup>

$$\dot{\vec{p}} = m \frac{d(\gamma \vec{v})}{dt} = q \left( \vec{E} + \vec{v} \times \vec{B} \right), \quad (23.28)$$

so if we take  $\vec{E} = E\hat{x}$ ,  $\vec{B} = 0$ , and  $\vec{v} = v\hat{x}$ , then we have

$$m \frac{d(\gamma \vec{v})}{dt} = m \left( \gamma \dot{v} + \gamma^3 \frac{v^2}{c^2} \dot{v} \right) = m \dot{v} \gamma^3 = qE. \quad (23.29)$$

This is a first order differential equation for  $v$ ,

$$\frac{\dot{v}}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} = \frac{qE}{m}, \quad (23.30)$$

which we can integrate to find

$$\int_0^v \frac{dv'}{\left(1 - \frac{v'^2}{c^2}\right)^{3/2}} = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{qE}{m} t. \quad (23.31)$$

Solving for  $v$ , we find

$$v = \frac{qEt}{m} \frac{c}{\sqrt{c^2 + \left(\frac{qEt}{m}\right)^2}}. \quad (23.32)$$

Thus for a while the second factor is close to one and the velocity just grows with time like  $qEt/m$ , as we expect from non-relativistic mechanics with a constant force. As the velocity approaches  $c$  however, its growth slows down. It gets closer and closer to  $c$ , but never reaches it in any finite time.

## 24 Lorentz invariance of electromagnetism

In the last lecture we discussed Einstein's modification of Galilean relativity to arrange for the speed of light to be the same in all rest frames. In this lecture, our final one, we study in more detail how the Lorentz transformations we introduced last time act on the electric and magnetic fields.

### 24.1 Lorentz invariance of charge

We will first think about how the charges and currents which source the electric and magnetic fields transform under Lorentz transformations. Recall from last time that the time and space coordinates  $(t', \vec{r}')$  in a frame  $O'$  moving with velocity  $v$  in the  $x$  direction with respect to a frame  $O$  are related to the time and space coordinates  $(t, \vec{r})$  in  $O$  by the Lorentz transformation

$$\begin{aligned} t' &= \gamma \left( t - \frac{v}{c^2} x \right) \\ x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z. \end{aligned} \quad (24.1)$$

We now introduce a new hypothesis: **the electric charge of a particle is the same in any frame of reference**. In equations we can write

$$q' = q, \quad (24.2)$$

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<sup>37</sup>It is not obvious that this is the right generalization of the Lorentz force law to relativistic situations. For example why not throw in a factor of  $\gamma$  on the right-hand side? Next lecture we will work out the Lorentz transformations of the electric and magnetic fields, and once you know these then you can show that this is the only option for the force law which preserves Lorentz invariance.

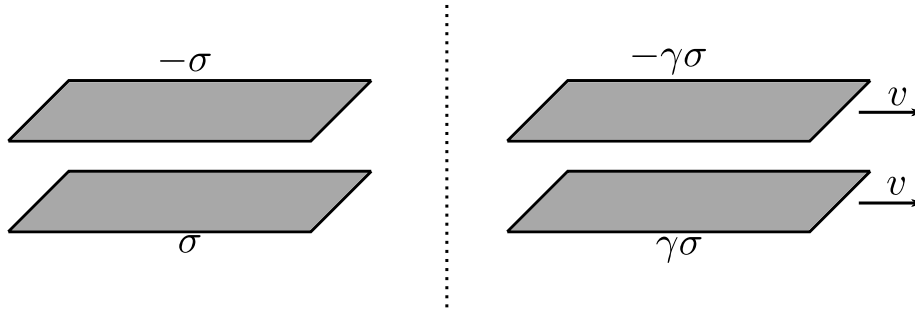


Figure 85: Parallel charged plates in two different reference frames. Note that the surface charge density in the moving frame is enhanced by a factor of  $\gamma$ .

so this hypothesis is sometimes called the “Lorentz invariance of charge”. This hypothesis is not obvious, for example the *energy* of a particle is certainly different in different reference frames. Moreover it is not really necessary: we could simply *postulate* the Lorentz transformations of the electric and magnetic fields we will find below, and then argue that Maxwell’s equations can only be Lorentz invariant if charge is Lorentz invariant. Indeed historically that is the way things proceeded. You will see however that the field transformations are somewhat complicated, so the Lorentz invariance of charge gives us a nice way to motivate them. However you choose to run the logic, there is no question experimentally that charge is Lorentz invariant. For example in the Large Hadron Collider protons are accelerated to velocities where the Lorentz factor  $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$  is about 7000, but their charge is still just that of a proton at rest. Moreover the highest-energy cosmic rays have Lorentz factors that can be as large as  $10^{10}$ , and even for these ridiculously energetic particles we still have seen no deviation from the rest charge.

## 24.2 Transformation of electric and magnetic fields

We will now use the Lorentz invariance of charge to derive the Lorentz transformation rules for the electric and magnetic fields. We begin by considering a pair of parallel charged plates of infinite size, one with surface charge density  $\sigma$  at location  $z = 0$  and the other with surface charge density  $-\sigma$  at location  $z = d$  (we take  $d, \sigma > 0$ ). In the rest frame  $O_0$  of the plates, the electric and magnetic fields between them are

$$\begin{aligned}\vec{E}_0 &= \frac{\sigma_0}{\epsilon_0} \hat{z} \\ \vec{B}_0 &= 0.\end{aligned}\tag{24.3}$$

Now let’s consider a different frame  $O$ , where the plates are moving in the  $y$  direction with velocity  $v$  (see figure 85). What are the electric and magnetic fields? We can figure out the electric field by noting that in the  $O$  frame, the surface charge density of the plates is  $\gamma\sigma_0$ . This enhancement of the surface charge density is due to length contraction: the charges in the plates look closer together in the  $y$  direction, so the density is higher. Therefore the electric field between the plates is

$$\vec{E} = \frac{\gamma\sigma_0}{\epsilon_0} \hat{z} = \gamma\vec{E}_0,\tag{24.4}$$

so apparently the electric field orthogonal to the direction of motion is enhanced compared to what it was in the rest frame of the plates.

What about the magnetic field? There is now a current flowing in the plates, due to the motion of the charges with velocity  $v$ , so there will also be a magnetic field. By symmetry, this magnetic field must be pointing in the  $x$  direction. We can determine its value using Ampere’s law: using a rectangular loop in the  $xz$  plane (see figure 86), we can see that the sheet at  $z = 0$  creates a magnetic field  $\vec{B} = \mu_0 v \sigma_0 \gamma / 2 \hat{x}$  between

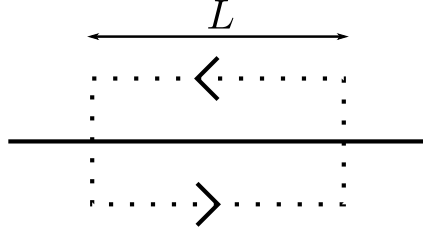


Figure 86: An Ampere loop for determining the magnetic field from a sheet of charge moving out of the page.

the plates. The upper sheet also produces a magnetic field of the same size and direction (outside the plates the contributions cancel), so the net magnetic field between the plates is

$$\vec{B} = \gamma\mu_0 v\sigma_0 \hat{x} = \frac{\gamma}{c^2} \vec{v} \times \vec{E}_0. \quad (24.5)$$

Thus an electric field in the rest frame which is orthogonal to the direction of motion in the new frame is converted into a magnetic field which is also orthogonal to the direction of motion.

What happens if we instead take the frame  $O$  to be such that the plates are moving in the positive  $z$  direction? There is now no change in the surface charge density  $\sigma_0$ , since the length contraction happens in a direction which is orthogonal to the plates, and so apparently there is also no change in the electric field:

$$\vec{E} = \vec{E}_0. \quad (24.6)$$

What about the magnetic field? It is not obvious, but actually the magnetic field continues to vanish in this frame. This may surprise you since right at the location of the plates there is now a current going upwards, but in fact this is precisely canceled in Maxwell's equations by the time derivative of the electric field so the curl of  $\vec{B}$  is zero. Here is one way to see the cancellation (don't worry about this if it is hard to follow): consider the current density in a cross sectional area  $A$  in the plane  $z = 0$  just as the lower plate is crossing this plane (remember now that it is moving upwards with velocity  $v$ ). The current density points in the  $z$  direction, so we have  $\vec{J} = J\hat{z}$ . We can compute the net charge which passes through this cross sectional area between  $t = -\epsilon$  and  $t = \epsilon$  by

$$\Delta Q = \int_{-\epsilon}^{\epsilon} dt JA = \sigma_0 A. \quad (24.7)$$

We can also compute the same integral for the displacement current:

$$\epsilon_0 \int_{\epsilon}^{\epsilon} dt A \dot{E}_z = \epsilon_0 A (E_z(t = \epsilon) - E_z(t = -\epsilon)) = -\sigma_0 A. \quad (24.8)$$

Since  $\epsilon$  can be as small as we like, this shows that  $\vec{J} = -\epsilon_0 \dot{\vec{E}}$  and thus that there is no net contribution to Maxwell's equations and indeed

$$\vec{B} = 0. \quad (24.9)$$

Thus an electric field in the rest frame which is parallel to the direction of motion in the new frame is unmodified in the new frame, and it makes no contribution to the magnetic field in the new frame either.

So far we have seen how an electric field in one frame of reference transforms into electric and magnetic fields in another frame of reference. We can summarize our understanding like this:

$$\begin{aligned} \vec{E}_{\perp} &= \gamma \vec{E}_{\perp,0} \\ \vec{E}_{\parallel} &= \vec{E}_{\parallel,0} \\ \vec{B}_{\perp} &= \frac{\gamma}{c^2} \vec{v} \times \vec{E}_{\perp,0} \\ \vec{B}_{\parallel} &= 0. \end{aligned} \quad (24.10)$$

Here  $\perp$  and  $\parallel$  indicate the parts of the electric and magnetic fields which are perpendicular or parallel to the direction of the velocity relating the two frames. But what if there is already a magnetic field in the starting frame of reference? To figure that out, we now consider the transformation from our frame  $O$  in which the plates are moving with velocity in the  $y$  direction to *another* frame  $O'$  in which they are moving with velocity  $v'$  in the  $y$  direction. In the  $O$  frame we already have the magnetic field (24.5), so we can ask what it transforms into in the  $O'$  frame. In fact we also know the electric and magnetic fields in the  $O'$  frame:

$$\begin{aligned}\vec{E}' &= \gamma' \frac{\sigma_0}{\epsilon_0} \hat{z} \\ \vec{B}' &= \gamma' \mu_0 v' \sigma_0 \hat{x},\end{aligned}\tag{24.11}$$

where

$$\gamma' \equiv \frac{1}{\sqrt{1 - \frac{v'^2}{c^2}}}.\tag{24.12}$$

What we want however is an expression for these electric and magnetic fields which is written entirely in terms of the fields in the  $O$  frame,

$$\begin{aligned}\vec{E} &= \gamma \frac{\sigma_0}{\epsilon_0} \hat{z} \\ \vec{B} &= \gamma \mu_0 v \sigma_0 \hat{x},\end{aligned}\tag{24.13}$$

and the relative velocity  $\vec{u} = u\hat{y}$  of the two frames. For example we want to convert  $\gamma'$  into

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}.\tag{24.14}$$

We can express  $u$  in terms of  $v$  and  $v'$  by using the relativistic addition of velocities law:

$$v = \frac{u + v'}{1 + \frac{uv'}{c^2}}.\tag{24.15}$$

Solving this for  $u$  we find

$$u = \frac{v - v'}{1 - vv'/c^2},\tag{24.16}$$

and we can also represent  $v'$  as

$$v' = \frac{v - u}{1 - \frac{uv}{c^2}}.\tag{24.17}$$

It takes a bit of tedious algebra to check (I'll leave it to you), but these various Lorentz factors are related by the convenient identity

$$\gamma' = \gamma_u \gamma \left(1 - \frac{uv}{c^2}\right).\tag{24.18}$$

Therefore we have

$$\begin{aligned}\vec{E}' &= \gamma_u \left( \frac{\gamma \sigma_0}{\epsilon_0} - \frac{u \gamma v \sigma}{\epsilon_0 c^2} \right) \hat{z} = \gamma_u \left( \vec{E} - \vec{u} \times \vec{B} \right) \\ \vec{B}' &= \gamma_u \gamma \mu_0 \sigma_0 v' \left(1 - \frac{uv}{c^2}\right) \hat{x} = \gamma_u \gamma \mu_0 \sigma_0 (v - u) \hat{x} = \gamma_u \left( \vec{B} - \frac{\vec{u}}{c^2} \times \vec{E} \right).\end{aligned}\tag{24.19}$$

Thus the magnetic field in the  $O$  frame, which is orthogonal to the direction of relative motion of the  $O'$  frame, is enhanced in the  $O'$  frame and also generates an electric field in the  $O'$  frame.



Finally we should think about what happens if we have a magnetic field in the  $O$  frame which is parallel to direction of relative motion in the  $O'$  frame. The easiest way to think about this is to take our above configuration in the  $O$  frame and add a solenoid wrapping the  $y$  axis with linear coil density  $n$  and current  $I$ . In the  $O'$  frame this solenoid will be moving with velocity  $-u$  in the  $y$  direction, so we would like to know if this produces any additional electric and magnetic fields. In fact it does not: since the current is always flowing in a direction which is orthogonal to the direction of relative motion, there is no length contraction along the direction of current flow so the solenoid in the  $O'$  frame is still electrically neutral. The linear density of coils  $n$  will get bigger by a factor of  $\gamma_u$  due to length contraction, but the current will flow more slowly by a factor of  $\gamma$  due to time dilation. Therefore the magnetic field produced by the solenoid will be the same in the  $O$  and  $O'$  frames, and it will produce no electric field in either frame.

Let's summarize what we have learned in this section: the relation between the electric and magnetic fields in the  $O$  and  $O'$  frames is

$$\begin{aligned}\vec{E}'_{\perp} &= \gamma_u \left( \vec{E}_{\perp} + \vec{u} \times \vec{B}_{\perp} \right) \\ \vec{B}'_{\perp} &= \gamma_u \left( \vec{B}_{\perp} - \frac{\vec{u}}{c^2} \times \vec{E}_{\perp} \right) \\ \vec{E}'_{\parallel} &= \vec{E}_{\parallel} \\ \vec{B}'_{\parallel} &= \vec{B}_{\parallel},\end{aligned}\tag{24.20}$$

where  $\vec{u}$  is the velocity of the  $O'$  frame in the  $O$  frame. In other words the electric and magnetic fields which are orthogonal to the direction of relative motion are enhanced and mixed together, while those which are parallel to the direction of relative motion are unmodified. You are welcome to check that the vacuum Maxwell equations are invariant under these substitutions.

### 24.3 Electric and magnetic fields of a moving point charge

As an application of the Lorentz transformation (24.20) we can now fairly easily compute the electric and magnetic fields of a point charge moving at constant velocity. Let's say that in a frame  $O'$  it is moving with velocity  $v$  in the  $\hat{x}$  direction, and that at  $t' = 0$  it is at  $\vec{r}' = 0$ . This frame is thus moving with velocity  $\vec{u} = -v\hat{x}$  in the rest frame  $O$ . In the rest frame we of course have

$$\begin{aligned}\vec{E} &= \frac{q}{4\pi\epsilon_0} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2 + y^2 + z^2)^{3/2}} \\ \vec{B} &= 0,\end{aligned}\tag{24.21}$$

our task is to use (24.20) to determine the electric and magnetic fields are in the  $O'$  frame. The spacetime coordinates are related by

$$\begin{aligned}t &= \gamma \left( t' - \frac{v}{c^2} x' \right) \\ x &= \gamma (x' - vt') \\ y' &= y \\ z' &= z,\end{aligned}\tag{24.22}$$

so from the first and third lines of (24.20) we have

$$\vec{E}'(\vec{r}', t') = \frac{q\gamma}{4\pi\epsilon_0} \frac{(x' - vt')\hat{x} + y'\hat{y} + z'\hat{z}}{(\gamma(x' - vt')^2 + y'^2 + z'^2)^{3/2}}.\tag{24.23}$$

We can simplify this expression a bit by introduction an "instantaneous separation vector"

$$\vec{r} \equiv (x' - vt')\hat{x} + y'\hat{y} + z'\hat{z},\tag{24.24}$$

which at each time points from the charge location to the point  $\vec{r}'$ . In terms of  $\vec{\tau}$ , a little algebra shows that

$$\vec{E}'(\vec{r}', t') = \frac{q}{4\pi\epsilon_0} \frac{1 - \frac{v^2}{c^2}}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}} \frac{\vec{\tau}}{|\tau|^3}, \quad (24.25)$$

where  $\theta$  is the angle between  $\vec{v}$  and  $\vec{\tau}$ . This electric field is something like the Coulomb field, but for large  $v$  it is suppressed by a factor of  $1/\gamma^2$  “ahead/behind of” the particle, while it is enhanced by a factor of  $\gamma$  “to the side” of the particle. We can also work out the magnetic field, from (24.20) this is just given by

$$\vec{B}'(\vec{r}', t') = \frac{\vec{v}}{c^2} \times \vec{E}' = \frac{\mu_0 q}{4\pi} \frac{1 - \frac{v^2}{c^2}}{(1 - \frac{v^2}{c^2} \sin^2 \theta)^{3/2}} \frac{\vec{v} \times \vec{\tau}}{|\tau|^3}, \quad (24.26)$$

which “wraps around” the particle trajectory as suggested by the right-hand rule with the same enhancement to the sides and suppression in front/behind.

## 24.4 Electromagnetic field strength tensor

The discussion of the last section was somewhat painful, and the formula (24.20) we obtained for the Lorentz transformations may seem rather unpleasant. In fact this unpleasantness is mostly due to the primitive notation we have used to describe electric and magnetic fields and space and time in this class. Here in the last lecture it is too late to re-invent everything, but we can try to give some sense of what lies ahead. The first thing to notice is that we can rewrite our Lorentz transformation (24.1) of space and time as

$$X' = \Lambda X, \quad (24.27)$$

where  $X$  is the “four-vector”

$$X \equiv \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad (24.28)$$

and  $\Lambda$  is the matrix

$$\Lambda \equiv \begin{pmatrix} \gamma & -v\gamma/c^2 & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (24.29)$$

Now let’s define the **electromagnetic field strength tensor**, which is the anti-symmetric matrix

$$F \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \quad (24.30)$$

You may wonder why anyone would define such a strange matrix, but the answer is apparent as soon as we observe that the Lorentz transformations (24.20) of the electric and magnetic fields directly imply that the electromagnetic field strength tensor has the simple transformation

$$F' = (\Lambda^{-1})^T F \Lambda^{-1}. \quad (24.31)$$

The inverse transformation looks even simpler:

$$F = \Lambda^T F' \Lambda. \quad (24.32)$$

Things get even better if we define a row vector

$$\partial \equiv \left( -\frac{1}{c^2} \partial_t \quad \partial_x \quad \partial_y \quad \partial_z \right) \quad (24.33)$$

and act on the electromagnetic field strength tensor with it:

$$\partial F = \begin{pmatrix} -\frac{1}{c^2}\partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \quad (24.34)$$

$$= \begin{pmatrix} \vec{\nabla} \cdot \vec{E} \\ \mu_0\epsilon_0\dot{E}_x - (\vec{\nabla} \times \vec{B})_x \\ \mu_0\epsilon_0\dot{E}_y - (\vec{\nabla} \times \vec{B})_y \\ \mu_0\epsilon_0\dot{E}_z - (\vec{\nabla} \times \vec{B})_z \end{pmatrix}. \quad (24.35)$$

The right-hand side here are precisely the combinations of  $\vec{E}$  and  $\vec{B}$  which appear in Gauss's law and Ampere's law! So in particular if we define a **current one-form**

$$J \equiv \begin{pmatrix} -\frac{\rho}{\epsilon_0} \\ \mu_0 J_x \\ \mu_0 J_y \\ \mu_0 J_z \end{pmatrix} \quad (24.36)$$

we can rewrite Gauss and Ampere's laws simply as

$$\partial F = -J. \quad (24.37)$$

How nice! What about the rest of Maxwell's equations? These can be obtained by defining the **dual electromagnetic field strength tensor**

$$\tilde{F} \equiv \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -\frac{1}{c^2}E_z & \frac{1}{c^2}E_y \\ B_y & \frac{1}{c^2}E_z & 0 & -\frac{1}{c^2}E_x \\ B_z & -\frac{1}{c^2}E_y & \frac{1}{c^2}E_x & 0 \end{pmatrix}, \quad (24.38)$$

in terms of which we have

$$\partial \tilde{F} = \begin{pmatrix} \vec{\nabla} \cdot \vec{B} \\ \frac{1}{c^2}(\dot{B}_x + (\vec{\nabla} \times \vec{E})_x) \\ \frac{1}{c^2}(\dot{B}_y + (\vec{\nabla} \times \vec{E})_y) \\ \frac{1}{c^2}(\dot{B}_z + (\vec{\nabla} \times \vec{E})_z) \end{pmatrix}. \quad (24.39)$$

Therefore all of Maxwell's equations can be written as

$$\begin{aligned} \partial F &= -J \\ \partial \tilde{F} &= 0. \end{aligned} \quad (24.40)$$

How wonderful! If only we had known earlier, we could have re-organized our entire presentation of electromagnetism around the electromagnetic field strength tensor instead of these silly quantities  $\vec{E}$  and  $\vec{B}$ ! Alas we've reached the end of the class, so if you want to learn more you will have to go on in physics :-).

## A Lecture Demonstrations:

- Coulomb's Law: D1, D2, D8
- Electric Field/Gauss's Law: D16
- Energy in electrostatics: D20
- Vector calculus: D29, D32
- Electric Potential: E6
- Conductors: D27, F5
- Image charges/capacitors: E3, D7
- Current density/ohmic materials: F1, F4
- Resistance/batteries: F2, F7
- Circuits/Kirchoff's rules: F12, E10
- Lorentz force law: G1, G9
- Ampere's law: G12, G8
- Vector potential/Biot-Savart: G16, G19
- Faraday's law: H2, H3, H4
- Inductors/magnetic energy: H5, H13, H17
- RLC circuits: L2
- AC circuits/resonance: L6
- Impedance: G7, L7
- Displacement current: J10, J7
- Waves: C27
- EM Waves: K1, K2, K3, A4
- Energy in waves: K4, K5