6.S890: Topics in Multiagent Learning

Lecture 6 – Prof. Daskalakis

Fall 2023

Nash Equilibrium Existence:
two-player zero-sum games

[von Neumann '28:]
In finite two-player zero-sum games $(R, C = -R)_{m \times n}$:

$$\min_{x \in \Delta^m} \max_{y \in \Delta^n} x^T C y = \max_{y \in \Delta^n} \min_{x \in \Delta^m} x^T C y$$

Corollary: A Nash equilibrium exists in finite two-player zero-sum games

[original proof used fixed point arguments]

0,0	-1,1	1,-1
1,-1	0,0	-1 , 1
-1,1	1 , -1	0,0

Min-max Equilibrium Computation ⟺ Linear Programming

[Danzig '47]

[Adler '13]
[Brooks-Reny'21]
[von Stengel'22]

No-regret Learning

Nash Equilibrium Existence:
general games

[John Nash ’50]: A Nash equilibrium exists in **every** finite game.

Deep influence in Economics, enabling other existence results.

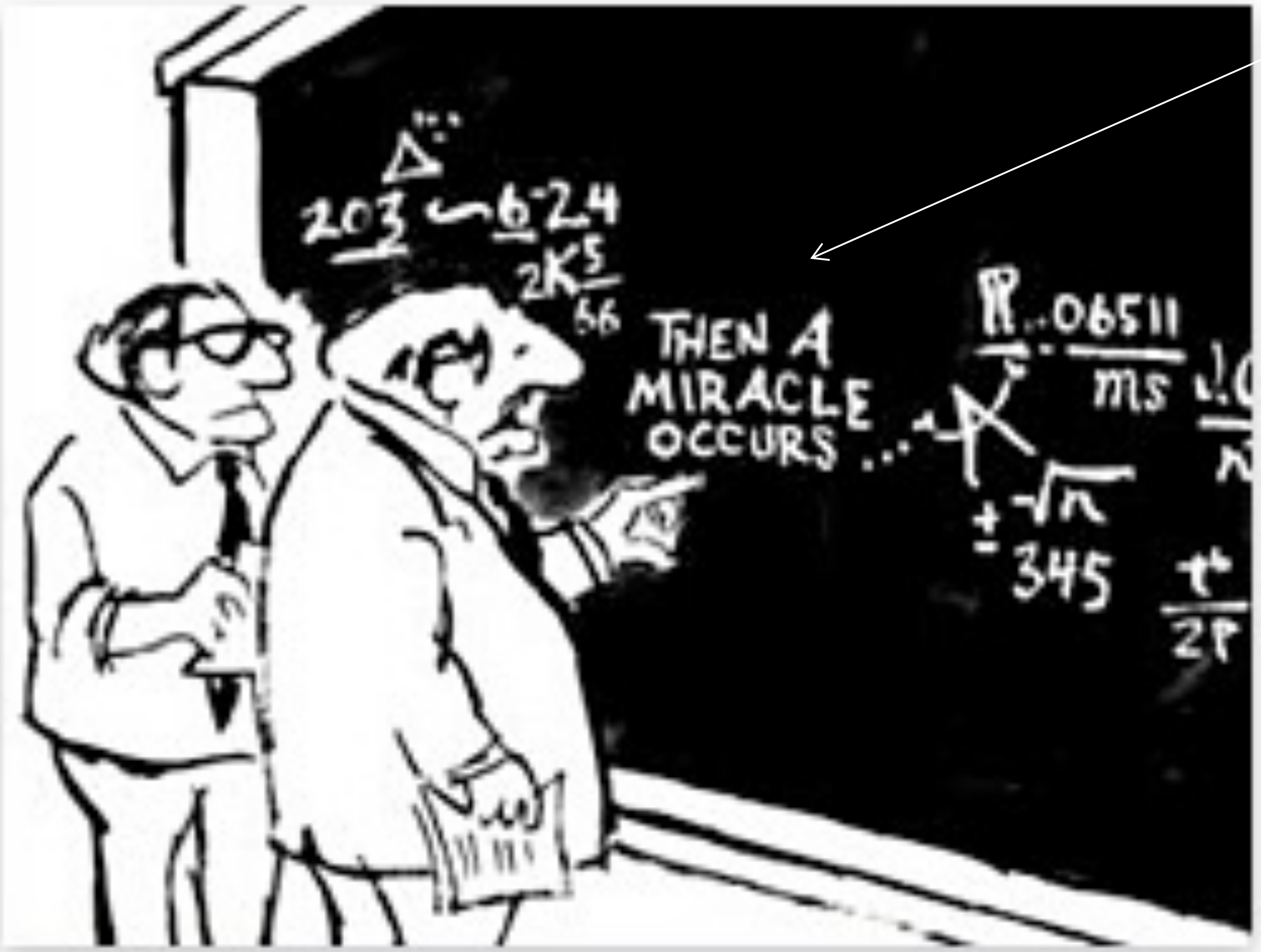
Proof non-constructive (uses Brouwer’s fixed point theorem)

No simpler proof has been discovered

[Daskalakis-Goldberg-Papadimitriou’06]: no simpler proof exists

i.e. **Nash Equilibrium Computation** ⟺ **Fixed Point Computation**

The non-constructive step?



what is the nature of non-constructiveness in the heart of Nash's theorem?

Menu

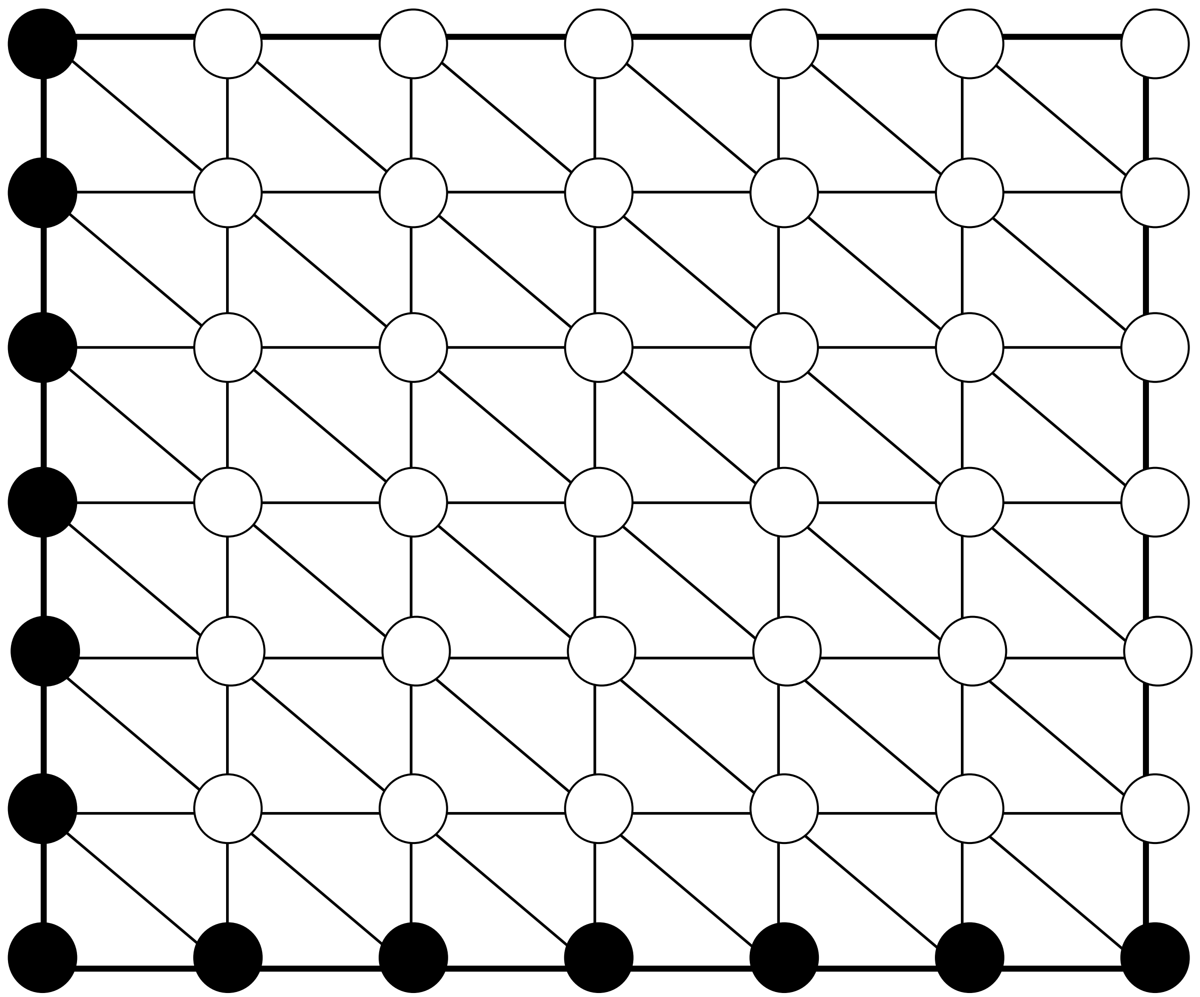
- **Refresher: Nash, von Neumann & Brouwer**
- Sperner's Lemma
- Brouwer via Sperner
- Sperner's Proof

Menu

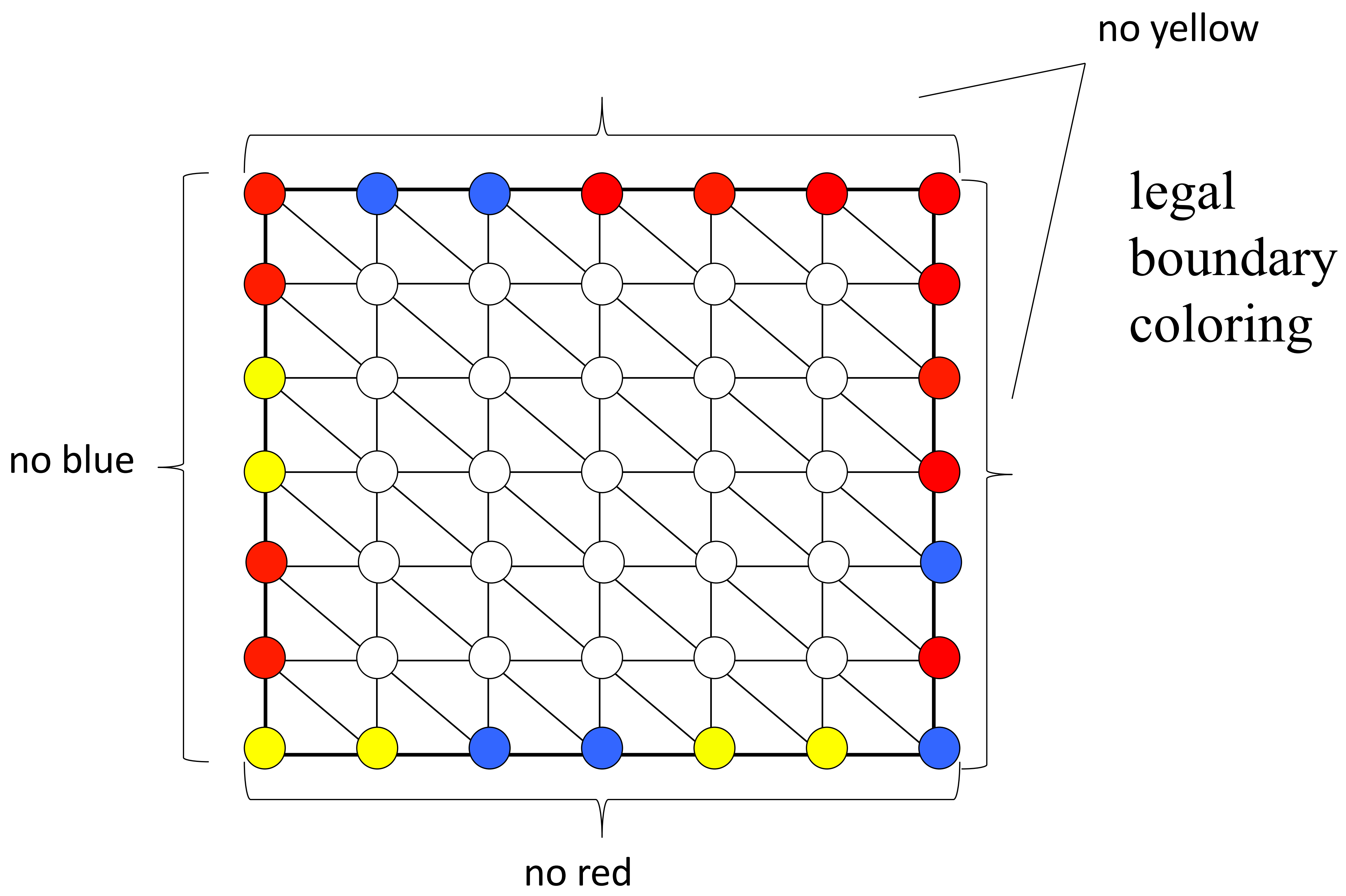
- **Refresher: Nash, von Neumann & Brouwer**
- **Sperner’s Lemma**
- Brouwer via Sperner
- Sperner’s Proof


Sperner

Sperner's Lemma (2-d)

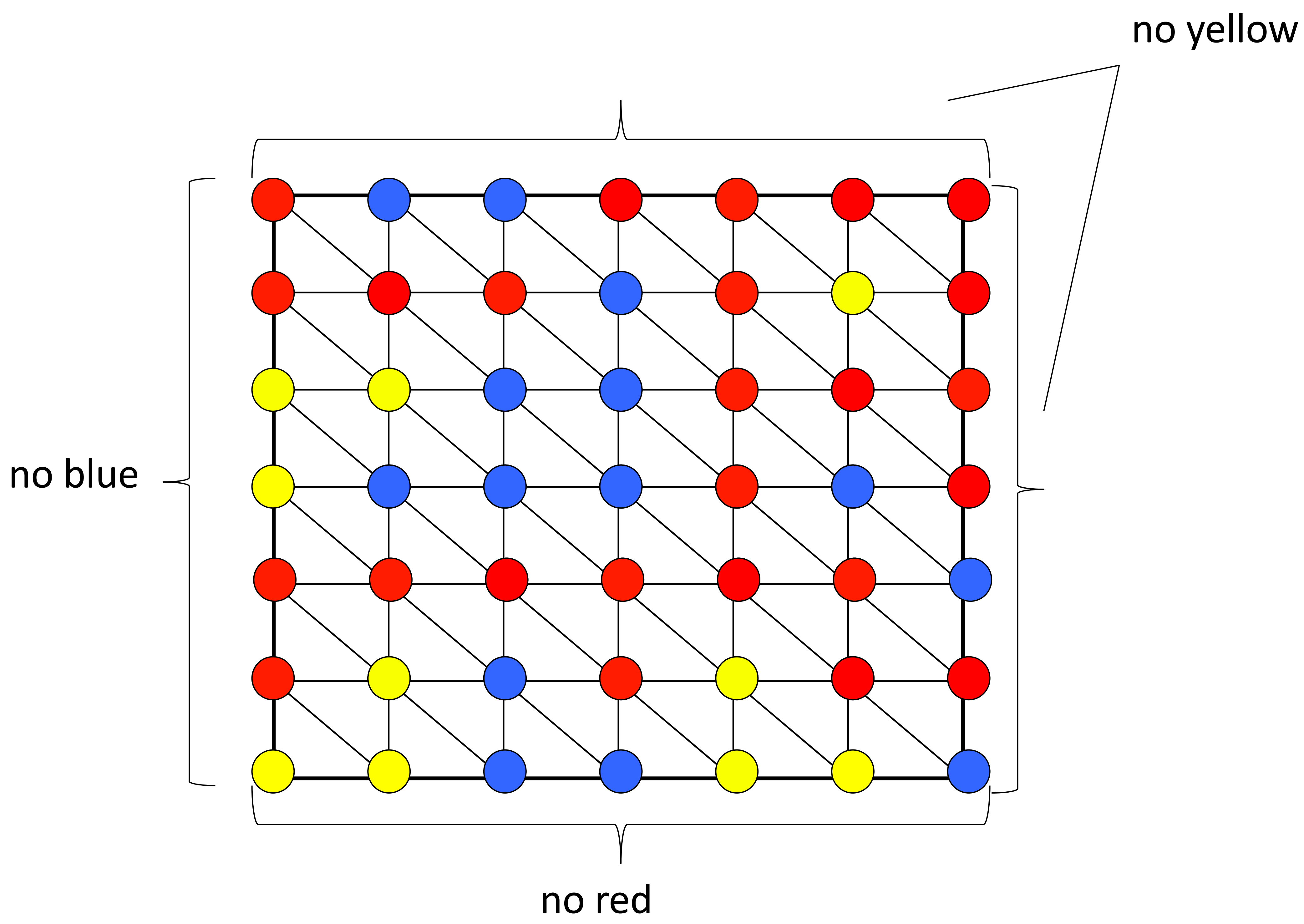


Sperner's Lemma (2-d)



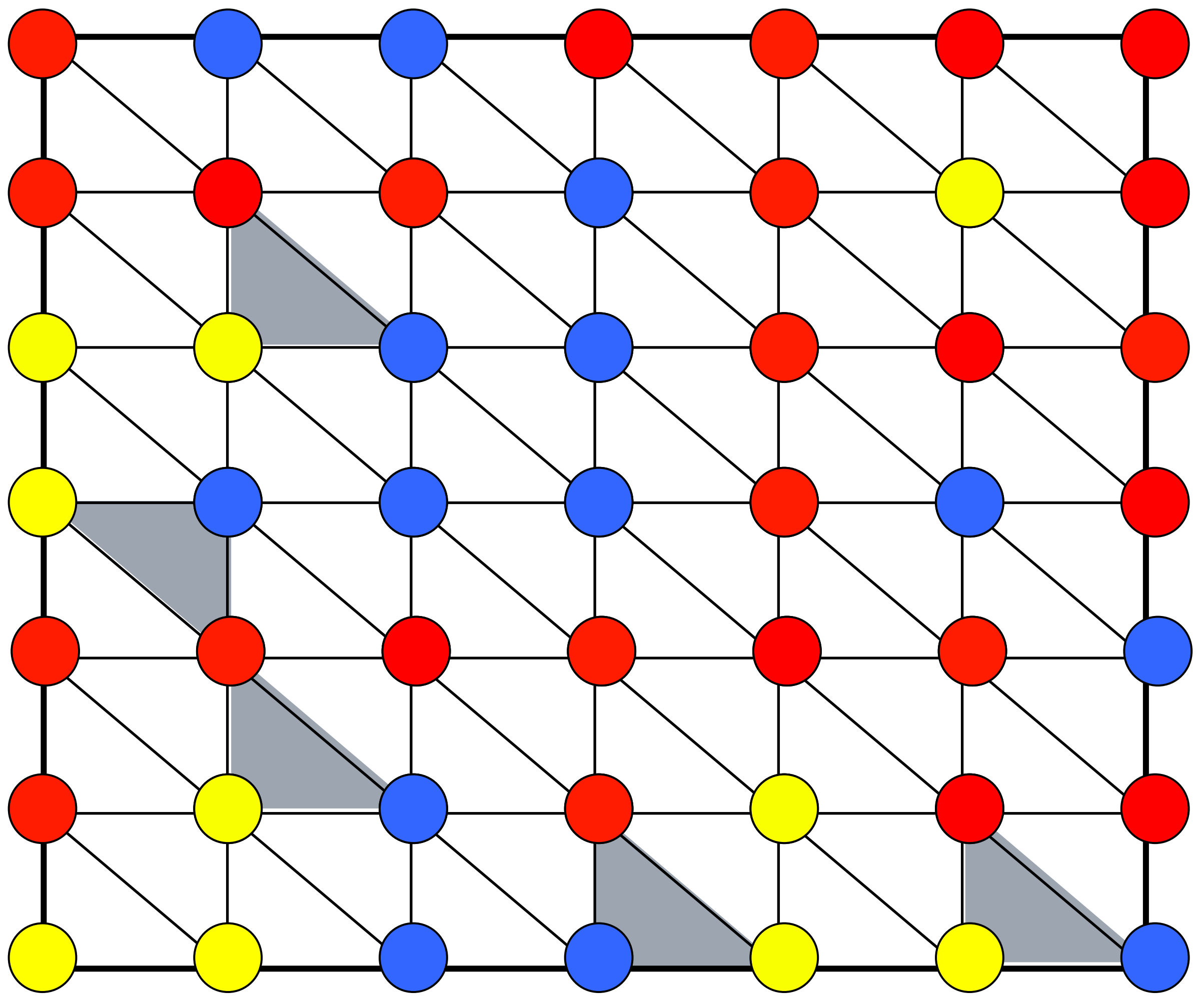
[Sperner 1928]: Color the boundary using three colors in a legal way.

Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Menu

- **Refresher: Nash, von Neumann & Brouwer**
- **Sperner’s Lemma**
- Brouwer via Sperner
- Sperner’s Proof

Menu

- **Refresher: Nash, von Neumann & Brouwer**
- **Sperner's Lemma**
- **Brouwer via Sperner**
- Sperner's Proof

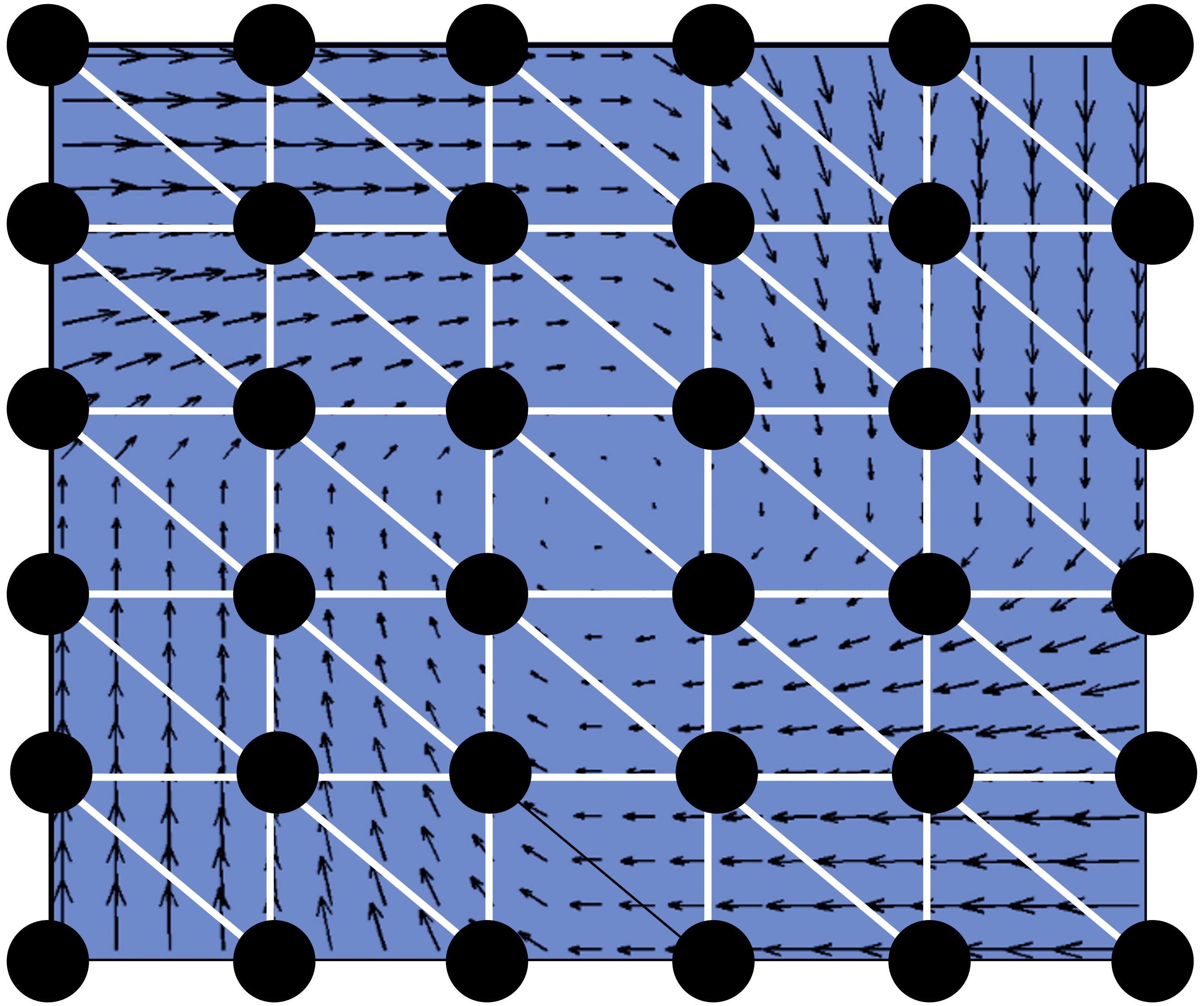

Sperner ⇒ Brouwer

Sperner $\Rightarrow$ Brouwer (High-Level)

Given continuous $f : [0,1]^2 \to [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$;

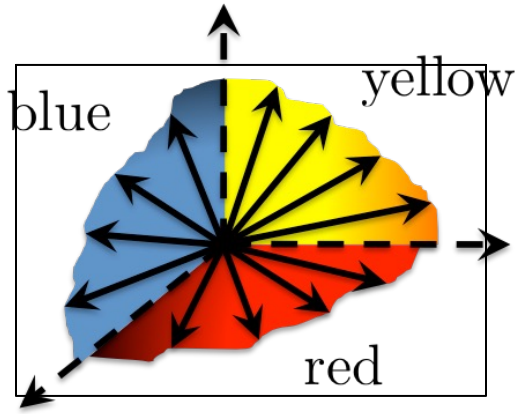


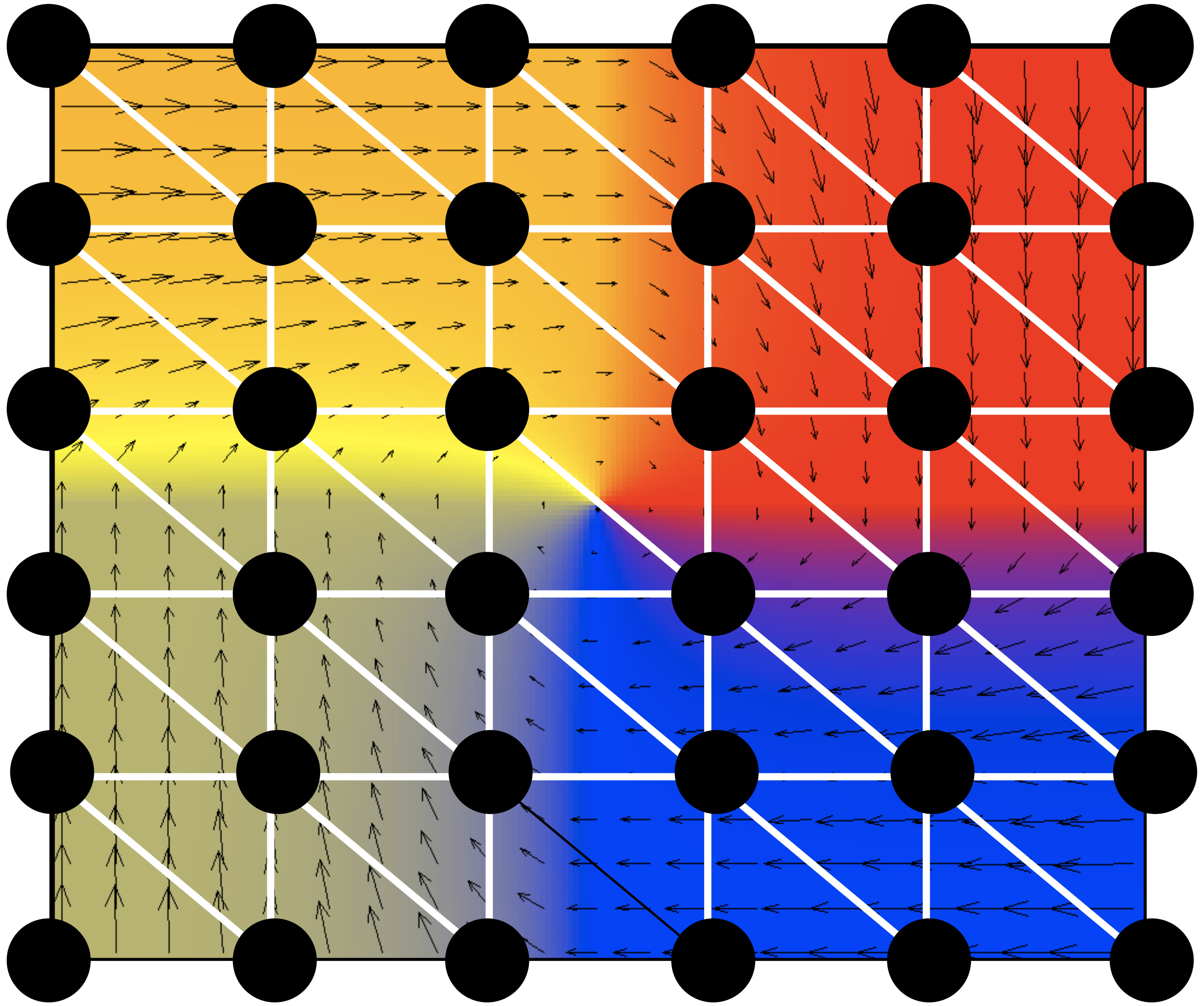
Sperner $\Rightarrow$ Brouwer (High-Level)

Given continuous $f : [0,1]^2 \to [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$; then color points according to the direction of $f(x)-x$;

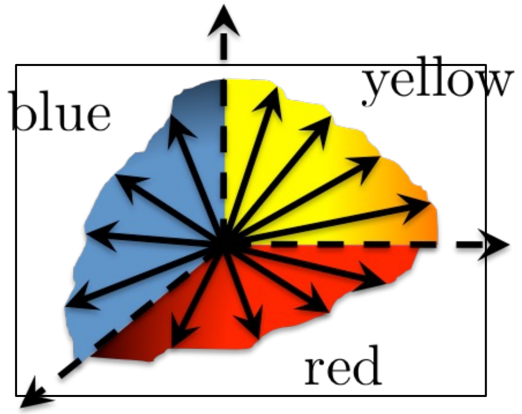


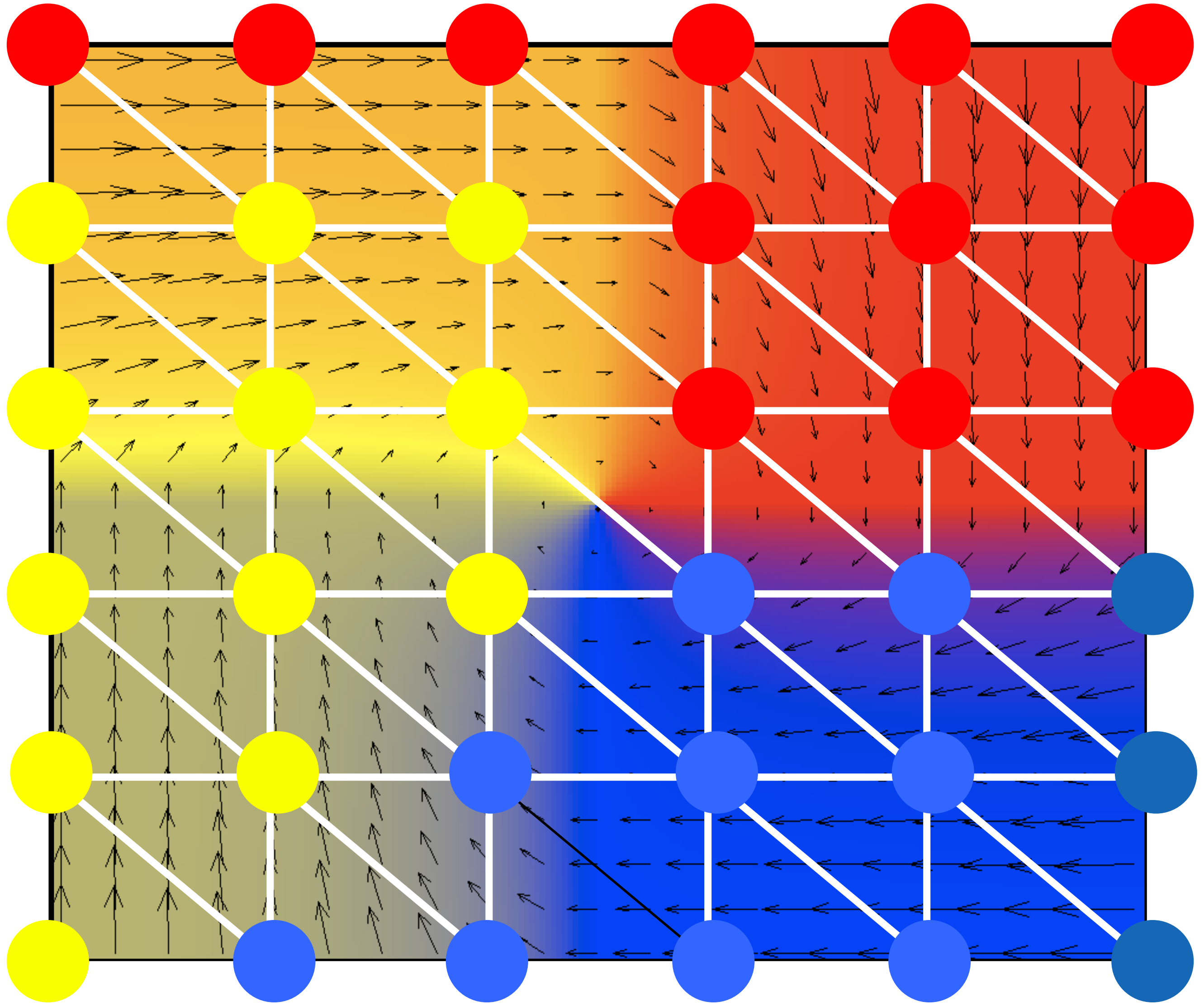


Sperner ⇒ Brouwer (High-Level)

Given continuous $f : [0,1]^2 \rightarrow [0,1]^2$
1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$; then color points according to the direction of $f(x)-x$;



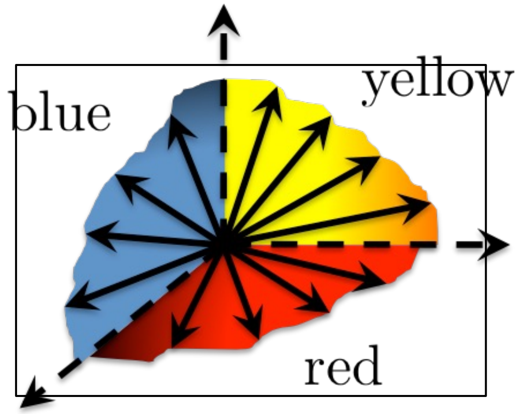


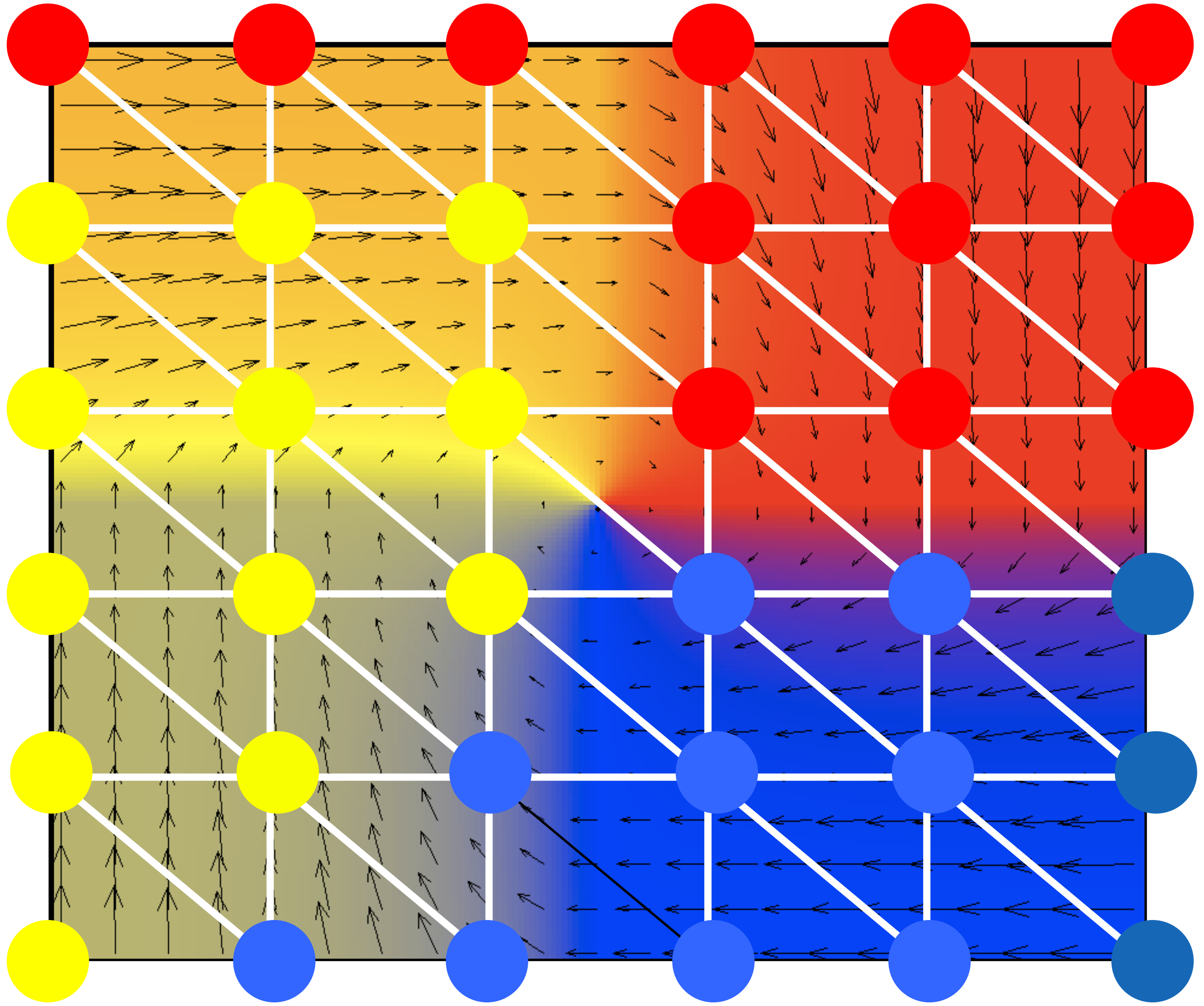
Sperner $\Rightarrow$ Brouwer (High-Level)

Given continuous $f : [0,1]^2 \to [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$; then color points according to the direction of $f(x)-x$; apply Sperner and argue trichromatic triangle contains approximate fixed points





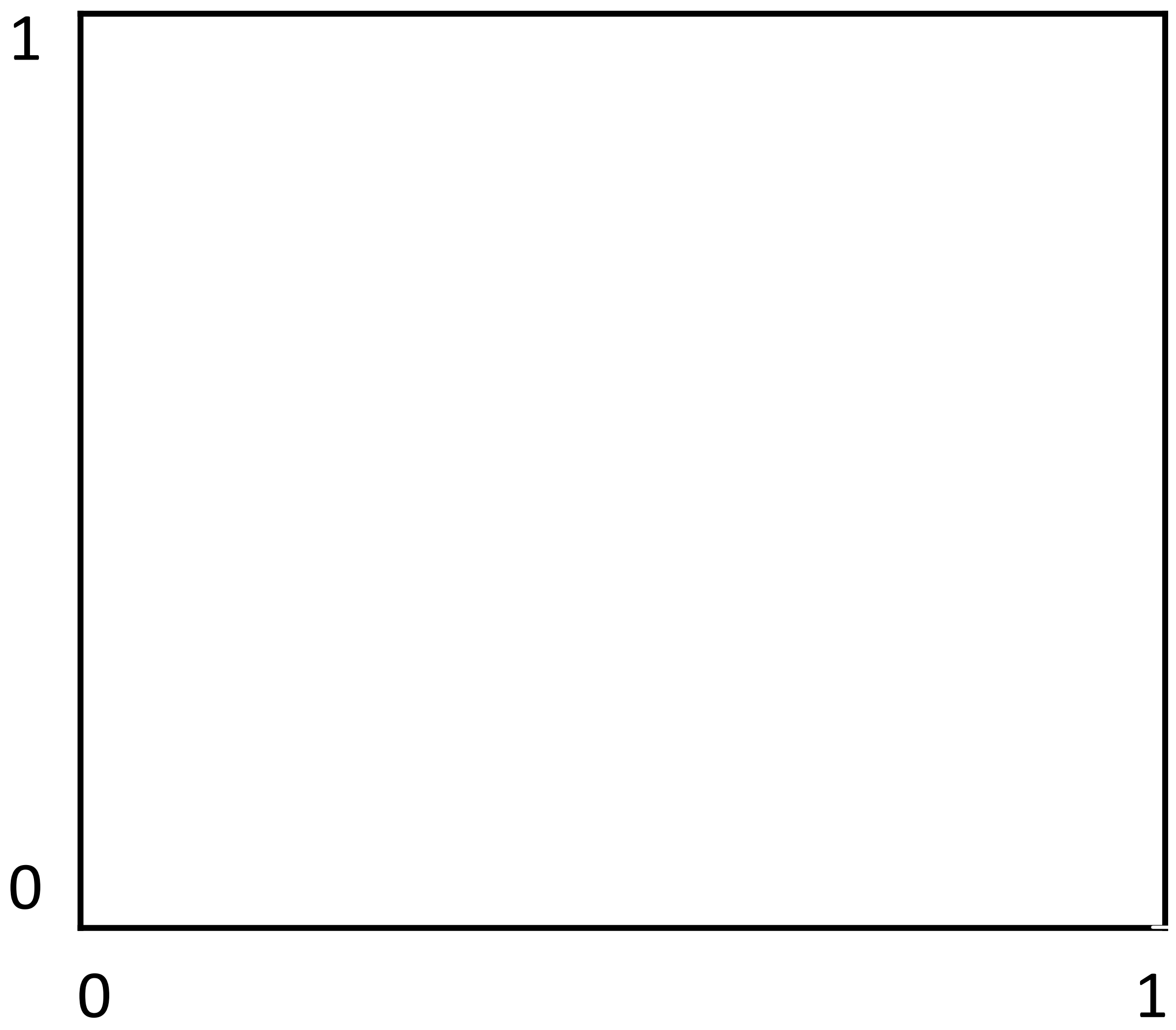
2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \to [0,1]^2$, continuous

↳ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z, w) < \delta(\epsilon) \Longrightarrow d(f(z), f(w)) < \epsilon$$



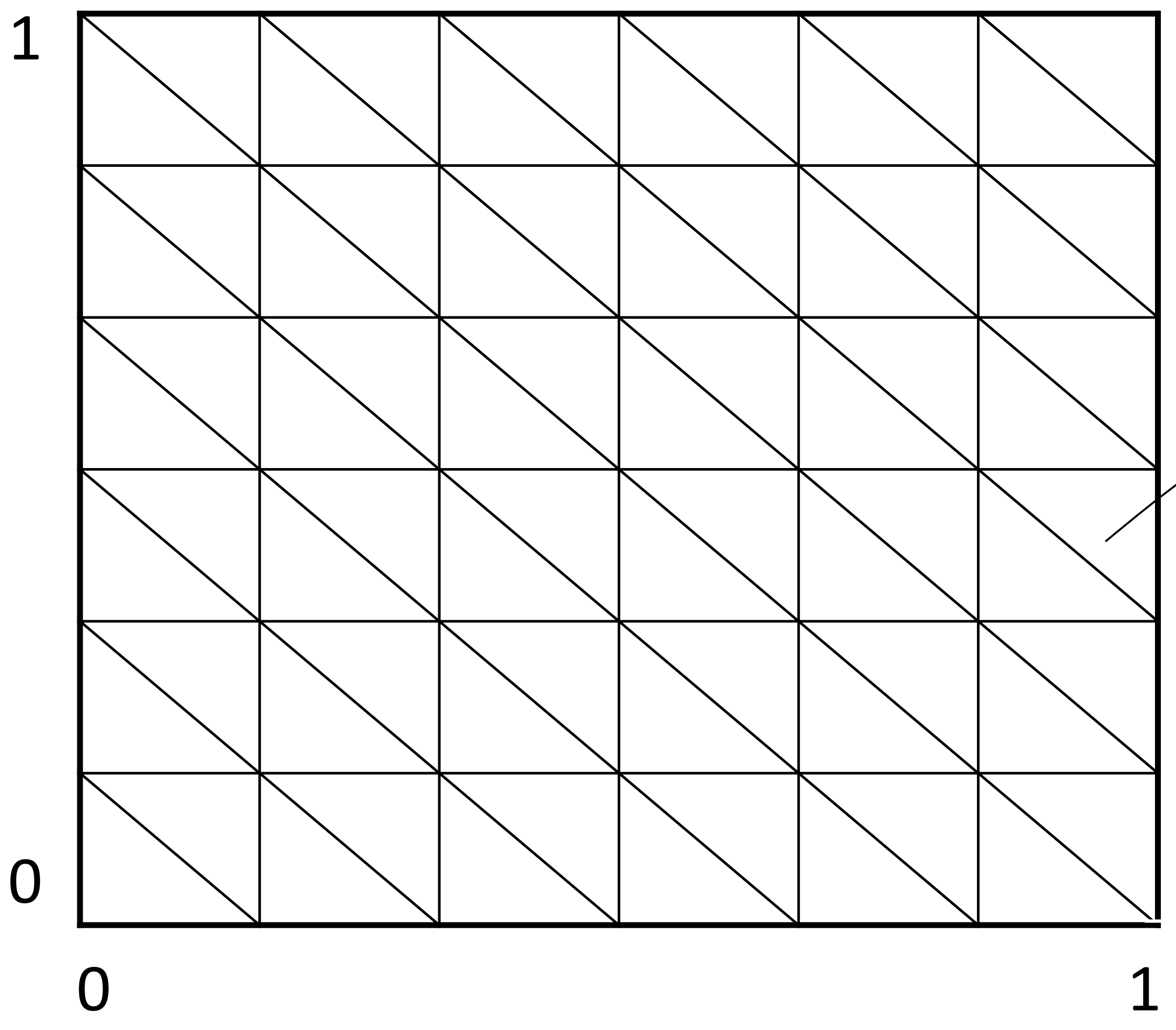
2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

→ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z, w) < \delta(\epsilon) \Longrightarrow d(f(z), f(w)) < \epsilon$$



choose some ϵ and triangulate so that the diameter of cells is

$\delta < \delta(\epsilon)$

2D-Brouwer on the Square

say d is the ℓ_∞ norm

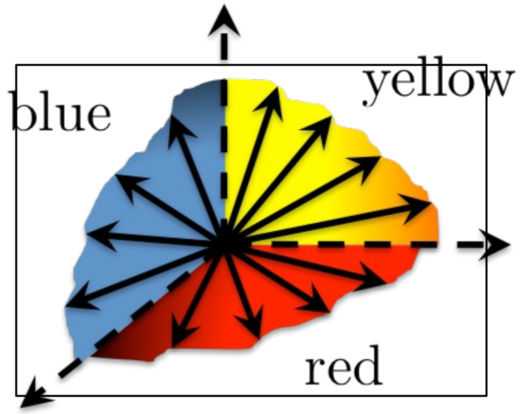
Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

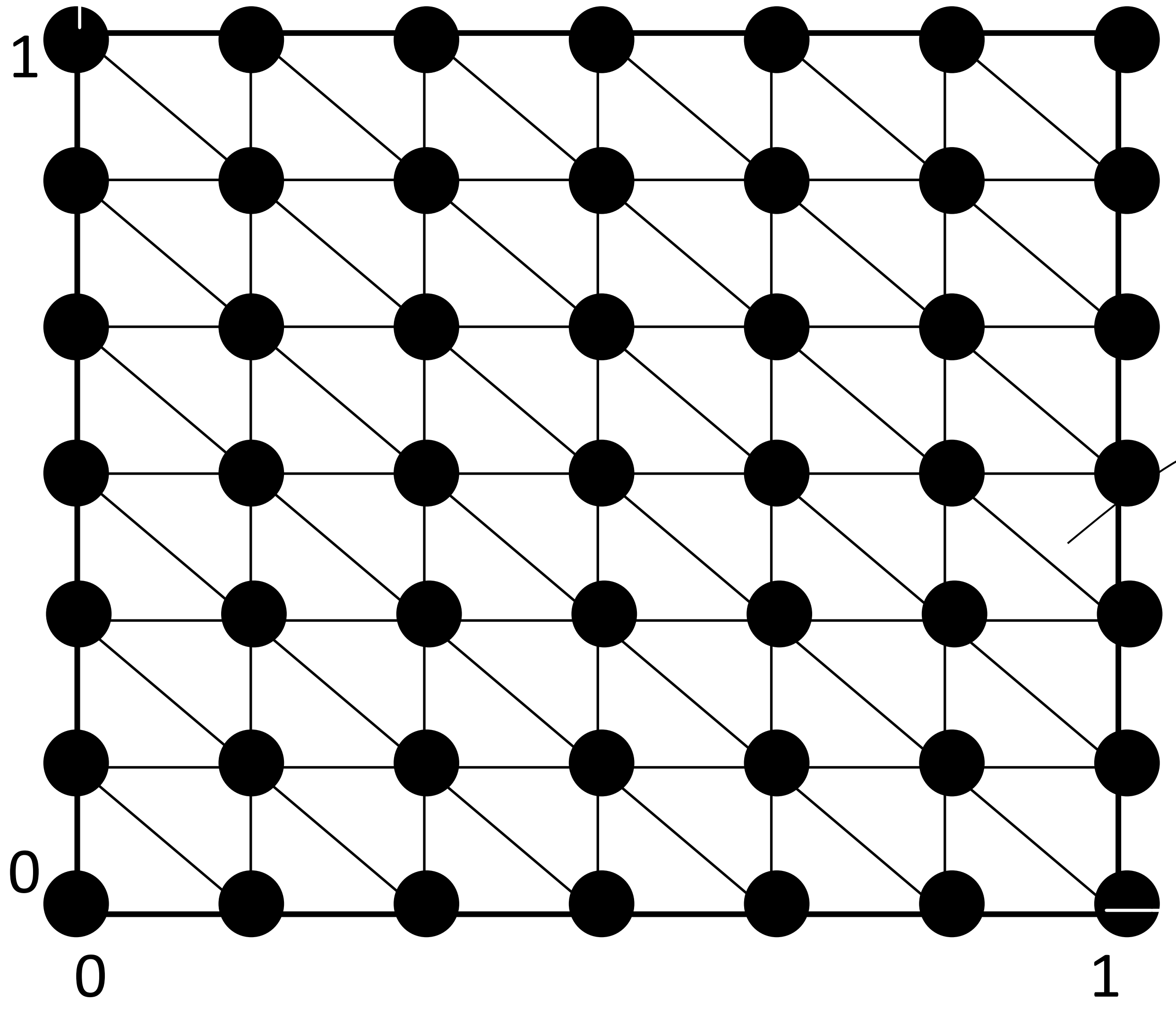
must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, s.t.$$

$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

color the nodes of the triangulation according to the direction of $f(x) - x$





choose some ϵ and triangulate so that the diameter of cells is $\delta < \delta(\epsilon)$

2D-Brouwer on the Square

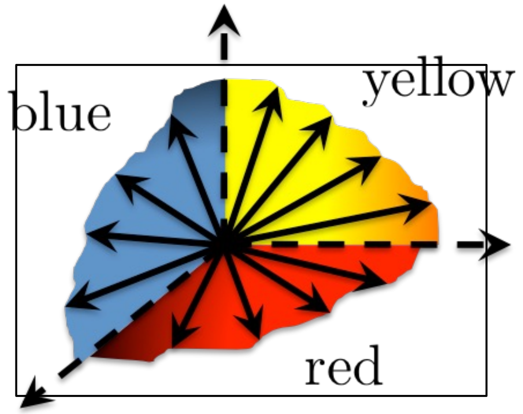
say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

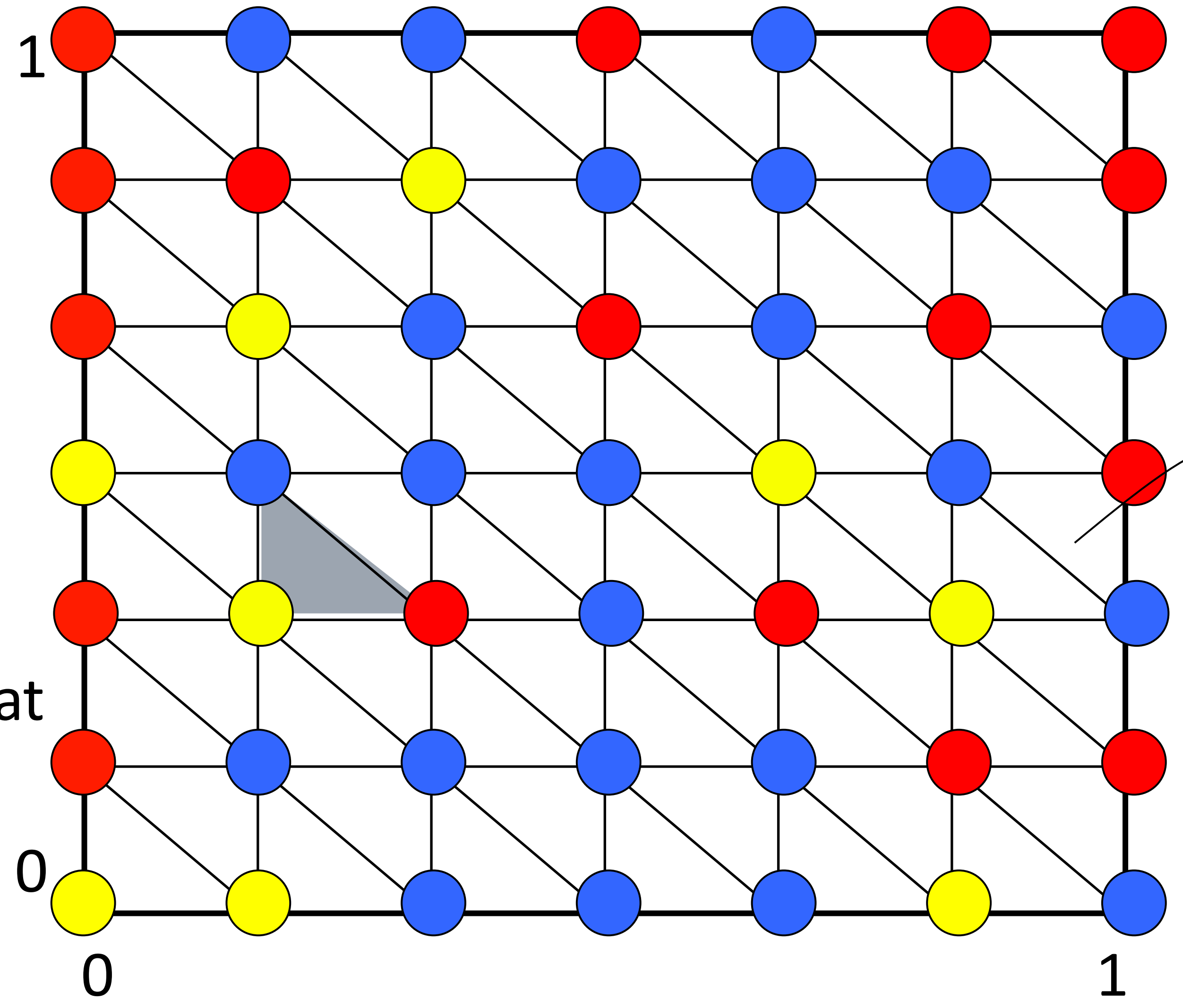
→ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

color the nodes of the triangulation according to the direction of $f(x) - x$



(tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions required by Sperner's lemma)



choose some ϵ and triangulate so that the diameter of cells is

$$\delta < \delta(\epsilon)$$

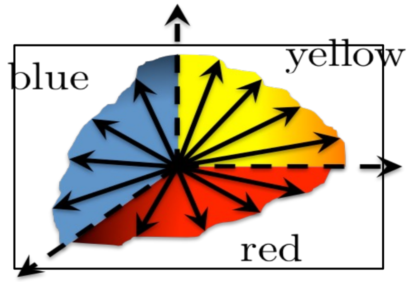
find a trichromatic triangle, guaranteed by Sperner

2D-Brouwer on the Square

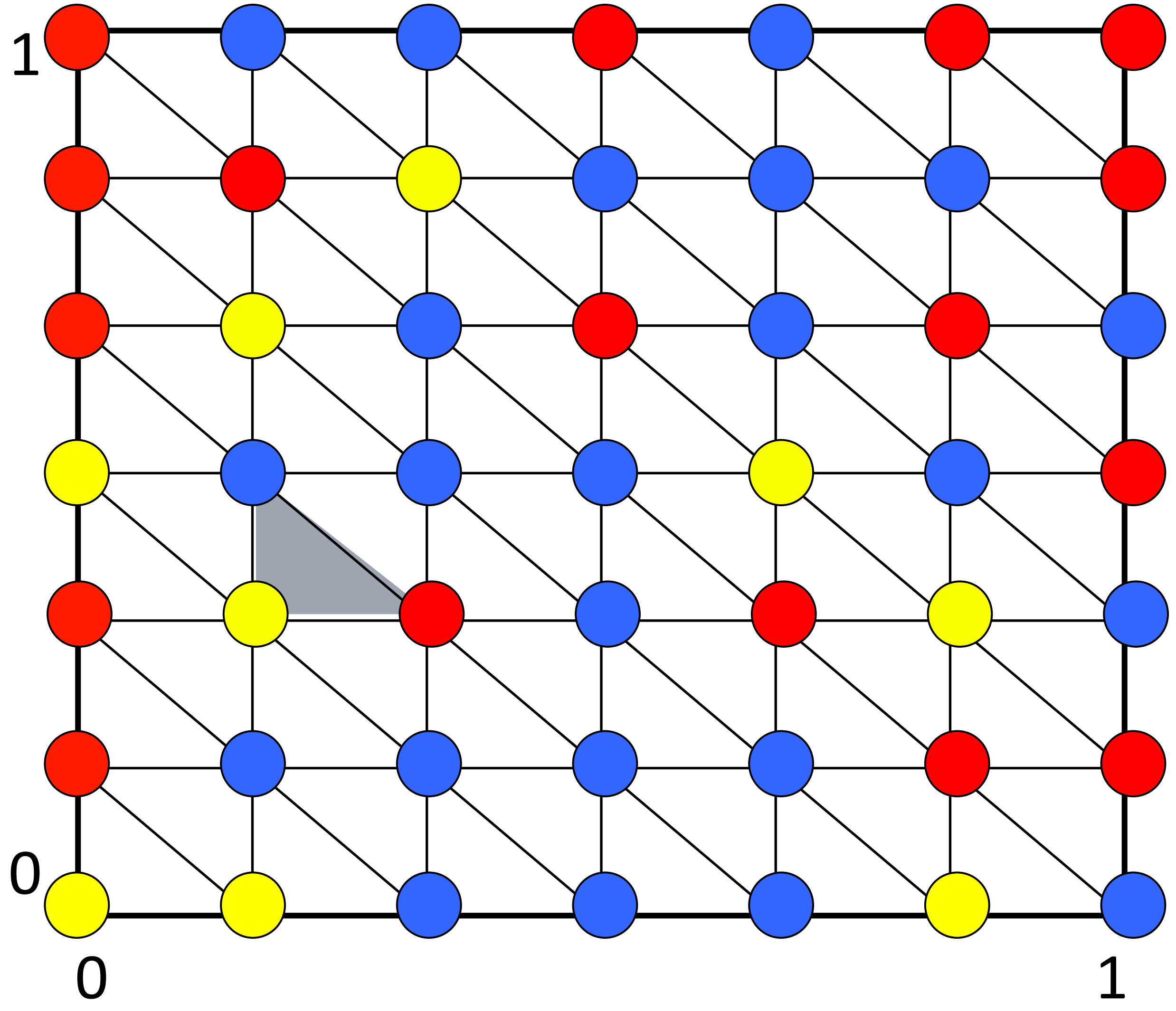
Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

say d is the ℓ_∞ norm

→ must be uniformly continuous (by the Heine-Cantor theorem)



$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$



Claim: If z^Y is the yellow corner of a trichromatic triangle, then

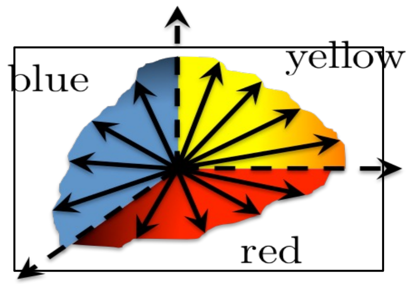
$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

Proof of Claim

Claim: If z^Y is the yellow corner of a trichromatic triangle, then $|f(z^Y) - z^Y|_\infty < \epsilon + \delta.$

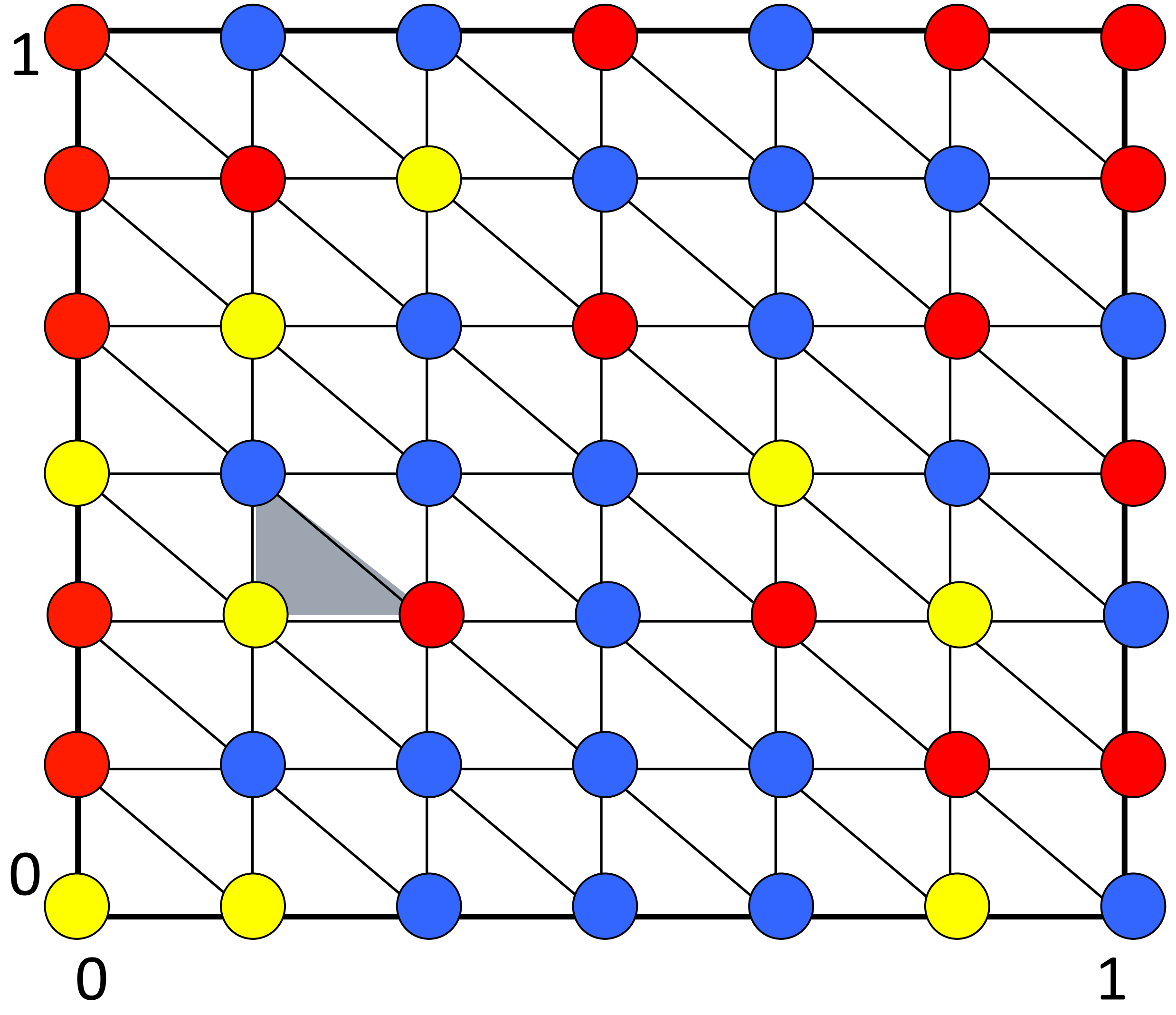
Proof: Let z^Y, z^R, z^B be the yellow/red/blue corners of a trichromatic triangle.

By the definition of the coloring, observe that the product of



$$(f(z^Y) - z^Y)_x \text{ and } (f(z^B) - z^B)_x \text{ is } \leq 0.$$

Hence:



$$\begin{aligned}
&|(f(z^Y) - z^Y)_x| \\
&\quad \leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x| \\
&\quad \leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x| \\
&\quad \leq d(f(z^Y), f(z^B)) + d(z^Y, z^B) \\
&\quad \leq \epsilon + \delta.
\end{aligned}$$

Similarly, we can show:

$$|(f(z^Y) - z^Y)_y| \leq \epsilon + \delta.$$

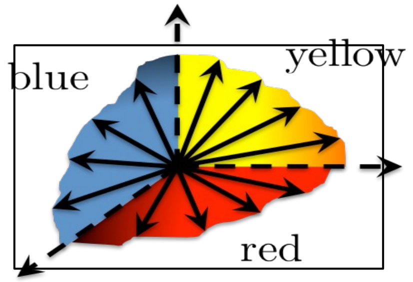
∎

2D-Brouwer on the Square

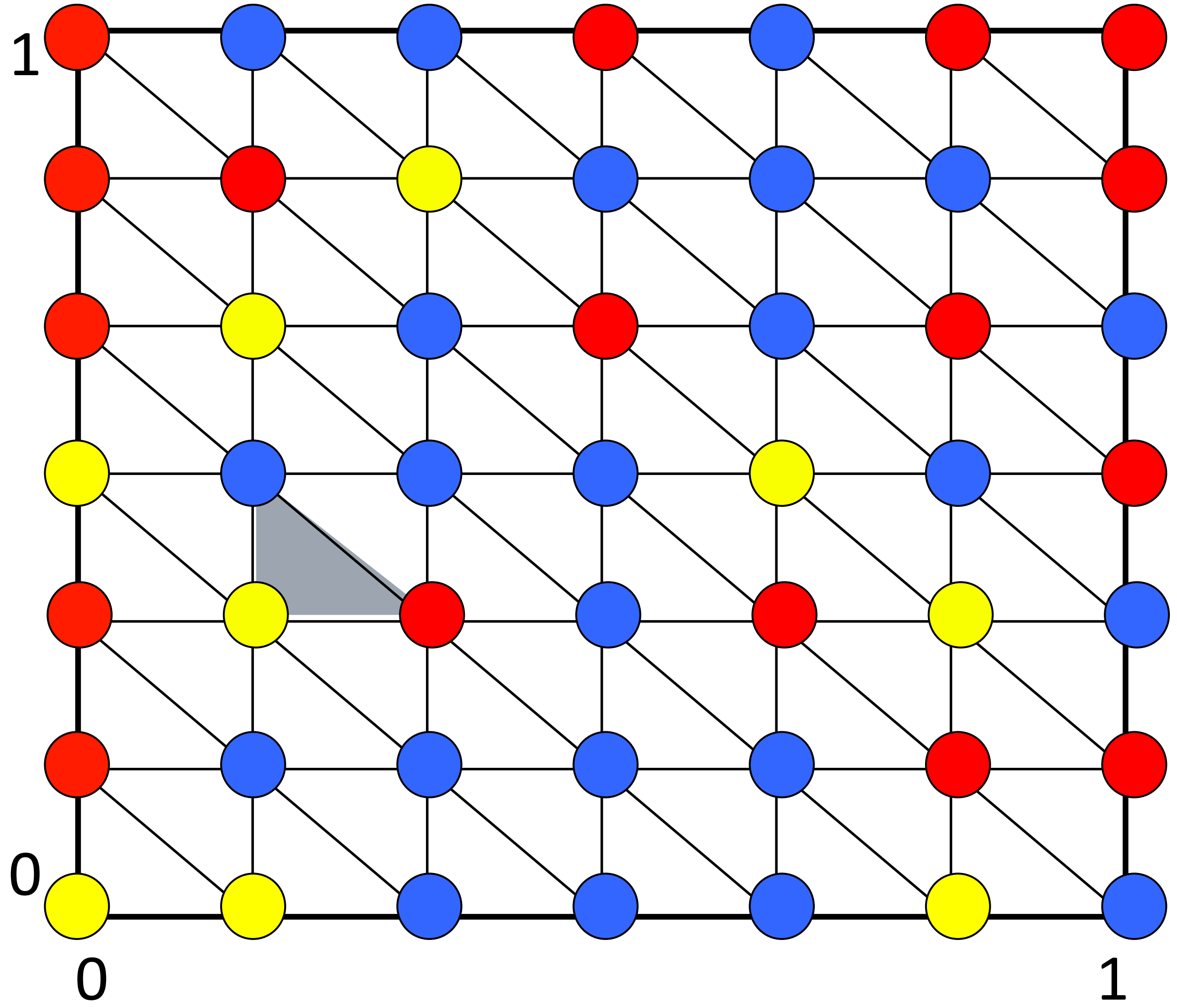
say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \to [0,1]^2$, continuous

→ must be uniformly continuous (by the Heine-Cantor theorem)



$$\forall \epsilon > 0,\ \exists \delta(\epsilon) > 0, s.t.$$
$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$



Claim: If z^Y is the yellow corner of a trichromatic triangle, then

$$|f(z^Y) - z^Y|_\infty < \epsilon + \delta.$$

Choosing $\delta = \min(\delta(\epsilon), \epsilon)$

$$|f(z^Y) - z^Y|_\infty < 2\epsilon.$$

2D-Brouwer on the Square

Finishing the proof of Brouwer's Theorem (Compactness):

- pick a sequence of epsilons: $\epsilon_i = 2^{-i}, i = 1, 2, \ldots$
- define a sequence of triangulations of diameter: $\delta_i = \min(\delta(\epsilon_i), \epsilon_i), i = 1, 2, \ldots$
- pick a trichromatic triangle in each triangulation, and call its yellow corner $z_i^{\mathrm{Y}}, i = 1, 2, \ldots$
- by compactness, this sequence has a converging subsequence $w_i, \quad i = 1, 2, \ldots$ with limit point w^*

Claim: $f(w^*) = w^*.$

Proof: Define the function $g(x) = d(f(x), x)$. Clearly, g is continuous since $d(\cdot, \cdot)$ is continuous and so is f. It follows from continuity that

$$g(w_i) \longrightarrow g(w^*), \text{ as } i \to +\infty.$$

But $0 \le g(w_i) \le 2^{-i+1}$. Hence, $g(w_i) \longrightarrow 0$. It follows that $g(w^*) = 0$.

Therefore, $d(f(w^*), w^*) = 0 \implies f(w^*) = w^*.$ ∎

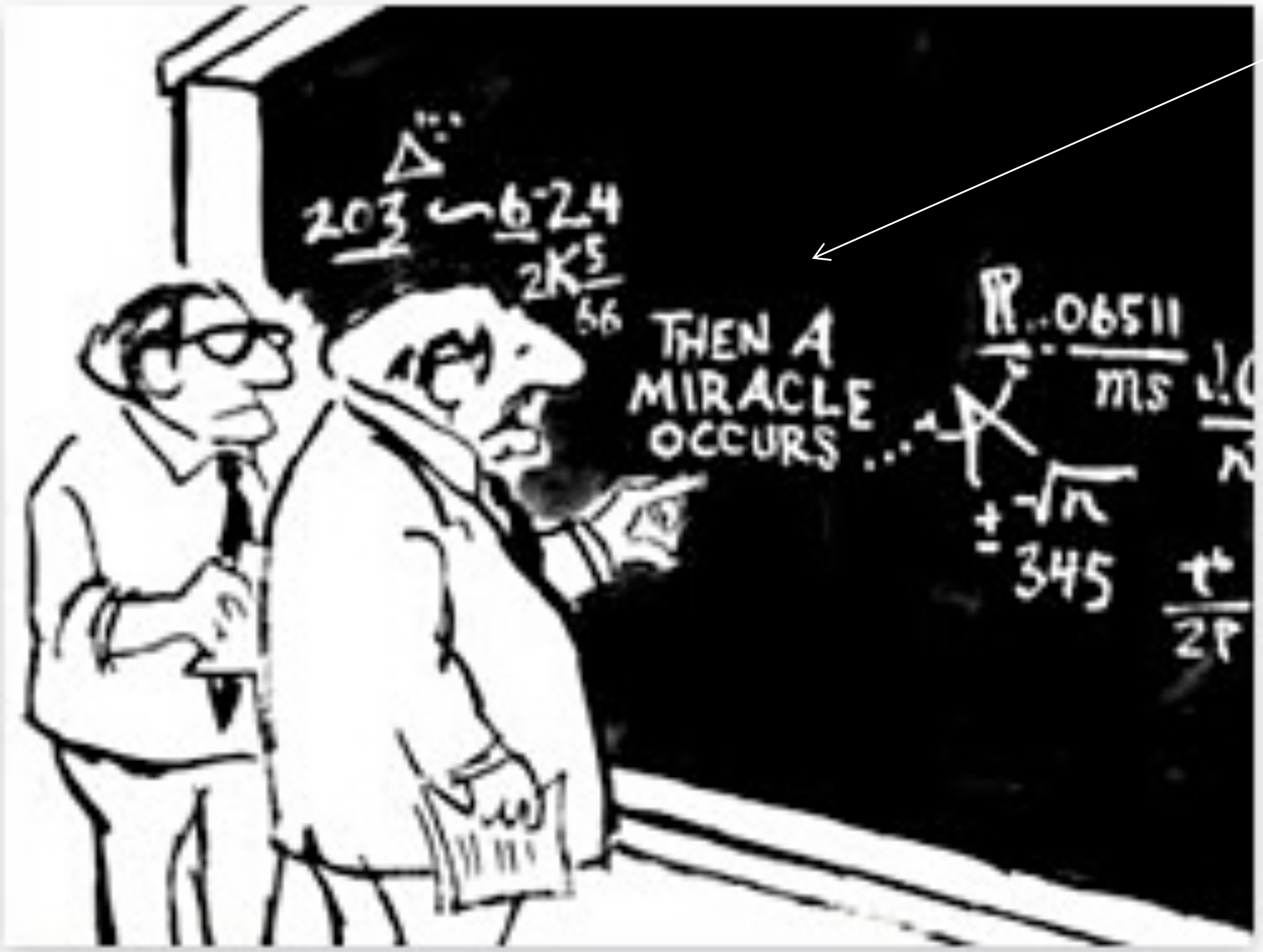
Menu

- Refresher: Nash, von Neumann & Brouwer
- **Sperner’s Lemma**
- **Brouwer via Sperner**
- Sperner’s Proof

Menu

- Refresher: Nash, von Neumann & Brouwer
- **Sperner's Lemma**
- **Brouwer via Sperner**
- **Sperner's Proof**

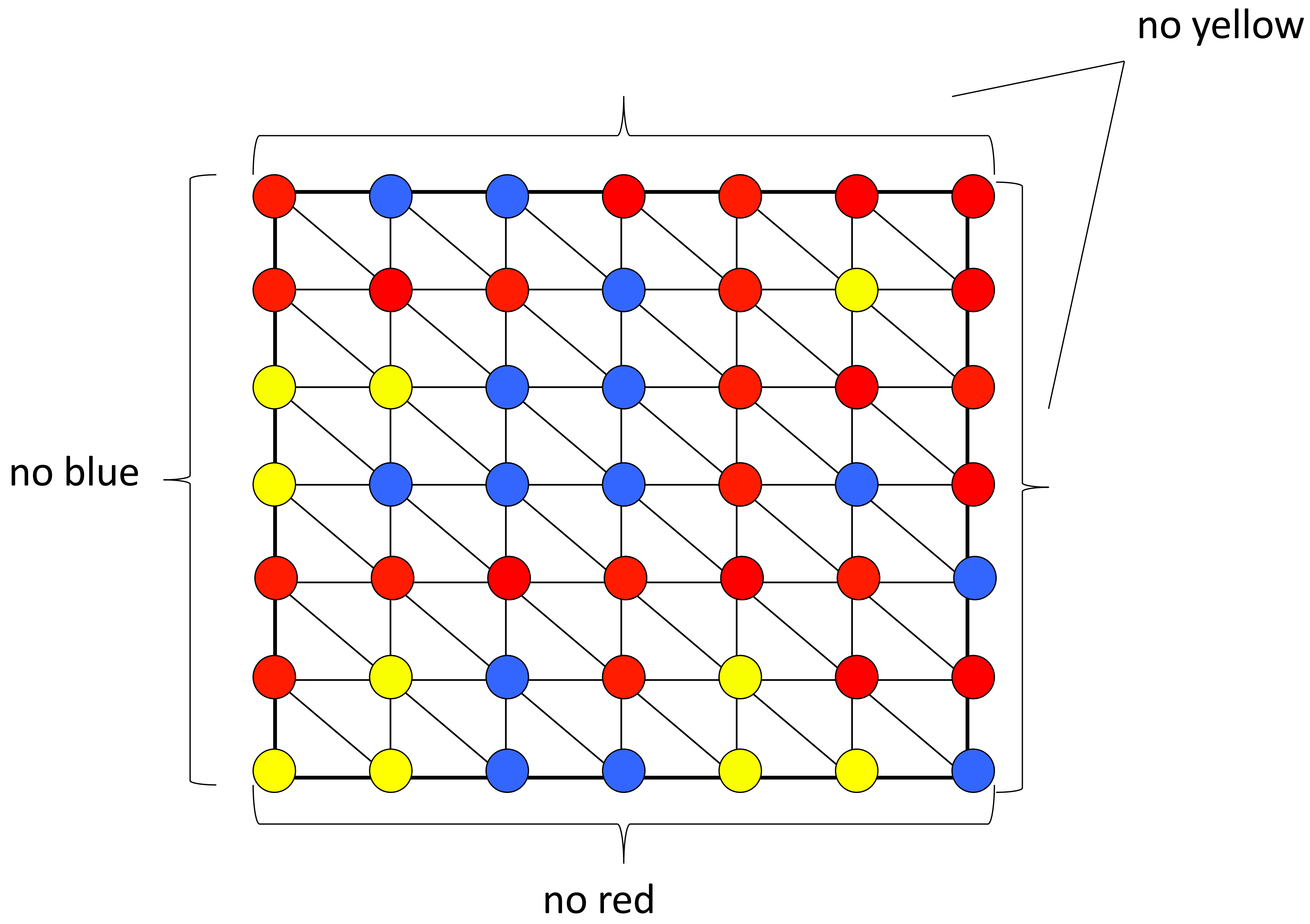
The non-constructive step?



what is the nature of non-constructiveness in the heart of Nash's theorem?

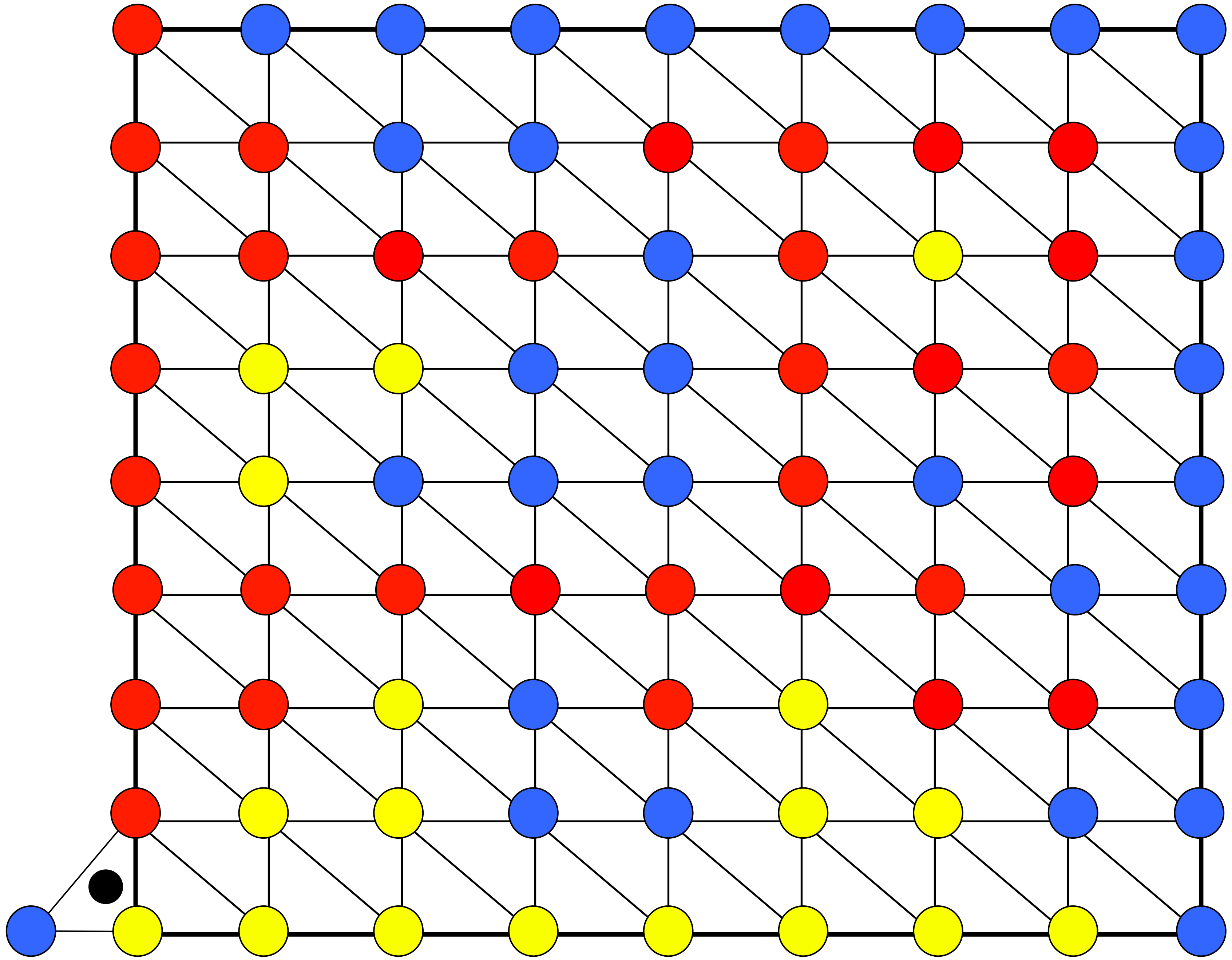
So far: Sperner's Theorem ⇒ Brouwer's Theorem ⇒ Nash's Theorem

Proof of Sperner's Lemma



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

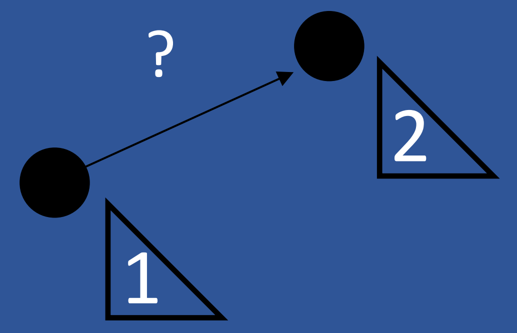
We also introduce an artificial tri-chromatic triangle.

Next we define a directed walk starting from the artificial tri-chromatic triangle.

[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Transition Rule: *If ∃ red - **yellow** door cross it with red on your left hand.*



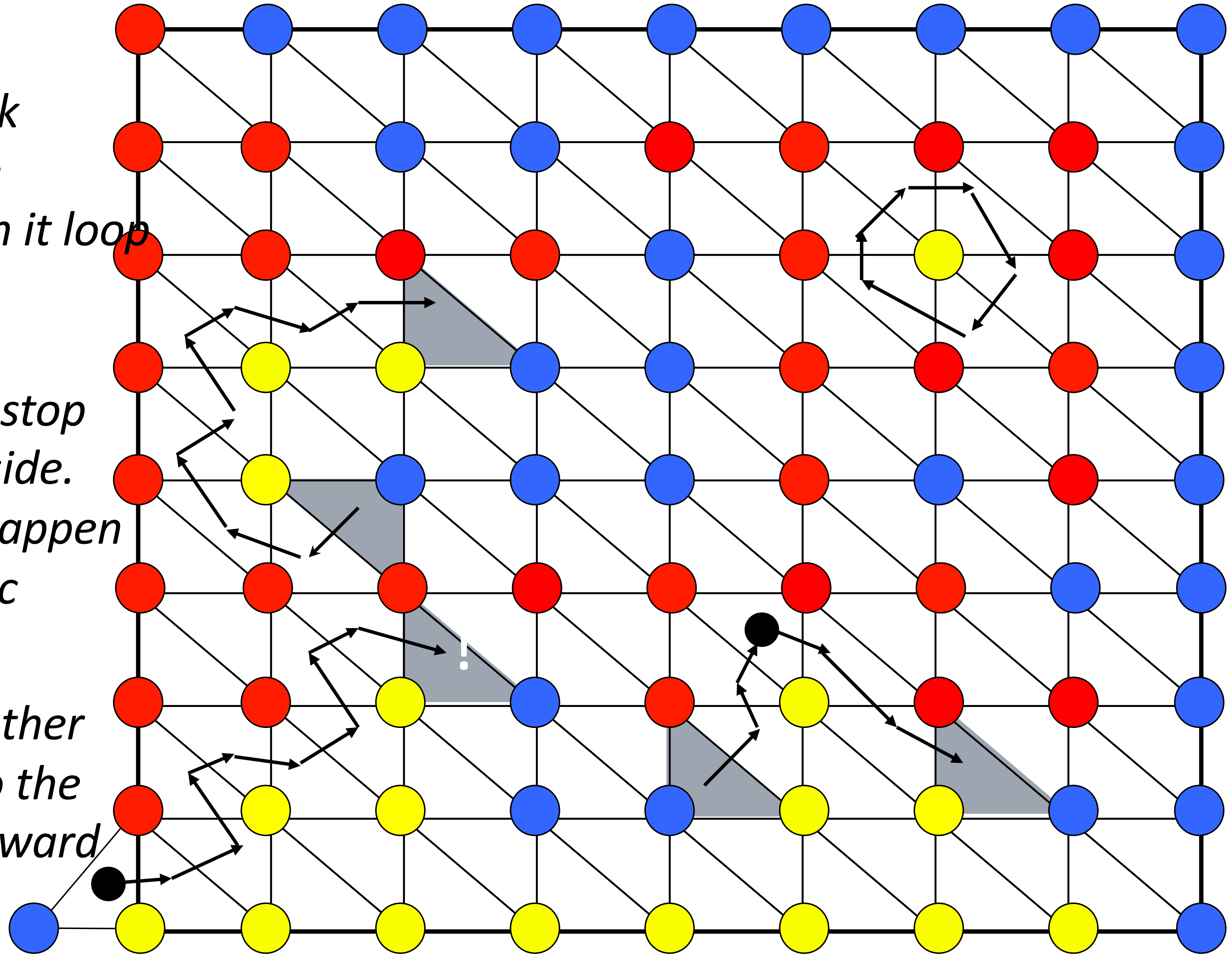
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Claim: *The walk cannot exit the square, nor can it loop into itself.*

Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…

Starting from other triangles we do the same going forward or backward.



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

We also introduce an artificial tri-chromatic triangle.

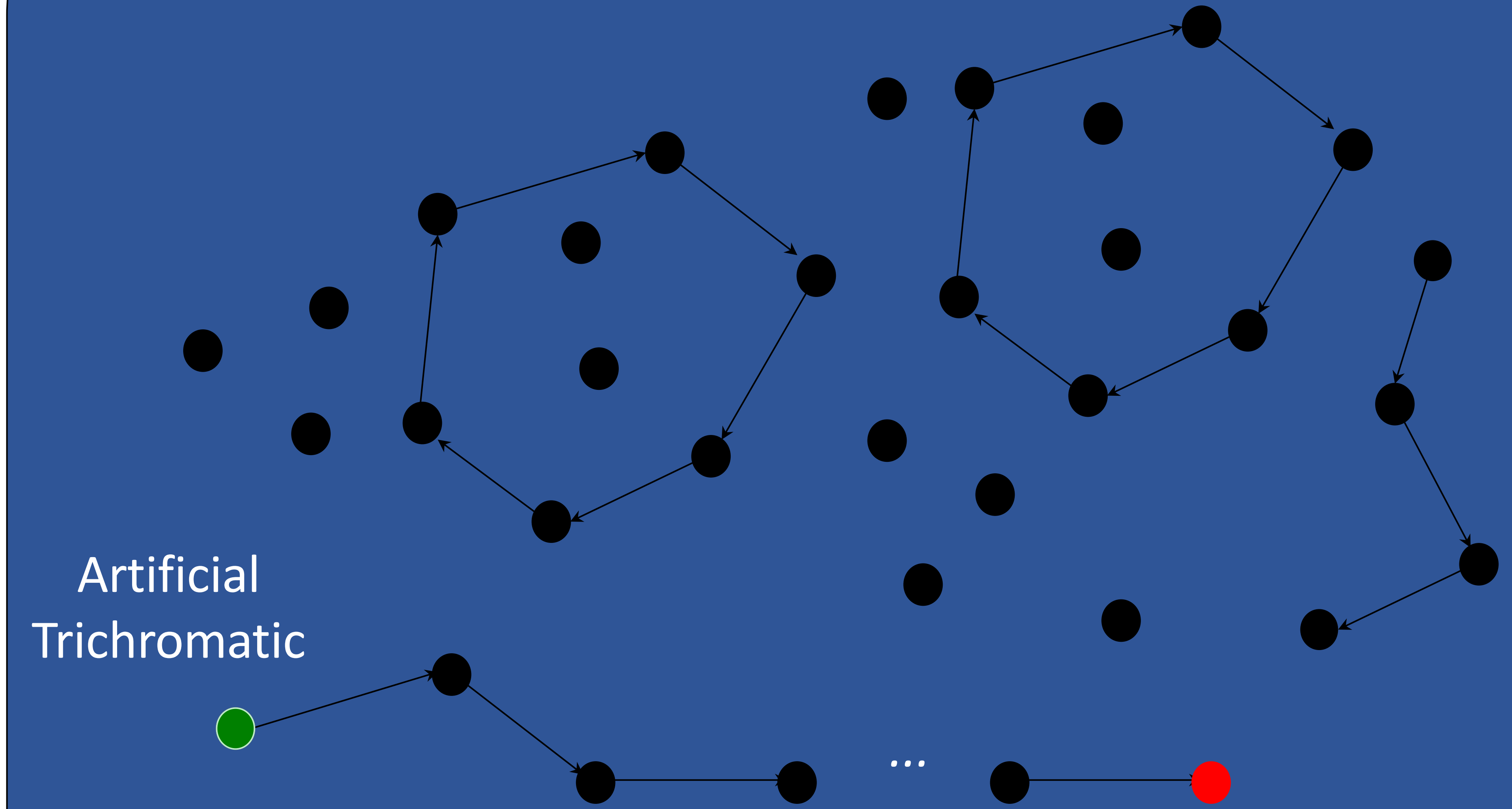
Next we define a directed walk starting from the artificial tri-chromatic triangle.

[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Structure of Proof:
A directed parity argument

Vertices of Graph ≡ Triangles

all vertices have in-degree, out-degree ≤ 1



degree 1 vertices: trichromatic triangles

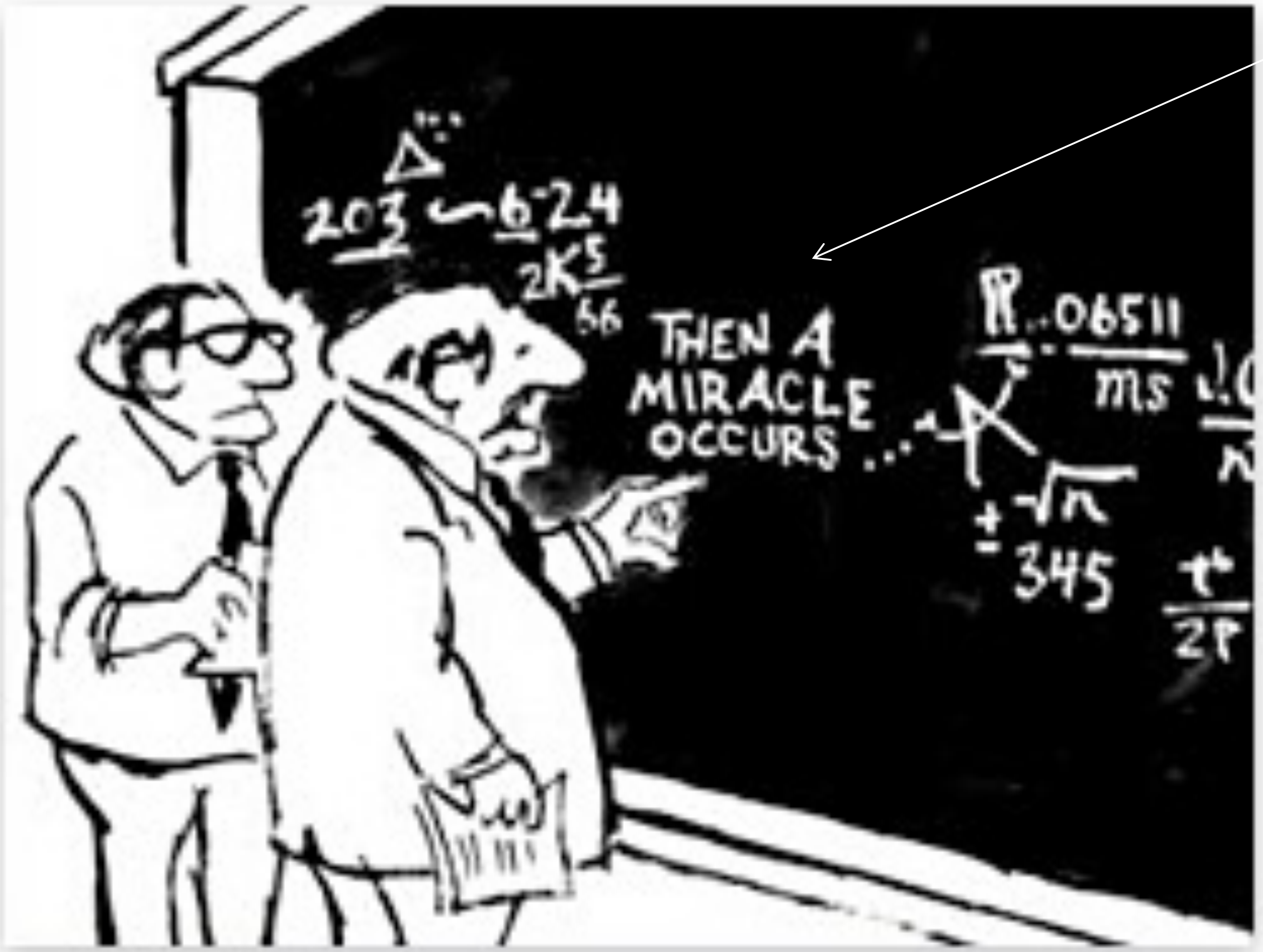
degree 2 vertices: no blue, non-trichromatic

degree 0 vertices: all other triangles

Proof: ∃ at least one trichromatic (artificial one) ⇒ ∃ another trichromatic

Also: degree 1 vertices are in pairs but one is fake ⇒ ∃ odd number of trichromatic!

So..what is the non-constructive step in Nash's proof?



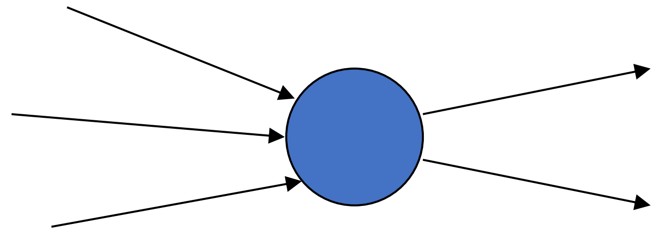
what is the nature of non-constructiveness in the heart of Nash's theorem?

We have shown: Sperner's Theorem ⇒ Brouwer's Theorem ⇒ Nash's Theorem

The Non-Constructive Step

An easy parity lemma:

A directed graph with an unbalanced node (a node with indegree ≠ outdegree) must have another.



But, wait, why is this non-constructive?

Given a directed graph and an unbalanced node, isn't it trivial to find another unbalanced node?

In some cases, the graph can be exponentially large in its succinct description…

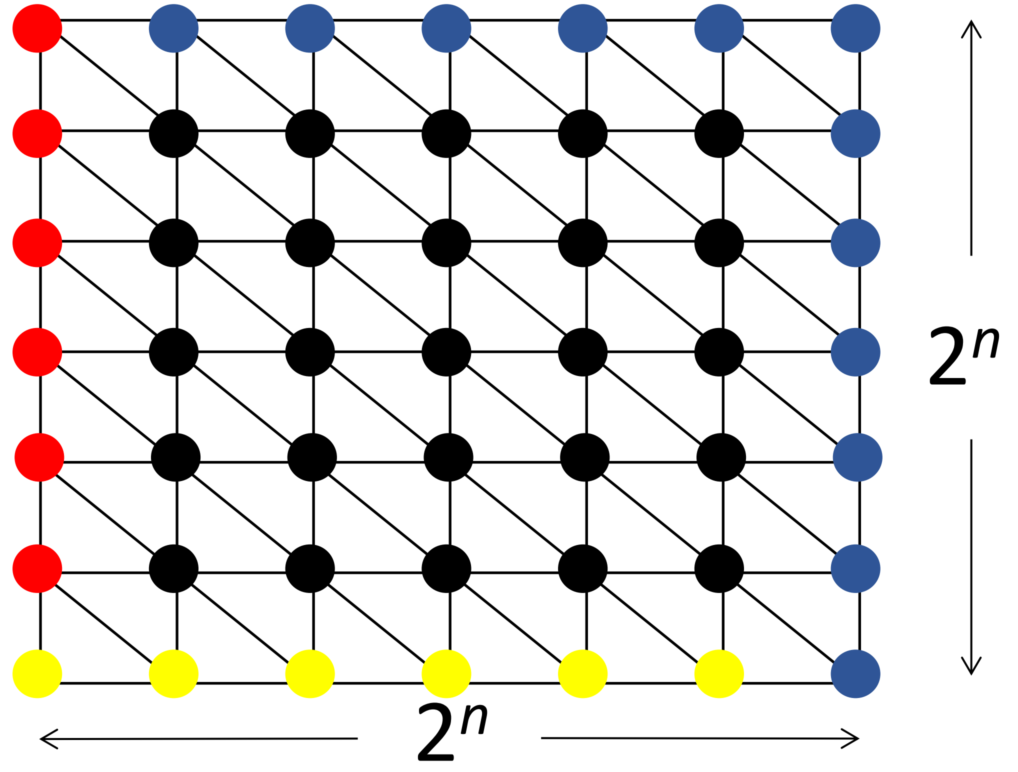
Example: next slide!

Computational Problem: SPERNER

INPUT:

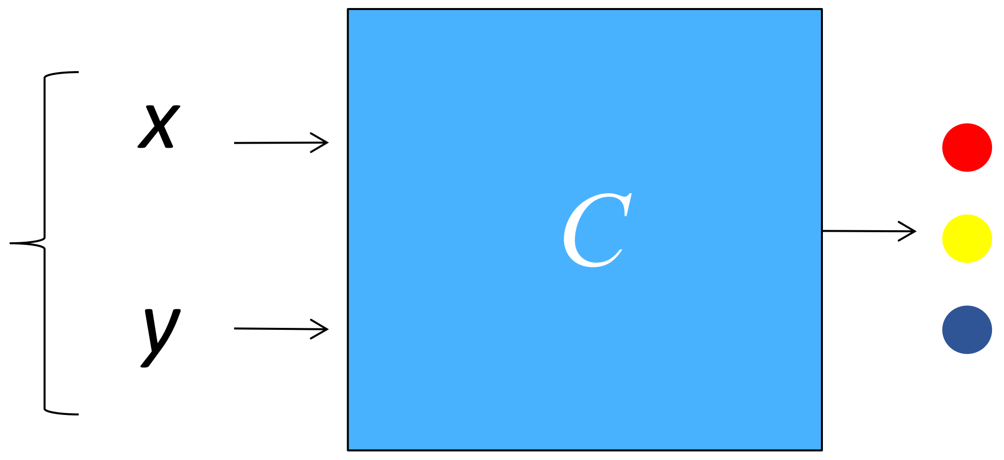
(i) n: specifies the size of a grid

(grid never written down!)



(ii) Imagine boundary has standard coloring shown above, while colors of internal vertices are given by a circuit:

input: the coordinates of a point *(n bits each)*



OUTPUT: A tri-chromatic triangle

exists because boundary coloring satisfies Sperner lemma constraints

but doing walk through grid to find one may take exponential time in n