

Lecture 4A

Feasibility, optimization, and separation

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In Lecture 1, we discussed how nonlinear optimization problems can be generally computationally intractable. In Lecture 3, we introduced a class of functions—called *convex* functions—for which first-order optimality conditions are both necessary and sufficient (assuming a convex feasible set Ω). In this lecture, we continue our study of convex optimization, showing that *in many cases* the solution to a convex optimization problem

$$\begin{aligned} \min_x \quad & f(x) \text{ convex} \\ \text{s.t.} \quad & x \in \Omega \text{ convex} \end{aligned}$$

can be found in polynomial time. However, it is also important to keep in mind that *not all convex optimization problems* can be solved in polynomial time. For example, optimization over the *copositive cone* (a convex set) is known to be intractable; we will talk more about that in a later class.

1 Separating a point from a closed convex set

An important property of any convex set Ω is that whenever a point y is not in Ω , then we can *separate* y from Ω using a *hyperplane*. In other words, *flat* separation surfaces are enough for certifying that a point $y \notin \Omega$.

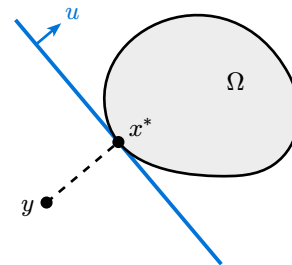
Theorem 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, closed, and convex set, and let $y \in \mathbb{R}^n$ be a point. If $y \notin \Omega$, then there exist $u \in \mathbb{R}^n, v \in \mathbb{R}$ such that

$$\langle u, y \rangle < v, \quad \text{and} \quad \langle u, x \rangle \geq v \quad \forall x \in \Omega.$$

Proof. The proof of the result rests on a very simple idea: the direction of the halfspace will be made orthogonal to the line that connects y to its projection x^* onto Ω , and the halfspace boundary will be set so that it passes through x^* . We now make the argument formal.

First, since Ω is nonempty and closed, a Euclidean projection x^* of y onto Ω exists,¹ as we discussed in Lecture 1. In other words, the nonlinear optimization problem

$$\begin{aligned} \min_x \quad & \frac{1}{2} \|x - y\|_2^2 \\ \text{s.t.} \quad & x \in \Omega \end{aligned}$$



^{*}These notes are class material that has not undergone formal peer review. The TAs and I are grateful for any reports of typos.

must have at least a solution $x^* \in \Omega$. Furthermore, since the objective function is differentiable and Ω is convex, from the first-order optimality conditions (see Lecture 2) we know that

$$\langle x^* - y, x - x^* \rangle \geq 0 \quad \forall x \in \Omega. \quad (1)$$

Let now

$$\begin{aligned} u &:= x^* - y, & [\triangleright \text{this is the direction that connects } y \text{ to } x^*] \\ \text{and } v &:= \langle u, x^* \rangle. & [\triangleright \text{so that the halfspace boundary passes through } x^*] \end{aligned}$$

Note that $u \neq 0$, since $x^* \in \Omega$ but $y \notin \Omega$. So, $\|u\| > 0$ and therefore

$$\langle u, y \rangle = \langle u, x^* - u \rangle = v - \|u\|_2^2 < v.$$

Thus, to complete the proof, we now need to show that $\langle u, x \rangle \geq v$ for all $x \in \Omega$. But this is exactly what (1) guarantees, since $u = x^* - y$ and $v = \langle u, x^* \rangle$ by definition. \square

The result above might not seem like much. After all, the proof is pretty straightforward, and the geometric intuition strong enough that one might be tempted to just take it for granted. Instead, the consequences of the result are deep, far-reaching, and intimately tied to some of the most significant breakthroughs in mathematical optimization theory.

1.1 Separation oracles

The result established in Theorem 1.1 justifies the following definition.

Definition 1.1 ((Strong) separation oracle). Let $\Omega \subseteq \mathbb{R}^n$ be convex and compact. A *strong separation oracle* for Ω is an algorithm that, given any point $y \in \mathbb{R}^n$, correctly outputs one of the following:

- “ $y \in \Omega$ ”, or
- “ $(y \notin \Omega, u, v)$ ”, where the pair $(u, v) \in \mathbb{R}^n \times \mathbb{R}$ is such that

$$\langle u, y \rangle < v, \quad \text{and} \quad \langle u, x \rangle \geq v \quad \forall x \in \Omega.$$

1.2 Finding separating hyperplanes in practice

Theorem 1.1 guarantees the *existence* of a separating hyperplane. In many problems of interest, *constructing* a separation oracle is simple.

Example 1.1 (Separation oracle for a convex polytope). Let Ω be a convex polytope, that is, the convex set defined by the intersection of a finite number of halfspaces (linear inequalities)

$$\Omega := \{x \in \mathbb{R}^n : Ax \leq b\}, \quad \text{where} \quad A = \begin{pmatrix} - & a_1^\top & - \\ & \vdots & \\ - & a_m^\top & - \end{pmatrix} \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m.$$

Then, given a point $y \in \mathbb{R}^n$, we can implement a separation oracle as follows:

- if $Ay \leq b$, return “ $y \in \Omega$ ”;

¹In fact, it is easy to prove that the projection is unique (see Homework 1, Exercise 4). However, we do not need uniqueness for the argument that follows.

- else, at least one of the inequalities $a_j^\top y \leq b_j$, $j \in \{1, \dots, m\}$ is violated. In other words, there exists j such that $a_j^\top y > b_j$, while by definition of Ω , $a_j^\top x \leq b_j$ for all x . This shows that the response “ $(y \notin \Omega, -a_j, -b_j)$ ” is a valid response.

Remark 1.1. Example 1.1 shows that whenever we have a finite number m of inequalities, a separation oracle for the polytope defined by those inequalities can be implemented in time that depends linearly on m and the dimension of the embedding space. This result establishes a *blanket* guarantee, but in some cases, one can do better: depending on the structure of the inequalities, sometimes one can get away with sublinear complexity in m . In some cases, one might be able to construct an efficient separation oracle even for polytopes that have an infinite number of inequalities!

We proceed with another classic example of a feasible set that admits a simple separation oracle.

Example 1.2 (Separation oracle for the semidefinite cone). Let $\Omega = \{M \in \mathbb{R}^{n \times n} : M \succcurlyeq 0\}$ be the set of semidefinite matrices, that is, all symmetric matrices such that $v^\top M v \geq 0$ for all $v \in \mathbb{R}^n$ —or, equivalently, such that all of M ’s eigenvalues are nonnegative.

Then, given a point $Y \in \mathbb{R}^{n \times n}$, we can implement a separation oracle as follows:

- if Y is *not* symmetric, then there exist $i, j \in \{1, \dots, n\}$ such that $Y_{ij} < Y_{ji}$; return “ $(Y \notin \Omega, E_{ij} - E_{ji}, 0)$ ”, where E_{ij} is the matrix of all zeros, except in position i, j where it has a 1.
- else, if Y is symmetric, we can compute all of its eigenvalues and eigenvectors. If one eigenvalue is negative, then the corresponding eigenvector w must be such that $w^\top Y w = \langle Y, ww^\top \rangle < 0$. Hence, return “ $(Y \notin \Omega, ww^\top, 0)$ ”.
- otherwise, return “ $Y \in \Omega$ ”.

As we show next, a fundamental result in optimization theory reveals that under mild hypotheses, if the feasible set admits an efficient separation oracle and the objective function is convex, then the solution can be computed efficiently.

2 Optimization via separation

In a major breakthrough in mathematical optimization, Khachiyan, L. G. [Kha80] proposed a polynomial-time algorithm for using separation oracles to find the minimum of a linear function. The algorithm, which goes under the name of *ellipsoid method* is actually more general, and applies to general convex objectives on sets for which separation oracles are available. The result builds on top of previous work by Šor, N. Z. [Šor77] and Yudin, D. B., & Nemirovskii, A. S. [YN76].

In particular, Khachiyan’s result was the first to show that linear programming problems can be solved in polynomial time. This was an unexpected result at the time, and in fact, the complexity of linear programming solvers was conjectured to be *not* polynomial (more on this in the next section). The result of Khachiyan stirred so much enthusiasm in the research community that the New York Times even advertised it on its first page.

Despite the enthusiasm, the ellipsoid method turned out to be very impractical. Still, it is a great theoretical idea, and its consequences are pervasive.

Similar Transportation Bond Issues Are Passed in New York and Jersey

Con Ed Takeover Action Fails in Westchester — Simon Wins in Bronx

By FRANK LYNN Voters approved mass transportation bond issues yesterday in New York and New Jersey, while rejecting a state college building bond issue in New Jersey and at least two constitutional amendments in New York.

Westchester County voters rejected a potential county takeover of the Consolidated Edison Company's electrical distribution system. (Page B3.) In contests for public office, Borough President Stanley Simon of the Bronx, a Democrat, and Dewey F. Coblan, the Republican candidate for Suffolk County Executive, were elected in landslide.

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CARTER SAID TO ASK FOR LOYALTY CHECK OF AIDES ON POLITICS

Ouster Implied for Those Unwilling to Help His Drive — Strauss to Head Re-election Unit

By TERENCE SMITH WASHINGTON, Nov. 6 — President Carter instructed his Cabinet officers last night to check the political "dependability" of their subordinates and strongly implied that those who were not prepared to campaign actively in his behalf should be dismissed, White House sources said today.

The President, at a bluntly political dinner in the family dining room of the White House last night, said that he expected all the political appointees in his Administration to be "actively engaged" in the campaign. Although Mr. Carter did not explicitly say that those unwilling to campaign should resign, one of the dinner guests said: "That was his whole message."

In another development, Robert S. Strauss, the President's special Middle East envoy, confirmed today that he would resign soon to become chairman of the Carter-Mondale Presidential Committee. The White House announced that Sol M. Linowitz, who helped negotiate the Panama Canal treaties, would replace Mr. Strauss in the Middle East post.

Political Atmosphere Reflected The White House dinner last night was the first full-scale discussion of Mr. Carter's forthcoming campaign that the President has held with his Cabinet and top members of his staff, and it reflected the intensely political atmosphere that has gripped the Carter White House since the announcement of the challenge for the Democratic Presidential nomination from Senator Edward M. Kennedy, who plans to announce his candidacy formally in Boston tomorrow.

Mr. Carter accepted an invitation to participate in a joint appearance with Senator Kennedy at an "open forum" in Des Moines in January. The Des Moines Register and Tribune, sponsor of the forum, said that Mr. Kennedy had also accepted the President's acceptance today appeared to be an effort to preempt Mr. Kennedy, who is reportedly planning to challenge Mr. Carter tomorrow to debate him before the primaries.

In the Iowa forum, Mr. Carter and Mr. Kennedy will share a platform and answer questions from a panel of reporters and editors and from the public. James Gannon, the newspaper's executive editor, said that Gov. Edmund G. Brown Jr. of California was not being invited to join the forum "because he has not mounted a serious campaign in Iowa."

The Iowa Democratic caucus on Jan. 21 will be the first major test of the election year. The switch to Mr. Strauss, a Cleveland former chairman of the Democratic National Committee, as head of the Carter-

IRAN'S CIVIL GOVERNMENT OUT; HOSTAGES FACE DEATH THREAT; OIL EXPORTS BELIEVED HALTED



In Teheran, a demonstrator held up pictures with the faces of President Carter and the Shah of Iran superimposed on those of victims of the firing squad.

STUDENTS WARN U.S. Ayatollah Instructs Secret Revolutionary Council to Form a Cabinet

By JOHN KIEFER Special to The New York Times TEHRAN, Iran, Nov. 6 — Prime Minister Mehdi Bazargan's provisional revolutionary government dissolved today, conceding power to the Islamic authority of Ayatollah Ruhollah Khomeini. The United States had been counting on Mr. Bazargan's Government to insure the safety of 60 or so American hostages seized Sunday at the American Embassy. The Government's abrupt collapse, after months of frustration and impotence, appeared to further dampen the already dim hope for a negotiated release of the hostages.

Militant Islamic students holding the embassy said today that they would kill the hostages if the United States used military force in a rescue attempt. The students are demanding that the United States hand over Shah Mohammed Reza Pahlavi, who is undergoing medical treatment in New York City. (In Washington, President Carter met with his foreign policy advisers and decided to maintain a nonprovocative posture toward Iran in the hope that the hostages would eventually be freed by Iranian religious authorities. And at the United Nations, a Palestine Liberation Organization spokesman said that Yasser Arafat, head of the guerrilla group, was sending a delegation to Teheran to "secure the safety of the Americans" and others held hostage. Page A14.)

Council's Membership Is Secret Accepting Mr. Bazargan's resignation, Ayatollah Khomeini ordered the Revolutionary Council, whose membership is secret but is believed to consist of Islamic leaders, to take over the government.

The political change appeared to mark a break between the West-oriented intellectuals, who opposed the Shah and had hoped to establish a parliamentary system, and their ostensible allies in the Islamic clergy, determined to impose the Islamic moral authority of the Koran. Since the seining of the Shah in mid-February, real power here has been concentrated in the person of Ayatollah Khomeini, the white-bearded ascetic of the Shiite branch of Islam who emerged from 15 years in exile to become the symbol of the Iranian revolution.

Since they assumed their offices in Teheran this morning the government buildings, the lawyers and other specialists who served in Mr. Bazargan's Government had been overwhelmed by the mullahs who gathered, in turbans and robes, around Ayatollah Khomeini in the dusty holy city of Qum. "You are weak old man," the Ayatollah chided Mr. Bazargan early on. The friction was constant. Mr. Bazargan often went on television to announce that

Continued on Page A14, Column 4

Main Iran Oil Port Reported Closed And Prices of Spot Petroleum Soar

By ANTHONY J. PARISI Iranian demand that the United States return the Shah to Teheran for trial. Later yesterday, the State Department said it was advising companies operating in Iran to start withdrawing American employees. Earlier, many had said they were not planning to do so. (Page D1.) Because almost all of Iran's oil leaves the country via Kharg Island, a stoppage there would shut off all but Iranian oil exports. That, according to oil experts interviewed yesterday, would soon put consumption in the same squeeze they faced this time last year, when Iranian production had fallen.

The run-up in spot market oil prices began early in the day, after reports that the Shah's new oil minister was ready to limit the amount of oil available to America. The buying spree was apparently bolstered by new rises in Persian Gulf and North Sea contract prices. (Page D5.) The situation turned into what some in

Continued on Page D5, Column 2

F.A.A. Seeks Fine of \$1.5 Million For 'Unairworthy' Braniff Flights

By RICHARD WITKIN The Federal Aviation Administration moved yesterday to impose a record fine of \$1.5 million against Braniff International Airways for conducting hundreds of flights with planes that were allegedly "unairworthy."

The agency said that the violations, in flights of Boeing 747 jumbo jets, were of conventional narrow-body planes, "appear to reflect a basic pattern of continuing disregard" of aircraft-maintenance rules by the management of the Texas-based carrier.

Specifically, Braniff was accused of using "improper and unapproved maintenance procedures," operating aircraft that had not been given the required inspections and of failing to keep adequate records.

Braniff Disputes Charges A Braniff spokesman, Jere L. Cox, said that the company felt it was the finest and "most meticulous" care of any airline. Braniff contended that the F.A.A.'s charges were "not in context with the facts" and voiced confidence that it would be able to "satisfy any questions they might have about any alleged discrepancies."

Braniff, which operates a vast domestic network and flights to Europe, Asia and South America, has one of the best safety records in the industry. Mr. Cox said the last major accident was the 1968 crash of a British-made BAC 1-11. The last two before that were crashes of Lockheed Electra turboprop planes about the age of the proposed \$1.5 million fine. But compromises are often negotiated. Ultimately, the Federal agency could take the issue to court.

The proposed penalties are about four times that of the fine levied last year against Braniff for similar violations.

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Interior Dept. Assailed On Missing a Deadline For Species Protection

By PHILIP SHABROFF Special to The New York Times WASHINGTON, Nov. 6 — To the dismay and anger of environmentalists, about 1,700 plants and a hundred animals are being dropped from consideration for protection as endangered species because a deadline Congress set a year ago for Interior Department action is about to pass.

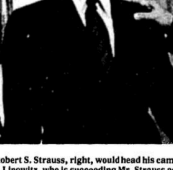
Department officials say that they have no choice. Last Nov. 10, President Carter signed amendments to the Endangered Species Act requiring, among other things, that if a species had been proposed for protection and the agency had not acted on it for two years or more, the agency then had one year to place it on the endangered or threatened species list or it would automatically be dropped.

That year is up next Monday, and the agency has met the amendments' requirements for only 37 species.

Concerned environmentalists, warning that many of the species being dropped may face extinction, charge that bureaucratic slowness and ineptitude are the reason the plants and animals are losing any chance of protection.

The Endangered Species Act has generated controversy since it was passed in 1973. The most recent political turmoil surrounding the law is the battle over the mall starter, an endangered three-toed sloth.

Continued on Page B7, Column 1



President Carter after announcing that Robert S. Strauss, right, would head his campaign. At left are Secretary of State Cyrus R. Vance and Sol M. Linowitz, who is succeeding Mr. Strauss as Middle East negotiator.

Energy Dept. Accuses 9 Refiners Of \$1.18 Billion Oil Overcharges

By RICHARD D. LYONS Special to The New York Times WASHINGTON, Nov. 6 — Department of Energy investigators announced today the filing of formal charges against nine major oil refiners, accusing them of having overcharged the public by \$1.18 billion from 1973 to 1976.

Moreover, Administration officials said the investigation of refiner overcharging since the 1973 energy embargo might eventually result in allegations that consumers paid up to \$25 billion too much for petroleum products in the last six years.

In general, spokesmen for the nine companies took the position that the allegations were both unwarranted and unfair. While some said they would have to study the charges, some conceded that the Federal allegations were correct.

The specific complaints, listing the alleged overcharges and known technically as notices of probable violation, were issued today against the Mobil Oil Corporation, \$72.7 million; the Shell Company, \$211.6 million; the Standard Oil Company of California, \$17.2 million; the Amstar Refining Corporation, \$88 million; the Gulf Oil Corporation, \$60.1 million; the Atlantic Richfield Company, \$55.8 million; and Conoco Inc., \$46.1 million.

According to the American Journal of Mathematics, the Russian discovery offers a way by which the number of steps in a solution can be dramatically reduced. It also offers the mathematician a way of learning quickly whether a problem has a solution or not, without having to complete the entire immense computation that may be required.

Continued on Page A26, Column 3

A Soviet Discovery Rocks World of Mathematics

By MALCOLM M. BROWNE A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer scientists, and experts have begun exploring its practical applications.

Mathematicians describe the discovery by L. G. Khachiyan as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of hit-or-miss basis.

Apart from its profound theoretical interest, the discovery may be applicable

in weather prediction, complicated industrial processes, petroleum refining, the scheduling of workers in large factories, secret codes and many other things.

"I have been deluged with calls from virtually every department of government for an interpretation of the significance of this," a leading expert on computer methods, Dr. George S. Dantzig of Stanford University, said in an interview.

The solution of mathematical problems that computers must be broken down into a series of steps. One class of problem sometimes involves no more than that

could take billions of years to compute.

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2.1 The intuition behind the ellipsoid method

Formalizing the details of the ellipsoid method is rather complex. A major source of difficulty is the fact that the algorithm needs to approximate square roots using fractions to be implementable on a finite-precision machine, and that causes all sorts of tricky analyses that the approximation error can indeed be kept under control. These details are certainly important, but are notoriously tedious, and fundamentally they are just that, *details*. If you are curious to read a formal account, I recommend the authoritative book by Grötschel, M., Lovász, L., & Schrijver, A. [GLS93]. For this lecture, we just focus on the *idea* behind the ellipsoid method.

The idea behind the ellipsoid method is rather elegant. At its core, it is a generalization of *binary search* from one dimension to multiple dimensions. At every iteration of the algorithm, the space is “cut” by using a separating hyperplane.

Feasibility. To build intuition, ignore for now the objective function, and consider the following problem: given a separation oracle for Ω (closed and convex), either find $x \in \Omega$, or determine that Ω is empty. You are given two radiuses:

- the radius $R > 0$ guarantees that if Ω is not empty, then $\Omega \cap \mathbb{B}_R(0) \neq \emptyset$;
- the radius $r > 0$ guarantees that if Ω is not empty, then it contains a ball of radius r in its interior.

If this problem were one-dimensional, then Ω would be either empty or an interval, and a separation oracle would be an algorithm that, given any $y \in \mathbb{R}$, would return whether $y \in \Omega$, or one of the statements “ y is too small” / “ y is too large”. Solving the problem now appears easy: start from the interval $[-R, R]$, and perform a binary search using the separation oracle to guide the search. Once the size of the search interval drops below r , we know that Ω is empty.

The ellipsoid method generalizes this idea to multiple dimensions. At every iteration, it keeps track of a “search space” (the generalization of the search interval above). Then, it queries the separation oracle for the center of this search space. If the point does not belong to Ω , and the separation oracle returns the separating hyperplane $\langle u, x \rangle = v$, then the search space is cut by considering now only the subset of the search space that intersects $\{x \in \mathbb{R}^n : \langle u, x \rangle \geq v\}$. The process continues until the volume of the search space becomes smaller than the radius r . The reason why this method is called the “ellipsoid method” is that the search space in the multi-dimensional case is not kept in the form of an interval, but rather as an ellipsoid. This is mostly for computational reasons, since we need to have an internal way of representing the search domain that is convenient to use.

Incorporating the objective. The above idea can be extended to incorporate an objective function $f(x)$. To do that, we will need to start cutting not only the search ellipsoid, but also the feasible set to make sure we end up at the optimum. In other words, you can think of this extended ellipsoid method as having “two modes”: while it has not found a feasible point in Ω , it cuts the search ellipsoid; then, once feasible points are found, it cuts the feasible set to exclude all values above the current value.

- Initialize at time $t = 1$ with the starting point $y_1 := 0 \in \mathbb{R}^n$, starting ellipsoid $\mathcal{E}_1 := \mathbb{B}_R(0)$, and starting feasible set $\Omega_1 := \Omega$.
- At each time t , we ask a separation oracle for Ω_t whether the center $c_t \in \mathbb{R}^n$ of the search ellipsoid \mathcal{E}_t belongs to Ω_t or not.² There are only two cases:
 - ▶ If the center c_t is *not* feasible, then set $\Omega_{t+1} := \Omega_t$, and cut the search space by setting \mathcal{E}_{t+1} to an ellipsoid that contains the intersection between \mathcal{E}_t and the halfspace containing Ω_t returned by the separation oracle.
 - ▶ If the center c_t is feasible, then we know for sure that all points $x \in \Omega_t$ such that $\langle \nabla f(c_t), x - c_t \rangle \geq 0$ are such that $f(x) \geq f(c_t)$. This follows trivially from the linear lower bound property of convex functions (Theorem 1.1 of Lecture 3):

$$\langle \nabla f(c_t), x - c_t \rangle \geq 0 \quad \implies \quad f(x) \geq f(c_t) + \langle \nabla f(c_t), x - c_t \rangle \geq f(c_t).$$

Hence, we can cut *both* the search ellipsoid \mathcal{E}_t and the feasible set Ω_t by considering their intersection with the halfspace $H_t := \{x \in \mathbb{R}^n : \langle \nabla f(c_t), x - c_t \rangle \leq 0\}$. In particular, we set $\Omega_{t+1} := \Omega_t \cap H_t$, and set \mathcal{E}_{t+1} to a smaller ellipsoid that contains $\mathcal{E}_t \cap H_t$.

- ▶ Finally, after the volume of the search ellipsoid has gotten sufficiently small (this happens after $T = O(n^2) \log(R/r)$ iterations), we output the following:
 - If we never encountered a center c_t that was feasible, then we report that Ω was infeasible.
 - Else, we output the c_t that minimizes f , out of those that were feasible.

Assuming that we can ignore all sorts of tedious rounding issues, the following guarantee can be shown [Gup20].

Theorem 2.1. Let R and r be as above, and let the range of the function f on Ω be bounded by $[-B, B]$. Then, the ellipsoid method described above run for $T \geq 2n^2 \log(R/r)$ steps either correctly reports that $\Omega = \emptyset$, or produces a point x^* such that

$$f(x^*) \leq f(x) + \frac{2BR}{r} \exp\left(-\frac{T}{2n(n+1)}\right) \quad \forall x \in \Omega.$$

2.2 Takeaway message: Separation implies optimization

If you squint your eyes, what the ellipsoid method proves constructively is the following: if we know how to construct a separation oracle for a set Ω , then we can optimize over Ω . Of course, this is a bit of a simplification (and there are all sorts of little conditions here and there as we have seen above), but nonetheless it is a good first approximation of the general message.

The opposite direction is also known to be true, even when by “optimization” we simply mean optimization of linear objective functions.

Further readings and bibliography

If you want to read more about the ellipsoid method, the book by Grötschel, M., Lovász, L., & Schrijver, A. [GLS93] is a standard and accessible reference on the topic. The bound on the approximation error incurred by the ellipsoid method was taken from Gupta, A. [Gup20].

- [Kha80] L. G. Khachiyan, “Polynomial algorithms in linear programming,” *USSR Computational Mathematics and Mathematical Physics*, vol. 20, no. 1, pp. 53–72, 1980.
- [Šor77] N. Z. Šor, “Cut-off method with space extension in convex programming problems,” *Cybernetics*, vol. 13, no. 1, pp. 94–96, 1977.
- [YN76] D. B. Yudin and A. S. Nemirovskii, “Informational complexity and efficient methods for the solution of convex extremal problems,” *Matekon*, vol. 13, no. 2, pp. 22–45, 1976.
- [GLS93] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*. Berlin, Germany: Springer, 1993. [Online]. Available: <https://link.springer.com/book/10.1007/978-3-642-78240-4>

²There is a caveat here: technically, we are assuming as given a separation oracle for Ω , *not* Ω_t . Yet, because Ω_t is obtained from Ω by intersecting with halfspaces, it is easy to see that one separation oracle for Ω_t can be constructed efficiently starting from that for Ω and the description of the intersected hyperplanes. Try working out the details!

[Gup20] A. Gupta, "The Centroid and Ellipsoid Algorithms." [Online]. Available: <https://www.cs.cmu.edu/~15850/notes/lec21.pdf>