MIT 6.7220/15.084 — Nonlinear Optimization (Spring '25) Thu, Feb 20th 2025

Lecture 5
The driver of duality: Separation

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In Lecture 3 we guessed the expression for the normal cone to the intersection of halfspaces. We then saw that our guess, natural in its graphical intuition, quite surprisingly implied immediately strong linear programming duality. In light of this, whatever proof techniques confirms our guess for the expression of the normal cone correct, rightfully deserves our attention, as it must encode the grain of duality.

As it turns our, the key idea behind the proof is the concept of *separation*.

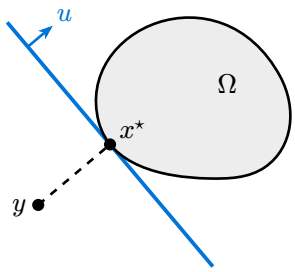
L5.1 Separating a point from a closed convex set

An important property of any convex set Ω is that whenever a point y is not in Ω, then we can *separate* y from Ω using a *hyperplane*. In other words, *flat* separating surfaces are enough for certifying that a point $y \notin \Omega$.

Theorem L5.1. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set, and let $y \in \mathbb{R}^n$ be a point. If $y \notin \Omega$, then there exist $u \in \mathbb{R}^n, v \in \mathbb{R}$ such that

$$\langle u, y \rangle < v, \qquad \text{and} \qquad \langle u, x \rangle \geq v \quad \forall x \in \Omega.$$

Proof. The proof of the result rests on a very simple idea: the direction of the halfspace will be made orthogonal to the line that connects y to its projection $x^\star$ onto Ω, and the halfspace boundary will be set so that it passes through $x^\star$. We now make the argument formal.



First, since Ω is nonempty and closed, a Euclidean projection $x^\star$ of y onto Ω exists,[1] as we discussed in Lecture 1. In other words, the nonlinear optimization problem

$$\begin{aligned} \min_x \ & \tfrac{1}{2}\|x - y\|_2^2 \\ \text{s.t.} \ & x \in \Omega \end{aligned}$$

*These notes are class material that has not undergone formal peer review. The TAs and I are grateful for any reports of typos.

[1] In fact, it is easy to prove that the projection is unique (see strict convexity, Lecture 4). However, we do not need uniqueness for the argument that follows.

must have at least a solution $x^\star \in \Omega$. Furthermore, since the objective function is differentiable and Ω is convex, from the first-order optimality conditions (see Lecture 2) we know that

$$\langle x^\star - y, x - x^\star \rangle \geq 0 \quad \forall x \in \Omega. \tag{1}$$

Let now

$$u := x^\star - y, \qquad [\triangleright \text{ this is the direction that connects } y \text{ to } x^\star]$$
$$\text{and } v := \langle u, x^\star \rangle. \qquad [\triangleright \text{ so that the halfspace boundary passes through } x^\star]$$

Note that $u \neq 0$, since $x^\star \in \Omega$ but $y \notin \Omega$. So, $\|u\| > 0$ and therefore

$$\langle u, y \rangle = \langle u, x^\star - u \rangle = v - \|u\|_2^2 < v.$$

Thus, to complete the proof, we now need to show that $\langle u, x \rangle \geq v$ for all $x \in \Omega$. But this is exactly what (1) guarantees, since $u = x^\star - y$ and $v = \langle u, x^\star \rangle$ by definition. □

The result above might not seem like much. After all, the proof is pretty straightforward, and the geometric intuition strong enough that one might be tempted to just take it for granted. Instead, the consequences of the result are deep, far-reaching, and intimately tied to some of the most significant breakthroughs in mathematical optimization theory.

Remark L5.1. The result of Theorem L5.1 holds even if we insist on only having strict inequalities, that is $\langle u, y \rangle < v$ and $\langle u, x \rangle > v$ for all $x \in \Omega$. We can see this in two ways:

- Graphically, in the proof we could have chosen v so that the halfspace would pass through the midpoint of the line connecting y and $x^\star$.
- Algebraically, let u, v be as in Theorem L5.1. We will show that we can always perturb v to make both inequalities hold strictly. The key is the observation that

$$\langle u, y \rangle = \frac{1}{2}\langle u, y \rangle + \frac{1}{2}\langle u, y \rangle < \frac{1}{2}(v + \langle u, y \rangle)$$
$$\langle u, x \rangle \geq v = \frac{1}{2}(v + v) > \frac{1}{2}(v + \langle u, y \rangle) \qquad \forall x \in \Omega.$$

  Hence, in both cases we have that the scalar $v' := \frac{1}{2}(v + \langle u, y \rangle)$ satisfies $\langle u, y \rangle < v'$, $\langle u, x \rangle > v'$ for all $x \in \Omega$.

L5.1.1 Separating a point from a convex cone

Before we prove the expression for the normal cone at the intersection of halfspaces, we will find it helpful to use the following corollary of separation for *convex cones*. A *cone* is a set with the property that the ray $\{\lambda \cdot x : \lambda \geq 0\}$ generated by any point x in the set is fully contained in the set.

Definition L5.1 (Cone)**.** A set S is a *cone* if, for any $x \in S$ and $\lambda \in \mathbb{R}_{\geq 0}$, the point $\lambda \cdot x \in S$.

Convex cones are among the simplest convex sets, and they appear all the time in optimization theory.[2] In particular, in the next theorem we show that separation of a point from a nonempty closed convex cone can always be achieved using a hyperplane passing through the origin.

Theorem L5.2. Let $S \subseteq \mathbb{R}^n$ be a nonempty closed convex cone, and $y \notin S$ be a point in $\mathbb{R}^n$. Then, there exists a hyperplane *passing through the origin* that separates y from S; formally, there exists $u \in \mathbb{R}^n$ such that

$$\langle u, y \rangle < 0 \quad \text{and} \qquad \langle u, x \rangle \geq 0 \quad \forall x \in S.$$

Proof. We already know from Theorem L5.1 that there exist $u \in \mathbb{R}^n, v \in \mathbb{R}$ such that

$$\langle u, y \rangle < v \quad \text{and} \qquad \langle u, x \rangle \geq v \quad \forall x \in S. \tag{2}$$

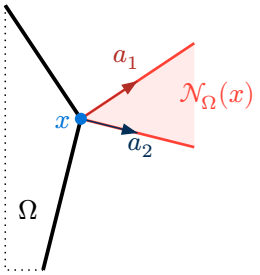
Consider any point $a \in S$. By definition of cone, $\lambda \cdot a \in S$ for all $\lambda \geq 0$. Thus, the separation condition on the right in (2) implies that $v \leq \lambda \cdot \langle u, a \rangle$ for all $\lambda \geq 0$. In particular, by plugging $\lambda = 0$, we find that $v \leq 0$, yielding $\langle u, y \rangle < 0$. Furthermore, dividing by λ we find that

$$\langle u, a \rangle \geq \frac{v}{\lambda} \qquad \forall \lambda \geq 0 \qquad \Longrightarrow \qquad \langle u, a \rangle \geq \sup_{\lambda \to \infty} \frac{v}{\lambda} = 0.$$

Since $a \in S$ was arbitrary, the statement follows. □

L5.2 A second look at the normal cone of linear constraints

In Lecture 3, we considered normal cones at the intersection of halfspaces. On that occasion, we drew a picture and were convinced that the normal cone at a point at the intersection of halfspaces was given by the conic hull of the directions orthogonal to those halfspaces (see the figure on the right).



This led to the following guess, which was left unproven.

Theorem L5.3. Let $\Omega \subseteq \mathbb{R}^n$ be defined as the intersection of m linear inequalities

$$\Omega := \{x \in \mathbb{R}^n : Ax \leq b\}, \quad \text{where} \quad A = \begin{pmatrix} — & a_1^\top & — \\ & \vdots & \\ — & a_m^\top & — \end{pmatrix} \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m.$$

Given a point $x \in \Omega$, define the index set of the "active" constraints

$$I(x) := \left\{ j \in \{1, ..., m\} : a_j^\top x = b_j \right\}.$$

[2] Hiriart-Urruty, J.-B., & Lemaréchal, C. [HL01], referring to convex cones, write: *"they are important in convex analysis (the "unilateral" realm of inequalities), just as subspaces are important in linear analysis (the "bilateral" realm of equalities)".*

Then, the normal cone at any $x \in \Omega$ is given by

$$\mathcal{N}_\Omega(x) = \left\{ \sum_{j \in I(x)} \lambda_j a_j : \lambda_j \geq 0 \right\} = \left\{ A^\top \lambda : \lambda^\top (b - Ax) = 0, \lambda \in \mathbb{R}^m_{\geq 0} \right\},$$

where the second equality rewrites the condition $j \in I(x)$ via the *complementary slackness* (see Lecture 3).

We now give the proof of the above result. We will do that by invoking the machinery of separation to argue that a direction outside of the normal cone must form an acute angle with at least one direction that remains in the feasible set Ω.

Proof of Theorem L5.3. Fix any $x \in \Omega$ and let

$$\mathcal{C}(x) := \left\{ \sum_{j \in I(x)} \lambda_j a_j : \lambda_j \geq 0 \right\}.$$

We will show that $\mathcal{N}_\Omega(x) = \mathcal{C}(x)$ by proving the two directions of inclusion separately.

- We start by showing that any $d \in \mathcal{C}(x)$ belongs to $\mathcal{N}_\Omega(x)$, that is,

  $$\langle d, y - x \rangle \leq 0 \text{ for all } y \in \Omega.$$

  Let d be expressed as $\sum_{j \in I(x)} \lambda_j a_j$ with $\lambda_j \geq 0$. Then, for any $y \in \Omega$,

  $$\begin{aligned} \left\langle \sum_{j \in I(x)} \lambda_j a_j, y - x \right\rangle &= \sum_{j \in I(x)} \lambda_j \langle a_j, y - x \rangle \\ &= \sum_{j \in I(x)} \lambda_j \big(\langle a_j, y \rangle - b_j \big) && \text{(by definition of } I(x), \langle a_j, x \rangle = b_j) \\ &\leq \sum_{j \in I(x)} \lambda_j \big(b_j - b_j \big) = 0. && \text{(since } y \in \Omega \text{ and } \lambda_j \geq 0) \end{aligned}$$

  This shows that $d \in \mathcal{N}_\Omega(x)$ and concludes the proof of this direction of the inclusion.
- We now look at the other direction. Take any $d \notin \mathcal{C}(x)$. Since $\mathcal{C}$ is a nonempty closed convex cone [▷ you should verify this claim], by the conic separation result of Theorem L5.2, there must exist $u \in \mathbb{R}^n$ such that

  $$\langle u, d \rangle < 0, \quad \text{and} \qquad \langle u, w \rangle \geq 0 \quad \forall w \in \mathcal{C}(x). \tag{3}$$

  We argue that for $\delta > 0$ small enough, the point $y := x - \delta \cdot u$ belongs to Ω. We do so by showing that it satisfies all the inequalities $a_j^\top x \leq b_j$ that define Ω:
  - if $j \in I(x)$, then $\langle a_j, x - \delta \cdot u \rangle = b_j - \delta \cdot \langle a_j, u \rangle \leq b_j$ since $\langle a_j, u \rangle \geq 0$ by (3).
  - if $j \notin I(x)$, then $b_j - \langle a_j, x \rangle > 0$. By continuity, small enough perturbations of x, in any direction, will not affect the strict inequality.

  Thus, the direction $\delta \cdot u$ remains inside of Ω starting from x. We now argue that it forms a strictly positive inner product with d. Indeed, note that from (3)

  $$\langle d, y - x \rangle = \langle d, -\delta \cdot u \rangle = -\delta \cdot \langle d, u \rangle > 0.$$

This shows that $d \notin \mathcal{C}(x) \implies d \notin \mathcal{N}_\Omega(x)$, completing the proof. □

L5.3 Separation oracles

The result established in Theorem L5.1 justifies the following definition.

Definition L5.2 ((Strong) separation oracle)**.** Let $\Omega \subseteq \mathbb{R}^n$ be convex and closed. A *strong separation oracle* for Ω is an algorithm that, given any point $y \in \mathbb{R}^n$, correctly outputs one of the following:

- "$y \in \Omega$", or
- "$(y \notin \Omega, u)$", where the vector $u \in \mathbb{R}^n$ is such that

$$\langle u, y \rangle < \langle u, x \rangle \qquad \forall x \in \Omega.$$

L5.3.1 Finding separating hyperplanes in practice

Theorem L5.1 guarantees the *existence* of a separating hyperplane. In many problems of interest, *constructing* a separation oracle is simple.

Example L5.1 (Separation oracle for a convex polytope)**.** Let Ω be a convex polytope, that is, the convex set defined by the intersection of a finite number of halfspaces (linear inequalities)

$$\Omega \coloneqq \{x \in \mathbb{R}^n : Ax \leq b\}, \quad \text{where} \quad A = \begin{pmatrix} — & a_1^\top & — \\ & \vdots & \\ — & a_m^\top & — \end{pmatrix} \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m.$$

Then, given a point $y \in \mathbb{R}^m$, we can implement a separation oracle as follows:

- if $Ay \leq b$, return "$y \in \Omega$";
- else, at least one of the inequalities $a_j^\top y \leq b_j$, $j \in \{1, ..., m\}$ is violated. In other words, there exists j such that $a_j^\top y > b_j$, while by definition of Ω, $a_j^\top x \leq b_j$ for all x. This shows that the response "$\big(y \notin \Omega, -a_j\big)$" is a valid response.

Remark L5.2. Example L5.1 shows that whenever we have a finite number m of inequalities, a separation oracle for the polytope defined by those inequalities can be implemented in time that depends linearly on m and the dimension of the embedding space. This result establishes a *blanket* guarantee, but in some cases, one can do better: depending on the structure of the inequalities, sometimes one can get away with sublinear complexity in m. In some cases, one might be able to construct an efficient separation oracle even for polytopes that have an infinite number of inequalities!

We proceed with another classic example of a feasible set that admits a simple separation oracle.

Example L5.2 (Separation oracle for the semidefinite cone). Let $\Omega = \{M \in \mathbb{R}^{n \times n} : M \succeq 0\}$ be the set of semidefinite matrices, that is, all symmetric matrices such that $v^\top M v \geq 0$ for all $v \in \mathbb{R}^n$—or, equivalently, such that all of M's eigenvalues are nonnegative.

Then, given a point $Y \in \mathbb{R}^{n \times n}$, we can implement a separation oracle as follows:

- if Y is *not* symmetric, then there exist $i, j \in \{1, ..., n\}$ such that $Y_{ij} < Y_{ji}$; return "$\big(Y \notin \Omega, E_{ij} - E_{ji}\big)$", where E_{ij} is the matrix of all zeros, except in position i, j where it has a 1.
- else, if Y is symmetric, we can compute all of its eigenvalues and eigenvectors. If one eigenvalue is negative, then the corresponding eigenvector w must be such that $w^\top Y w = \langle Y, ww^\top \rangle < 0$. Hence, return "$\big(Y \notin \Omega, ww^\top\big)$".
- otherwise, return "$Y \in \Omega$".

As we show next, a fundamental result in optimization theory reveals that under mild hypotheses, if the feasible set admits an efficient separation oracle and the objective function is convex, then the solution can be computed efficiently.

L5.4 Optimization via separation

In a major breakthrough in mathematical optimization, Khachiyan, L. G. [Kha80] proposed a polynomial-time algorithm for using separation oracles to find the minimum of a linear function. The algorithm, which goes under the name of *ellipsoid method* is actually more general, and applies to general convex objectives on sets for which separation oracles are available. The result builds on top of previous work by Šor, N. Z. [Šor77] and Yudin, D. B., & Nemirovskii, A. S. [YN76].

In particular, Khachiyan's result was the first to show that linear programming problems can be solved in polynomial time. This was an unexpected result at the time, and in fact, the complexity of linear programming solvers was conjectured to be *not* polynomial (more on this in the next section). The result of Khachiyan stirred so much enthusiasm in the research community that the New York Times even advertised it on its first page.

Despite the enthusiasm, the ellipsoid method turned out to be very impractical. Still, it is a great theoretical idea, and its consequences are pervasive.

L5.4.1 The intuition behind the ellipsoid method

Formalizing the details of the ellipsoid method is rather complex. A major source of difficulty is the fact that the algorithm needs to approximate square roots using fractions to be implementable on a finite-precision machine, and that causes all sorts of tricky analyses that the approximation error can indeed be kept under control. These details are certainly important, but are notoriously tedious, and fundamentally they are just that, *details*. If you are curious to read a formal account, I recommend the authoritative book by Grötschel, M., Lovász, L., & Schrijver, A. [GLS93]. For this lecture, we just focus on the *idea* behind the ellipsoid method.

"All the News That's Fit to Print"

The New York Times

LATE CITY EDITION

Weather: Mostly sunny, cool today; clear, cold tonight. Sunny tomorrow. Temperature range: today 43-51; yesterday 44-56. Details on page A28.

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Similar Transportation Bond Issues Are Passed in New York and Jersey

Con Ed Takeover Action Fails in Westchester — Simon Wins in Bronx

By FRANK LYNN

Voters approved mass transportation bond issues yesterday in New York and New Jersey, while rejectingg a state college building bond issue in New Jersey and at least two constitutional amendments in New York.

In Westchester County, voters rejected a potential county takeover of the Consolidated Edison Company's electrical distribution system. [Page B5.]

In contests for public office, Borough President Stanley Simon of the Bronx, a Democrat, and Peter F. Cohalan, the Republican candidate for Suffolk County Executive, were elected in landslides.

Simon Getting 55% of Vote

With 80 percent of the vote counted, Mr. Simon was polling 55 percent of the vote, while Assemblyman G. Oliver Koppell, the Liberal candidate, and State Senator Joseph L. Galiber, an independent, were neck and neck for second place with about 17 percent each. In a potential embarrassment for the Conservative Party, the Right to Life Party candidate, Jeronimo Dominquez, was well ahead of the Conservative, Terrence J. Dwyer.

In Suffolk County, Mr. Cohalan, who had defeated the Republican incumbent, County Executive John V. N. Klein, in the Republican primary on Sept. 11, was defeating his Democratic opponent, Dr. Martin J. Feldman, by a margin of better than 2 to 1 and was leading a Republican sweep in the county.

In three other significant contests, District Attorney Thomas R. Sullivan was winning re-election on Staten island, Erie County Executive Edward J. Rutkowski, a Republican serving an interim term,

Continued on Page B4, Column 4

Democrats in Kentucky and Mississippi Victors in Races for Governor

By ADAM CLYMER

Democrats overcame stiff Republican challenges and held onto the governorships in Kentucky and Mississippi yesterday. But a Republican ousted Cleveland's angry young Democratic Mayor, Dennis J. Kucinich.

The Kentucky and Mississippi results signified that a national Republican resurgence, seen a year ago in a net gain of six governorships and 300 legisative seats, had been checked. The Republicans, looking toward next year's Presidential race, had campaigned hard for the two governorships because President Carter carried those states only narrowly in 1976. He won then with major help from Democratic Governors who were ineligible to run again this year.

The governorships, the Cleveland mayoral race and the City Hall victory in Philadelphia of former Representative William J. Green, a Democrat, were the key races in scattered off-year elections that included contests for several other major city halls and ballot propositions on taxes, spending and returnable bottles and cans.

Brown Wins in Kentucky

In Kentucky, John Y. Brown Jr., a veteran Democratic fund-raiser making his first run for public office, beat former Gov. Louie B. Nunn, who had waged a hard personal campaign against him. Mr. Brown cut into his opponent's strength in Republican areas and rolled up heavy majorities in Democratic regions of the state.

In Mississippi, William Winter, a Democratic former Lieutenant Governor, defeated Gil Carmichael, a Meridian

Continued on Page A17, Column 1

CARTER SAID TO ASK FOR LOYALTY CHECK OF AIDES ON POLITICS

Ouster Implied for Those Unwilling to Help His Drive — Strauss to Head Re-election Unit

By TERENCE SMITH
Special to The New York Times

WASHINGTON, Nov. 6 — President Carter instructed his Cabinet officers last night to check the political "dependability" of their subordinates and strongly implied that those who were not prepared to campaign actively in his behalf should be dismissed, White House sources said today.

The President, at a bluntly political dinner in the family dining room of the White House last night, said that he expected all the political appointees in his Administration to be "actively engaged" in the campaign. Although Mr. Carter did not explicitly say that those unwilling to campaign should resign, one of the dinner guests said: "That was his whole message."

In another development, Robert S. Strauss, the President's special Middle East envoy, confirmed today that he would resign soon to become chairman of the Carter-Mondale Presidential Committee. The White House announced that Sol M. Linowitz, who helped negotiate the Panama Canal treaties, would replace Mr. Strauss in the Middle East post.

Political Atmosphere Reflected

The White House dinner last night was the first full-scale discussion of Mr. Carter's forthcoming campaign that the President has held with his Cabinet and top members of his staff, and it reflected the intensely political atmosphere that has gripped the Carter White House in anticipation of the challenge for the Democratic Presidential nomination from Senator Edward M. Kennedy, who plans to announce his candidacy formally in Boston tomorrow.

Mr. Carter accepted an invitation to participate in a joint appearance with Senator Kennedy at an "open forum" in Des Moines in January. The Des Moines Register and Tribune, sponsor of the forum, said that Mr. Kennedy had also accepted. The President's acceptance today appeared to be an effort to preempt Mr. Kennedy, who is reportedly planning to challenge Mr. Carter tomorrow to debate him before the primaries.

In the Iowa forum, Mr. Carter and Mr. Kennedy will share a platform and answer questions from a panel of reporters and editors and from the public. James Gannon, the newspaper's executive editor, said that Gov. Edmund G. Brown Jr. of California was not being invited to join the forum "because he has not mounted a serious campaign in Iowa." The Iowa Democratic caucuses on Jan. 21 will be the first major test of the election year.

The switch to Mr. Strauss, a 61-year-old former chairman of the Democratic National Committee, as head of the Carter-

Continued on Page A19, Column 1

IRAN'S CIVIL GOVERNMENT OUT; HOSTAGES FACE DEATH THREAT; OIL EXPORTS BELIEVED HALTED

United Press International

In Teheran, a demonstrator held up pictures with the faces of President Carter and the Shah of Iran superimposed on those of victims of the firing squad.

STUDENTS WARN U.S.

Ayatollah Instructs Secret Revolutionary Council to Form a Cabinet

By JOHN KIFNER
Special to The New York Times

TEHERAN, Iran, Nov. 6 — Prime Minister Mehdi Bazargan's provisional revolutionary Government dissolved today, conceding power to the Islamic authority of Ayatollah Ruhollah Khomeini.

The United States had been counting on Mr. Bazargan's Government to insure the safety of 60 or so American hostages seized Sunday at the American Embassy. The Government's abrupt collapse, after months of frustration and impotence, appeared to further dampen the already dim hopes for a negotiated release of the hostages.

Militant Islamic students holding the embassy said today that they would kill the hostages if the United States used military force in a rescue attempt. The students are demanding that the United States hand over Shah Mohammed Riza Pahlevi, who is undergoing medical treatment in New York City.

[In Washington, President Carter met with his foreign policy advisers and decided to maintain a nonprovocative posture toward Iran in the hope that the hostages would eventually be freed by Iranian religious authorities. And at the United Nations, a Palestine Liberation Organization spokesman said that Yasir Arafat, head of the guerrilla group, was sending a delegation to Teheran to "secure the safety of the Americans" and others held hostage. Page A14.]

Council's Membership Is Secret

Accepting Mr. Bazargan's resignation, Ayatollah Khomeini ordered the Revolutionary Council, whose membership is secret but is believed to consist of Islamic leaders, to take over the government.

The political change appeared to mark a break between the Western-oriented intellectuals, who opposed the Shah and had hoped to establish a parliamentary system, and their onetime allies in the Islamic clergy, determined to impose the law and moral authority of the Koran.

Since the deposing of the Shah in mid-February, real power here has been concentrated in the person of Ayatollah Khomeini, the white-bearded ascetic of the Shiite branch of Islam who emerged from 15 years in exile to become the symbol of the Iranian revolution.

Since they assumed their offices in Teheran's imposing Government buildings, the lawyers and other specialists who served in Mr. Bazargan's Government had been overshadowed by the mullahs who gathered, in turbans and robes, around Ayatollah Khomeini in the dusty holy city of Qum.

"You are weak, old man," the Ayatollah chided Mr. Bazargan early on. The friction was constant. Mr. Bazargan often went on television to announce that

Continued on Page A14, Column 4

F.A.A. Seeks Fine of $1.5 Million For 'Unairworthy' Braniff Flights

By RICHARD WITKIN

The Federal Aviation Administration moved yesterday to impose a record fine of $1.5 million against Braniff International Airways for conducting hundreds of flights with planes that were allegedly "unairworthy."

The agency said that the violations, in flights of Boeing 747 jumbo jets as well as conventional narrow-body planes, "appear to reflect a basic pattern of continuing disregard" of aircraft-maintenance rules by the management of the Texas-based carrier.

Specifically, Braniff was accused of using "improper and unapproved maintenance procedures," of operating planes that had not been given the required inspections and of failing to keep adequate records.

Braniff Disputes Charges

A Braniff spokesman, Jere L. Cox, said that the company felt it gave its fleet the finest and "most meticulous" care of any airline. Braniff contended that the F.A.A.'s charges were "not in context with the facts" and voiced confidence that it would be able to "satisfy any questions they might have about any alleged discrepancies."

Braniff, which operates a vast domestic network and flights to Europe, Asia and South America, has one of the best safety records in the industry. Mr. Cox said the last major accident was the 1968 crash of a British-made BAC 1-11. The last two before that were crashes of Lockheed Electra turboprop planes about a decade earlier.

The airline has 20 days to file a formal answer to the 16 pages of accusations that form the basis of the proposed $1.5 million fine. But compromises are often negotiated. Ultimately, the Federal agency could take the issue to court.

The proposed penalties are about four

Continued on Page A28, Column 3

Main Iran Oil Port Reported Closed And Prices of Spot Petroleum Soar

By ANTHONY J. PARISI

Iran was reported yesterday to have stopped loading all oil tankers at the country's major port at Kharg Island in the Persian Gulf, and oil prices soared in the spot market as a result. The report, which stemmed from oil industry sources, was later circulated by the Central Intelligence Agency, the State Department and White House aides.

The aides did not make it clear whether the reported action was connected to the Iranian demand that the United States return the Shah to Teheran for trial.

Later yesterday, the State Department said it was advising companies operating in Iran to start withdrawing American employees. Earlier, many had said they were not planning to do so. [Page D1.]

Because almost all of Iran's oil leaves the country via Kharg Island, a stoppage there would all but halt Iranian oil exports. That, according to oil experts interviewed yesterday, would soon put consuming nations in the same squeeze they faced this time last year, when Iranian production first faltered.

The run-up in spot market oil prices began early in the day, after reports that Iran's new oil minister was ready to limit the amount of oil available to America. The buying spree was apparently bolstered by new rises in Persian Gulf and North Sea contract prices. [Page D5.]

The situation turned into what some in

Continued on Page D5, Column 3

Interior Dept. Assailed On Missing a Deadline For Species Protection

By PHILIP SHABECOFF
Special to The New York Times

WASHINGTON, Nov. 6 — To the dismay and anger of environmentalists, about 1,700 plants and a hundred animals are being dropped from consideration for protection as endangered species because a deadline Congress set a year ago for Interior Department action is about to pass.

Department officials say that they have no choice. Last Nov. 10, President Carter signed amendments to the Endangered Species Act requiring, among other things, that if a species had been proposed for protection and the agency had not acted on it for two years or more, the agency then had one year to place it on the endangered or threatened species list or it would automatically be dropped.

That year is up next Monday, and the agency has met the amendments' requirements for only 37 species.

Snail Darter vs. a Dam

Concerned environmentalists, warning that many of the species being dropped may face extinction, charge that bureaucratic slowness and ineptitude are the reasons the plants and animals are losing any chance of protection.

The Endangered Species Act has generated controversy since it was passed in 1973. The most recent political turmoil surrounding the law was the battle over the snail darter, an endangered three-

Continued on Page B7, Column 1

The New York Times/Teresa Zabala

President Carter after announcing that Robert S. Strauss, right, would head his campaign. At left are Secretary of State Cyrus R. Vance and Sol M. Linowitz, who is succeeding Mr. Strauss as Mideast negotiator.

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

Mathematicians describe the discovery by L.G. Khachian as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of hit-or-miss basis.

Apart from its profound theoretical interest, the discovery may be applicable in weather prediction, complicated industrial processes, petroleum refining, the scheduling of workers at large factories, secret codes and many other things.

"I have been deluged with calls from virtually every department of government for an interpretation of the significance of this," a leading expert on computer methods, Dr. George B. Dantzig of Stanford University, said in an interview.

The solution of mathematical problems by computer must be broken down into a series of steps. One class of problem sometimes involves so many steps that it could take billions of years to compute.

The Russian discovery offers a way by which the number of steps in a solution can be dramatically reduced. It also offers the mathematician a way of learning quickly whether a problem has a solution or not, without having to complete the entire immense computation that may be required.

According to the American journal Sci-

Continued on Page A20, Column 3

Energy Dept. Accuses 9 Refiners Of $1.18 Billion Oil Overcharges

By RICHARD D. LYONS
Special to The New York Times

WASHINGTON, Nov. 6 — Department of Energy investigators announced today the filing of formal charges against nine major oil refiners, accusing them of having overcharged the public by $1.18 billion from 1973 to 1976.

Moreover, Administration officials said the investigation of refiner overcharging since the 1973 Arab embargo might eventually result in allegations that consumers paid up to $25 billion too much for petroleum products in the last six years.

In general, spokesmen for the nine companies took the position that the allegations were both unwarranted and incorrect. While some said they would have to study the charges, none conceded that the Federal allegations were correct.

The specific complaints, listing the alleged overcharges and known technically as notices of probable violation, were issued today against the Mobil Oil Corporation, $274.7 million; the Shell Oil Company, $211.6 million; the Standard Oil Company (Ohio), $179 million; the Sun Company, $129.7 million; Chevron USA, a subsidiary of the Standard Oil Company of California, $117.2 million; the Amerada Hess Corporation, $88 million; the Gulf Oil Corporation, $80.1 million; the Atlantic Richfield Company, $58.5 million, and Conoco Inc., $46.1 million.

Energy Department investigators previously charged 15 major refiners with having overcharged by $6.4 billion from 1973 through 1976.

"This is the largest legal offensive and counteroffensive in the history of the

Continued on Page D5, Column 4

INSIDE

$25,000 Scotto Gift Cited

A dock union official testified that Anthony M. Scotto gave him a $25,000 cash contribution last year for Governor Carey's election campaign. Page B1.

CUNY Finds Tuition Hurts

The percentage of top high school students applying to the City University of New York dropped sharply after tuition was imposed in 1976. Page B1.

Figure 1: https://timesmachine.nytimes.com/timesmachine/1979/11/07/issue.html

The idea behind the ellipsoid method is rather elegant. At its core, it is a generalization of *binary search* from one dimension to multiple dimensions. At every iteration of the algorithm, the space is "cut" by using a separating hyperplane.

■ **Feasibility.** To build intuition, ignore for now the objective function, and consider the following problem: given a separation oracle for Ω (closed and convex), either find $x \in \Omega$, or determine that Ω is empty. You are given two radiuses:

- the radius $R > 0$ guarantees that if Ω is not empty, then $\Omega \cap \mathbb{B}_R(0) \neq \emptyset$;
- the radius $r > 0$ guarantees that if $\Omega \cap \mathbb{B}_R(0)$ is not empty, then it contains a ball of radius r in its interior.

If this problem were one-dimensional, then Ω would be either empty or an interval, and a separation oracle would be an algorithm that, given any $y \in \mathbb{R}$, would return whether $y \in \Omega$, or one of the statements "y is too small" / "y is too large". Solving the problem now appears easy: start from the interval $[-R, R]$, and perform a binary search using the separation oracle to guide the search. Once the size of the search interval drops below r, we know that Ω is empty.

The ellipsoid method generalizes this idea to multiple dimensions. At every iteration, it keeps track of a "search space" (the generalization of the search interval above). Then, it queries the separation oracle for the center c_t of this search space. If the point does not belong to Ω, and the separation oracle returns the separating direction $u \in \mathbb{R}^n$, then the search space is cut by considering now only the subset of the search space that intersects $\{x \in \mathbb{R}^n : \langle u, x - c_t \rangle \geq 0\}$. The process continues until the volume of the search space becomes smaller than the radius r. The reason why this method is called the "ellipsoid method" is that the search space in the multi-dimensional case is not kept in the form of an interval, but rather as an ellipsoid. This is mostly for computational reasons, since we need to have an internal way of representing the search domain that is convenient to use.

■ **Incorporating the objective.** The above idea can be extended to incorporate an objective function $f(x)$. To do that, we will need to start cutting not only the search ellipsoid, but also the feasible set to make sure we end up at the optimum. In other words, you can think of this extended ellipsoid method as having "two modes": while it has not found a feasible point in Ω, it cuts the search ellipsoid; then, once feasible points are found, it cuts the feasible set to exclude all values above the current value.

- Initialize at time $t = 1$ with the starting point $y_1 := 0 \in \mathbb{R}^n$, starting ellipsoid $\mathcal{E}_1 := \mathbb{B}_R(0)$, and starting feasible set $\Omega_1 := \Omega$.
- At each time t, we ask a separation oracle for Ω_t whether the center $c_t \in \mathbb{R}^n$ of the search ellipsoid $\mathcal{E}_t$ belongs to Ω_t or not.[3] There are only two cases:
  - If the center c_t is *not* feasible, then set $\Omega_{t+1} := \Omega_t$, and cut the search space by setting $\mathcal{E}_{t+1}$ to an ellipsoid that contains the intersection between $\mathcal{E}_t$ and the halfspace containing Ω_t returned by the separation oracle.

[3]There is a caveat here: technically, we are assuming as given a separation oracle for Ω, *not* Ω_t. Yet, because Ω_t is obtained from Ω by intersecting with halfspaces, it is easy to see that one separation oracle for Ω_t can be constructed efficiently starting from that for Ω and the description of the intersected hyperplanes. Try working out the details!

- If the center c_t is feasible, then we know for sure that all points $x \in \Omega_t$ such that $\langle \nabla f(c_t), x - c_t \rangle \geq 0$ are such that $f(x) \geq f(c_t)$. This follows trivially from the linear lower bound property of convex functions (Theorem L4.1 of Lecture 4):

$$\langle \nabla f(c_t), x - c_t \rangle \geq 0 \qquad \Longrightarrow \qquad f(x) \geq f(c_t) + \langle \nabla f(c_t), x - c_t \rangle \geq f(c_t).$$

  Hence, we can cut *both* the search ellipsoid $\mathcal{E}_t$ and the feasible set Ω_t by considering their intersection with the halfspace $H_t := \{x \in \mathbb{R}^n : \langle \nabla f(c_t), x - c_t \rangle \leq 0\}$. In particular, we set $\Omega_{t+1} := \Omega_t \cap H_t$, and set $\mathcal{E}_{t+1}$ to a smaller ellipsoid that contains $\mathcal{E}_t \cap H_t$.

- Finally, after the volume of the search ellipsoid has gotten sufficiently small (this happens after $T = O(n^2) \log(R/r)$ iterations), we output the following:
  - If we never encountered a center c_t that was feasible, then we report that Ω was infeasible.
  - Else, we output the c_t that minimizes f, out of those that were feasible.

Assuming that we can ignore all sorts of tedious rounding issues, the following guarantee can be shown [Gup20].

Theorem L5.4. Let R and r be as above, and let the range of the function f on Ω be bounded by $[-B, B]$. Then, the ellipsoid method described above run for $T \geq 2n^2 \log(R/r)$ steps either correctly reports that $\Omega = \emptyset$, or produces a point $x^\star$ such that

$$f(x^\star) \leq f(x) + \frac{2BR}{r} \exp\left(-\frac{T}{2n(n+1)}\right) \qquad \forall x \in \Omega.$$

L5.4.2 Takeaway message: Separation implies optimization

If you squint your eyes, what the ellipsoid method proves constructively is the following: if we know how to construct a separation oracle for a set Ω, then we can optimize over Ω. Of course, this is a bit of a simplification (and there are all sorts of little conditions here and there as we have seen above), but nonetheless it is a good first approximation of the general message.

In a later lecture, we will discuss how the opposite direction is also known to be true, even when by "optimization" we simply mean optimization of linear objective functions.

Further readings and bibliography

If you want to read more about the ellipsoid method, the book by Grötschel, M., Lovász, L., & Schrijver, A. [GLS93] is a standard and accessible reference on the topic. The bound on the approximation error incurred by the ellipsoid method was taken from Gupta, A. [Gup20].

[HL01] Hiriart-Urruty, J.-B., & Lemaréchal, C. (2001). *Fundamentals of Convex Analysis*. Springer. https://link.springer.com/book/10.1007/978-3-642-56468-0

[Kha80] Khachiyan, L. G. (1980). Polynomial algorithms in linear programming. *USSR Computational Mathematics and Mathematical Physics*, *20*(1), 53–72.

[Šor77] Šor, N. Z. (1977). Cut-off method with space extension in convex programming problems. *Cybernetics*, *13*(1), 94–96.

[YN76] Yudin, D. B., & Nemirovskii, A. S. (1976). Informational complexity and efficient methods for the solution of convex extremal problems. *Matekon*, *13*(2), 22–45.

[GLS93] Grötschel, M., Lovász, L., & Schrijver, A. (1993). *Geometric Algorithms and Combinatorial Optimization*. Springer. https://link.springer.com/book/10.1007/978-3-642-78240-4

[Gup20] Gupta, A. (2020). *The Centroid and Ellipsoid Algorithms*. https://www.cs.cmu.edu/~15850/notes/lec21.pdf

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Changelog

- Feb 20, 2025: Added Remark L5.1 and edited Definition L5.2.
- Feb 24, 2025: Changed compact → closed in Definition L5.2. (Thanks https://piazza.com/class/m6lg9aspoutda/post/m7jzmkp7jyv6xd)
- Mar 15, 2025: Renamed variable for clarity in (3). (Thanks Khizer Shahid!)