# Sign-stability of Positive Markov Jump Linear Systems

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**Abstract**

This paper investigates stability of Positive Markov Jump Linear Systems (PMJLSs) in the absence of a numerical realization. It considers the situation when only signs (and not magnitudes) of the entries of the subsystem matrices and the Markov transition matrices are known. The result is an analysis of a qualitative notion of stability known as sign-stability. Although the notions of sign-stability of PMJLSs are natural extensions of standard stochastic stability concepts such as exponential almost sure, mean and mean-square stability, the sign-stability notions are proven equivalent, which is not the case for their corresponding standard concepts. Moreover, for irreducible Markov chains, the particular structure of the Markov chain is shown to have no bearing on sign-stability.

*Key words:* Sign-stability; Positive Markov Jump Linear Systems

## 1 Introduction

Linear systems invariant with respect to the nonnegative orthant are said to be *positive* (Farina & Rinaldi 2000). Numerous examples of such dynamics involving intrinsically nonnegative quantities are found in biology (Jacques 1996), epidemics (Ogura & Preciado 2017), chemistry (De Leenheer et al. 2006), and air transportation (Gopalakrishnan et al. 2016).

In real-world systems, components and parameters vary, configurations change, and failures suddenly occur, resulting in changes in their dynamics (Costa et al. 2005, 2013). Positive Markov Jump Linear Systems (PMJLS) lie at the intersection of positive systems and stochastic switching systems, and allow for random transitions between candidate continuous dynamics, called *discrete modes* or *subsystems*. PMJLSs combine analytical tractability with rich behavior by assuming that each discrete mode has positive and linear dynamics, and that the switching between modes is governed by a Markov chain. These assumptions lead to appealing theoretical properties as well as practical applications (Bolzern et al. 2014*a*,*b*, Guo 2016, Ogura & Preciado 2017). Stability of a PMJLS, however, is a more subtle matter than that of its deterministic subsystems. For instance, a PMJLS whose discrete modes are all unstable can still be stable, and similarly, having only stable modes is not sufficient to guarantee stability of a PMJLS (Bolzern et al. 2014*b*).

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These counterintuitive properties result from the entanglement between the subsystems and Markov chain of a PMJLS, as evidenced by several of the stability criteria that have previously been determined for a general MJLS (Bolzern et al. 2014*b*, Fang & Loparo 2002).

To verify stability criteria for a PMJLS typically requires knowledge of both the subsystem matrices and the Markov chain. In practice, this implicitly assumes system *observability and identifiability* (Vidal et al. 2002); not always true. Furthermore, identifying modes and estimating transition probabilities is subjected to implementation challenges: memory availability to obtain accurate parameters, "*the possibility of a new, previously unseen dynamical behavior*" (Fox et al. 2011), and so on. Indeed, model uncertainty afflicts many disciplines of science and engineering, and this fueled the development of robust control theory (Zhou & Doyle 1999). It would seem natural, therefore, to approach the problem using well-established robust control methods, and in fact, positive linear systems lend themselves to, e.g., tight convex robust stability and performance characterizations that do not hold for general linear systems (Colombino & Smith 2016, Shafai et al. 1997, Son & Hinrichsen 1996). Nevertheless, this approach is still essentially numerical, typically assuming a nominal model and bounded variations around it. Alternatively, it is often easier to determine the signs of interactions with confidence, even when their magnitudes cannot be reliably estimated. As a result, in the case of PMJLSs, one obtains models that capture the feasibility of mode transitions, and the qualitative relationships between states (that is, the signs of entries in the subsystem matrices). The question then remains as to what extent such qualitative information can characterize the stability of a PMJLS. For linear systems, the notion of sign-stability was introduced to analyze systems described by matrices whose sign structure guarantees negative eigenvalues (Jeffries et al. 1977). Seminal papers that pursued such a qualitative approach to the study of dynamical systems led to graph characterizations of fundamental properties such as structural stability (Jeffries et al. 1977) and structural controllability (Lin 1974). Such characterizations emphasize that linear system models are also useful in the study of networked systems since the interactions between the continuous states of a linear system can be represented as a graph. In other words, the vertices of the network correspond to the continuous states, the edges correspond to the interactions between them, and the adjacency matrix is the system matrix. This analogy extends to PMJLSs, where each discrete mode describes a different network topology. Recent work has considered the sign-stability of linear systems in which the system matrices are *Metzler*, that is, characterized by nonnegative off-diagonal entries (Briat 2017). By contrast, (Wang et al. 2014) focused on sets of sign patterns under which *arbitrary switching* could allow asymptotic stability; in other words, it considered the stability of every state trajectory, and for any initial conditions.

This paper examines how the sign patterns of PMJLSs relate to common notions of stochastic stability. Its main contribution are twofold: *i*) necessary and sufficient conditions on the graph of a PMJLS to guarantee, or allow, stability; *ii*) establishing the introduced structural notions of stability are equivalent—not true for the standard stability ideas from which they derive. There exits no prior literature with tight conditions on the potential stability (stochastic and deterministic) of general dynamical systems, or even PMJLSs in particular (Catral et al. 2009). The approach adopted in this paper to characterize graphs of PMJLSs that are always stable is not an extension of results from either positive or switching systems (Briat 2017, Wang et al. 2014). Also, the analysis of sign-stability of PMJLS is not amenable to techniques adopted in prior work (Briat 2017, Jeffries et al. 1977) because the matrices' entries may not be independent. For example, one may require the spectra of individual matrix structures to be located in the left half-plane, which is neither sufficient nor necessary for the stability of general PMJLSs (Briat 2017). Moreover, in contrast to techniques for general switching systems (Wang et al. 2014), the proofs presented herein rely on the Metzler property of positive systems, and the specific way in which PMJLS transition between modes. Not surprisingly, even when particularized to positive systems, the results for arbitrary switching remain too demanding compared to the criteria to be presented. Namely, guaranteeing stability under arbitrary switching for any realization of a system requires *every* mode to be stable, which is not the case for PMJLS. Our results are qualitative analogues to prior results on the equivalence between PMJLS stability and particular matrices being stable (Bolzern et al. 2014*b*). It is our belief that qualitative results are a valuable and insightful complement to traditional approaches to robust stability, typically numerically-based. Valuable in the sense that such numerical data may not be available, and obtaining even rough approximations could prove costly; Insightful in the sense that the approach emphasizes what are the mechanisms that cause instability, which could enable *fixing* unstable systems, and also *designing* structurally stable systems.

## 2 Preliminaries

### *2.1 Notation*

A set that gets special notation is $[N] := \{1, 2, \cdots, N\}$, $N$ a positive integer. More generally, uppercase letters in calligraphic font denote sets; math font is reserved for special sets, e.g., the real numbers $\mathbb{R}$. Otherwise, uppercase letters denote matrices, whether in $\mathbb{R}^{n\times n}$, $\mathbb{S}^{n\times n}$, or $\mathbb{E}^{n\times n}$. For example, $I_n$ is the identity matrix of dimension $n$, whose domain can be inferred from the context in which it appears. Also, given an $n$-by-$m$ matrix and a subset $\mathcal{I} \subseteq [n] \times [m]$, $A_{\mathcal{I}}$ denotes the principal submatrix of $A$ with rows and columns indexed by $\mathcal{I}$. Lowercase letters have no usage constraints in general, but $e_i$ are reserved to denote column vectors with all entries equal to 0 except for the $i$-th one, which is either 1 or '+', depending on the context. Another vector that receives special notation is $\mathbf{1}$, all of whose entries are 1's. $A \succcurlyeq B$ and $A \succ B$ mean $A_{i,j} \geq B_{i,j}$ and $A_{i,j} > B_{i,j}$, respectively, for all $i, j$. Matrix $\operatorname{diag}(A)$ preserves diagonal entries of $n$-by-$n$ matrix $A$ and sets the others to 0,

$$(\operatorname{diag}(A))_{i,j} = \begin{cases} A_{i,i}, & i \in [n], \\ 0, & i \neq j. \end{cases}$$

Given matrices $A_i$, $i \in [N]$, $\operatorname{blockdiag}(A_1, \cdots, A_N)$ denotes a block-diagonal matrix with upper-left block $A_1$, followed by diagonal block $A_2$, up to bottom-right block $A_N$. Moreover,

$$\mathbb{1}_A(x) = \begin{cases} 1, x \in A, \\ 0, x \notin A, \end{cases}$$

represents the indicator function.

### *2.2 Positive Markov Jump Linear Systems*

Consider a system given by

$$\dot{x}(t) = A_{\sigma(t)} x(t),\ t \geq 0 \tag{1}$$

$$\Pr\{\sigma(t+h) = j | \sigma(t) = i\} = \pi_{i,j} h + o(h),\ i \neq j, \tag{2}$$

where $x(t)$ is a real $n$-dimensional vector representing the continuous state, $\sigma(t)$ belongs to the set $\{1, \cdots, N\}$, and $A_i$ are real $n$-by-$n$ Metzler matrices (i.e., all off-diagonal entries of $A_i$ are nonnegative) (Farina & Rinaldi 2000). Each $A_i$ is a system matrix that represents different continuous dynamics, known as the *discrete mode* or *subsystem*. Since the $A_i$s are all Metzler matrices, given that the initial state $x(0)$ is positive, $x(t)$ evolves in the positive orthant for all positive $t$. Mode transitions are driven by changes in $\sigma$, assumed a time-homogeneous Markov process given by (2), where $h > 0$, and $\pi_{i,j} \geq 0$ is the transition rate from mode $i$ at time $t$ to mode $j$ at time $t + h$. Assuming $k = 0$ at time $t = 0$, the sojourn time after the $k$-th jump is denoted $\tau_k$. Systems described by (1)-(2) are called Positive Markov Jump Linear Systems (PMJLSs) (Bolzern et al. 2014*b*).

The infinitesimal generator $\Pi$ of (2) is defined by a real $N$-by-$N$ matrix with entries $\pi_{i,j}$ such that

$$\pi_{i,j} \begin{cases} \geq 0, & i \neq j, \\ = -\sum_{j=1, j\neq i}^{N} \pi_{i,j}, & i = j, \end{cases} \tag{3}$$

where $\mathcal{M}^{N\times N}$ denotes the set of all such $\Pi$ satisfying (3) that are also irreducible. $\Pi$ is irreducible if there does not exist permutation matrice $P$ s.t. $P\Pi P^T$ is upper-triangular. Effectively, irreducibility is equivalent to accessibility, in the sense that every mode can be reached from any other mode. The randomness of PMJLS mode transitions implies that initial conditions alone are not enough to determine state trajectories, thereby motivating the use of stochastic notions of stability that consider, for example, the expected value of the continuous state.

**Definition 1** *The PMJLS* (1)-(2) *is said to be exponentially mean stable (EM stable) if, for any positive initial condition* $x(0)$ *and distribution* $\pi(0)$*, there exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$E[x(t)] \prec \alpha e^{-\beta t} \|x(0)\| \, \mathbf{1}.$$

**Definition 2** *The PMJLS* (1)-(2) *is said to be exponentially mean-square stable (EMS stable) if there exist positive real scalars* $\alpha$ *and* $\beta$ *such that*

$$E\left[\|x(t)\|^2\right] < \alpha e^{-\beta t} \|x(0)\|^2 ,$$

*for any positive initial condition* $x(0)$*, and any initial probability distribution* $\pi(0)$*.*

**Definition 3** *For any real scalar* $\delta > 0$*, the PMJLS* (1)-(2) *is called exponentially* $\delta$*-moment stable if there exist positive real scalars* $\alpha$ *and* $\beta$ *such that*

$$E\left[\|x(t)\|^\delta\right] < \alpha e^{-\beta t} \|x(0)\|^\delta$$

*for any initial condition* $x(0) > 0$*, and any initial probability distribution* $\pi(0)$*.*

**Definition 4** *The PMJLS* (1)-(2) *is called exponentially almost-sure stable (EAS stable) if there exists a positive real scalar* $\rho$ *such that, for any initial condition* $x(0) > 0$ *and any initial probability distribution* $\pi(0)$*,*

$$P\left\{\limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\| \leq -\rho\right\} = 1.$$

In switched systems such as (1)-(2), subsystem matrices and mode transitions govern the continuous state dynamics, and in the case of PMJLSs, knowledge of both suffices to test for various forms of stochastic stability. The theorems below are derived in Bolzern et al. (2014*b*), Ogura & Martin (2014), and offer such characterizations for EM and EMS stability. Results are expressed in terms of Kronecker's operators: given $A$ in $\mathbb{R}^{n\times n}$ and $B$ in $\mathbb{R}^{m\times m}$, $\otimes$ denotes Kronecker's product

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{n,1}B & \cdots & a_{n,n}B \end{bmatrix},$$

and $\oplus$ is Kronecker's sum

$$A \oplus B = A \otimes I_m + I_n \otimes B.$$

**Theorem 5** *The following statements are equivalent:*

i. *System* (1)-(2) *is EM stable.*
ii. *The* $nN$*-by-*$nN$ *matrix*

$$\tilde{A} = blockdiag\,(A_1, \cdots, A_N) + \Pi^T \otimes I_n \tag{4}$$

*is stable.*

**Theorem 6** *The following statements are equivalent:*

i. *System* (1)-(2) *is EMS stable.*
ii. *The* $n^2N$*-by-*$n^2N$ *matrix*

$$\hat{A} = blockdiag\,(A_1 \oplus A_1, \cdots, A_N \oplus A_N) + \Pi^T \otimes I_{n^2} \tag{5}$$

*is stable.*

Although exponential stability is amenable to the concise characterizations above, in reality exponential decay might be too much to ask. The stochastic jumps between modes break the equivalence between asymptotic and exponential stability of each linear mode, and that justifies introducing EAS stability; as long as the expected state norm "contracts" in expectation after a certain number of jumps, even momentary unstable dynamics is allowed. EAS stability, therefore, emphasizes the analysis of (1) given $k$ jumps, which isolates the randomness of a PMJLS to the time elapsed until the $k$-th jump occurs, expressed as $T_k = \sum_{i=0}^{k-1} \tau_i$, and the sequence of modes $\sigma_i = \sigma(T_i)$ visited in the $k$ jumps, described by a discrete-time *embedded Markov chain* (Bremaud 1998). The statistics of $T_k$ derive from that of exponentially distributed individual sojourn times $\tau_i$, whereas the embedded Markov chain has distribution

$$p_{i,j} = P\{\sigma_{k+1} = j | \sigma_k = i\} = \begin{cases} -\pi_{i,j}/\pi_{i,i}, & i \neq j, \\ 0, & i = j. \end{cases} \quad (6)$$

Moreover, under the assumption the Markov process of (1) is irreducible, both $\sigma(t)$ and $\sigma_k$ have unique steady-state distributions, $\bar{\pi}$ and $\bar{p}$, respectively, where

$$\bar{p}_i = \frac{\bar{\pi}_i \pi_{i,i}}{\sum_{j=1}^{N} \bar{\pi}_j \pi_{j,j}}. \quad (7)$$

The mapping of initial conditions into the current state is naturally described in terms of state transition matrices. Assuming $k$ jumps occurred over a time interval $[t_0, t_1]$, given initial condition $x(t_0)$, consider the state transition matrix $\Phi(t_1, t_0)$ of (1), given by

$$\Phi(t_1, t_0) = \Phi(t_1, T_k) \prod_{0 \leq i \leq k-1} \Phi(T_{i+1}, T_i) \quad (8)$$

where $x(t) = \Phi(t, T_i)\, x(T_i)$ is the solution to linear system [1]

$$\dot{x}(t) = A_{\sigma(t)} x(t), t \in [T_i, T_{i+1}],$$

with initial condition $x(T_i) = \Pi_{k=0}^{i-1} \Phi(T_{k+1}, T_k)\, x(T_0)$. The next Theorem is from Bolzern et al. (2006).

**Theorem 7** *System (1) is EAS-stable if and only if there exists some positive integer $m$ such that*

$$E\left[\log \|\Phi(T_m, 0)\| \mid x(0) = \bar{p}\right] < 0. \quad (9)$$

### *2.3 Qualitative characterization of PMJLS*

In this paper, numerical realizations of $\Pi$ and $A_i$s are considered unknown. Hypotheses are the Markov chain is irreducible, and that $A_i$ belong to sets of structured matrices whose entries have known signs. The following definition formalizes what a "structured matrix" is, and provides a unifying object that is used to specify $\Pi$, the $A_i$s, and other important matrices.

[1] Since $\sigma(t)$ is constant over $[T_i, T_{i+1}]$ for all $0 \leq i \leq k-1$, $A_{\sigma(t)}$ is constant over the same time interval and (1) reduces to a linear system within $[T_i, T_{i+1}]$.

Table 1
Set reference list.

| Symbol | Set description |
|---|---|
| $\mathbb{S}^{n\times n}$ | $n$-by-$n$ matrices with entries in $\{0, -, +\}$ |
| $\mathbb{E}^{n\times n}$ | $n$-by-$n$ matrices with entries in $\{0, -, +, *\}$ |
| $\mathbb{M}_{R^{n\times n}}$ | Metzler $n$-by-$n$ real matrices |
| $\mathbb{M}_{S^{n\times n}}$ | Metzler $n$-by-$n$ sign matrices |
| $\mathcal{M}^{N\times N}$ | Irreducible real $N$-by-$N$ matrices satisfying (3) |
| $\mathcal{Q}(A)$ | Qualitative class associated with real matrix $A$ |
| $\mathcal{Q}_S$ | $\mathcal{Q}(A)$ s.t. $\text{sgn}(A_{i,j}) = S_{i,j}$ for some $A$ |
| $\mathcal{Q}_M^{\Pi}$ | $\mathcal{M}^{N\times N} \cap \mathcal{Q}_M$ |

**Definition 8** *A sign-matrix is a matrix, each of whose entries take values in the set $\mathbb{S} := \{-, 0, +\}$. The set of all $n$-by-$m$ sign-matrices is denoted by $\mathbb{S}^{n\times m}$.*

Certain classes of real matrices whose properties do not depend on the particular values of its entries admit a natural correspondence with sign-matrices. Metzler matrices are one such class, so let $\mathbb{M}_R^{n\times n}$ represent the set of real $n$-by-$n$ Metzler matrices, and denote by $\mathbb{M}_S^{n\times n}$ the set of sign-matrices whose off-diagonal entries are nonnegative.

For every real matrix $A$, there exists a unique sign-matrix obtained by taking entries in $\mathbb{S}$ according to the sign of $A_{i,j}$. Since two distinct real matrices can produce the same sign-matrix, one can define the set of all real matrices that correspond to the same sign-matrix.

**Definition 9** *The qualitative class of a real $n$-by-$m$ matrix $A$ is given by the set*

$$\mathcal{Q}(A) := \left\{ B \in \mathbb{R}^{n\times m} \;\middle|\; \begin{matrix} sgn(B_{i,j}) = sgn(A_{i,j}), \\ i = 1, \cdots, n, \quad j = 1, \cdots, m \end{matrix} \right\}.$$

The above definition implies that there is a unique qualitative class corresponding to any sign-matrix, and vice versa. For a given sign-matrix $S$, let $\mathcal{Q}_S$ denote the qualitative class $\mathcal{Q}(A)$ such that $\text{sgn}(A_{i,j}) = S_{i,j}$ for some real matrix $A$ of appropriate dimensions. Similarly, given an $N$-by-$N$ sign-matrix $M$, let $\mathcal{Q}_M^{\Pi} := \mathcal{M}^{N\times N} \cap \mathcal{Q}_M$. Note that if $M$ does not conform to the sign structure implied by (3), then $\mathcal{Q}_M^{\Pi}$ is empty. Table 1 summarizes the above sets, for convenience.

Subsystem matrices with known sign structures, as well as the Markov chain of a PMJLS, can now be described in terms of their sign-matrices. This allows us to write

(1)-(2) as

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t), \forall t \geq 0, \\ A_i &\in \mathcal{Q}_{S_i} \text{ (known)},\ S_i \in \mathbb{M}_S^{n\times n},\ \forall i \in [N], \\ \Pi &\in \mathcal{Q}_M^{\Pi},\ \text{for some irreducible } M \in \mathbb{M}_S^{n\times n}.\end{aligned} \tag{10}$$

In the above, $A_i$s are unknown, but belong to known qualitative class $\mathcal{Q}_{S_i}$. Similarly, $\mathcal{Q}_M^{\Pi}$ is known but not $\Pi$.

**Definition 10** *If there exist $A_i \in \mathcal{Q}_{S_i}$, and $\Pi \in \mathcal{Q}_M^{\Pi}$, such that* (1)-(2) *is EM stable, then PMJLS* (10) *is called potentially EM sign-stable (PEM sign-stable). If this holds for every $\Pi$ in $Q_M^{\Pi}$ and $A_i$ in $Q_{S_i}$, then* (10) *is said to be EM sign-stable.*

**Definition 11** *If there exist $A_i \in \mathcal{Q}_{S_i}$ and $\Pi \in \mathcal{Q}_M^{\Pi}$ such that* (1)-(2) *is EMS stable, then PMJLS* (10) *is called potentially EMS sign-stable (PEMS sign-stable). If this holds for every $\Pi$ in $Q_M^{\Pi}$ and $A_i$ in $Q_{S_i}$, then* (10) *is said to be EMS sign-stable.*

### *2.4 Representative matrices*

The qualitative PMJLS (10) admits various subsystem prototypes given by sign-matrices $S_i$, which could represent different qualitative classes. To capture this feature, $\mathbb{S}$ must be extended to handle ambiguity arising from sign-matrices $S_i$ with different corresponding nonzero entries. This will enable synthesizing in a single matrix the possible multitude of subsystem structures for PMJLS (10).

**Definition 12** *Put $\mathbb{E} := \mathbb{S} \cup \{*\}$. An $m$-by-$n$ matrix whose entries belong to $\mathbb{E}$, is called extended sign-matrix.*

Extending $\mathbb{S}$ by including indeterminate entries '$*$' allows "adding" different $S_i$ having different nonzero signs for the same entry through an operation $+_e$:

$$\begin{aligned}&+_e : \mathbb{E} \times \mathbb{E} \to \mathbb{E} \\ &(s_1, s_2) \mapsto \begin{cases} -, & \{s_1, s_2\} \subseteq \{-, 0\}, \{s_1, s_2\} \not\subseteq \{0\}, \\ 0, & s_i = 0, i = 1, 2, \\ +, & \{s_1, s_2\} \subseteq \{0, +\}, \{s_1, s_2\} \not\subseteq \{0\}, \\ *, & \textit{otherwise}. \end{cases}\end{aligned}$$

In turn, "adding" sign-matrices of $Q_{S_i}$ synthesizes the sign information accross modes in a single matrix.

**Definition 13** *Given a finite set of sign-matrices $\mathcal{S}$,*

$$R := \sum_{S \in \mathcal{S}} S$$

*is called the representative matrix of $\mathcal{S}$.*

Although multiplication can be defined in $\mathbb{S}$ without the element '$*$', it is also useful to define the product $\times_e$ in $\mathbb{E}$:

$$\begin{aligned}&\times_e : \mathbb{E} \times \mathbb{E} \to \mathbb{E} \\ &(s_1, s_2) \mapsto \begin{cases} -, & \{s_1, s_2\} = \{-, +\}, \\ 0, & \{0\} \subseteq \{s_1, s_2\}, \\ +, & s_1 = s_2, \{s_1, s_2\} \subseteq \{-, +\}, \\ *, & \{*\} \subseteq \{s_1, s_2\}. \end{cases}\end{aligned}$$

**Remark 14** *To preserve consistency between sign-matrices and representative matrices nomenclature, elements of $\mathbb{E}$ are said to be nonnegative and nonpositive when they belong to $\{0, +\}$ and $\{-, 0\}$, respectively. Nonzero elements, on the other hand, can refer to both $\{-, +\}$ or $\{-, +, *\}$, depending on the context.*

### *2.5 Graph-theoretic concepts*

The graph $\mathcal{G}_R$ associated with an $n$-by-$n$ matrix $R$ taking entries in $\mathbb{E}$ is defined as the pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \cdots, n\}$ denotes the set of *vertices* and $\mathcal{E}$ denotes the set of *edges* of the graph. The set of edges, $\mathcal{E}$, is a subset of $\{1, 2, \cdots n\} \times \{1, 2, \cdots n\} \times \{-, +, *\}$ containing triplets $(i, j, R_{j,i})$. An edge $(i, i, \cdot)$ is called a *loop*. A sequence $v_1 e_2 v_2 \cdots v_k$ is called a *walk*. Since for each pair of vertices there can only be one edge, they are dropped in expressing walks, which are represented by a sequence $v_1 v_2 \cdots v_k$ instead. In particular, walks such that no vertex is visited more than once are called *paths*. Moreover, adding an edge to a path of length $k-1$ between its last and first edges gives a *k-cycle* or simply *cycle*, represented as $(v_1 v_2 \cdots v_{k-1})$. Because cycles visit vertices only once, for every cycle in $\mathcal{G}_R$, there is a corresponding permutation $\sigma$ in the symmetric group of order $n$. The sign of $\sigma$ is determined by its parity:-1 when $\sigma$ is odd; 1 otherwise. A permutation is odd when it can be written as an odd-numbered product of transpositions (permutations of two elements). For example, the permutation corresponding to (243) is odd (it can be decomposed into two transpositions). Thus, define the sign of a cycle $(v_1 v_2 \cdots v_k)$ as the product of the sign of its corresponding permutation $\sigma$ and its edge signs, i.e.,

$$\operatorname{sgn}(v_1 v_2 \cdots v_k) = \operatorname{sgn}(\sigma) \Pi_{m=1}^{k-1} R_{i_m, i_{m+1}}, \tag{11}$$

where the product symbol '$\times_e$' is omitted. A cycle whose sign is positive (negative, respectively) is called a positive (negative, respectively) cycle.

A graph $\mathcal{G}_{sub}$ given by $(\mathcal{V}_{sub}, \mathcal{E}_{sub})$ is called a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, if $\mathcal{V} \supseteq \mathcal{V}_{sub}$, and $\mathcal{E} \supseteq \mathcal{E}_{sub}$. In general, subgraphs are assumed induced by their vertex set, i.e., $\mathcal{E}_{sub}$ is given by the set of edges $(i, j, \cdot)$ such that $v_i, v_j$ both belong to $\mathcal{V}_{sub}$ and $(i, j, \cdot)$ belongs to $\mathcal{E}$. Two subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are *disjoint* if $\mathcal{V}_1$, $\mathcal{V}_2$ (and $\mathcal{E}_1$, $\mathcal{E}_2$) are disjoint. A set of disjoint subgraphs $\{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_k\}$ is called a decomposition of $\mathcal{G}$ if $\mathcal{V}_i \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_k$ contains $\mathcal{V}$. A Hamiltonian decomposition consists solely of disjoint cycles, and if the cycles are all $k$-cycles, it is called a Hamiltonian $k$-decomposition, denoted by $\mathcal{H}_k$.

**Remark 15** *The above definition of a graph $\mathcal{G}$ is suited to matrices in $\mathbb{E}^{m\times n}$. Thus, for a representative matrix $R$ and a sign-matrix $S$, $\mathcal{G}_R$ and $\mathcal{G}_S$ are both well-defined.*

To avoid wordy arguments, statemens about extended matrix $R$ actually refer to $\mathcal{G}_R$: an edge $R_{i,j}$ instead of $e = (i, j, R_{j,i})$; cycles in $R$ instead of cycles in $\mathcal{G}_R$. Accordingly, an off-diagonal edge $R_{j,i}$ means $i \neq j$. The same applies to sign-matrices $S_i$, which are also extended sign-matrices.

## 3 Illustrative examples

The following examples contrast notions of stability, in anticipation of the main results of the paper.

In prior work, (Bolzern et al. 2014*b*) showed EMS, EM (1-moment) and Exponential Almost-Sure (EAS) stability are not equivalent in PMJLSs using a series of counterexamples. The same counterexamples are now revisited, to show that these results may not hold from a structural point of view.

**Example 1.** (Bolzern et al. 2014*b*) Let $n = 1$, $N = 2$, $A_1 = \alpha$, $A_2 = -4$ and

$$\Pi = \begin{bmatrix} -1.5 & 1.5 \\ 1.5 & -1.5 \end{bmatrix}.$$

The system is EAS stable if and only if $\alpha < 4$, EM stable if and only if $\alpha < 12/11$, and EMS stable if and only if $\alpha < 12/19$. Thus, EAS does not imply EM stability (nor 1-moment stability), and the latter is not sufficient to establish EMS stability. In the structural framework, however, there is no distinction between them. For example, let $S_1 = +$, $S_2 = -$, and

$$M = \begin{bmatrix} - & + \\ + & - \end{bmatrix},$$

which represents the sign-PMJLS of the previous system for $\alpha > 0$. Potential sign-stability in EAS, EM and EMS sense follow from Theorem 17 and Theorems 22 and 23, because the one-dimensional mode $S_2$ is negative, but EMS, EM and EAS sign-stability don't. Indeed, e.g., the system is EMS (therefore EM and EAS) stable for $\alpha < 12/19$, but unstable for $\alpha > 4$.

Now suppose $S_1 = 0$, and consider an arbitrary realization with $A_1 = 0$, $A_2 = -\gamma$, where $\gamma > 0$, and

$$\Pi = \begin{bmatrix} -\pi_{1,2} & \pi_{1,2} \\ \pi_{2,1} & -\pi_{2,1} \end{bmatrix}.$$

Because $\pi_{1,2}$, $\pi_{2,1}$, and $\gamma$ are all positive, regardless of their particular values, the largest eigenvalues of $\hat{A}$ (see (5)) is negative, and the system is EMS (therefore EM and EAS) stable.

**Example 2.** (Bolzern et al. 2014*b*) Let $n = 2$, $N = 2$, and

$$A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

Both modes are unstable, but $\hat{A}$ is stable, and the system is EMS (therefore EM and EAS) stable.

By contrast, if $(A_1)_{1,1} = 1$, then $\tilde{A}$ is not stable, excluding EM stability, which rules out EMS stability, too. All this agrees with Theorems 19 and 22 below. In fact, even if both modes had all-negative diagonals, the system could still become unstable for off-diagonal entries large enough. For example, if

$$A_1 = \begin{bmatrix} -1 & 0 \\ 2 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 2 \\ 0 & -2 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix},$$

the system is unstable.

## 4 Exponentially Mean (EM) sign-stability

Given sign-matrices $S_i$ and $M$ of PMJLS (10), denote by $R$ the representative matrix of $\mathcal{S} = \{S_1, \cdots, S_N\}$, and let

$$\tilde{S} := \begin{bmatrix} S_1 +_e \tilde{M}_{1,1} & \tilde{M}_{2,1} & \cdots & \tilde{M}_{N,1} \\ \tilde{M}_{1,2} & S_2 +_e \tilde{M}_{2,2} & \cdots & \tilde{M}_{N,1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{M}_{1,N} & \tilde{M}_{2,N} & \cdots & S_N +_e \tilde{M}_{N,N} \end{bmatrix}, \tag{12}$$

where $\tilde{M}_{i,j} = M_{i,j} \times_e I_n$. Note that $\tilde{S}$ is the extended sign-matrix analogue of real matrix (4), whose graph exposes how the Markov chain interconnects and meshes with individual subsystems in a PMJLS.

To illustrate this interplay, consider PMJLS (10) with

$$S_1 = \begin{bmatrix} - & + \\ 0 & + \end{bmatrix}, \quad S_2 = \begin{bmatrix} + & 0 \\ + & - \end{bmatrix}, \quad M = \begin{bmatrix} - & + \\ + & - \end{bmatrix}. \tag{13}$$

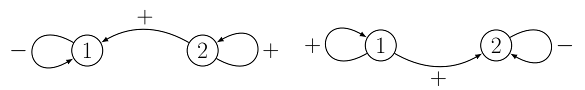


Fig. 1. (Left) The graph of sign-matrix $S_1$, $\mathcal{G}_{S_1}$. (Right) The graph of sign-matrix $S_2$, $\mathcal{G}_{S_2}$.

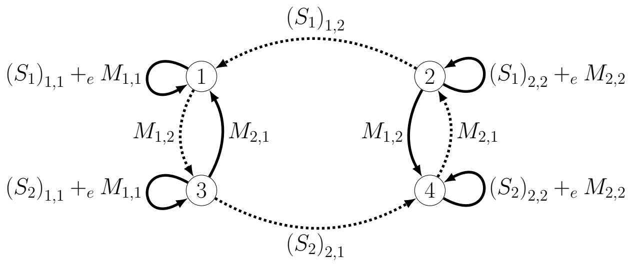


Fig. 2. The graph of sign-matrix $\tilde{S}$, and non-$M$ cycle $\tilde{c}$ (dotted edges).

In this case, (12) takes the form of

$$\tilde{S} = \begin{bmatrix} - & + & + & 0 \\ 0 & * & 0 & + \\ + & 0 & * & 0 \\ 0 & + & + & - \end{bmatrix}, \tag{14}$$

where both the second and third diagonal entries of (14) equal '$*$' because $S_i$ have positive diagonal entries, in contrast with negative diagonal sign-matrices $M_{i,i} \times_e I_2$. In the graph of $S_1$ there are no cycles, only loops, as it can be seen in Figure 1. Similarly, $\mathcal{G}_{S_2}$ has loops but not cycles because there is no edge from vertex 2 to 1. In $\mathcal{G}_{\tilde{S}}$, however, the Markov chain connects these two otherwise acyclic graphs, creating higher-order cycles that include edges from vertex 2 to 1 in $\mathcal{G}_{S_1}$ and from vertex 1 to 2 in $\mathcal{G}_{S_2}$, and are "closed" by edges that do not belong to $\mathcal{G}_{S_i}$. Figure 2 illustrates this point, as $\mathcal{G}_{\tilde{S}}$ is shown with each edge labeled by its associated entry in $\tilde{S}$. Namely, $\mathcal{G}_{\tilde{S}}$ has a cycle $\tilde{c} = (1342)$, whose edges are dotted in Figure 2 and associated with entries $\tilde{S}_{3,1}$, $\tilde{S}_{4,3}$, $\tilde{S}_{2,4}$, and $\tilde{S}_{1,2}$, which are equal to $M_{1,2}$, $(S_2)_{2,1}$, $M_{2,1}$ and $(S_1)_{1,2}$, respectively.

In the previous example, $\tilde{c}$ was a loop with both $S_i$ and $M$ edges, but if either $S_1$ or $S_2$ had a loop, $\tilde{S}$ would have a loop as well—(12) or (34), respectively. In either case, cycles in $\tilde{S}$ with $S_i$ edges are in 1-to-1 correspondence with $R$ cycles. This fact is repeatedly used in upcoming proofs, so it is formally restated as the next proposition. Loops such as $\tilde{c}$ are critical because $A_i$ entries can be large, resulting in positive cycles with high gain, which cause instability in positive systems. This is not the case for $M$ cycles because high gains are "compensated" by increasing negative loops, by the construction of $M$. Thus, these loops are referred to as *non-$M$ cycles.*

**Proposition 16** *The representative matrix $R$ of a set $\mathcal{S} \subset \mathbb{E}^{n\times n}$ has a cycle of order $k$ if and only if $\tilde{S}$ has a non-$M$ cycle of order $h \geq k$.*

**Proof.** Suppose $R$ has a cycle of order $k$, given by $c = (i_1 i_2 \cdots i_k)$. For each edge in $c$, e.g., $(i_1, i_2, \cdot)$, there exists a nonzero entry $R_{i_2,i_1}$ given by a nonzero entry of some $S_{j_1}$ in $\mathcal{S}$, i.e., for every $R_{i_2,i_1}$ edge there is an $(S_{j_i})_{i_2,i_1}$ edge such that $R_{i_2,i_1} = (S_{j_1})_{i_2,i_1}$. Let $j_1, j_2, \cdots, j_{k-1}$ denote the distinct consecutive indices of such elements $S_{j_i}$ in $\mathcal{S}$. For example, if $R_{i_2,i_1}$, $R_{i_3,i_2}$ and $R_{i_4,i_3}$ are given by $(S_1)_{i_2,i_1}$, $(S_1)_{i_3,i_2}$ and $(S_3)_{i_4,i_3}$, respectively, then $j_1$ equals 1, and $j_2$ equals 3. By construction of $\tilde{S}$, for every edge $(S_j)_{i_2,i_1}$ in $c$, $\tilde{S}$ has a nonzero edge $\tilde{S}_{nj-n+i_2,nj-n+i_1}$. Thus, if all edges in $c$ are related to the same mode $S_j$, then $\tilde{c} = ((nj-n+i_1)\cdots(nj-n+i_k))$ is a non-$M$ cycle and we are done. Otherwise, suppose there are at least two distinct modes associated with the edges of $c$.

The Markov chain of (10) is irreducible. Therefore, because each sign-matrix $S_i$ represents some mode $i$, there is a shortest path between any two vertices $j_1$ and $j_2$ in $\mathcal{G}_M$ such that no vertex is visited twice, and consequently, there is a shortest walk $j_1 v_2 \cdots j_2 \cdots j_{k-1} \cdots j_1$ starting at $j_1$ and visiting $j_2$, then $j_3$, and so on, which ends at $j_1$ and no vertex is visited more than $k-1$ times. Moreover, by construction of $\tilde{S}$ in (12), if there exists a mode transition $M_{j,i}$, then there is an edge in $\mathcal{G}_{\tilde{S}}$ between all vertices of $S_i$ and $S_j$ that are equal modulo $n$, for every entry of these sign-matrices. For instance, if there is a nonzero probability of a transition from mode 2 to mode 1, and $S_i$ are $n$-by-$n$ sign-matrices, then $\mathcal{G}_{\tilde{S}}$ has edges from vertex 1 to $n+1$, 2 to $n+2$, and so on, up to an edge from vertex $n$ to $2n$. Hence, the sequence $\tilde{c}$ given by $(nj_1 - n + i_1)\,(nj_1 - n + i_2)\,(nv_2 - n + i_2)\cdots(nj_1 - n + i_1)$ is a non-$M$ cycle. Indeed, by hypothesis $c$ is a cycle, so every vertex in $c$ is visited exactly once, and even if there exist repeated vertices in $j_1 v_2 \cdots j_{k-1} \cdots j_1$—which occurs when $c$ has two nonconsecutive $S_i$ edges, for the same $i$—these repeated vertices correspond to different vertices in $\mathcal{G}_{\tilde{S}}$, so every vertex is visited exactly once in $\tilde{c}$.

Conversely, assuming $\mathcal{G}_R$ acyclic, suppose $\tilde{S}$ has a non-$M$ cycle $\tilde{c} = (i_1 i_2 \cdots i_k)$. Then, consider the following cases:

i. $\tilde{c}$ has only $S_i$ edges. Let $\mathcal{J} = \{j_1, \cdots, j_{k-1}\}$ denote the set of indices of $S_j$ for which there is an $S_j$ edge in $\tilde{c}$. Since there are no $M$ edges in $\tilde{c}$, then $\mathcal{J}$ must be a singleton. Indeed, $S_i$ are diagonal blocks in $\tilde{S}$, by construction, implying there cannot be $S_i$ edges between different $S_i$ vertices. Because $\mathcal{J}$ is a singleton, $\tilde{c}_{\mathrm{mod}} := ((i_1 \mod n)\,(i_2 \mod n)\cdots(i_{k-1} \mod n))$ is in fact an $S_{j_1}$ cycle. Then, by construction, $R_{i_2 \mod n, i_1 \mod n}$, $R_{i_3 \mod n, i_2 \mod n}$, up to $R_{i_{k-1} \mod n, i_{k-2} \mod n}$ are nonzero, so $\tilde{c}_{\mathrm{mod}}$ is an $R$ cycle, a contradiction.

ii. $\tilde{c}$ has $M$ edges. Let $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}$ be an $nN$-by-$nN$ permutation matrix, $P_i$ of dimension $nN \times N$ whose columns are the canonical column vectors $e_j$, as in $P_i = \begin{bmatrix} e_{j_1^i} & e_{j_2^i} & \cdots & e_{j_N^i} \end{bmatrix}$, and $\{j_1^i, j_2^i, \cdots, j_N^i\}$ such that $j_h^i = i + (h-1)\,n$. The action of $P$ on $\tilde{S}$ yields

$$P^T \tilde{S} P = \begin{bmatrix} M^T + D_{1,1} & D_{1,2} & \cdots & D_{1,n} \\ D_{2,1} & M^T + D_{2,2} & \cdots & D_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ D_{n,1} & D_{n,2} & \cdots & M^T + D_{n,n} \end{bmatrix}, \tag{15}$$

where $D_{i,j} = \mathrm{blockdiag}((S_1)_{i,j}, \cdots, (S_N)_{i,j})$. $\tilde{S}$ has a non-$M$ cycle if and only if $P^T \tilde{S} P$ has a non-$M$ cycle, so let $\tilde{c}_P = (i'_1 i'_2 \cdots i'_k)$ be the cycle given by the vertices of $\tilde{c}$ permuted by $P$. It can be seen from (15) that the $S_h$ edges in $\tilde{c}_P$ are in fact $D_{i,j}$ edges, $i \neq j$, between vertices $i'_m$ and $i'_{m+1}$ such that $\left|i'_m - i'_{m+1}\right| \geq N$. But since $\tilde{c}_P$ is a cycle, for every $D_{i,j}$ edge, $\tilde{c}_P$ must also have an edge $D_{j,l}$ and $D_{h,i}$, for some $l$ and $h$. Let $\mathcal{J} = \{(j_1, j_2), (j_2, j_3), \cdots, (j_m, j_1)\}$ denote the consecutive index pairs for which there is an $D_{j_i, j_l}$ index in $\tilde{c}_P$. For every index pair in $\mathcal{J}$, $(j_1, j_2)$ e.g., there is some $i$ such that $(S_i)_{j_1, j_2}$ is nonzero, which implies $R_{j_1, j_2}$ is also nonzero. Therefore, $c = (j_1 j_2 \cdots j_m)$ is an $R$ cycle, a contradiction. □

**Theorem 17** *Let $R$ be the representative matrix for the set of sign-matrices of the PMJLS described by* (10)*. The following statements are equivalent:*

*i. PMJLS* (10) *is EM sign-stable.*

*ii. $\mathcal{G}_R$ is acyclic, $R$ only has negative diagonal entries.*

**Proof.** Suppose (10) is EM sign-stable. By definition, (10) is EM stable for every choice of $A_i$ in $\mathcal{Q}_{S_i}$, so Theorem 5 implies that $\tilde{A}$ must be stable for every $A_i$ in $\mathcal{Q}_{S_i}$. By contradiction, assume the first condition of (ii) not satisfied, i.e., $\mathcal{G}_R$ contains a cycle. Then, let $\tilde{c}$ be the corresponding non-$M$ cycle of $\mathcal{G}_{\tilde{S}}$ given by Proposition 16. Let $m$ denote the number of $S_i$ edges in $\tilde{c}$, and assume $\tilde{c}$ has order $h$. $\tilde{c}$ is a non-$M$ cycle so $h$ and $m$ are both greater than 1. The sign of every $\tilde{c}$ edge is given by an off-diagonal entry of $\tilde{S}$, which are all positive because both $S_i$ and $M$ are Metzler. Thus, for each $S_i$, consider $A_i$ in $\mathcal{Q}_{S_i}$ such that $(A_i)_{j,k}$ equals $\gamma > (\sqrt{n}+2)^h$ if $(S_i)_{j,k}$ is an edge of $\tilde{c}$, otherwise let the other nonzero elements $(A_i)_{j,k}$ equal $\pm\gamma^{-1}$, following the sign of $(S_i)_{j,k}$. Similarly, consider $\Pi$ in $\mathcal{Q}_M^{\Pi}$ such that all nonzero entries $\pi_{j,k}$ equal 1 if $M_{j,k}$ is an edge of $\tilde{c}$, and equal $\gamma^{-1}$, otherwise.

Given that every cycle in $\tilde{S}$ corresponds to a permutation of one of its submatrices, Leibniz's determinant formula provides an explicit connection between cycles and determinants. In particular, there is some $\mathcal{J} \subseteq [nN]$ with $|\mathcal{J}| = h$ for which there is some permutation $\tilde{\sigma}$ in $Sym_h$ (the $h$-th order symmetric group) corresponding to $\tilde{c}$, such that

$$\begin{aligned} (-1)^h \det \tilde{A}_{\mathcal{J}} &= (-1)^h \sum_{\sigma \in Sym_h} \left( \mathrm{sgn}\,(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i} \right) \\ &= -\gamma^m + (-1)^h \sum_{\substack{\sigma \in Sym_h, \\ \sigma \neq \tilde{\sigma}}} \left( \mathrm{sgn}\,(\sigma) \prod_{i=1}^{k} a_{i,\sigma_i} \right) \\ &\leq ((k!-1) - \gamma)\, \gamma^{m-1}, \end{aligned} \tag{16}$$

where $\tilde{A}$ is the matrix given by (4) for this particular choice of $A_i$ and $\Pi$. That inequality (16) holds follows from the fact that any other Hamiltonian decomposition $\mathcal{H}_h$ of $\tilde{S}_h$ can have at most $h-2$ coincident edges with $\tilde{c}$, otherwise $\mathcal{H}_h$ would be equal to $\tilde{c}$. These coincident edges cannot form a cycle on their own, since $\tilde{c}$ is a cycle, and therefore must be multiplied by some power of $\gamma^{-1}$ greater than or equal to 2. In the case that there are exactly $h-2$ coincident edges, the remaining two are either $S_i$ or $M$ off-diagonal, and the corresponding product of $\tilde{A}$ edges is upper-bounded by $\gamma^{m-2}$. For elements in the determinant of $\tilde{A}_{\mathcal{J}}$ that are multiples of $k$ diagonal entries, for some $k \leq h$, because there are at most $n-1$ entries equal to $\gamma^{-1}$ for every row in $\Pi$ and at most one diagonal entry equal to $\gamma^{-1}$ for every $A_i$, every diagonal entry of $\tilde{A}$ is upper bounded by $1 + n\gamma^{-1}$. Noting that

$$\left(1 + n\gamma^{-1}\right)^k < \gamma^{m-1}$$

is equivalent to

$$n < \gamma \left( \gamma^{\frac{m-1}{k}} - 1 \right),$$

the hypothesis that $\gamma > (\sqrt{n}+2)^h$ implies $\gamma > \sqrt{n}$, $\gamma^{\frac{m-1}{k}} - 1 > \sqrt{n} + 1$, and

$$n < \sqrt{n}\left(\sqrt{n}+1\right) < \gamma \left( \gamma^{\frac{m-1}{k}} - 1 \right),$$

because $\frac{1}{h} < \frac{m-1}{k}$ and $\gamma > 1$.

Inequality (16) implies $\tilde{A}$ is unstable for large enough $\gamma$. Indeed, let $p(\tilde{A}) = s^{nN} + p_1 s^{nN-1} + \cdots + p_{nN}$ be the characteristic polynomial of $\tilde{A}$, with

$$p_h = (-1)^h \sum_{X \subseteq [n], |X| = h} \det \tilde{A}_X.$$

A similar rationale as the one used to establish inequality (16) can be used to see that for each $\mathcal{X} \subseteq [nN]$ such that $|\mathcal{X}| = h$ and $\mathcal{X} \neq \mathcal{J}$, $(-1)^h \det \tilde{A}_{\mathcal{X}} \leq k! \gamma^{m-1}$. Thus, since there at most $\binom{n}{h}$ such subsets, for $\gamma > \left(\binom{n}{h} + 1\right) k!$, $p_h$ is negative, $p(\tilde{A})$ has a positive root, and (1) is unstable. Therefore, $\mathcal{G}_R$ must be acyclic.

Now assume the second condition of (ii) does not hold, and $R$ has some diagonal entry belonging to the set $\{0, +, *\}$. Accordingly, there is a positive diagonal entry $(S_i)_{k,k}$ for some $i$ and $k$, or $(S_i)_{k,k}$ is 0 for some $k$ and every $i$. Assuming that the first is true, consider $A_i$ in $\mathcal{Q}_{S_i}$ such that every entry has unitary magnitude, except for $(A_i)_{k,k}$, taken as $n$. $A_i$ has positive trace, so it has at least one positive eigenvalue. Also, consider $\pi_{i,j}$ and $\pi_{j,i}$ equal to $\epsilon$, for every $j$ in $[n]$. Then, because polynomial roots are continuous functions of the polynomial coefficients, by taking $\epsilon$ sufficiently small, the spectrum of $\tilde{A}$ can be made arbitrarily close to that of a block diagonal matrix, one of whose blocks is $A_i$. Hence, for $\epsilon$ small enough, $\tilde{A}$ has a positive eigenvalue, and therefore is not stable. Otherwise, assume every element $S_i$ of $\mathcal{S}$ has a zero diagonal entry $(S_i)_{k,k}$, for some $k$, and consider arbitrary $A_i \in \mathcal{Q}_{S_i}$ for $i = 1, \cdots, N$, and some $\Pi \in \mathcal{Q}_M$. Because $R$ is acyclic, $\mathcal{G}_{\tilde{S}}$ does not have any non-$M$ cycles, by Proposition 16. Hence, $\tilde{A}$ has the same eigenvalues of

$$\tilde{A}_{diag} = \mathrm{diag}(\mathrm{blockdiag}\,(A_1, \cdots, A_N)) + \Pi^T \otimes I_n. \tag{17}$$

Denote by $v$ the vector $[e_k^T \ \cdots \ e_k^T]^T$ in $\mathbb{R}^{nN}$, $e_k$ in $\mathbb{R}^n$. Then, $\tilde{A}_{diag} v = [\tilde{v}_1^T \ \cdots \ \tilde{v}_N^T]^T$, where $\tilde{v}_i$ are vectors in $\mathbb{R}^n$ given by

$$\tilde{v}_i = \begin{bmatrix} 0 & 0 & \cdots & (A_i)_{k,k} + \pi_{i,i} + \sum_{j \neq i}^N \pi_{i,j} & \cdots & 0 \end{bmatrix}^T = 0.$$

Therefore, $\tilde{A}_{diag}$ has a zero eigenvalue, and so does $\tilde{A}$, which cannot be stable. In other words, $R$ must have only negative diagonal entries.

Conversely, assume (ii). $\tilde{A}$ is stable if and only if (17) is stable, because $\tilde{A}$ has no non-$M$ cycles. Define

$$P = [P_1 \ \cdots \ P_N], \tag{18}$$

a permutation matrix with $P_i = [e_{j_1^i} \ \cdots \ e_{j_N^i}]$ such that $j_h^i = i + (h-1)\, n$. $P$ transforms $\tilde{A}_{diag}$ into a block diagonal matrix $P^T \tilde{A}_{diag} P = \mathrm{diag}\,(X_i)$, where

$$X_i = \Pi^T + D_{i,i}, \tag{19}$$

and $D_{i,j} = \mathrm{blockdiag}((A_1)_{i,j}, (A_2)_{i,j}, \cdots, (A_N)_{i,j})$.

Similarity transformations preserve eigenvalues, therefore $\tilde{A}$ is stable if and only if $P^T \tilde{A}_{diag} P$ is stable, which in turn is stable if and only if each of its block diagonal matrices $X_i$ is stable. Put

$$\alpha := \max_{i \in [N]} -\pi_{i,i} + \max_{i \in [N]} -D_{i,i},$$

to decompose $\Pi^T$ as in

$$\Pi^T = U - \alpha I.$$

$U$ is nonnegative because $\pi_{i,j} \geq 0$, $i \neq j$. Moreover, irreducibility of $\Pi$ implies the same of $U$, and Perron-Frobenius Theorem implies the largest eigenvalue of $U$ is positive and equal to its spectral radius $\rho\,(U)$. By means of (3) and Gershgorin's Circle Theorem, $\Pi^T$ must have nonpositive eigenvalues, the largest of which is 0. Since the largest eigenvalue of $U$ is equal to that of $\Pi^T$ shifted by $\alpha$, it must be positive because $U$ is irreducible; $\rho\,(U) = \alpha$ follows. Now, $\alpha$ is so large that $U + D_{i,i}$ is also nonnegative, therefore $\rho(U - D_{i,i}) < \rho\,(U)$ results from monotonicity of the Perron-Frobenius eigenvalue (Vandergraft 1968). $X_i$ is stable. $\square$

**Remark 18** *For positive systems, positive loops and cycles are the mechanisms that cause instability. Theorem 17 generalizes this fact to PMJLS, stating it is also necessary that no cycles be formed by "superposition" of modes, i.e., no cycles may be formed by edges from different modes. Individual modes, however, need not be stable; they can be marginally stable, as long as for every diagonal entry, at least one mode is negative, since irreducibility implies every mode is visited infinitely many times. In this sense, $\Pi$ is crucial to the matter of stability, since irreducibility implicitly plays a key role in the case where individual modes are not stable.*

## 5 Potentially EM (PEM) sign-stability

In the previous section, positive cycles and loops have been shown destabilizing, becoming dominant for large enough entries, and making PMJLS (1)-(2) unstable. On the other hand, by continuity of eigenvalues of $\tilde{A}$ with respect to its entries, the mere presence of such destabilizing mechanisms does not prevent stability, as long as there is a stable system spanned by $\mathcal{G}_{\tilde{S}}$. The theorem below formalizes this idea.

**Theorem 19** *Let $R$ be the representative matrix for the set of sign-matrices of the PMJLS given by* (10)*. The following statements are equivalent:*

*i. PMJLS* (10) *is PEM sign-stable.*
*ii. Each diagonal entry of $R$ is negative or indeterminate.*

**Proof.** Suppose every diagonal entry of the representative sign-matrix $R$ is either negative or indeterminate. As a consequence, there exist $i_1, i_2, \cdots, i_n$ such that $(S_{i_1})_{1,1}$, $(S_{i_2})_{2,2}$, $\cdots$, $(S_{i_n})_{n,n}$ are all negative. Assuming that every remaining entry of every $S_i$ is zero, Theorem 5 implies PMJLS (10) is PEM sign-stable for any choice of $\Pi$. Therefore, the choice of $A'_i$ such that $\left(A'_{i_1}\right)_{1,1}, \cdots, \left(A'_{i_n}\right)_{n,n}$ are negative, with all other entries equal to zero, results in an EM-stable PMJLS. To address the case where $S_i$ have nonzero entries other than $\left(S_{i_j}\right)_{j,j}$, let $\epsilon$ be a positive number and define

$$(A_i)_{h,k} = \begin{cases} \left(A'_{i_j}\right)_{j,j} & , i = i_j, \\ \epsilon & , (S_i)_{h,k} = +, \\ -\epsilon & , (S_i)_{h,k} = -. \end{cases}$$

Such matrices belong to $\mathcal{Q}_{S_i}$, therefore $(A_1, \cdots, A_n)$ along with $\Pi$ make up a realization of (10). For $\epsilon$ sufficiently small, the eigenvalues of $\tilde{A}$ can be taken arbitrarily close to those of $\tilde{A}'$, by continuity of polynomial roots with respect to its coefficients. $\tilde{A}$ is stable.

Conversely, assume (10) PEM sign-stable, but suppose $R$ has a nonnegative diagonal entry. Accordingly, for some integer $q$ in $[n]$, $(S_i)_{q,q}$ are all nonnegative for every $i$ in $[N]$, i.e., $(A_i)_{q,q}$ are all nonnegative for any $A_i$ in $\mathcal{Q}_{S_i}$. Thus, consider an arbitrary realization of (10), with modes $A_i \in \mathcal{Q}_{S_i}$ and infinitesimal generator $\Pi \in \mathcal{Q}^{\Pi}_{M}$. Let $P$ be permutation matrix (18), and $X_i$ be the matrix (19). Along the lines of the proof of Theorem 5, $\Pi^T$ has a unique largest eigenvalue equal to 0. By monotonicity of the Perron-Frobenius eigenvalue (Vandergraft 1968), the largest eigenvalue of $X_q$ must be nonnegative—since $D_{q,q} \geq 0$. If $D_{i,j} = 0$, $i \neq j$, $\tilde{A}$ is not stable, because $P^T \tilde{A} P$ is block diagonal, and at least one of the blocks, $X_q$, is not stable. Hence, $\tilde{A}$ is amenable to decomposition

$$\tilde{A} = U - \alpha I,$$

$U$ nonnegative and irreducible, such that $\rho(U) > \alpha > 0$. Relaxing the hypothesis that off-diagonal $D_{i,j}$ are 0, then

$$\tilde{A} = \tilde{U} - \alpha I,$$

with $\tilde{U} = U + \text{blockdiag}(A_i - \text{diag}(A_i))$, i.e., adding the off-diagonal elements of $A_i$ to $\tilde{A}$. But then again, by monotonicity of Perron-Frobenius, $\rho(\tilde{U}) \geq \rho(U)$; $\tilde{A}$ is not stable, (10) is not PEM sign-stable. □

## 6 EM, EMS and EAS are sign-equivalent

Exponential mean-square stability has been shown stronger than exponential mean stability for positive Markov jump linear systems because the former implies $\delta$-stability with $\delta = 1$, and for PMJLSs the notions of EM stability and 1-stability are equivalent (Bolzern et al. 2014*b*). These implications led the authors in (Bolzern et al. 2014*b*) to investigate whether the converse was also true. By exhibiting a series of counterexamples, this conjecture has been proven false. The question remains of whether there are equivalent qualitative stability notions in the structural framework. In the positive answer below, EM, EMS and EAS (PEM, PEMS and PEAS) sign-stability are shown equivalent.

**Theorem 20** *PMJLS* (10) *is EM sign-stable if and only if it is EMS sign-stable.*

**Proof.** Every EMS stable PMJLS is also EM stable (Bolzern et al. 2014*b*), proving necessity.

Conversely, let $R'$ be the representative matrix of $S'_i = S_i \oplus S_i$ of a PMJLS (10). Sufficiency follows because $\mathcal{G}_{R'}$ inherits graphical properties of $\mathcal{G}_R$ that guarantee the modified PMJLS with $S'_i$ is EM sign-stable, which means the original PMJLS is EMS sign-stable. To see this, consider an EM sign-stable PMJLS (10), with $N$-dimensional $M$ and $n$-dimensional sign-matrices $S_i$, and define the extended sign-matrices

$$S'_i = \begin{bmatrix} (S_i)_{1,1} \times_e I_n + S_i & \cdots & (S_i)_{1,n} \times_e I_n \\ \vdots & \ddots & \vdots \\ (S_i)_{n,1} \times_e I_n & \cdots & (S_i)_{n,n} \times_e I_n + S_i \end{bmatrix}. \tag{20}$$

Let $R$ and $R'$ denote the representative matrices of $\mathcal{S} = \{S_1, \cdots, S_n\}$ and $\mathcal{S}' = \{S'_1, \cdots, S'_n\}$, respectively. From (20), every diagonal entry in $R$ is negative if and only if $R'$ has an all-negative diagonal. Similarly, $\mathcal{G}_R$ is acyclic if and only if $\mathcal{G}_{R'}$ is acyclic. Indeed, $R' = R \oplus R$ so if $R$ has a cycle, the diagonal $n$-dimensional blocks of $R'$ have cycles, and so does $R'$. Conversely, suppose $R'$ has a cycle, $c'$. $R'$ has the same structure as (12), with $S_i$ and $M$ replaced by $R$. If $c'$ is a non-$M$ cycle, since $R +_e \cdots +_e R = R$, by Proposition 16, $\mathcal{G}_R$ has a cycle. Otherwise, $c'$ is an $M$-cycle, i.e., $\mathcal{G}_R$ has a cycle.

Hence, extending an EM sign-stable PMJLS as above yields a well-defined PMJLS for which every realization $A'_i$ and $\Pi$ is EM stable. But Theorem 5 then implies every PMJLS with infinitesimal generator $\Pi$ and modes $A_i$ such that $A'_i = A_i \oplus A_i$ is EMS stable. □

**Theorem 21** *PMJLS* (10) *is EM sign-stable if and only if it is EAS sign-stable.*

**Proof.** Every EM stable PMJLS (1) is also EAS stable (Bolzern et al. 2014*b*), proving necessity.

Conversely, suppose (10) is EAS sign-stable but not EM sign-stable. In light of Theorem 17, either:

i. $(S_m)_{i_m,i_m} = +$, for some mode $m$.

Without loss of generality, suppose $m = N$, $i_m = n$. Also, suppose $S_i = -I_n$, $i \neq N$,

$$S_N = \begin{bmatrix} -I_{n-1} & 0 \\ 0 & + \end{bmatrix}, \tag{21}$$

and let

$$M = \begin{bmatrix} - & + & 0 & 0 & \cdots & 0 & 0 \\ + & - & + & 0 & \cdots & 0 & 0 \\ + & 0 & - & + & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ + & 0 & 0 & 0 & \cdots & 0 & - \end{bmatrix}. \tag{22}$$

i.e., mode 1 is accessible from each mode $i$, which in turn is accessible from mode $i-1$. Consider then $A_i = -\epsilon I_n$, with $1 > \epsilon > 0$, and

$$A_N = \begin{bmatrix} -\epsilon I_{n-1} & 0 \\ 0 & \gamma \end{bmatrix}, \tag{23}$$

with $\gamma > 0$, and $\Pi$ such that

$$\pi_{i,i+1} = \epsilon\mu, \quad \pi_{i+1,i} = (1-\epsilon)\,\mu, \quad \pi_{N,1} = \alpha,$$

for $i = 2, \cdots, N-1$. Accordingly, transition probabilities (6) are given by

$$p_{i,j} = P\left\{\sigma_{k+1} = j | \sigma_k = i\right\} = \begin{cases} \epsilon, & j = 1, \\ 1-\epsilon, & j = i+1, \\ 0, & o.w., \end{cases}$$

for $i = 2, \cdots, N-1$, otherwise $p_{1,2} = p_{N,1} = 1$, as shown in Figure 3.

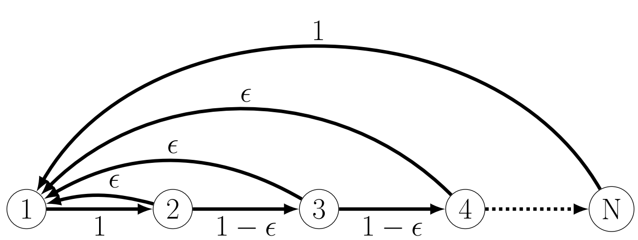


Fig. 3. Embedded Markov chain.

Given $k$ jumps occurred, let

$$\upsilon_{-N} = \Sigma_{i=0}^{k-1}\tau_i \mathbb{1}_{[N-1]}(\sigma_i), \quad \upsilon_N = \Sigma_{i=0}^{k-1}\tau_i \mathbb{1}_N(\sigma_i),$$

denote the total sojourn time in modes $[N-1]$ and mode $N$, respectively. The state transition matrix is

$$\Phi(T_k, 0) = \exp\begin{bmatrix} -T_k\epsilon & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \gamma\upsilon_N - \epsilon\upsilon_{-N} \end{bmatrix},$$

with norm

$$\|\Phi(T_k, 0)\| = \exp\left[\gamma\upsilon_N - \epsilon\upsilon_{-N}\right]. \tag{24}$$

Now, let $k = M(N-1) + h$, where $M$ and $h$ are positive integers such that $k$ satisfies (9). Then,

$$E\left[\log\|\Phi(T_k, 0)\| \mid x(0) = \bar{p}\right] < 0$$

if and only if $E\left[\gamma\upsilon_N - \epsilon\upsilon_{-N} \mid x(0) = \bar{p}\right] < 0$, in view of (24). Since $\tau_i$ are independent random variables, and the expected number of times mode $N$ is visited is minimized if $\sigma_0 = 1$, the *law of the unconscious statistician* (LOTUS) (Durrett 2011) implies

$$\begin{aligned} E\left[\gamma\upsilon_N - \epsilon\upsilon_{-N} \mid x(0) = \bar{p}\right] & \\ \geq E\left[\gamma\upsilon_N - \epsilon\upsilon_{-N} \mid x(0) = e_1\right] & \\ > M\left[\gamma\varrho - \epsilon(1-\varrho)\right] & \qquad (25) \\ > 0, & \qquad (26) \end{aligned}$$

where $\varrho = (1-\epsilon)^{N-2}$, given $\gamma > \epsilon(1-\varrho)\varrho^{-1}$.

At this point, several comments are in order. By construction, $\Sigma_{i=0}^{k-1}\mathbb{1}_N(\sigma_i) \geq X$, where $X \sim \text{Binom}(M, \varrho)$, and inequality (25) follows. That is because

$$P\left(\sigma_{i+(N-m)} = N | \sigma_i = m\right) = \begin{cases} (1-\epsilon)^{N-2}, & m = 1 \\ (1-\epsilon)^{N-m}, & m \geq 2 \end{cases}$$

implies $P(\sigma_i = N, i \in I_m) \geq \varrho$, where

$$I_m = \{m(N-1), \cdots, (m+1)(N-1) - 1\}.$$

In words, every $N-1$ mode transitions the probability of mode $N$ being reached is at least $(1-\epsilon)^{N-2}$, which holds with equality if the initial mode is 1, otherwise the current mode is "closer" to $N$, and the probability of $\sigma_i = m$ reaching $N$ within $N-m$ steps is $(1-\epsilon)^{N-m}$.

Also, note that Markov chain (22) is a worst-case structure in the sense that it is the one for which the expected number of times mode $N$ is visited is minimzed *given the freedom to choose transition probabilities from one state to the other*. Hence, for any general structure, transition probabilities can be chosen such that $M\left[\gamma\varrho - \epsilon(1-\varrho)\right]$ remains a lower bound to $E\left[\gamma\upsilon_N - \epsilon\upsilon_{-N} | \, x(0) = \bar{p}\right]$.

Similarly, (23) can be seen as best-case realizations in the sense that if $S_i$ differ from $-I_n$ and $S_N$ differs from (21), the discrepancies must either be '0' or '+' entries, in which case realizations whose corresponding entries are either $\epsilon$ or $2\epsilon$ can be chosen, so that contradiction (26) still holds since

$$E\left[\|\Phi\left(T_k,0\right)\| - \left(\gamma \upsilon_N - \epsilon \upsilon_{-N}\right) \middle| \, x(0)=\bar{p}\right] \geq 0.$$

ii. $\mathcal{G}_R$ has a cycle.
The construction of a counterexample follows that of the first case, except that instead of a single mode, a chain of modes has to be visited, in a particular order, as in Figure 4. Since the remaining arguments remain almost the same, for the sake of brevity, they are not presented again. □

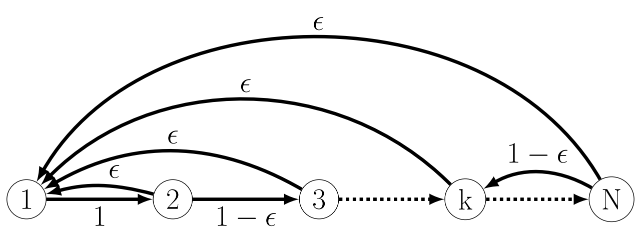


Fig. 4. Embedded Markov chain.

In the same vein, one can consider the equivalence between notions of potential sign-stability.

**Theorem 22** *PMJLS* (10) *is PEM sign-stable if and only if it is PEMS sign-stable.*

**Proof.** Once again, the equivalence follows from shared properties of $R$ and $R'$, namely having only negative or indeterminate diagonal entries. The rest of the argument is the same: PMJLS (1)-(2) with modes $A_i$ and infinitesimal generator $\Pi$ is EMS stable if and only if the PMJLS given by modes $A_i' = A_i \oplus A_i$ and same infinitesimal generator $\Pi$ is EM stable. Because of the similarity of arguments, a detailed proof of this theorem is omitted. □

**Theorem 23** *PMJLS* (10) *is PEM sign-stable if and only if it is PEAS sign-stable.*

**Proof.** If there exists a realization of (10) that is EM stable, that realization must also be EAS stable, which implies necessity.

Conversely, assume (10) PEAS sign-stable, but suppose it is not PEM sign-stable. In view of Theorem 19, it must be that some diagonal element $j$ of every mode $S_m$ is either positive or zero, therefore for any realization with modes $A_i$, of which all off-diagonal elements are positive, given $k$ jumps occurred,

$$\Phi\left(T_k,0\right) \succcurlyeq \prod_{i=0}^{k-1} \exp\left(\operatorname{diag}\left(A_{\sigma_i}\tau_i\right)\right) \succcurlyeq \operatorname{diag}\left(e_j e_j^T\right)$$

so that $\|\Phi\left(T_k,0\right)\| \geq 1$ for all $k$, and by monotonicity

$$E\left[\log\|\Phi\left(T_k,0\right)\| \,\middle|\, x(0)=\bar{p}\right] \geq 0,$$

a contradiction. □

*An interpretation of the Representative Matrix*

In its original statement (Fang & Loparo 2002), Theorem 6 has an additional condition equivalent to EMS stability:

iii. There exist positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, $i=1,2,\cdots,N$ such that inequalities

$$P_i A_i + A_i^T P_i + \sum_{j=1}^{N} \pi_{i,j} P_j < 0 \tag{27}$$

hold for all $i=1,2,\cdots,N$.

Consider an EMS sign-stable PMJLS (10), and $A_i, \Pi$ a realization of such PMJLS. Multiplying $\Pi$ by $\alpha > 0$, obtains realizations $\Pi_\alpha$ parameterized by $\alpha$. $\Pi_\alpha$ and $\Pi$ share the sign structure, therefore $\Pi_\alpha$ must also satisfy (27) for $P_{i,\alpha} > 0$, because (10) is assumed EMS sign-stable. By taking $\alpha \to +\infty$, it must be that $P_{i,\alpha} - P_{j,\alpha} \to 0$, otherwise for some $i$ the sum $\sum_{j=1}^N \pi_{i,j} P_{j,\alpha}$ dominates $P_{i,\alpha}A_i + A_i^T P_{i,\alpha}$, contradicting (27). Denoting the stationary distribution of $\Pi$ by $\beta$, since $\beta^T\Pi = 0$, then

$$\sum_{i=1}^{N} \beta_i \left(\sum_{j=1}^{N} \pi_{i,j} P_{j,\alpha}\right) = \sum_{j=1}^{N}\sum_{i=1}^{N} \left(\beta^T \Pi\right)_i P_{j,\alpha} = 0.$$

Multiplying (27) by $\beta_i$, and summing over $i$ obtains

$$\lim_{\alpha\to+\infty} \sum_{i=1}^{N} \beta_i \left(P_{i,\alpha}A_i + A_i^T P_{i,\alpha}\right) = P A_{avg} + A_{avg}^T P < 0.$$

Therefore, *average matrix* $A_{avg} := \sum_{i=1}^N \beta_i A_i$ is stable, corroborating Theorems 17 and 20, because $A_{avg}$ is in fact *a realization of representative matrix* $R$. The converse is corollary of Theorem 6: sign-stability of $A_{avg}$ implies (10) EMS sign-stable. The same is true of EM and EAS sign-stability because they are sign-equivalent to EMS stability, after Theorems 20 and 21.

## 7 Conclusions

What can be said about stability of a Positive Markov Jump Linear System solely based on feasibility of mode transitions and signs of each entry of each mode? By introducing structural notions of stability based on the concept of sign-stability that parallel standard notions of stochastic stability, this paper provides a clear answer to this question, and establishes equivalence of the introduced notions, which is not true in the standard setting.

There do not exist Exponentially Mean-Square (EMS) stable, Exponentially Mean (EM) stable or Exponentially Almost Sure (EAS) stable positive Markov Jump Linear Systems with irreducible Markov chains, such that a diagonal entry of $A_i$ is nonnegative for every mode. This is a necessary condition on the structure of the matrix, and does not depend on particular realizations of a PMJLS. It is sufficient for EMS, EM and EAS stability that the representative matrix of the system be acyclic and have all-negative diagonal, regardless of the realization. These results imply that if there are no positive diagonal entries, and for every diagonal entry there is at least one mode in which it is negative, and the graph of $\tilde{A}$ and $\hat{A}$ have no cycles, then irrespective of particular $A_i$ the system is EMS, EM and EAS stable.

Although EM stability does not imply EMS stability for general PMJLS, if a PMJLS (with irreducible Markov chain) is EM stable for every realization of sign-matrices, the same is true for EMS stability. Conversely, if no realization of sign-matrices is EMS stable, then none will be EM stable. The same holds for EAS and EM stability: the first does not imply the last, in general, but if a PMJLS is always (never) EAS stable it will always (never) be EM stable. In other words, the three stability notions are not equivalent for particular realizations of a PMJLS, but they are *structurally* equivalent. Finally, one notes the subtle role of the Markov chain in sign-stability of a PMJLS: irreducibility is the key, and although the particular structure of the Markov chain is crucial to quantitative notions of stability, it is irrelevant to their sign-stability counterparts. Indeed, assuring each mode is recurrent is what allows individual modes to be marginally stable even considering the stronger notion of sign-stability. This fact is not true for a general linear hybrid system with arbitrary switching, for it could stay indefinitely in one marginally stable mode. The above observations lead to the conjecture that similar versions of Theorems 17 and 19 hold for general linear hybrid systems with finitely constant modes and "irreducibility" property, to be further investigated.

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## References

Bolzern, P., Colaneri, P. & De Nicolao, G. (2006), 'On Almost sure stability of continuous-time Markov jump linear systems', *Automatica* **42**, 983–988.

Bolzern, P., Colaneri, P. & De Nicolao, G. (2014*a*), Stabilization via switching of positive Markov jump linear systems, *in* 'IEEE Conference on Decision and Control (CDC)', pp. 2359–2364.

Bolzern, P., Colaneri, P. & De Nicolao, G. (2014*b*), 'Stochastic stability of Positive Markov Jump Linear Systems', *Automatica* **50**(4), 1181–1187.

Bremaud, P. (1998), *Markov chains: Gibbs fields, Monte Carlo simulation, and queues.*, New York.

Briat, C. (2017), 'Sign properties of Metzler matrices with applications', *Linear Algebra and its Applications* **515**, 53–86.

Catral, M., Olesky, D. D. & van den Driessche, P. (2009), 'Allow problems concerning spectral properties of sign pattern matrices: A survey', *Linear Algebra and Its Applications* **430**(11-12), 3080–3094.

Colombino, M. & Smith, R. (2016), 'Convex characterization of robust stability analysis and control synthesis for positive linear systems', *Proceedings of the IEEE Conference on Decision and Control* **61**(February), 4379–4384.

Costa, O., Fragoso, M. & Marques, R. (2005), *Discrete-Time Markov Jump Linear Systems*, Springer-Verlag.

Costa, O., Fragoso, M. & Todorov, M. (2013), *Continuous-time Markov jump linear systems*, Springer-Verlag.

De Leenheer, P., Angeli, D. & Sontag, E. (2006), 'Monotone chemical reaction networks', *Journal of Mathematical Chemistry* **41**(3).

Durrett, R. (2011), *Probability: Theory and Examples*, Cambridge Univ. Press.

Fang, Y. & Loparo, K. (2002), 'Stabilization of Continuous-Time Jump Linear', *IEEE Transactions on Automatic Control* **47**(10), 1590–1603.

Farina, L. & Rinaldi, S. (2000), *Positive linear systems—theory and applications*, Wiley & Sons.

Fox, E., Sudderth, E., Jordan, M. & Willsky, A. (2011), 'Bayesian nonparametric inference of switching dynamic linear models', *IEEE Transactions on Signal Processing* **59**(4), 1569–1585.

Gopalakrishnan, K., Balakrishnan, H. & Jordan, R. (2016), Stability of Networked Systems with Switching Topologies, *in* 'IEEE Conference on Decision and Control (CDC)'.

Guo, Y. (2016), 'Stabilization of positive Markov jump systems', *Journal of the Franklin Institute* **353**(14), 3428–3440.

Jacques, J. (1996), *Compartmental Analysis in Biology and Medicine*, 3rd edn, BioMedware.

Jeffries, C., Klee, V. & Van den Driessche, P. (1977), 'When is a matrix sign stable?', *Canadian Journal of Mathematics* (2), 315–326.

Lin, C.-T. (1974), 'Structural Controllability', *IEEE Transactions on Automatic Control* **19**(3), 201–208.

Ogura, M. & Martin, C. (2014), 'Stability Analysis of Positive Semi-Markovian Jump Linear Systems with State Resets', *SIAM Journal on Control and Optimization* **52**(3), 1809–1831.

Ogura, M. & Preciado, V. (2017), 'Optimal Design of Switched Networks of Positive Linear Systems via Geometric Programming', *IEEE Transactions on Control of Network Systems* **4**(2), 213–222.

Shafai, B., Chen, J. & Kothandaraman, M. (1997), 'Explicit formulas for stability radii of nonnegative and Metzlerian matrices', *IEEE Transactions on Automatic Control* **42**(2), 265–270.

Son, N. & Hinrichsen, D. (1996), 'Robust Stability of Positive Continuous-time Systems', *Numerical Functional Analysis and Optimization* **17**(5-6), 1–13.

Vandergraft, J. (1968), 'Spectral properties of matrices which have invariant cones', *SIAM Journal of Applied Mathematics* **16**(6).

Vidal, R., Chiuso, A. & Soatto, S. (2002), Observability and identifiability of jump linear systems, *in* 'IEEE Conference on Decision and Control (CDC)', pp. 3614–3619.

Wang, Q., Wang, T., Hou, Y. & Dong, C. (2014), 'Sign Stability for Switched Linear Systems and Its Application in Flight Control', *Journal of Applied Mathematics* .

Zhou, K. & Doyle, J. (1999), *Essentials of Robust Control*.