HIGHER THETA SERIES FOR UNITARY GROUPS OVER FUNCTION FIELDS

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Abstract. In previous work, we defined certain virtual fundamental classes for special cycles on the moduli stack of Hermitian shtukas, and related them to the higher derivatives of non-singular Fourier coefficients of Siegel-Eisenstein series. In the present article, we construct virtual fundamental classes in greater generality, including those expected to relate to the higher derivatives of *singular* Fourier coefficients. We assemble these classes into "higher" theta series, which we conjecture to be modular. Two types of evidence are presented: structural properties affirming that the cycle classes behave as conjectured under certain natural operations such as intersection products, and verification of modularity in several special situations. One innovation underlying these results is a new approach to special cycles in terms of derived algebraic geometry.

Contents

1. Introduction

The earliest examples of theta functions were generating series for the number of representations of integers by quadratic forms. It has been known at least since the work of Jacobi that theta functions enjoy remarkable symmetry properties, which later became known as *modularity*, that underlie many of their applications. An incarnation of theta functions in arithmetic algebraic geometry was discovered by Kudla, who named them *arithmetic theta series*. This paper is about modularity in the context of arithmetic theta series.

The earliest examples of arithmetic theta series were constructed by Kudla as generating series with coefficients being cycle classes in the Chow groups of Shimura varieties [Kud04]. Kudla envisioned a conjectural *arithmetic Siegel–Weil formula* [Kud97], which would further require extending the special cycles to good integral models of Shimura varieties. A significant difficulty is the task of defining the appropriate cycle classes in the arithmetic Chow group indexed by singular Fourier coefficients. For example, for unitary Shimura varieties Kudla and Rapoport constructed the cycle classes on their integral models indexed by non-singular Fourier coefficients in [KR11, KR14], while Li and the third author [LZ20] proved an arithmetic Siegel–Weil formula for the non-singular Fourier coefficients. However, the definition of the singular terms, and therefore also the full arithmetic theta series, remains open (except in some lower dimensional case, see [KRY06]).

In [FYZ21] we proposed a function field analogue of this story: we defined special cycles on the moduli stack of Hermitian shtukas, constructed certain virtual fundamental classes for the cycles indexed by *non-singular* Fourier coefficients, and related them to the Taylor expansion of Fourier coefficients of corresponding Siegel-Eisenstein series. A novel feature of the function field version is that cycle classes can be defined for each non-negative integer r, and related to the r^{th} derivative of the Fourier coefficients of Siegel-Eisenstein series, whereas only the cases $r = 0$ and $r = 1$ seem to be witnessed over number fields (at least for the time being).

In this paper, we will construct cycle classes in general, going beyond the non-singular cases considered in [FYZ21], and assemble them into full "higher" arithmetic theta series (so named because they are related to higher derivatives of Siegel-Eisenstein series). The form of the singular terms exhibits interesting complexities that will be discussed further in §1.1. We formulate a conjecture about the modularity of such theta series, and then give evidence for this conjecture.

1.1. The modularity conjecture.

We now introduce notation so as to be able to describe our conjecture and the main results with more precision. Let X be a smooth, proper and geometrically connected curve over $k = \mathbf{F}_q$ of characteristic $p \neq 2$, and let $\nu\colon X' \to X$ be a connected étale double cover, with the non-trivial automorphism denoted $\sigma \in \mathrm{Aut}(X'/X)$. Let F be the function field of X and let F' be the ring of rational functions on X'. In [FYZ21] we defined the moduli stack $\mathrm{Sht}^r_{U(n)}$ parametrizing rank n "Hermitian shtukas" with r legs. We also defined certain special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$ indexed by $\mathcal{E}$, a vector bundle of rank m with $1 \leq m \leq n$ on X', and a Hermitian map $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ where $\mathcal{E}^\vee := \underline{\mathrm{Hom}}(\mathcal{E}, \omega_{X'})$ is the Serre dual of $\mathcal{E}$. The space of such a was called $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$ in [FYZ21], but is called $\mathcal{A}_{\mathcal{E}}(k)$ in this paper. (Everything in [FYZ21] works in a slightly more general setup allowing a similitude factor, but for simplicity we omit this from our introduction.)

To define the higher theta series, we construct an appropriate virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a))$ for *every* $a \in \mathcal{A}_{\mathcal{E}}(k)$.

This was done in [FYZ21] when a is *non-singular* (meaning that $a\colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ is injective as a map of coherent sheaves) and either $\operatorname{rank}\mathcal{E} = n$ or $\mathcal{E}$ is a direct sum of line bundles, by taking derived intersections from the situation where $\operatorname{rank}\mathcal{E} = 1$, following the ideas of [KR14] in the number field case. However, even in the non-singular case, to handle general m and $\mathcal{E}$ we must take a new approach based on Hitchin stacks (Definition 4.4). The dissimilarity to the number field situation comes from the fact that not every vector bundle on a proper curve splits as a sum of line bundles, while every vector bundle over the ring of integers of a number field splits as a direct sum of line bundles.

For singular a, the construction of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is more complicated. The cycle $\mathcal{Z}^r_{\mathcal{E}}(a)$ admits an open-closed decomposition according to the possible kernels of the map a, and the contribution from each stratum is the product of a virtual fundamental class constructed from a Hitchin stack and the top Chern class of a certain tautological bundle. The construction is completed in Definition 4.7. It may be a useful guide for the number field case, where no definition of special cycle classes in the arithmetic Chow group is currently known, for singular Fourier coefficients, at the time of this writing.

Having defined $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for each a, we then assemble them into higher theta series. More precisely, if $\operatorname{rank}\mathcal{E} = m$, then we consider the quasi-split unitary group (with respect to the double cover X'/X) of rank $2m$ over X, abbreviated $U(2m)$, and the standard Siegel parabolic P_m. (In the main body of the paper, starting in §9.1, we use the notation H_m for $U(2m)$.) We write down a function on $U(2m)(\mathbb{A})$ valued in $\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)})$:

$$\widetilde{Z}^r_m : U(2m)(\mathbb{A}) \longrightarrow \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)})$$

characterized by the following properties

(1) $\widetilde{Z}^r_m$ is left invariant under $P_m(F)$ and right invariant under $K = U(2m)(\widehat{\mathcal{O}})$;
(2) for any point in $P_m(F)\backslash P_m(\mathbb{A})/K \cap P_m(\mathbb{A}) \simeq P_m(F)\backslash U(2m)(\mathbb{A})/K$ represented by $(\mathcal{G}, \mathcal{E})$, where $\mathcal{G}$ is a rank $2m$ vector bundle on X' with a Hermitian structure $h : \mathcal{G} \simeq \sigma^*\mathcal{G}^*$ and $\mathcal{E}$ is a Lagrangian

sub-bundle of $\mathcal{G}$, we have a "Fourier expansion" (in the sense of [FYZ21, §2.6])

$$\widetilde{Z}_m^r(\mathcal{G},\mathcal{E}) = \chi(\det \mathcal{E}) q^{n(\deg \mathcal{E} - \deg \omega_X)/2} \sum_{a \in \mathcal{A}_{\mathcal{E}}(k)} \psi_0(\langle e_{\mathcal{G},\mathcal{E}}, a\rangle)\zeta_*[\mathcal{Z}_{\mathcal{E}}^r(a)]. \tag{1.1}$$

Here we refer to §4.5 for the undefined notation in the right hand side. We note that, in the special case $\mathcal{E} = \mathcal{O}_{X'}^{\oplus m}$ the trivial bundle of rank m, the set of all such $(\mathcal{G},\mathcal{E})$ is naturally isomorphic to $N_m(F)\backslash N_m(\mathbb{A})/K \cap N_m(\mathbb{A})$, where N_m denotes the unipotent radical of P_m. Then $\mathcal{A}_{\mathcal{E}}(k)$ is naturally isomorphic to the Pontryagin dual of $N_m(F)\backslash N_m(\mathbb{A})/K \cap N_m(\mathbb{A})$ (depending on the choice of a non-trivial character $\psi_0 : k \to \mathbb{C}^\times$). For this $\mathcal{E}$, (1.1) more closely resembles the expressions for arithmetic theta series on Shimura varieties, as one finds for example in [Kud04, (5.4)].

Conjecture 1.1 (Modularity conjecture). *The function $\widetilde{Z}_m^r$ descends to a function*

$$Z_m^r : U(2m)(F)\backslash U(2m)(\mathbb{A}) \longrightarrow \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}_{U(n)}^r),$$

i.e., $\widetilde{Z}_m^r$ is left $U(2m)(F)$-invariant.

In other words, the class $\widetilde{Z}_m^r(\mathcal{G},\mathcal{E}) \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}_{U(n)}^r)$ should depend only on the Hermitian bundle $\mathcal{G}$ and not on its Lagrangian sub-bundle $\mathcal{E}$.

When $r = 0$, $\mathrm{Ch}_0(\mathrm{Sht}_{U(n)}^0)$ is simply the space of $\mathbf{Q}$-valued functions on $\mathrm{Bun}_{U(n)}(k)$ and the evaluation map turns $\widetilde{Z}_m^r$ into a two-variable function

$$U(2m)(\mathbb{A}) \times U(n)(\mathbb{A}) \longrightarrow \mathbf{Q}.$$

In this case, we obtain the classical theta function and the modularity conjecture essentially follows from the Poisson summation formula.

Remark 1.2. A conjecture can also be formulated in the case $X' = X \coprod X$. The special cycles then live on the more familiar moduli stack of $\mathrm{GL}(n)$-shtukas, and we refer to §4.8 for the details.

1.2. **Main results.** Our main results give some evidence towards the modularity conjecture.

One type of evidence, considered in Part III, is of numerical nature: we prove modularity of the functions obtained by intersecting our arithmetic series with classes analogous to what would be called *CM (Complex Multiplication) cycles* for unitary Shimura varieties. In particular, this entails proving the modularity of our arithmetic series for rank 1 unitary groups.

A second type of evidence, studied in Part II, concerns more abstract "coherence properties" of the special cycles. For example, we prove that the product of special cycle classes in the Chow ring behaves as predicted in [Kud04]. Perhaps surprisingly, the proofs rely crucially on the methods of *derived algebraic geometry*, and in particular on a construction of *derived special cycles* which yield our virtual fundamental classes. This will be discussed more in §1.3. This is a novelty of the singular terms; derived algebraic geometry has not played a role so far in studying the non-singular terms. It leads us to suspect that derived algebraic geometry may also prove useful in the more classical Shimura variety context of the Kudla program.

1.2.1. *Linear invariance.* We establish compatibility properties of the special cycles under various natural operations. Here we state an example (Theorem 7.1), which we call the *linear invariance* following the analog in the number field case considered by Howard in [How12].

Theorem 1.3. *Given a decomposition $\mathcal{E} \approx \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \ldots \mathcal{E}_j$, and $a_i \in \mathcal{A}_{\mathcal{E}_i}(k)$, the intersection product*

$$[\mathcal{Z}_{\mathcal{E}_1}^r(a_1)] \cdot_{\mathrm{Sht}_{U(n)}^r} [\mathcal{Z}_{\mathcal{E}_2}^r(a_2)] \cdot_{\mathrm{Sht}_{U(n)}^r} \ldots \cdot_{\mathrm{Sht}_{U(n)}^r} [\mathcal{Z}_{\mathcal{E}_j}^r(a_j)]$$

coincides with the sum of $[\mathcal{Z}_{\mathcal{E}}^r(a)]$ over all $a \colon \mathcal{E} \to \sigma^*\mathcal{E}^\vee$ satisfying the condition that

$$\text{the composition } \mathcal{E}_i \to \mathcal{E} \xrightarrow{a} \sigma^*\mathcal{E}^\vee \to \sigma^*\mathcal{E}_i^\vee \text{ is } a_i \text{ for each } 1 \le i \le j. \tag{1.2}$$

Although in principle both sides of the equality may be expressed in terms of elementary constructions, our proof relies on the derived algebraic geometry interpretation of the special cycle classes and we do not know a proof without derived methods.

1.2.2. *A refinement of the main result of* [FYZ21]. The stack $\mathrm{Sht}^r_{U(n)}$ is a disjoint union of two open-closed substacks and the modularity conjecture predicts that the restriction of the generating series $\widetilde{Z}^r_m$ to each of them should also be modular. In [FYZ21] we identified the degree of the $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_0(\mathcal{Z}^r_{\mathcal{E}}(a))$ for non-singular a with the r^{th} central derivative of the (suitably normalized) a^{th} Fourier coefficient of Siegel–Eisenstein series. In Theorem 9.5, we refine this result and show that the restriction of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for non-singular a to each of the two open-closed substacks has equal degree. The proof turns out to be non-elementary.

1.2.3. *The case* $n = m = 1$.

Theorem 1.4. *The modularity conjecture holds when* $n = m = 1$.

In this case the higher theta series are valued in the Chow group of proper zero-cycles, and are therefore essentially determined by their degrees. We show that the degrees are given by explicit automorphic functions, namely higher derivatives of a suitably normalized Eisenstein series. In fact this was already established for non-singular Fourier coefficients in [FYZ21], so the remaining work is to calculate the singular term, which turns out to be the Chern class of a certain tautological bundle, and to relate it to the Taylor expansion of the corresponding L-function. This computation is carried out in §10. Analogous results over number fields (for $r = 1$) were established by Kudla-Rapoport-Yang [KRY99].

1.2.4. *Intersection with "CM cycles".* For $\theta \colon Y \to X$ a degree n cover (possibly ramified), we have a "CM cycle" $\mathrm{Sht}^r_{U(1)_Y}$ of dimension r and a finite morphism $\Theta : \mathrm{Sht}^r_{U(1)_Y} \to \mathrm{Sht}^r_{U(n)}$. We consider the intersection number of the resulting cycle class $\Theta_*[\mathrm{Sht}^r_{U(1)_Y}]$ with the arithmetic theta series $\widetilde{Z}^r_{m=1}(g)$ in codimension r (i.e. the generating series of corank $m = 1$ special cycles).

Theorem 1.5. *For any* n*, the function* $U(2)(\mathbb{A}) \ni g \mapsto \left\langle \widetilde{Z}^r_{m=1}(g), \Theta_*[\mathrm{Sht}^r_{U(1)_Y}] \right\rangle \in \mathbb{C}$ *is left invariant under* $U(2)(F)$.

In fact, we can identify the intersection number with the r^{th} derivative of an explicit Eisenstein series. For the non-singular terms, this could be thought of as proving a higher-derivative, function-field analogue of [How12]. For the singular terms, it could be thought of as a higher-derivative, function-field analogue of the proof of the "averaged Colmez Conjecture" in [AGHMP18] (also obtained by a different method in [YZ18]).

One reason that we are limited to the corank $m = 1$ case is that, in order to intersect with $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ in the corank $m > 1$ case, we need to construct natural proper cycles of higher (than r) dimension on $\mathrm{Sht}^r_{U(n)}$. Some candidates are given by the analogs of basic loci on unitary Shimura varieties, which may reach nearly (but nevertheless strictly smaller than) half of the dimension of $\mathrm{Sht}^r_{U(n)}$. If we demand proper cycles that are surjective to the base X'^r, then we only know how to construct examples of dimension r but not higher, see Example 4.17 and Example 4.18.

1.2.5. *Geometric properties of special cycles.* In §8.3, we study the geometric properties of the special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$ in the special case where $\operatorname{rank} \mathcal{E} = m = 1$. We show that if a is non-singular then it is LCI of the correct dimension, and that the virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ coincides with the naive fundamental class. This fulfills a result promised in [FYZ21, Remark 7.10], and allows us to prove that the general constructions of cycle classes considered in this paper recovers the more naïve definitions studied in [FYZ21].

1.3. **Some remarks on the derived algebro-geometric method.** Although we are able to give an explicit formula for the special cycle classes in Part I using only "classical" algebraic geometry, the key foundation for the structural results proved in Part II is another interpretation of these classes in terms of *derived algebraic geometry*. We emphasize that the formulation of the modularity conjecture itself requires no input from derived algebraic geometry, while the evidence does.

To summarize, in §5 we define derived enhancements of the special cycles and show (Theorem 6.5) that their intrinsic derived fundamental cycles coincide with the virtual classes defined earlier. One advantage of this approach is that it does not involve separating the non-singular and singular cases, and so gives a uniform, conceptual derivation of the virtual fundamental classes for special cycles indexed by all Fourier coefficients. We find this to be compelling philosophical evidence for our definition of the singular terms.

Let us elaborate on the role of derived algebraic geometry. A derived scheme/stack has an underlying classical scheme/stack which we call its *classical truncation*, and in this sense the derived object can be thought of as "enhancing" the classical object with some kind of "derived structure". For example, a *quasi-smooth* (i.e., derived analogue of LCI) derived scheme provides a "perfect obstruction theory", in the sense of Behrend-Fantechi, for its classical truncation. Now, the process of classical truncation can lose good geometric properties; for example, *any* (arbitrarily singular) finite type affine scheme can arise as the classical truncation of a derived scheme which is quasi-smooth. The "hidden smoothness" philosophy of Deligne, Drinfeld, and Kontsevich holds that many naturally occurring singular moduli spaces are the classical truncations of natural quasi-smooth derived moduli spaces[1], and this was one of the early motivations to consider derived algebraic geometry.

In fact, it has been understood since the seminal work of Kudla-Rapoport [KR14] that the special cycles comprising arithmetic theta series need to be defined in a "derived" way. The physical special cycles are often not even of the "correct" dimension, and may be quite singular, so instead of considering their naive fundamental classes one wants to construct *virtual* fundamental classes. Kudla-Rapoport did this for the non-singular terms on unitary Shimura varieties, by presenting the cycles as a "derived intersection" of classical schemes with the correct "expected dimension". Then the virtual fundamental class was defined as a refined intersection product in Fulton's sense. Our construction of the non-singular terms on Hermitian shtukas also fits this mold.

For singular terms, we do not know of a presentation that may be used to carry out a similar strategy. What we shall see, however, is that all special cycles (even for singular coefficients) can be promoted to derived stacks in a natural way, which always have the correct dimension in the derived sense, and are always quasi-smooth. This gives another example of the "hidden smoothness" philosophy. Moreover, quasi-smooth derived stacks have an intrinsic notion of fundamental class, which can be viewed as a virtual fundamental class of the underlying classical stack. This gives an intrinsic construction of a virtual fundamental class to each special cycle, which is uniform with respect to the Fourier coefficient (whether singular or not).

From this perspective, the reason that cycles indexed by non-singular Fourier coefficients can be defined more easily is that the derived structure on such cycles can be constructed in an elementary way, by taking the derived intersection of classical stacks. We do not know of such an elementary construction for singular coefficients, nor is it necessary for us. This suggests that derived algebraic geometry may also be relevant for the classical Kudla program (over number fields), where the cycles indexed by singular coefficients had previously been defined in a more ad hoc manner. However, the methods we use to construct the derived special cycles do not have an obvious analogue in the number field situation.

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1.4. **Notation.** Throughout this paper, $k = \mathbf{F}_q$ is a finite field of odd characteristic p. Let $\ell \neq p$ be a prime. Let $\psi_0 : k \to \overline{\mathbf{Q}}_\ell^\times$ be a nontrivial character. For any space over $\mathbf{F}_q$, we denote by $\mathrm{Frob} = \mathrm{Frob}_q$ the q-power Frobenius endomorphism.

1.4.1. Let X denote a smooth, projective, geometrically connected curve over k, of genus g_X. Let ω_X be the line bundle of 1-forms on X.

Let $F = k(X)$ denote the function field of X. Let $|X|$ be the set of closed points of X. For $v \in |X|$, let $\mathcal{O}_v$ be the completed local ring of X at v with fraction field F_v and residue field k_v. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adèles of F, and $\widehat{\mathcal{O}} = \prod_{v\in|X|} \mathcal{O}_v$. Let $\deg(v) = [k_v : k]$, and $q_v = q^{\deg(v)} = \#k_v$. Let $|\cdot|_v : F_v^\times \to q_v^{\mathbf{Z}}$ be the absolute value such that $|\varpi_v|_v = q_v^{-1}$ for any uniformizer ϖ_v of $\mathcal{O}_v$. Let $|\cdot|_F : \mathbb{A}_F^\times \to q^{\mathbf{Z}}$ be the absolute value that is $|\cdot|_v$ on $F_v^\times$.

1.4.2. Let X' be another smooth curve over k and $\nu : X' \to X$ be a finite map of degree 2 that is generically étale. We denote by σ the non-trivial automorphism of X' over X. The case where X' is geometrically disconnected is allowed unless stated otherwise; it is usually allowed in Parts 1 and 2 but not in Part 3. Let F' be the ring of rational functions on X', which is either a quadratic extension of F or $F \times F$. We let k' be the ring of constants in F', which may be $\mathbf{F}_q$, $\mathbf{F}_{q^2}$ or $\mathbf{F}_q \times \mathbf{F}_q$. The notations $\omega_{X'}, |X'|, F'_{v'}, \mathcal{O}_{v'}, k_{v'}, \mathbb{A}_{F'}, |\cdot|_{v'}, |\cdot|_{F'}, q_{v'}$ and $\deg(v')$ (for $v' \in |X'|$) are defined similarly as their counterparts

[1] In modern terms, "hidden quasi-smoothness" would be a more accurate name for this philosophy. As far as we know, the name "quasi-smooth" is due to Lurie.

for X. Additionally, for $v \in |X|$, we use $\mathcal{O}'_v$ to denote the completion of $\mathcal{O}_{X'}$ along $\nu^{-1}(v)$, and define F'_v to be its total ring of fractions.

1.4.3. *Notation for cycle classes.* For a stack $\mathcal{Y}$, $\mathrm{Ch}(\mathcal{Y})$ denotes its *rationalized* Chow group in the sense of [Jos02]. We denote by $[\mathcal{Y}]^{\mathrm{naive}} \in \mathrm{Ch}(\mathcal{Y})$ the fundamental class of $\mathcal{Y}$. Typically we will work with "virtual fundamental classes" in $\mathrm{Ch}(\mathcal{Y})$ which do not (at least a priori) coincide with the naïve ones, and we shall denote such by $[\mathcal{Y}] \in \mathrm{Ch}(\mathcal{Y})$, although they will in fact depend on some auxiliary construction, such as a realization of $\mathcal{Y}$ as a fibered product or as the classical truncation of a derived stack $\mathscr{Y}$.

1.4.4. *Derived notation.* In §5 – §8, we adopt some notational conventions that differ from the rest of the paper. Namely, in those sections we operate within ∞-categories, so fibered products mean "derived fibered products", limits mean "homotopy limits", etc. unless noted otherwise. We refer to §5 for the precise explanation of the notation used in those sections.

1.4.5. *Some notational departures from* [FYZ21]*.* We emphasize that some notation has changed from our first paper [FYZ21] regarding Hitchin spaces and Hitchin bases. There we introduced certain Hitchin stacks $\mathcal{M} \subset \mathcal{M}^{\mathrm{all}}$ and Hitchin bases $\mathcal{A} \subset \mathcal{A}^{\mathrm{all}}$, decorated by indices, but in this paper they would be denoted $\mathcal{M}^{\circ} \subset \mathcal{M}$ and $\mathcal{A}^{\mathrm{ns}} \subset \mathcal{A}$. This will be explained more precisely when it comes up in the text.

Part 1. Formulation of the conjecture

2. Some (more) special cycles on moduli of shtukas

In this section we introduce a variant and a generalization of the special cycles defined in [FYZ21]. The variant, which plays a technical role in later definitions and proofs, is obtained by replacing $U(n)$ with the general linear group. For the generalization of special cycles, we consider Hermitian shtukas with similitude line bundles. Later we will formulate the modularity conjecture in this generality.

2.1. Shtukas for $\mathrm{GL}(n)'$.

We denote $\mathrm{GL}(n)' := \mathrm{Res}_{X'/X}\,\mathrm{GL}(n)$, a group scheme over X. In this subsection we define stacks $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$ parametrizing certain special types of shtukas for $\mathrm{GL}(n)'$, and establish their basic geometric properties. Their role in the study of Hermitian shtukas is of a somewhat technical nature, stemming from the fact that the Hitchin spaces corresponding to $\mathrm{GL}(n)'$ have better technical properties. They appear in an intermediate stage in the construction of cycle classes labeled by singular Fourier coefficients.

We begin by explicating the appropriate notion of bundles and Hecke correspondences. Let $\mathrm{Bun}_{\mathrm{GL}(n)'}$ be the moduli stack of $\mathrm{GL}(n)'$-bundles on X. By general properties of Weil restriction, there is a canonical equivalence of groupoids

$$\{\mathrm{GL}(n)'\text{-bundles on } X \times S\} \cong \{\mathrm{GL}(n)\text{-bundles on } X' \times S\}.$$

Hence the datum of a $\mathrm{GL}(n)'$-bundle on $X \times S$ is equivalent to the datum of a rank n vector bundle on $X' \times S$, and $\mathrm{Bun}_{\mathrm{GL}(n)'}$ is simply equivalent to the moduli stack of $\mathrm{GL}(n)$-bundles on X'.

Definition 2.1. Let $r \geq 0$ be an integer. The *Hecke stack* $\mathrm{Hk}^r_{\mathrm{GL}(n)'}$ has as S-points the groupoid of the following data:

(1) $x'_i \in X'(S)$ for $i = 1, \ldots, r$, with graphs denoted $\Gamma_{x'_i} \subset X' \times S$.
(2) A sequence of vector bundles $\mathcal{F}_0, \ldots, \mathcal{F}_r$ of rank n on $X' \times S$.
(3) Isomorphisms $f_i \colon \mathcal{F}_{i-1}|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma x'_i}} \xrightarrow{\sim} \mathcal{F}_i|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma x'_i}}$, for $1 \leq i \leq r$, which are lower of length 1 at x'_i and upper of length 1 at $\sigma x'_i$, in the terminology of [FYZ21, Definition 6.5].

Warning 2.2. The stack $\mathrm{Hk}^r_{\mathrm{GL}(n)'}$ is different from the usual iterated Hecke stack for rank n vector bundles on X', for example as considered for $n = 2$ in [YZ17], because we have demanded modifications to occur over conjugate pairs of points on the curve.

Lemma 2.3. *The (Artin) stack* $\mathrm{Bun}_{\mathrm{GL}(n)'}$ *is smooth.*

Proof. This follows from the standard obstruction theory argument, cf. [Hei10, Prop. 1]. □

Definition 2.4. Let $r \geq 0$ be an integer. We define $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$ by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathrm{GL}(n)'} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathrm{GL}(n)'} \arrow[d] \\ \mathrm{Bun}_{\mathrm{GL}(n)'} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathrm{Bun}_{\mathrm{GL}(n)'} \times \mathrm{Bun}_{\mathrm{GL}(n)'} \end{tikzcd}$$

A point of $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$ will be called a "$\mathrm{GL}(n)'$-shtuka". (But see Warning 2.5.)

Concretely, the S-points of $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$ are given by the groupoid of the following data:

(1) $x'_i \in X'(S)$ for $i = 1, \ldots, r$, with graphs denoted $\Gamma_{x'_i} \subset X \times S$.
(2) A sequence of vector bundles $\mathcal{F}_0, \ldots, \mathcal{F}_r$ of rank n on $X' \times S$.
(3) Isomorphisms $f_i \colon \mathcal{F}_{i-1}|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma x'_i}} \xrightarrow{\sim} \mathcal{F}_i|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma x'_i}}$, which are lower of length 1 at x'_i and upper of length 1 at $\sigma x'_i$.
(4) An isomorphism of vector bundles $\varphi \colon \mathcal{F}_r \cong {}^\tau\mathcal{F}_0 = (\mathrm{Id}_{X'} \times \mathrm{Frob}_S)^* \mathcal{F}_0$.

Warning 2.5. For the same reason as Warning 2.2, the stack $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$ is different from the usual iterated stack of rank n shtukas on X', for example as considered for $n = 2$ in [YZ17].

Lemma 2.6. *(1) The projection map* $(\mathrm{pr}_X, \mathrm{pr}_r) : \mathrm{Hk}^r_{\mathrm{GL}(n)'} \to (X')^r \times \mathrm{Bun}_{\mathrm{GL}(n)'}$ *recording* $\{x_i\}_{i=1}^r$ *and* $\mathcal{F}_r$ *is smooth of relative dimension* $2r(n-1)$.

(2) $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$ *is a smooth Deligne-Mumford stack, locally of finite type, and separated, of pure dimension* $r(2n-1)$.

Proof. The proof of (1) is similar to the proof of [FYZ21, Lemma 6.9(1)], except that in the case $r = 1$ the upper and lower modifications are independent, so the map $\mathrm{Hk}^1_{\mathrm{GL}(n)'} \to X' \times \mathrm{Bun}_{\mathrm{GL}(n)'}$ is (étale locally on target) a $\mathbf{P}^{n-1}$-fibration over a $\mathbf{P}^{n-1}$-fibration.

Part (2) follows from (1) upon applying [Laf18, Lemma 2.13]. □

2.2. **Special cycles.** We will define some special cycles on $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$.

Definition 2.7. Let $\mathcal{E}$ be a rank m vector bundle on X'.

We define the stack $\mathcal{Z}^r_{\mathcal{E},\mathrm{GL}(n)'}$ whose S-points are the groupoid of the following data:

- A $\mathrm{GL}(n)'$-shtuka $(\{x'_1, \ldots, x'_r\}, \{\mathcal{F}_0, \ldots, \mathcal{F}_r\}, \{f_1, \ldots, f_r\}, \varphi) \in \mathrm{Sht}^r_{\mathrm{GL}(n)'}(S)$.
- Maps of coherent sheaves $t_i \colon \mathcal{E} \boxtimes \mathcal{O}_S \to \mathcal{F}_i$ on $X' \times S$ such that the isomorphism $\varphi \colon \mathcal{F}_r \cong {}^\tau\mathcal{F}_0$ intertwines t_r with ${}^\tau t_0$, and the maps t_{i-1}, t_i are intertwined by the modification $f_i \colon \mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ for each $i = 1, \ldots, r$, i.e. the diagram below commutes.

$$\begin{tikzcd} \mathcal{E} \boxtimes \mathcal{O}_S \arrow[r, equal] \arrow[d, "t_0"] & \mathcal{E} \boxtimes \mathcal{O}_S \arrow[r, equal] \arrow[d, "t_1"] & \ldots \arrow[r, equal] \arrow[d] & \mathcal{E} \boxtimes \mathcal{O}_S \arrow[r, "\sim"] \arrow[d, "t_r"] & {}^\tau(\mathcal{E} \boxtimes \mathcal{O}_S) \arrow[d, "{}^\tau t_0"] \\ \mathcal{F}_0 \arrow[r, dashed, "f_0"] & \mathcal{F}_1 \arrow[r, dashed, "f_1"] & \ldots \arrow[r, dashed, "f_r"] & \mathcal{F}_r \arrow[r, "\varphi"] & {}^\tau\mathcal{F}_0 \end{tikzcd}$$

In the sequel, when writing such diagrams we will usually just omit the "$\boxtimes\mathcal{O}_S$" factor from the notation.

We define $\mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'} \subset \mathcal{Z}^r_{\mathcal{E},\mathrm{GL}(n)'}$ to be the open substack where the maps $\{t_i\}$ are all injective over every geometric point of S (equivalently, any one of $\{t_i\}$ is injective).

We will call the $\mathcal{Z}^r_{\mathcal{E},\mathrm{GL}(n)'}$, $\mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'}$ (or unions of their irreducible components) *special cycles of corank* m (with r legs) on $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$.

Proposition 2.8. *Let* $\mathcal{E}$ *be any vector bundle of rank* m *on* X'. *Then the projection map* $\mathcal{Z}^r_{\mathcal{E},\mathrm{GL}(n)'} \to \mathrm{Sht}^r_{\mathrm{GL}(n)'}$ *is finite.*

Proof. This follows from similar argument as for [FYZ21, Proposition 7.5]. □

2.3. **Hermitian shtukas with similitude.** In [FYZ21, §6] we worked with Hermitian shtukas based on the notion of a Hermitian bundle, which there was defined as a vector bundle $\mathcal{F}$ with a Hermitian structure $h \colon \mathcal{F} \xrightarrow{\sim} \sigma^* \underline{\mathrm{Hom}}(\mathcal{F}, \omega_{X'})$.

In this section we consider a more general situation, where the notion of Hermitian structure is expanded to include maps $h \colon \mathcal{F} \xrightarrow{\sim} \sigma^* \underline{\mathrm{Hom}}(\mathcal{F}, \omega_{X'} \otimes \nu^* \mathfrak{L})$ for any line bundle $\mathfrak{L}$ on X. These can be seen as torsors for a similitude unitary group. Most of the arguments of [FYZ21] already work at this level and generality, and it encompasses interesting situations not seen in the case $\mathfrak{L} = \mathcal{O}_X$; for example, when n is odd and $\mathfrak{L}$ is not a norm from X', the moduli space of shtukas with an odd number of legs is non-empty. The methods of [FYZ21] then give "Kudla-Rapoport style" identities between odd order Taylor coefficients of Siegel-Eisenstein series, whose functional equation has sign -1, and special cycles with an odd number of legs; see §9.4 for the precise statements.

Definition 2.9. Let $\mathfrak{L}$ be a line bundle on X.

(1) We define $\mathrm{Bun}_{U(n),\mathfrak{L}}$ analogously to [FYZ21, Definition 6.1] but with the appearances of "$\mathcal{F}^\vee$" ($= \underline{\mathrm{Hom}}(\mathcal{F}, \omega_{X'})$) in *loc. cit.* replaced by $\underline{\mathrm{Hom}}(\mathcal{F}, \omega_{X'} \otimes \nu^*\mathfrak{L})$. Similarly, for an integer $r \geq 0$, we define $\mathrm{Hk}^r_{U(n),\mathfrak{L}}$ analogously to [FYZ21, Definition 6.3], $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ analogously to [FYZ21, Definition 6.6]. For a rank m vector bundle $\mathcal{E}$ on X, we define $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}$ analogously to [FYZ21, Definition 7.1].

We will call the $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}$ (or their connected components) *special cycles of corank* m (with r legs), where we remind that $m = \operatorname{rank} \mathcal{E}$.

(2) The $\mathfrak{L}$-*twisted Hitchin base* $\mathcal{A}_{\mathcal{E},\mathfrak{L}}$ parametrizes maps $a \colon \mathcal{E} \to \sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \omega_{X'} \otimes \nu^*\mathfrak{L})$ such that $\sigma^* a^\vee = a$ where $a^\vee$ is the map obtained by dualizing a and then twisting by $\omega_{X'} \otimes \nu^*\mathcal{L}$. The open subscheme $\mathcal{A}^{\mathrm{ns}}_{\mathcal{E},\mathfrak{L}} \subset \mathcal{A}_{\mathcal{E},\mathfrak{L}}$ parametrizes a whose restriction to all geometric points of the test scheme are injective as maps of coherent sheaves.

Note when $\mathfrak{L} = \mathcal{O}_X$, $\mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)$ is what was denoted $\mathcal{A}^{\mathrm{all}}_{\mathcal{E}}(k)$ in [FYZ21, Definition 7.2]; $\mathcal{A}^{\mathrm{ns}}_{\mathcal{E},\mathfrak{L}}(k)$ is what was denoted $\mathcal{A}_{\mathcal{E}}(k)$ in *loc. cit.*.

(3) We have a decomposition $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}} = \coprod_{a \in \mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)} \mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$. For $a \in \mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)$, define $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)^\circ$ and $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)^*$ analogously to [FYZ21, Definition 7.4] (i.e., $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)^\circ$ is the open substack of $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$ when t_i are injective; $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)^*$ is the open substack where t_i are nonzero).

We will spell out some of these definitions. The S-points of the moduli stack $\mathrm{Bun}_{U(n),\mathfrak{L}}$ is the groupoid of pairs $(\mathcal{F}, h)$ where X is a rank n vector bundle on X', h is an isomorphism $\mathcal{F} \xrightarrow{\sim} \sigma^*\underline{\mathrm{Hom}}(\mathcal{F}, \omega_{X'} \otimes \nu^*\mathfrak{L})$ satisfying $\sigma^* h^\vee = h$ (which we call an $\mathfrak{L}$-*twisted Hermitian structure*), and morphisms $(\mathcal{F}, h) \xrightarrow{\sim} (\mathcal{F}', h')$ are isomorphisms $\mathcal{F} \to \mathcal{F}'$ intertwining h with h'.

The Hecke stack (with r legs) $\mathrm{Hk}^r_{U(n),\mathfrak{L}}$ has S-points being the groupoid of the following data:

(1) $x'_i \in X'(S)$ for $i = 1, \ldots, r$, with graphs denoted by $\Gamma_{x'_i} \subset X' \times S$.
(2) A sequence of vector bundles $\mathcal{F}_0, \ldots, \mathcal{F}_r$ of rank n on $X' \times S$, each equipped with $\mathfrak{L}$-*twisted Hermitian structures* $h_0, \ldots, h_r$.
(3) Isomorphisms $f_i \colon \mathcal{F}_{i-1}|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma(x'_i)}} \xrightarrow{\sim} \mathcal{F}_i|_{X' \times S - \Gamma_{x'_i} - \Gamma_{\sigma(x'_i)}}$, for $1 \leq i \leq r$, compatible with the h_i, which are lower of length 1 at x'_i and upper of length 1 at $\sigma x'_i$ (cf. [FYZ21, Remark 6.4] for the terminology).

The stack $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is defined by the Cartesian diagram

$$\begin{tikzcd}
\mathrm{Sht}^r_{U(n),\mathfrak{L}} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{U(n),\mathfrak{L}} \arrow[d, "(\mathrm{pr}_0, \mathrm{pr}_r)"] \\
\mathrm{Bun}_{U(n),\mathfrak{L}} \arrow[r, "(\mathrm{Id}, \mathrm{Frob})"] & \mathrm{Bun}_{U(n),\mathfrak{L}} \times \mathrm{Bun}_{U(n),\mathfrak{L}}
\end{tikzcd}$$

where $\mathrm{pr}_i : \mathrm{Hk}^r_{U(n),\mathfrak{L}} \to \mathrm{Bun}_{U(n),\mathfrak{L}}$ records $(\mathcal{F}_i, h_i)$.

Let $\eta : \mathbb{A}^\times / F^\times \to \{\pm 1\}$ be the quadratic character associated to F'/F by class field theory. Since X'/X is étale, the character descends to $\eta : \mathrm{Pic}_X(k)/\mathrm{Pic}_{X'}(k) \to \{\pm 1\}$, and for $\mathfrak{L} \in \mathrm{Pic}_X(k)$ we have $\eta(\mathfrak{L}) = 1$ if and only if $\mathfrak{L}$ is a norm from X'.

Lemma 2.10. *With notation as above,* $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ *is non-empty if and only if* $(-1)^r = \eta(\mathfrak{L})^n$.

Proof. The case $n = 1$ is established later in Lemma 2.15. Here we shall assume this case and then establish the general case.

Note that taking determinants induces a map $\mathrm{Sht}^r_{U(n),\mathfrak{L}} \to \mathrm{Sht}^r_{U(1),\mathfrak{L}^{\otimes n} \otimes \omega_X^{\otimes(n-1)}}$. By the result for the $n = 1$ case, this shows that $\mathrm{Sht}^r_{U(n),\mathfrak{L}} = \emptyset$ if $(-1)^r \neq \eta(\mathfrak{L})^n$. It remains to prove that whenever $(-1)^r = \eta(\mathfrak{L})^n$, then $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is non-empty.

If $\eta(\mathfrak{L}) = 1$, then $\mathrm{Sht}_{U(n),\mathfrak{L}}$ is isomorphic to $\mathrm{Sht}_{U(n)}$ by twisting, so the result follows from [FYZ21, Lemma 6.7].

Suppose $\eta(\mathfrak{L}) = -1$. With $\mathrm{Sht}^r_{\mathrm{GL}(1)/X'}$ defined as in [YZ17, (5.4)], there is a map $\mathrm{Sht}^r_{\mathrm{GL}(1)/X'} \to \mathrm{Sht}^r_{U(2),\mathfrak{L}}$ sending $\mathcal{F}_0 \dashrightarrow \ldots \dashrightarrow {}^\tau\mathcal{F}_r \cong {}^\tau\mathcal{F}_0$ to $\mathcal{F}_0 \oplus (\sigma^*\mathcal{F}_0^\vee \otimes \nu^*\mathfrak{L}) \dashrightarrow \ldots \dashrightarrow \mathcal{F}_r \oplus (\sigma^*\mathcal{F}_r^\vee \otimes \nu^*\mathfrak{L}) \cong {}^\tau(\mathcal{F}_0 \oplus (\sigma^*\mathcal{F}_0^\vee \otimes \nu^*\mathfrak{L}))$. Since $\mathrm{Sht}^r_{\mathrm{GL}(1)/X'}$ is non-empty whenever r is even, we find that $\mathrm{Sht}^r_{U(2),\mathfrak{L}}$ is non-empty whenever r is even. Taking direct sums induces a map

$$\mathrm{Sht}^0_{U(2),\mathfrak{L}} \times \mathrm{Sht}^r_{U(n-2),\mathfrak{L}} \to \mathrm{Sht}^r_{U(n),\mathfrak{L}} \tag{2.1}$$

which then inductively shows that $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is non-empty whenever r and n are even.

It remains to show that if $\eta(\mathfrak{L}) = -1$ and n is odd, then $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ is non-empty whenever r is odd. Since we are assuming the $n = 1$ case, we know that $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ is non-empty for all odd r. Then iterating (2.1) shows that $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is non-empty for all odd n and r. □

Let $\mathcal{E}$ be a rank m vector bundle on X'. The S-points of the stack $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}$ form the groupoid of the following data:

- An S-point $(\{x'_1, \ldots, x'_r\}, \{\mathcal{F}_0, \ldots, \mathcal{F}_r\}, \{f_1, \ldots, f_r\}, \varphi) \in \mathrm{Sht}^r_{U(n),\mathfrak{L}}(S)$.
- Maps of coherent sheaves $t_i \colon \mathcal{E} \boxtimes \mathcal{O}_S \to \mathcal{F}_i$ on $X' \times S$ such that the isomorphism $\varphi\colon \mathcal{F}_r \cong {}^\tau\mathcal{F}_0$ intertwines t_r with ${}^\tau t_0$, and the maps t_{i-1}, t_i are intertwined by the modification $f_i \colon \mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ for each $i = 1, \ldots, r$, i.e. the diagram below commutes.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, equal] \arrow[d, "t_0"] & \mathcal{E} \arrow[r, equal] \arrow[d, "t_1"] & \ldots \arrow[r, equal] \arrow[d] & \mathcal{E} \arrow[r, "\sim"] \arrow[d, "t_r"] & {}^\tau\mathcal{E} \arrow[d, "{}^\tau t_0"] \\
\mathcal{F}_0 \arrow[r, dashed, "f_0"] & \mathcal{F}_1 \arrow[r, dashed, "f_1"] & \ldots \arrow[r, dashed, "f_r"] & \mathcal{F}_r \arrow[r, "\varphi"] & {}^\tau\mathcal{F}_0
\end{tikzcd}$$

For properties of the objects in Definition 2.9 whose proofs are the same for general $\mathfrak{L}$ as written in the case $\mathfrak{L} = \mathcal{O}_X$ in [FYZ21], we will just cite the statements from [FYZ21]. For example, by the same proofs as for [FYZ21, Lemma 6.8, Lemma 6.9], we have the following geometric properties.

Lemma 2.11. *Let $\mathfrak{L}$ be any line bundle on X.*

(1) The stack $\mathrm{Bun}_{U(n),\mathfrak{L}}$ is smooth and equidimensional of dimension $n^2(g_X - 1)$.
(2) The projection map $(\mathrm{pr}_X, \mathrm{pr}_r) : \mathrm{Hk}^r_{U(n),\mathfrak{L}} \to (X')^r \times \mathrm{Bun}_{U(n),\mathfrak{L}}$ recording $\{x_i\}$ and $(\mathcal{F}_r, h_r)$ is smooth of relative dimension $r(n-1)$.
(3) $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is a Deligne-Mumford stack locally of finite type. The map $\mathrm{Sht}^r_{U(n),\mathfrak{L}} \to (X')^r$ is smooth, separated, equidimensional of relative dimension $r(n-1)$.

Forgetting the Hermitian structures give maps $\mathrm{Bun}_{U(n),\mathfrak{L}} \to \mathrm{Bun}_{\mathrm{GL}(n)'}$ and $\mathrm{Hk}^r_{U(n),\mathfrak{L}} \to \mathrm{Hk}^r_{\mathrm{GL}(n)'}$, which induce a map over $(X')^r$

$$\mathrm{Sht}^r_{U(n),\mathfrak{L}} \to \mathrm{Sht}^r_{\mathrm{GL}(n)'}$$

Lemma 2.12. *Let $\mathcal{E}$ be any vector bundle of rank m on X'. Then we have*

$$\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}} \cong \mathcal{Z}^r_{\mathcal{E},\mathrm{GL}(n)'} \times_{\mathrm{Sht}^r_{\mathrm{GL}(n)'}} \mathrm{Sht}^r_{U(n),\mathfrak{L}},$$
$$\mathcal{Z}^{r,\circ}_{\mathcal{E},\mathfrak{L}} \cong \mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'} \times_{\mathrm{Sht}^r_{\mathrm{GL}(n)'}} \mathrm{Sht}^r_{U(n),\mathfrak{L}},$$

as stacks over $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$.

Proof. Immediate from the definitions. □

2.4. **The case** $n = 1$. We now undertake a closer analysis of $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ for $n = 1$. We first set up some notation. Let $\mathrm{Prym}_{\mathfrak{N}} = \mathrm{Bun}_{U(1),\mathfrak{N}}$ be $\mathrm{Nm}^{-1}(\mathfrak{N})$ where $\mathrm{Nm} : \mathrm{Pic}_{X'} \to \mathrm{Pic}_X$ is the norm map. When $\mathfrak{N} = \mathcal{O}_X$ we omit the subscript $\mathfrak{N}$. In this case, we let Prym^0 denote its neutral component, and Prym^1 for the other component (both are defined over k).

Recall that Prym^0 and Prym^1 are also geometrically connected. Since $\mathrm{Prym}_{\mathfrak{N}}$ is a torsor under Prym, $\mathrm{Prym}_{\mathfrak{N}}$ also has two *geometric* connected components. However, its number of (k-rational) connected components depends on $\eta(\mathfrak{N})$, as explained below.

Lemma 2.13. *If $\eta(\mathfrak{N}) = 1$ then $\mathrm{Prym}_{\mathfrak{N}}$ has two connected components. If $\eta(\mathfrak{N}) = -1$ then $\mathrm{Prym}_{\mathfrak{N}}$ has one connected component.*

Proof. When $\eta(\mathfrak{N}) = 1$, i.e., $\mathfrak{N}$ is a norm, $\mathrm{Prym}_{\mathfrak{N}}$ has a k-point hence is a trivial Prym-torsor, therefore both geometric components of $\mathrm{Prym}_{\mathfrak{N}}$ are defined over k. When $\eta(\mathfrak{N}) = -1$, $\mathrm{Prym}_{\mathfrak{N}}$ has no k-point, which implies that the two geometric components of $\mathrm{Prym}_{\mathfrak{N}}$ are permuted by $\mathrm{Gal}(\overline{k}/k)$ (for otherwise a Frobenius stable geometric component, being a torsor under the connected group Prym^0, would contain a k-point by Lang's theorem), hence $\mathrm{Prym}_{\mathfrak{N}}$ is connected. $\square$

Lemma 2.14. *Let $\epsilon(\mathfrak{N}) \in \{0, 1\}$ be such that $\eta(\mathfrak{N}) = (-1)^{\epsilon(\mathfrak{N})}$. Then the Lang map*

$$\begin{aligned} \mathrm{Lang}\colon \mathrm{Prym}_{\mathfrak{N}} &\to \mathrm{Prym} \\ \mathcal{F} &\mapsto {}^{\tau}\mathcal{F} \otimes \mathcal{F}^{-1} \end{aligned}$$

lands in $\mathrm{Prym}^{\epsilon(\mathfrak{N})}$.

Proof. Given $\mathfrak{N}_1, \mathfrak{N}_2 \in \mathrm{Pic}_X(k)$ such that $\mathfrak{N}_2 \otimes (\mathfrak{N}_1)^{-1} = \mathrm{Nm}(\mathfrak{N}')$ for some $\mathfrak{N}' \in \mathrm{Pic}_{X'}(k)$, twisting by $\mathfrak{N}'$ induces an isomorphism $\mathrm{Prym}_{\mathfrak{N}_1} \xrightarrow{\sim} \mathrm{Prym}_{\mathfrak{N}_2}$. Hence if $\mathfrak{N}$ is a norm, then $\mathrm{Prym}_{\mathfrak{N}} \xrightarrow{\sim} \mathrm{Prym}_{\mathcal{O}_X}$, in which the claim is a result of Wirtinger explained in [Mum71, §2].

If $\mathfrak{N}$ is not a norm, by the twisting argument above, it suffices to show the statement for a single choice of $\mathfrak{N}$. We take $\mathfrak{N} = \mathcal{O}(x)$ for a closed point $x \in |X|$ which is inert in X'.

We claim that it suffices to check that the statement for a single geometric point $\mathcal{F} \in \mathrm{Prym}_{\mathfrak{N}}$. Indeed, since $\mathrm{Prym}_{\mathfrak{N}}$ is a Prym-torsor, any geometric point of $\mathrm{Prym}_{\mathfrak{N}}$ is of the form $\mathcal{F} \otimes \mathcal{F}'$ for some $\mathcal{F}' \in \mathrm{Prym}$, and $\mathrm{Lang}(\mathcal{F} \otimes \mathcal{F}') \cong \mathrm{Lang}(\mathcal{F}) \otimes \mathrm{Lang}(\mathcal{F}')$ lies in the same component of Prym as $\mathrm{Lang}(\mathcal{F})$ since $\mathrm{Lang}(\mathrm{Prym}) \subset \mathrm{Prym}^0$.

To describe a geometric point $\mathcal{F} \in \mathrm{Prym}_{\mathfrak{N}}(\overline{k})$, write $x \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k} = \{x_1, \ldots, x_d\}$ such that $\mathrm{Frob}(x_i) = x_{i+1 \pmod d}$, etc. Denoting x' the point of X' lying above x, we have $x' \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k} = \{x'_1, x'_2, \ldots, x'_{2d}\}$ where $\mathrm{Frob}(x'_i) = x'_{i+1 \pmod{2d}}$ and $\sigma x'_i = x'_{i+d \pmod{2d}}$, etc. Then $\mathcal{F} := \mathcal{O}(x'_1 + x'_2 + \ldots + x'_d)$ lies in $\mathrm{Prym}_{\mathfrak{N}}(\overline{k})$, and ${}^{\tau}\mathcal{F} \otimes \mathcal{F}^{-1} = \mathcal{O}(x'_{d+1} - x'_1) = \mathcal{O}(\sigma x'_1 - x'_1)$, which lies in the non-neutral component of Prym. $\square$

Let r be even (resp. odd) if $\epsilon(\mathfrak{N}) = 0$ (resp. $\epsilon(\mathfrak{N}) = 1$). By unwinding definitions one sees directly that the diagram below is Cartesian:

$$\begin{tikzcd} \mathrm{Sht}^r_{U(1),\mathfrak{N}} \arrow[r, "p"] \arrow[d, "p_{[1,r]}:=(p_1,\cdots,p_r)"'] & \mathrm{Prym}_{\mathfrak{N}} \arrow[d, "\mathrm{Lang}"] \\ X'^r \arrow[r, "\mathrm{AJ}^r"] & \mathrm{Prym}^{\epsilon(\mathfrak{N})} \end{tikzcd} \tag{2.2}$$

Here $\mathrm{AJ}^r : X'^r \to \mathrm{Prym}$ is the map $(x_1, \cdots, x_r) \mapsto \mathcal{O}(\sum_{i=1}^r (\sigma x_i - x_i))$. The map $p_i : \mathrm{Sht}^r_{U(1)} \to X'$ records the i-th leg $(1 \leq i \leq r)$, $p_{[1,r]} := (p_1, \cdots, p_r) : \mathrm{Sht}^r_{U(1)} \to (X')^r$ and $p : \mathrm{Sht}^r_{U(1)} \to \mathrm{Prym}_{\mathfrak{N}}$ records $\mathcal{F}_0$.

Lemma 2.15. *$\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ is non-empty if and only if $(-1)^r = \eta(\mathfrak{L})$.*

Proof. Combine Lemma 2.14 and (2.2). $\square$

Lemma 2.16. *If $r > 0$ and $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ is non-empty, then $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ has two geometric connected components. Under these same assumptions, $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ is connected if and only if r is odd.*

Proof. Let $\mathfrak{N} = \omega_X \otimes \mathfrak{L}$. We know that $\mathrm{Sht}^r_{U(1),\mathfrak{L}} \neq \varnothing$ if and only if $\eta(\mathfrak{N}) = \eta(\mathfrak{L}) = (-1)^r$. We assume this in the following.

First we establish that there are two geometric connected components. Consider the Cartesian square (2.2). For any $\epsilon \in \mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$ (a torsor for $\mathbf{Z}/2\mathbf{Z}$), let $\mathrm{Sht}^{r,\epsilon}_{U(1),\mathfrak{L}}$ be the preimage of $\mathrm{Prym}^{\epsilon}_{\mathfrak{N}}$ under p. We need to show that $\mathrm{Sht}^{r,\epsilon}_{U(1),\mathfrak{L}}$ is geometrically connected.

As a $\mathrm{Prym}^0(k)$-torsor over $X'^r_{\overline{k}}$ (cf. (2.2)), $\mathrm{Sht}^{r,\epsilon}_{U(1),\mathfrak{L},\overline{k}}$ is given by the homomorphism

$$\pi_1(X'^r_{\overline{k}}) \xrightarrow{\mathrm{AJ}^r_*} \pi_1(\mathrm{Prym}^{\underline{r}}_{\overline{k}}) \xrightarrow{\Lambda_{\mathrm{Prym}}} \mathrm{Prym}^0(k) \tag{2.3}$$

where the first map is induced by AJ^r (using notation of §10.2; and $\underline{r} = r \mod 2 \in \{0, 1\}$), and the second map is given by the Lang torsor $\mathrm{Lang} : \mathrm{Prym}^{\epsilon}_{\mathfrak{N}} \to \mathrm{Prym}^{\underline{r}}$. It suffices to show that (2.3) is surjective. Since $\mathrm{Prym}^{\epsilon}_{\mathfrak{N}}$ is geometrically connected, Λ_{Prym} is surjective. It remains to show that AJ^r_* is surjective.

Fixing $z = (z_1, \cdots, z_{r-1}) \in X'^{r-1}(\overline{k})$ and letting $\Delta_z = \mathrm{AJ}^{r-1}(z)$, we have a commutative diagram

$$\begin{tikzcd} X'_{\overline{k}} \arrow[r, "\mathrm{AJ}_{X'}"] \arrow[d, "i_z"] & \mathrm{Pic}^1_{X',\overline{k}} \arrow[r, "\sigma-1"] & \mathrm{Prym}^1_{\overline{k}} \arrow[d, "\otimes\Delta_z"] \\ X'^r_{\overline{k}} \arrow[rr, "\mathrm{AJ}^r"] & & \mathrm{Prym}^r_{\overline{k}} \end{tikzcd} \tag{2.4}$$

Here $\mathrm{AJ}_{X'} : X' \to \mathrm{Pic}^1_{X'}$ is the Abel-Jacobi map for X', $i_z(x) = (x, z_1, \cdots, z_{r-1})$. It induces a commutative diagram on fundamental groups

$$\begin{tikzcd} \pi_1(X'_{\overline{k}}) \arrow[r, "\mathrm{AJ}_{X',*}"] \arrow[d, "i_{z*}"] & \pi_1(\mathrm{Pic}^1_{X',\overline{k}}) \arrow[r, "(\sigma-1)_*"] & \pi_1(\mathrm{Prym}^1_{\overline{k}}) \arrow[d, "\cong"] \\ \pi_1(X'^r_{\overline{k}}) \arrow[rr, "\mathrm{AJ}^r_*"] & & \pi_1(\mathrm{Prym}^r_{\overline{k}}) \end{tikzcd} \tag{2.5}$$

By geometric class field theory, $\mathrm{AJ}_{X',*} : \pi_1(X'_{\overline{k}}) \to \pi_1(\mathrm{Pic}^1_{X',\overline{k}})$ is surjective, realizing the latter as the abelianization of the former. On the other hand, $\sigma - 1 : \mathrm{Pic}^1_{X',\overline{k}} \to \mathrm{Prym}^1_{\overline{k}}$ is a torsor under $\mathrm{Pic}^0_{X,\overline{k}}$ which is connected, it induces a surjection on π_1. These then imply that the top row of the above diagram is surjective. Therefore the bottom row is surjective as well, i.e., AJ^r_* is surjective.

To prove the last assertion in the Lemma, we show that Frob swaps the two geometric connected components of $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ if and only if r is odd. For $\mathcal{F} \in \mathrm{Sht}^r_{U(1),\mathfrak{L}}(\overline{k})$, ${}^\tau\mathcal{F} \otimes \mathcal{F}^{-1}$ is the tensor product of r line bundles of the form $\mathcal{O}(x - \sigma x)$, each of which lies in Prym^1, so the tensor product lies in Prym^0 if and only if r is even.

□

3. Hitchin stacks

In this section we introduce certain stacks which will be used to analyze special cycles, generalizing the constructions in [FYZ21, §8].

3.1. **Moduli of sections of gerbes.**

In order to encompass the moduli stacks $\mathrm{Bun}_{U(n)}$ and $\mathrm{Bun}_{U(n),\mathfrak{L}}$ in a common framework, it will be advantageous to adopt a more general perspective of moduli stacks of sections of gerbes.

Example 3.1. Let G be a group scheme over any scheme S. Then the relative classifying stack BG is equipped with the structure of a gerbe over S, and the groupoid of sections of BG over S is equivalent to the groupoid over G-torsors over S. In particular, for a group scheme G over the curve X, Bun_G can be interpreted as a moduli stack of sections of the gerbe BG over X.

In the context of this paper, the moduli stack of Hermitian bundles $\mathrm{Bun}_{U(n)}$ over X play a more fundamental role than the group scheme $U(n)$ itself. Indeed, to recover $U(n)$ from $\mathrm{Bun}_{U(n)}$ we need to choose a base point $(\mathcal{F}, h) \in \mathrm{Bun}_{U(n)}(k)$ and define $U(n)$ to be the group scheme of automorphisms of $(\mathcal{F}, h)$. Better yet, we should consider the gerbe $BU(n)$ over X rather than the group scheme $U(n)$ over X. Then sections of the gerbe $BU(n)$ are equivalent to $U(n)$-torsors. This point of view generalizes better to include spaces like $\mathrm{Bun}_{U(n),\mathfrak{L}}$, which are not moduli stacks of torsors for a group scheme, but can be seen as moduli stacks of sections of a gerbe $BU(n)_{\mathfrak{L}}$, which will be defined next.

Definition 3.2. Let $\mathscr{G}$ be a gerbe over X. We define the stack $\mathrm{Bun}_{\mathscr{G}}$ over k to be

$$\mathrm{Bun}_{\mathscr{G}} := \mathrm{Sect}(X, \mathscr{G}) = R_{X/k}\mathscr{G}. \tag{3.1}$$

In other words, the S-points of $\mathrm{Bun}_{\mathscr{G}}$ form the groupoid of maps $X \times S \to \mathscr{G}$ over X.

In view of Example 3.1, we have $\mathrm{Bun}_{BG} = \mathrm{Bun}_G$ for a group scheme G over X.

3.1.1. *Unitary gerbes.* Fix a line bundle $\mathfrak{L}$ over X. We define the gerbe $BU(n)_{\mathfrak{L}}$ over X to represent the following moduli problem: for any scheme S with a map $s : S \to X$, liftings of s to $BU(n)_{\mathfrak{L}}$ form the groupoid of Hermitian vector bundles $(\mathcal{F}, h)$ over $S' := S \times_X X'$ valued in $s^*\mathfrak{L}$, i.e., h is an isomorphism $\mathcal{F} \xrightarrow{\sim} \sigma_S^* \underline{\mathrm{Hom}}(\mathcal{F}, \nu_S^* s^*(\omega_X \otimes \mathfrak{L}))$ satisfying $h = \sigma^* h^\vee$ (here $\sigma_S : S' \to S'$ and $\nu_S : S' \to S$ are induced from σ and ν). Forgetting the datum of h defines the *standard map* $BU(n)_{\mathfrak{L}} \to B\,\mathrm{GL}(n)'$.

We call $BU(n)_{\mathfrak{L}}$ the *unitary gerbe over X of rank n and similitude line bundle* $\mathfrak{L}$. With this definition and Definition 3.2, we have

$$\mathrm{Bun}_{BU(n)_{\mathfrak{L}}} = \mathrm{Bun}_{U(n),\mathfrak{L}}\,. \tag{3.2}$$

For most of the paper, the only gerbes that will concern us are $BU(n)_{\mathfrak{L}}$ or $B\,\mathrm{GL}(n)'$. However, in §7.2 and §11 it will be necessary to deal with a more general class of gerbes, which we introduce next.

3.1.2. *Gerbes of unitary type.* We define a class of gerbes over X that we call *gerbes of unitary type over X*.

Let Y be another smooth projective curve over k (not assumed to be geometrically connected) and $\theta : Y \to X$ be a finite morphism (possibly ramified). Let $\mathrm{Irr}(Y)$ be the set of irreducible components of Y and $\underline{n} : \mathrm{Irr}(Y) \to \mathbf{Z}_{>0}$ be a function. For $Y_\alpha \in \mathrm{Irr}(Y)$ we denote $\underline{n}(Y_\alpha)$ by n_α. Let

$$n = \sum_{Y_\alpha \in \mathrm{Irr}(Y)} n_\alpha [Y_\alpha : X] \tag{3.3}$$

where $[Y_\alpha : X]$ is the degree of $\theta_\alpha := \theta|_{Y_\alpha} : Y_\alpha \to X$.

Let $\mathfrak{L} \in \mathrm{Pic}(X)$ and $\mathfrak{L}_\alpha = \theta_\alpha^* \mathfrak{L}$. Consider the unitary gerbe $BU(n_\alpha)_{\mathfrak{L}_\alpha}$ over Y_α with similitude line bundle $\mathfrak{L}_\alpha$ defined using the double cover $Y'_\alpha = Y_\alpha \times_X X'$.

We claim there is a canonical map of gerbes over X

$$\prod_{Y_\alpha} R_{Y_\alpha/X} BU(n_\alpha)_{\mathfrak{L}_\alpha} \to BU(n)_{\mathfrak{L}}. \tag{3.4}$$

We describe the map the level of S-points. For $s : S \to X$, $(R_{Y_\alpha/X} BU(n_\alpha)_{\mathfrak{L}_\alpha})(S)$ is the groupoid of Hermitian bundles $(\mathcal{F}_\alpha, h_\alpha)$ over $S' \times_X Y_\alpha$ with similitude line bundle the pullback of $s^*\mathfrak{L}$ to $S_\alpha := S \times_X Y_\alpha$. Given such S-points $(\mathcal{F}_\alpha, h_\alpha)$ for each Y_α, (3.4) sends them to the direct sum $\mathcal{F} = \oplus_\alpha \theta'_{S_\alpha *} \mathcal{F}_\alpha$ (where $\theta'_{S_\alpha} : S'_\alpha = S' \times_X Y_\alpha \to S'$ is the projection). The pushforward of h_α induces a map

$$\theta'_{S_\alpha *} h_\alpha : \theta'_{S_\alpha *} \mathcal{F}_\alpha \xrightarrow{\sim} \theta'_{S_\alpha *} \sigma_S^* \underline{\mathrm{Hom}}(\mathcal{F}_\alpha, (\omega_{Y_\alpha} \otimes \mathfrak{L})|_{S'}). \tag{3.5}$$

The relative dualizing sheaves satisfy $\omega_{S'_\alpha/S'} \cong \omega_{Y'_\alpha/X'}|_{S'} \cong \omega_{Y_\alpha/X}|_{S'}$. Grothendieck-Serre duality gives

$$\theta'_{S_\alpha *} \underline{\mathrm{Hom}}(\mathcal{F}_\alpha, \omega_{S'_\alpha/S'}) = \underline{\mathrm{Hom}}(\theta'_{S_\alpha *} \mathcal{F}_\alpha, \mathcal{O}_{S'}). \tag{3.6}$$

Therefore the right side of (3.5) is isomorphic to

$$\begin{aligned} & \sigma_S^* \theta'_{S_\alpha *} \underline{\mathrm{Hom}}(\mathcal{F}_\alpha, \omega_{Y'_\alpha/X'}|_{S'} \otimes \theta'^*_{S_\alpha}(\omega_X \otimes \mathfrak{L})|_{S'}) && (3.7)\\ \cong\ & \sigma_S^* \theta'_{S_\alpha *} \underline{\mathrm{Hom}}(\mathcal{F}_\alpha, \omega_{S'_\alpha/S'}) \otimes (\omega_X \otimes \mathfrak{L})|_{S'} && (3.8)\\ \cong\ & \sigma_S^* \underline{\mathrm{Hom}}(\theta'_{S_\alpha *} \mathcal{F}_\alpha, \mathcal{O}_{S'}) \otimes (\omega_X \otimes \mathfrak{L})|_{S'} && (3.9)\\ \cong\ & \sigma_S^* \underline{\mathrm{Hom}}(\theta'_{S_\alpha *} \mathcal{F}_\alpha, \nu_S^* s^*(\omega_X \otimes \mathfrak{L})). && (3.10) \end{aligned}$$

In other words, $\theta'_{S_\alpha *} h_\alpha$ is a Hermitian form on $\theta'_{S_\alpha *} \mathcal{F}_\alpha$ with similitude line bundle $\mathfrak{L}$. Then the direct sum of $\theta'_{S_\alpha *} h_\alpha$ gives a Hermitian form h on $\mathcal{F}$ with similitude line bundle $\mathfrak{L}$.

Definition 3.3. Let $\mathfrak{L} \in \mathrm{Pic}(X)$. A gerbe $\mathscr{G}$ over X together with a map $i : \mathscr{G} \to BU(n)_{\mathfrak{L}}$ over X is called *a gerbe of unitary type of rank n and similitude line bundle* $\mathfrak{L}$, if there exists the data $\theta : Y \to X$ and $\underline{n} : \mathrm{Irr}(Y) \to \mathbf{Z}_{>0}$ as above (satisfying (3.3)) such that $(\mathscr{G}, i)$ is isomorphic to $\prod_{Y_\alpha} R_{Y_\alpha/X} BU(n_\alpha)_{\mathfrak{L}_\alpha}$ (product over X) with the canonical map to $BU(n)_{\mathfrak{L}}$ defined in (3.4). The *standard map* $\mathscr{G} \to B\,\mathrm{GL}(n)'$ is inflated via i from the standard map for $BU(n)_{\mathfrak{L}}$.

3.1.3. *Hecke stacks and Shtukas for gerbes of unitary type.* Let $i : \mathscr{G} \cong \prod_{Y_\alpha} R_{Y_\alpha/X} BU(n_\alpha)_{\mathfrak{L}_\alpha}$ be a gerbe of unitary type. We set $Y' := Y \times_X X' \cong \coprod Y'_\alpha$ with involution $\sigma_Y = \mathrm{Id}_Y \times \sigma$. We have

$$\mathrm{Bun}_{\mathscr{G}} \cong \prod_\alpha \mathrm{Bun}_{U(n_\alpha)/Y_\alpha, \mathfrak{L}_\alpha} \tag{3.11}$$

where $\mathrm{Bun}_{U(n_\alpha)/Y_\alpha, \mathfrak{L}_\alpha}$ is the moduli of $\mathfrak{L}_\alpha$-twisted Hermitian bundles of rank n_α over Y'_α.

Then we define $\mathrm{Hk}^r_{\mathscr{G}}$ to be the moduli stack with S-points being the groupoid of the following data:

- $(y'_1, \cdots, y'_r) \in Y'(S)^r$,
- Hermitian bundles $(\mathcal{F}_i, h_i)_{i=0}^r$, with each $\mathcal{F}_i$ a vector bundle on $Y' \times S$, of rank n_α on Y'_α, and h_i is an $\mathfrak{L}_\alpha = \theta_\alpha^* \mathfrak{L}$-twisted Hermitian structure on $\mathcal{F}_i$, and
- Isomorphisms $f_i \colon \mathcal{F}_{i-1}|_{Y' \times S - \Gamma_{y'_i} - \Gamma_{\sigma_Y y'_i}} \xrightarrow{\sim} \mathcal{F}_i|_{Y' \times S - \Gamma_{y'_i} - \Gamma_{\sigma_Y y'_i}}$, for $1 \leq i \leq r$, which are lower of length 1 at y'_i and upper of length 1 at $\sigma y'_i$.

By recording how many of y'_i are lying over each component of Y, we have a decomposition

$$\mathrm{Hk}^r_{\mathscr{G}} \cong \coprod_{\underline{r}} \prod_{\alpha \in \mathrm{Irr}(Y)} \mathrm{Hk}^{r_\alpha}_{U(n_\alpha)/Y_\alpha, \mathfrak{L}_\alpha} \tag{3.12}$$

where $\underline{r}$ runs over the set of functions $\underline{r} : \mathrm{Irr}(Y) \to \mathbf{Z}_{\geq 0}$ such that $|\underline{r}| := \sum_\alpha r_\alpha$ is equal to r.

We define $\mathrm{Sht}^r_{\mathscr{G}}$ by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathscr{G}} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathscr{G}} \arrow[d, "(\mathrm{pr}_0, \mathrm{pr}_r)"] \\ \mathrm{Bun}_{\mathscr{G}} \arrow[r, "(\mathrm{Id}, \mathrm{Frob})"] & \mathrm{Bun}_{\mathscr{G}} \times \mathrm{Bun}_{\mathscr{G}} \end{tikzcd}$$

Similarly we have

$$\mathrm{Sht}^r_{\mathscr{G}} \cong \coprod_{|\underline{r}| = r} \prod_{\alpha \in \mathrm{Irr}(Y)} \mathrm{Sht}^{r_\alpha}_{U(n_\alpha)/Y_\alpha, \mathfrak{L}_\alpha}. \tag{3.13}$$

In order to make the notation more uniform, we will denote a gerbe of unitary type by $BG \to X$ (even if it does not arise as the classifying stack of a group scheme G). We will write Hk^r_G for Hk^r_{BG} (in the unitary type case) or if $G = \mathrm{GL}(n)'$.

3.2. **Hitchin stacks.** We introduce Hitchin stacks $\mathcal{M}_{H_1, H_2}$ for certain gerbes BH_1 and BH_2, generalizing the construction in [FYZ21, §8].

There is an equivalence of categories between the groupoid of $\mathrm{GL}(n)'$-torsors over X, and the groupoid of vector bundles of rank n on X' (with maps being isomorphisms). If $\mathcal{E}$ is a $\mathrm{GL}(n)'$-torsor, we denote by $\mathrm{V}(\mathcal{E})$ the vector bundle associated by this equivalence. We introduce this notation because we shall frequently need to talk about maps between vector bundles which are not isomorphisms (and so do not come from maps of torsors).

Because of Example 3.1, we will use the notation BH for a gerbe over a base S, and refer to a global section of BH over S as an "H-torsor over S", even when the gerbe does not actually come as the classifying space of a group scheme H. More generally, given a map of gerbes $BH \to B\,\mathrm{GL}(n)'$ over X, and an H-torsor $\mathcal{E}$ on $X \times S$, we will denoted by $\mathrm{V}(\mathcal{E})$ the associated rank n vector bundle on $X' \times S$.

Definition 3.4. Let $BH_1 \to B\,\mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be two maps of gerbes over X. We define the "Hitchin-type space" $\mathcal{M}_{H_1, H_2}$ whose S-points are the groupoid of data:

- $\mathcal{E}_{H_1}$, an H_1-torsor over $X \times S$.
- $\mathcal{F}_{H_2}$, an H_2-torsor over $X \times S$.
- A map of vector bundles $t \colon \mathrm{V}(\mathcal{E}_{H_1}) \to \mathrm{V}(\mathcal{F}_{H_2})$ over $X' \times S$.

We define $\mathcal{M}^\circ_{H_1, H_2} \subset \mathcal{M}_{H_1, H_2}$ to be the open substack where the map t is injective as a map of coherent sheaves. Note that the definition is in terms of the gerbes BH_1, BH_2 and their maps to $B\,\mathrm{GL}(m)', B\,\mathrm{GL}(n)'$, but in the notation we only put H_1, H_2 as a shorthand for BH_1, BH_2 and these maps (this is just notational shorthand – there may not be an actual group scheme H_i from which BH_i comes).

Remark 3.5. Let us comment on what generality of gerbes will appear. In all examples of interest, $BH_1 \to B\,\mathrm{GL}(m)'$ comes from a map of smooth group schemes over X, and BH_2 will be either a gerbe of unitary type with the standard map to $B\,\mathrm{GL}(n)'$, or simply $B\,\mathrm{GL}(n)'$ (with the identity map). The reader may focus on the cases where the gerbes arise as classifying stacks of smooth group schemes over X, without missing the main ideas.

Example 3.6. Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$ and $BH_2 = BU(n) \to B\,\mathrm{GL}(n)'$ be the standard map. In this case, the stack $\mathcal{M}_{H_1, H_2}$ (resp. $\mathcal{M}^\circ_{H_1, H_2}$) is the Hitchin stack denoted $\mathcal{M}^{\mathrm{all}}(m, n)$ (resp. $\mathcal{M}(m, n)$) in [FYZ21]. (Note the notational inconsistency with [FYZ21]: in this paper we do not use the superscript "all" to indicate all maps are allowed, and we use the superscript $\circ$ to indicate the substack where the map t is injective.)

More generally, for $\mathfrak{L}$ a line bundle on X we may take $BH_2 = BU(n)_{\mathfrak{L}} \to B\,\mathrm{GL}(n)'$ to be the standard map. We also denote the corresponding Hitchin stack $\mathcal{M}_{H_1,H_2}$ by $\mathcal{M}_{H_1,U(n),\mathfrak{L}}$. It parametrizes $\mathcal{E}_{H_1} \in \mathrm{Bun}_{H_1}$, $(\mathcal{F}, h) \in \mathrm{Bun}_{U(n),\mathfrak{L}}$, and a map of vector bundles

$$t\colon \mathrm{V}(\mathcal{E}_{H_1}) \to \mathcal{F}$$

over $X' \times S$. Its open substack $\mathcal{M}^{\circ}_{H_1,H_2} = \mathcal{M}^{\circ}_{H_1,U(n),\mathfrak{L}}$ is the locus where t is injective as a map of coherent sheaves (fiberwise over the test scheme S). *Henceforth when $\mathfrak{L}$ is understood, we may suppress it from the notation.*

Example 3.7. In this paper we shall also be interested in the case $H_1 = \mathrm{GL}(m_1)' \times \ldots \times \mathrm{GL}(m_j)'$ where $m = m_1 + \ldots + m_j$, the map $BH_1 \to B\,\mathrm{GL}(m)'$ is induced by the standard block diagonal inclusion, and $BH_2 = BU(n)_{\mathfrak{L}} \to B\,\mathrm{GL}(n)'$ induced by the standard embedding. This comes up, for example, in §7.3.

3.3. **Hitchin base.** We construct Hitchin bases for our Hitchin stacks, generalizing [FYZ21, §8.2].

Definition 3.8. Let $BH_1 \to B\,\mathrm{GL}(m)'$ be a map of gerbes over X and fix a line bundle $\mathfrak{L}$ on X. The *$\mathfrak{L}$-twisted Hitchin base* $\mathcal{A}_{H_1,\mathfrak{L}}$ is the stack whose S-points are the groupoid of the following data:

- $\mathcal{E}$ an H_1-torsor on $X \times S$.
- $a\colon \mathrm{V}(\mathcal{E}) \to \sigma^*\underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \omega_{X'} \otimes \nu^*\mathfrak{L}) = \sigma^*\,\mathrm{V}(\mathcal{E})^\vee \otimes \nu^*\mathfrak{L}$ is a map of coherent sheaves on $X' \times S$ such that $\sigma^*(a^\vee) = a$.

We define $\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}} \subset \mathcal{A}_{H_1,\mathfrak{L}}$ to be the open substack where a is injective fiberwise over the test scheme S. *When $\mathfrak{L}$ is understood, we will omit it from the notation in the future.*

Definition 3.9. Let $BH_1 \to B\,\mathrm{GL}(m)'$ be a map of gerbes over X and fix a line bundle $\mathfrak{L}$ on X. Take $BH_2 = BU(n)_{\mathfrak{L}}$ with the standard map to $B\,\mathrm{GL}(n)'$. We define the *Hitchin fibration* $f\colon \mathcal{M}_{H_1,H_2} \to \mathcal{A}_{H_1,\mathfrak{L}}$ sending $(\mathcal{E}, (\mathcal{F}, h), t)$ to the composition

$$a\colon \mathrm{V}(\mathcal{E}) \xrightarrow{t} \mathcal{F} \xrightarrow{h} \sigma^*\mathcal{F}^\vee \otimes \nu^*\mathfrak{L} \xrightarrow{\sigma^* t^\vee} \sigma^*\,\mathrm{V}(\mathcal{E})^\vee \otimes \nu^*\mathfrak{L}.$$

Note that the pre-image $\mathcal{M}_{H_1,H_2}|_{\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}}$ is contained in $\mathcal{M}^{\circ}_{H_1,H_2}$.

Example 3.10. For $H_1 \xrightarrow{=} \mathrm{GL}(m)'$, and $\mathfrak{L} = \mathcal{O}_X$, $\mathcal{A}_{H_1,\mathfrak{L}}$ (resp. $\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}$) coincides with the Hitchin base denoted $\mathcal{A}^{\mathrm{all}}(m)$ (resp. $\mathcal{A}(m)$) in [FYZ21, Definition 8.2]. Note the notational inconsistency with [FYZ21]: in this paper we do not use the superscript "all" to indicate that all maps are allowed, and we use the superscript $\circ$ to indicate the substack where the map t must be injective.

3.4. **Smoothness of some Hitchin stacks.** We will use the description of the tangent complex for the following general situation. Suppose $G \to X$ is a smooth group scheme acting linearly on a vector bundle $V \to X$. Then the relative Lie algebra $\mathrm{Lie}(G/X)$ acts on V, and the relative tangent complex for $V/G \to X$ at a point (x, v) (where $v \in V_x$) is represented by the complex

$$\begin{array}{rccc} \alpha_v : & \underbrace{\mathrm{Lie}\, G_x}_{\deg -1} & \longrightarrow & \underbrace{V_x}_{\deg 0} \\ & Y & \longrightarrow & Y \cdot v \end{array} \tag{3.14}$$

Let κ be a field. A κ-point of $\mathrm{Sect}(X, V/G)$ can be identified with the data of a G-bundle $\mathcal{E}$ over X_κ plus a G-equivariant map $s\colon \mathcal{E} \to V$ lying over the identity map on X_κ. It is explained in [Ngo10, §4.14] that the tangent space to $\mathrm{Sect}(X, V/G)$ at this κ-point is

$$\mathrm{H}^0(X_\kappa, \underbrace{\mathcal{E} \times^G \mathrm{Lie}(G/X)}_{\deg -1} \xrightarrow{\alpha_s} \underbrace{\mathcal{E} \times^G V}_{\deg 0}) \tag{3.15}$$

where the map $\alpha_s : \mathcal{E} \times^G \mathrm{Lie}(G/X) \to \mathcal{E} \times^G V$ is given by the action of $\mathrm{Lie}(G/X)$ on s (so that its fiber over $x \in X$ is identified with (3.14) upon choosing a trivialization of $\mathcal{E}$ at x), and the obstructions to deformation lie in

$$\mathrm{H}^1(X_\kappa, \underbrace{\mathcal{E} \times^G \mathrm{Lie}(G/X)}_{\deg -1} \xrightarrow{\alpha_s} \underbrace{\mathcal{E} \times^G V}_{\deg 0}). \tag{3.16}$$

In particular, $\mathrm{Sect}(X, V/G)$ is smooth at κ-points where (3.16) vanishes.

Proposition 3.11. *(1) Let $BH_1 \to B\,\mathrm{GL}(m)'$ be induced from any homomorphism of smooth group schemes $H_1 \to \mathrm{GL}(m)'$ over X, and $BH_2 \xrightarrow{=} B\,\mathrm{GL}(n)'$. Then the stack $\mathcal{M}^{\circ}_{H_1,H_2}$ is smooth.*

(2) Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$ and $BH_2 = BU(n)_{\mathfrak{L}} \to B\,\mathrm{GL}(n)'$ be the standard map. Then the stack $\mathcal{M}_{H_1,H_2}|_{\mathcal{A}^{\mathrm{ns}}_{H_1}}$ is smooth.

Proof. It is immediate from the definitions that $\mathcal{M}_{H_1,H_2}$ is a special case of $\mathrm{Sect}(X, V/G)$ where $G = H_1 \times H_2$, and V is the vector bundle of homomorphisms from the standard representation of $\mathrm{GL}(m)'$ inflated to H_1 via the given map $H_1 \to \mathrm{GL}(m)'$, to the standard representation of $\mathrm{GL}(n)'$ inflated to H_2 similarly.

We will show that the obstruction group to $\mathcal{M}_{H_1,H_2}$ vanishes at any $\overline{\kappa}$-point of $\mathcal{M}^{\circ}_{H_1,H_2}$. Consider a geometric point $\mathrm{Spec}\,\overline{\kappa} \to \mathcal{M}^{\circ}_{H_1,H_2}$, which is identified with the data of an H_1-torsor $\mathcal{E}$, an H_2-torsor $\mathcal{F}$ [2], and an injective map of the associated vector bundles $t\colon \mathrm{V}(\mathcal{E}) \to \mathrm{V}(\mathcal{F})$. Specializing (3.16) to this situation, the obstruction group is

$$\mathrm{H}^1(X_{\overline{\kappa}}, \underbrace{\mathcal{E} \times^{H_1} \mathrm{Lie}(H_1/X) \oplus \mathcal{F} \times^{H_2} \mathrm{Lie}(H_2/X)}_{\deg -1} \xrightarrow{\alpha_t} \underbrace{\underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathrm{V}(\mathcal{F}))}_{\deg 0}).$$

When $BH_2 = BU(n)_{\mathfrak{L}}$, $\mathrm{Lie}(H_2/X)$ is not a priori defined. In this case, $\mathrm{V}(\mathcal{F})$ is equipped with an $\mathfrak{L}$-twisted Hermitian form h, and we understand $\mathcal{F} \times^{H_2} \mathrm{Lie}(H_2/X)$ as the vector bundle $\underline{\mathrm{End}}^{asa}(\mathrm{V}(\mathcal{F}))$ (over $X_{\overline{\kappa}}$) of anti-self-adjoint endomorphisms of $\mathrm{V}(\mathcal{F})$, i.e., locally $B : \mathrm{V}(\mathcal{F}) \to \mathrm{V}(\mathcal{F})$ such that $h(Bx, y) + h(x, By) = 0$ for $x, y \in \mathrm{V}(\mathcal{F})$. The differential "$\alpha_t$" is given by $(A, B) \mapsto -tA + Bt$, where $A \in \mathcal{E} \times^{H_1} \mathrm{Lie}(H_1/X), B \in \mathcal{F} \times^{H_2} \mathrm{Lie}(H_2/X)$. Since the coherent cohomology of a torsion sheaf on a curve X vanishes in positive cohomological degrees, it therefore suffices to show that the cokernel of the differential "α_t" is torsion, or in other words α_t is generically surjective.

Let V (resp. U) be the generic fiber of $\mathrm{V}(\mathcal{F})$ (resp. $\mathrm{V}(\mathcal{E})$), a vector space of rank n (resp. m) over $K' = F' \otimes_k \overline{\kappa}$. Let $K = F \otimes_k \overline{\kappa}$. Let T be the generic fiber of t. By the assumption that the $\overline{k}$-point lies in $\mathcal{M}^{\circ}_{H_1,H_2}$, $T : U \to V$ is a K'-linear injective map.

In case (1), the generic fiber of $\mathcal{F} \times^{H_2} \mathrm{Lie}(H_2/X)$ is $\mathrm{End}_{K'}(V)$. The map $\mathrm{End}_{K'}(V) \to \mathrm{Hom}_{K'}(V, W)$ given by $B \mapsto BT$ is already surjective since T is injective. This shows that α_t is generically surjective, and the obstructions vanish, as desired.

In case (2), we argue as follows. In this case, upon trivializing the generic fiber of $\omega_X \otimes \mathfrak{L}$, V carries a Hermitian form $h : V \otimes_{K'} \sigma^* V \to K'$. The generic fiber of $\mathcal{E} \times^{H_1} \mathrm{Lie}(H_1/X)$ is $\mathrm{End}_{K'}(U)$ and the generic fiber of $\mathcal{F} \times^{H_2} \mathrm{Lie}(H_2/X)$ can be identified with the K-vector space $\mathrm{End}^{asa}_{K'}(V)$ of anti-self-adjoint endomorphisms $B : V \to V$. By the assumption that $\overline{\kappa}$-point lies over the non-degenerate locus $\mathcal{A}^{\mathrm{ns}}_{H_1} \hookrightarrow \mathcal{A}_{H_1}$, T is injective and $h|T(U)$ is non-degenerate. Therefore we may assume $(V, h) = (U, h_U) \oplus (W, h_W)$ is a direct sum of two non-degenerate Hermitian spaces, and T is the inclusion of U in V. We have

$$\mathrm{Hom}(U, U \oplus W) \cong \mathrm{End}(U) \oplus \mathrm{Hom}(U, W),$$

and

$$\mathrm{End}^{asa}(V) \cong \mathrm{End}^{asa}(U) \oplus \mathrm{Hom}(U, W) \oplus \mathrm{End}^{asa}(W),$$

where the last isomorphism is given by $B \mapsto (\mathrm{pr}_U(B|_U), \mathrm{pr}_W(B|_U), \mathrm{pr}_W(B|_W))$. Under these identifications, the generic fiber of α_t then takes the form

$$\begin{array}{rcl} \mathrm{End}(U) \oplus \mathrm{End}^{asa}(U) \oplus \mathrm{Hom}(U, W) \oplus \mathrm{End}^{asa}(W) & \to & \mathrm{End}(U) \oplus \mathrm{Hom}(U, W) \\ (A, B_1, B_2, B_3) & \mapsto & (-A + B_1, -B_2) \end{array}$$

from which we see that α_t is generically surjective.

□

Remark 3.12. The following variant will be used below in Lemma 3.15. Following the proof of [FYZ21, Lemma 8.14], we define an "$\mathfrak{L}$-twisted almost-Hermitian bundle with defect at $(x', \sigma(x'))$" to be the data of a vector bundle $\mathcal{F}^\flat$ on $X' \times S$ equipped with a Hermitian map $h\colon \mathcal{F}^\flat \hookrightarrow \sigma^*(\mathcal{F}^\flat)^\vee \otimes \nu^*\mathfrak{L}$ with cokernel an invertible sheaf on the union of the graphs of x' and $\sigma(x')$. Let $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L}}$ be the Hitchin stack parametrizing $x' \in X', \mathcal{E} \in \mathrm{Sect}(X, BH_1)$, and $\mathfrak{L}$-twisted almost-Hermitian bundle $\mathcal{F}$ with defect at $(x', \sigma(x'))$ and $t\colon \mathrm{V}(\mathcal{E}) \to \mathcal{F}^\flat$. There is a Hitchin fibration $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L}} \to \mathcal{A}_{H_1,\mathfrak{L}}$ defined completely analogously to §3.3.

[2] Here the notation differs slightly from §2.3, where $\mathcal{F}$ denoted the associated vector bundle.

Then the same argument as for Proposition 3.11 shows that the map $\mathcal{M}^{\flat}_{H_1,U(n),\mathfrak{L}}|_{\mathcal{A}^{\mathrm{ns}}_{H_1}} \to X'$ is smooth if BH_1 is the classifying stack of a smooth group scheme H_1/X.

3.5. **Hecke stacks for Hitchin stacks.**

Definition 3.13 (Hecke stacks for Hitchin spaces)**.** Let $BH_1 \to B\,\mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be as in Definition 3.4. Further assume that BH_2 is of unitary type or $B\,\mathrm{GL}(n)'$, so that $\mathrm{Hk}^r_{H_2}$ has been defined (cf. §3.1.3 for the first case, and §2.1 for the second case). For $r \geq 0$, we define $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$ to be the stack whose S-points are given by the groupoid of the following data:

(1) $(\{x'_i\}, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1 \dashrightarrow \ldots \dashrightarrow \mathcal{F}_r) \in \mathrm{Hk}^r_{H_2}(S)$.
(2) $\mathcal{E}$ an H_1-torsor on $X \times S$.
(3) Maps $t_i \colon \mathrm{V}(\mathcal{E}) \to \mathrm{V}(\mathcal{F}_i)$, fitting into the commutative diagram below.

$$\begin{tikzcd}
\mathrm{V}(\mathcal{E}) \arrow[r, equal] \arrow[d, "t_0"] & \mathrm{V}(\mathcal{E}) \arrow[r, equal] \arrow[d, "t_1"] & \ldots \arrow[r, equal] \arrow[d] & \mathrm{V}(\mathcal{E}) \arrow[d, "t_r"] \\
\mathrm{V}(\mathcal{F}_0) \arrow[r, dashed] & \mathrm{V}(\mathcal{F}_1) \arrow[r, dashed] & \ldots \arrow[r, dashed] & \mathrm{V}(\mathcal{F}_r)
\end{tikzcd}$$

(The dashed notation follows [FYZ21, Definition 6.5].) Let $\mathrm{pr}_i \colon \mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}} \to \mathcal{M}_{H_1,H_2}$ be the map recording $(\mathcal{E}, \mathcal{F}_i, t_i)$, for $0 \leq i \leq r$.

We define $\mathrm{Hk}^{r,\circ}_{\mathcal{M}_{H_1,H_2}} \subset \mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$ to be the fibered product of pr_0 (equivalently, any pr_i) with $\mathcal{M}^{\circ}_{H_1,H_2} \hookrightarrow \mathcal{M}_{H_1,H_2}$, and $\mathrm{pr}_i^{\circ} \colon \mathrm{Hk}^{r,\circ}_{\mathcal{M}_{H_1,H_2}} \to \mathcal{M}^{\circ}_{H_1,H_2}$ to be the restriction of pr_i.

Lemma 3.14. *Let $BH_1 \to B\,\mathrm{GL}(m)'$ be induced from any homomorphism of smooth group schemes $H_1 \to \mathrm{GL}(m)'$ over X, and $BH_2 \xrightarrow{=} B\,\mathrm{GL}(n)'$.*

(1) There is a canonical map $\mathrm{pr}_{1/2} : \mathrm{Hk}^1_{\mathcal{M}_{H_1,H_2}} \to \mathcal{M}_{H_1,H_2}$ such that $(\mathrm{pr}_{1/2}, \mathrm{pr}_{X'}) : \mathrm{Hk}^1_{\mathcal{M}_{H_1,H_2}} \to \mathcal{M}_{H_1,H_2} \times X'$ is smooth and of relative (equi)dimension $2(n-1)$. In particular, $\mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}}$ is smooth.

(2) For any geometric point $\xi \in \mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}}$, the local dimension of $\mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}}$ at ξ satisfies

$$\dim_{\xi} \mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}} - \dim_{\mathrm{pr}_i(\xi)} \mathcal{M}^{\circ}_{H_1,H_2} = 2n - 1 - m, \quad i = 0, 1. \tag{3.17}$$

Proof. (1) Let $(x', \mathcal{E}, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1, t_0, t_1)$ be an S-point of $\mathrm{Hk}^1_{\mathcal{M}_{H_1,H_2}}$. By means of the given rational isomorphism between $\mathcal{F}_0$ and $\mathcal{F}_1$, we may form the intersection $\mathcal{F}^{\flat}_{1/2} = \mathcal{F}_0 \cap \mathcal{F}_1$, which is an S-point of $\mathrm{Bun}_{\mathrm{GL}(n)'}$. By definition, the maps t_0, t_1 factor through a unique map $t^{\flat} \colon \mathrm{V}(\mathcal{E}) \to \mathcal{F}^{\flat}_{1/2}$. The data of $(\mathcal{E}, \mathcal{F}^{\flat}_{1/2}, t)$ determines an S-point of $\mathcal{M}_{H_1,H_2}$. We define $\mathrm{pr}_{1/2}(x', \mathcal{E}, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1, t_0, t_1) = (\mathcal{E}, \mathcal{F}^{\flat}_{1/2}, t)$.

To recover $(x', \mathcal{E}, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1, t_0, t_1)$ from its image $(\mathcal{E}, \mathcal{F}^{\flat}_{1/2}, t, x')$ under $(\mathrm{pr}_{1/2}, \mathrm{pr}_{X'})$ is equivalent to giving the datum of a line in each of the fibers of $\mathcal{F}^{\flat}_{1/2}$ at x' and $\sigma(x')$. Hence $(\mathrm{pr}_{1/2}, \mathrm{pr}_{X'})$ is a $\mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$-fiber bundle. In particular it is smooth of relative dimension $2(n-1)$.

Now $\mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}}$ is also the preimage of $\mathcal{M}^{\circ}_{H_1,H_2}$ under $\mathrm{pr}_{1/2}$. The smoothness of $\mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}}$ follows by combining the relative smoothness with Proposition 3.11.

(2) Let $\xi = (x'_1, \mathcal{E}, \mathcal{F}_0 \dashrightarrow \mathcal{F}_1, t_0, t_1)$ be a geometric point of $\mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}}$. Comparing the tangent complexes of $\mathcal{M}^{\circ}_{H_1,H_2}$ at $\mathrm{pr}_i(\xi) = (\mathcal{E}, \mathcal{F}_i, t_i) \in \mathcal{M}^{\circ}_{H_1,H_2}$ and at $\mathrm{pr}_{1/2}(\xi) = (\mathcal{E}, \mathcal{F}^{\flat}_{1/2}, t) \in \mathcal{M}^{\circ}_{H_1,H_2}$ given in the proof of Proposition 3.11, we see that (for $i = 0, 1$)

$$\begin{aligned}
&\dim_{\mathrm{pr}_i(\xi)} \mathcal{M}^{\circ}_{H_1,H_2} - \dim_{\mathrm{pr}_{1/2}(\xi)} \mathcal{M}^{\circ}_{H_1,H_2} \\
&= \deg \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) - \deg \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^{\flat}_{1/2}) = m.
\end{aligned} \tag{3.18}$$

On the other hand, by (1) we know that

$$\dim_{\xi} \mathrm{Hk}^1_{\mathcal{M}^{\circ}_{H_1,H_2}} = \dim_{\mathrm{pr}_{1/2}(\xi)} \mathcal{M}^{\circ}_{H_1,H_2} + 2n - 1. \tag{3.19}$$

Combining (3.18) and (3.19) we get (3.17).

□

When $BH_2 = BU(n)_{\mathfrak{L}}$, the composition $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}} \xrightarrow{\mathrm{pr}_i} \mathcal{M}_{H_1,H_2} \xrightarrow{f} \mathcal{A}_{H_1,\mathfrak{L}}$ is independent of i, so that $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$ has a well-defined map to $\mathcal{A}_{H_1,\mathfrak{L}}$.

Lemma 3.15. *Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$ and $BH_2 = BU(n)_{\mathfrak{L}}$ with the standard map to $B\,\mathrm{GL}(n)'$. Define $\mathcal{M}^{\mathrm{ns}}_{H_1,H_2} := \mathcal{M}_{H_1,H_2}|_{\mathcal{A}^{\mathrm{ns}}_{H_1}}$ and $\mathrm{Hk}^1_{\mathcal{M}_{H_1,H_2}} := \mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}}|_{\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}}$. Then:*

(1) The projection map $\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}} \to X'$ is smooth. In particular, $\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}}$ is smooth.

(2) For any geometric point $\xi \in \mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}}$, the local dimension of $\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}}$ at ξ satisfies

$$\dim_\xi \mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}} - \dim_{\mathrm{pr}_i(\xi)} \mathcal{M}^{\mathrm{ns}}_{H_1,H_2} = n - m, \quad i = 0, 1. \tag{3.20}$$

Proof. (1) We claim that $\mathrm{Hk}^1_{\mathcal{M}_{H_1,H_2}} \cong \mathcal{M}^\flat_{H_1,U(n),\mathfrak{L}}$, which is defined in Remark 3.12. If we admit this then the assertion follows from the smoothness of $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L}}|_{\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}} \to X'$ mentioned in Remark 3.12. So it suffices to establish the claim.

We define a map $\mathrm{Hk}^1_{\mathcal{M}_{H_1,H_2}} \to \mathcal{M}^\flat_{H_1,U(n),\mathfrak{L}}$. Let $(x', \mathcal{F}_0 \leftarrow \mathcal{F}^\flat_{1/2} \to \mathcal{F}_1) \in \mathrm{Hk}^1_{U(n),\mathfrak{L}}$. The generically compatible Hermitian structures on $\mathcal{F}_0$ and $\mathcal{F}_1$ equip $\mathcal{F}^\flat_{1/2}$ with an ($\mathfrak{L}$-twisted) almost Hermitian structure (cf. [FYZ21, Proof of Lemma 8.14] for the definition of "almost Hermitian") with defect at $(x', \sigma(x'))$.

Given $(\mathcal{F}^\flat, h^\flat)$ almost Hermitian with defect at $(x', \sigma(x'))$, define $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) as the upper modification of $\mathcal{F}^\flat$ at x' (resp. $\sigma(x')$) inside $\sigma^*(\mathcal{F}^\flat)^\vee \otimes \nu^*\mathfrak{L}$. It is easy to see that this defines the inverse map.

(2) Let $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L},x'}$ be the fiber of $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L}}$ over $x' \in X$. Consider a geometric point

$$\xi = (\mathcal{E}, (\mathcal{F}_{1/2}, h), t_{1/2} \colon \mathrm{V}(\mathcal{E}) \to \mathcal{F}_{1/2})$$

of $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L},x}|_{\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}}$. By the smoothness established in (1) and Proposition 3.11, the local dimensions at ξ and $\mathrm{pr}_i(\xi)$ may be computed as the Euler characteristic of the respective tangent complexes. Comparing the tangent complexes of $\mathcal{M}^\flat_{H_1,U(n),\mathfrak{L},x'}$ at ξ and of $\mathcal{M}^{\mathrm{ns}}_{H_1,H_2}$ at $\mathrm{pr}_i(\xi) = (\mathcal{E}, \mathcal{F}_i, t_i) \in \mathcal{M}^{\mathrm{ns}}_{H_1,H_2}$ using the proof of Proposition 3.11, we see that (for $i = 0, 1$)

$$\begin{aligned} &\dim_\xi \mathcal{M}^\flat_{H_1,U(n),\mathfrak{L},x'} - \dim_{\mathrm{pr}_i(\xi)} \mathcal{M}^{\mathrm{ns}}_{H_1,H_2} \\ &= -\deg \underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_{1/2}) + \deg \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^\flat_{1/2}) - \deg \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) \end{aligned}$$

where $\underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_{1/2})$ is the space of anti-self-adjoint morphisms with respect to the Hermitian map $h \colon \mathcal{F}^\flat_{1/2} \hookrightarrow \sigma^*(\mathcal{F}^\flat_{1/2})^\vee \otimes \mathfrak{L}$. Here we have used $\deg \underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_0) = 0$.

We have $\deg \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^\flat_{1/2}) - \deg \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) = -m$, as in the proof of Lemma 3.14. To compute $\deg \underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_{1/2})$, we reduce to the case where the double cover is split, by base changing along $X' \to X$. In that case, $X' = X \sqcup X$ and we may assume x' lies in the first copy of X and its image in X is denoted by x. Then the datum of $\mathcal{F}^\flat_{1/2}$ may be identified with a pair of vector bundles $\mathcal{F}^{(1)}, \mathcal{F}^{(2)}$ on X and a map $h \colon \mathcal{F}^{(1)} \to (\mathcal{F}^{(2)})^\vee \otimes \mathfrak{L}$ whose cokernel is flat of length 1 along the graph of x. Then $\underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_{1/2})$ consists of endomorphisms (B_1, B_2) of $\mathcal{F}^{(1)} \boxplus \mathcal{F}^{(2)}$ such that for every local section $v^{(1)} \in \mathcal{F}^{(1)}(U)$ and $v^{(2)} \in \mathcal{F}^{(2)}(U)$ on an open subset $U \subset X$, we have $h(B_1 v^{(1)}) = -B_2^\vee(hv^{(1)})$. Hence any such endomorphism is determined by its restriction to $\mathcal{F}^{(2)}$, giving an injection $\underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_{1/2}) \hookrightarrow \underline{\mathrm{End}}((\mathcal{F}^{(2)})^\vee \otimes \mathfrak{L})$. Let us abbreviate ${}'\mathcal{F}^{(2)} := (\mathcal{F}^{(2)})^\vee \otimes \mathfrak{L}$, which we remind is a rank n vector bundle on X. The image of the preceding injection consists of those maps in $\underline{\mathrm{End}}({}'\mathcal{F}^{(2)})$ preserving $\mathcal{F}^{(1)}$, viewed as a subsheaf of ${}'\mathcal{F}^{(2)}$, hence the image is equal to the kernel of the composition of arrows below

$$\begin{tikzcd} \underline{\mathrm{End}}({}'\mathcal{F}^{(2)}) \arrow[r, two heads] & \underline{\mathrm{Hom}}({}'\mathcal{F}^{(2)}, {}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)}) \arrow[d] \\ & \underline{\mathrm{Hom}}(\mathcal{F}^{(1)}, {}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)}) \end{tikzcd}$$

The sheaf $\underline{\mathrm{Hom}}(\mathcal{F}^{(1)}, {}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)})$ is torsion of degree n on X. The long exact sequence for $\underline{\mathrm{Hom}}(-, {}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)})$ shows that the image of the vertical map is the kernel of the surjection

$$\underline{\mathrm{Hom}}(\mathcal{F}^{(1)}, {}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)}) \to \underline{\mathrm{Ext}}^1({}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)}, {}'\mathcal{F}^{(2)}/\mathcal{F}^{(1)})$$

whose codomain is an invertible sheaf along the graph of x. As $\underline{\mathrm{End}}({}'\mathcal{F}^{(2)})$ has degree 0, we conclude that $\underline{\mathrm{End}}^{asa}(\mathcal{F}^\flat_{1/2})$ has degree $-(n-1)$.

□

3.6. **Hitchin shtukas.** We now discuss a notion of shtukas for Hitchin-type spaces $\mathcal{M}_{H_1,H_2}$.

Definition 3.16 (Shtukas for Hitchin spaces)**.** Let $BH_1 \to B\,\mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be as in Definition 3.13. For $r \geq 0$, we define $\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}$ as the fibered product

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}} \arrow[d, "(\mathrm{pr}_0,\mathrm{pr}_r)"] \\ \mathcal{M}_{H_1,H_2} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathcal{M}_{H_1,H_2} \times \mathcal{M}_{H_1,H_2} \end{tikzcd} \tag{3.21}$$

We define the open substack $\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}} \subset \mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}$ as the fibered product

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathcal{M}^\circ_{H_1,H_2}} \arrow[d, "(\mathrm{pr}_0,\mathrm{pr}_r)"] \\ \mathcal{M}^\circ_{H_1,H_2} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathcal{M}^\circ_{H_1,H_2} \times \mathcal{M}^\circ_{H_1,H_2} \end{tikzcd} \tag{3.22}$$

Note that $\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}} \hookrightarrow \mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}$ can be equivalently described as the base change of $\mathcal{M}^\circ_{H_1,H_2} \hookrightarrow \mathcal{M}_{H_1,H_2}$ against any of the projection maps $\mathrm{pr}_i \colon \mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}} \to \mathcal{M}_{H_1,H_2}$.

Example 3.17. Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$ and $BH_2 = BU(n)_{\mathfrak{L}} \to B\,\mathrm{GL}(n)'$ the standard map. Then

$$\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} \mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}.$$

and

$$\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} \mathcal{Z}^{r,\circ}_{\mathcal{E},\mathfrak{L}}. \tag{3.23}$$

When $\mathfrak{L} = \mathcal{O}_X$, $\mathrm{Hk}^{r,\circ}_{\mathcal{M}_{H_1,H_2}}$ (resp. $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$) is the stack denoted by $\mathrm{Hk}^r_{\mathcal{M}(m,n)}$ (resp. $\mathrm{Hk}^r_{\mathcal{M}^{\mathrm{all}}(m,n)}$) in [FYZ21, §8], and $\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}}$ (resp. $\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}$) is the stack denoted by $\mathrm{Sht}^r_{\mathcal{M}(m,n)}$ (resp. $\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{all}}(m,n)}$) in [FYZ21, §8].

Example 3.18. Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$ and $BH_2 \xrightarrow{=} B\,\mathrm{GL}(n)'$. Then

$$\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} \mathcal{Z}^r_{\mathcal{E},\mathrm{GL}(n)'},$$

and

$$\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} \mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'}. \tag{3.24}$$

Remark 3.19. Note that if we take $H_1 \xrightarrow{=} \mathrm{GL}(0)'$, then $\mathcal{M}_{H_1,H_2} = \mathrm{Bun}_{H_2} = \mathrm{Sect}(X, BH_2)$. Furthermore, if BH_2 is of unitary type or $B\,\mathrm{GL}(n)'$ then the definition of $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$ (resp. $\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}$) above specializes to $\mathrm{Hk}^r_{H_2}$ (resp. $\mathrm{Sht}^r_{H_2}$) as defined in §3.1.3.

3.7. **Cycle classes from Hitchin shtukas.**

Definition 3.20. For any stack S over $\mathbf{F}_q$ we define a morphism

$$\Phi^r_S : S^r \to S^{2r} \tag{3.25}$$

by the formula $\Phi^r_S(\xi_0, \cdots, \xi_{r-1}) = (\xi_0, \xi_1, \xi_1, \xi_2, \xi_2, \cdots, \xi_{r-1}, \mathrm{Frob}(\xi_0))$. When S is fixed in the context, we simply write Φ^r.

We rewrite $\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}}$ as the fiber product

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}} \arrow[r] \arrow[d] & (\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,H_2}})^r \arrow[d, "(\mathrm{pr}_0^\circ,\mathrm{pr}_1^\circ)^r"] \\ (\mathcal{M}^\circ_{H_1,H_2})^r \arrow[r, "\Phi^r_{\mathcal{M}^\circ_{H_1,H_2}}"] & (\mathcal{M}^\circ_{H_1,H_2})^{2r} \end{tikzcd} \tag{3.26}$$

Definition 3.21. Let $BH_1 \to B\,\mathrm{GL}(m)'$ be induced from any homomorphism of smooth group schemes $H_1 \to \mathrm{GL}(m)'$ over X, and $BH_2 \xrightarrow{=} B\,\mathrm{GL}(n)'$. By Lemma 3.14, the fundamental class of $(\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})^r$ is defined, which we denote by $[(\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})^r]^{\mathrm{naive}} \in \mathrm{Ch}_*((\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})^r)$. Then we define the cycle class $[\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}}] \in \mathrm{Ch}_{r(2n-1-m)}(\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})$ as the the image of $[(\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})^r]^{\mathrm{naive}}$ under the refined Gysin map along $\Phi^r : (\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'})^r \to (\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'})^{2r}$ (which is defined since $\mathcal{M}^\circ_{H_1,H_2}$ is smooth and its connected components are equidimensional by Proposition 3.11 and the description of the tangent spaces in (3.15); see [YZ17, §A.1.4])

$$[\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}}] := (\Phi^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})^![(\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})^r]^{\mathrm{naive}} \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}}). \tag{3.27}$$

In particular, when $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$, the dimension formula in Lemma 3.14 implies that $[\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}}] \in \mathrm{Ch}_{r(2n-1)-rm}(\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})$.

Remark 3.22. Definition 3.21 will be used in the next section to define cycle classes $[\mathcal{Z}^r_{\mathcal{E}}(a)]$. Even though we are in some sense more interested in the case $BH_2 = BU(n)_{\mathfrak{L}}$, for the purpose of constructing cycle classes corresponding to singular a, it was crucial to take $BH_2 = B\,\mathrm{GL}(n)'$ in Definition 3.21, because Proposition 3.11 gives smoothness of the $\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}$ even over the singular part of the Hitchin base. Because we lack such control when $BH_2 = BU(n)_{\mathfrak{L}}$, we cannot make an analogous definition in that case.

4. Formulation of the modularity conjecture

Let $\mathcal{E}$ be a vector bundle on X' of rank m, and let $\mathfrak{L}$ be a line bundle on X. For any $a \in \mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)$, we have defined a special cycle $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a) \to \mathrm{Sht}^r_{U(n),\mathfrak{L}}$, cf. Definition 2.9. The goal of this section is to construct a virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a))$ for every a, and formulate a conjecture that a generating series of such cycle classes is modular. We note that $\dim \mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$ can differ significantly from $r(n-m)$ in general, so we really need a virtual fundamental class.

It turns out that when a is non-singular, $[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)]$ can be defined directly using Hitchin stacks. For possibly singular a, we define $[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)]$ in two steps. First, we define the cycle class on the open-closed substack $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)^\circ$ consisting of generically injective maps from $\mathcal{E}$. Next, on the rest of the connected components of $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$, we reduce to the case of an already-defined cycle class (of smaller corank), and cap it with an appropriate Chern polynomial coming from tautological bundles over $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$. (Later in §6, specifically Theorem 6.5, we will see how this recipe arises from a natural derived enhancement of $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$.)

In this section, we fix a similitude line bundle $\mathfrak{L}$ on X and consider $\mathfrak{L}$-twisted Hermitian bundles. When there is no confusion we will omit $\mathfrak{L}$ from the notation, e.g., we write $\mathcal{A}_{\mathcal{E}}$ and $\mathcal{Z}^r_{\mathcal{E}}(a)$ for $\mathcal{A}_{\mathcal{E},\mathfrak{L}}$ and $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$.

4.1. **Decomposition according to kernel.** Let $\mathcal{K} \subset\subset \mathcal{E}$ be a sub-bundle of $\mathcal{E}$ (the notation $\mathcal{K} \subset\subset \mathcal{E}$ means that $\mathcal{K}$ is a sub-bundle of $\mathcal{E}$, i.e., the quotient $\mathcal{E}/\mathcal{K}$ is a vector bundle) and $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{K}$ be the quotient bundle. We define the closed substack $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \subset \mathcal{Z}^r_{\mathcal{E}}$ to parametrize those $(\mathcal{E} \xrightarrow{t_i} \mathcal{F}_i)$ such that $\mathcal{K} \subset \ker(t_i)$. We define $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ \subset \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]$ to be the open substack where $\ker(t_i) = \mathcal{K}$.

Each $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ is locally closed in $\mathcal{Z}^r_{\mathcal{E}}$. It is clear that $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ for varying $\mathcal{K}$ form a partition of $\mathcal{Z}^r_{\mathcal{E}}$. In particular,

$$\mathcal{Z}^r_{\mathcal{E}}[0] = \mathcal{Z}^r_{\mathcal{E}}; \quad \mathcal{Z}^r_{\mathcal{E}}[0]^\circ = \mathcal{Z}^{r,\circ}_{\mathcal{E}} \tag{4.1}$$

$$\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ = \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \setminus (\bigcup_{\mathcal{K} \subsetneq \mathcal{K}'} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}']). \tag{4.2}$$

We will show that $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ are in fact open-closed in $\mathcal{Z}^r_{\mathcal{E}}$.

Lemma 4.1. *The substack $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \subset \mathcal{Z}^r_{\mathcal{E}}$ is open-closed.*

Proof. Consider the natural map

$$r_{\mathcal{K}} : \mathcal{Z}^r_{\mathcal{E}} \to \mathcal{Z}^r_{\mathcal{K}} \tag{4.3}$$

by restricting $t_i : \mathcal{E} \to \mathcal{F}_i$ to $\mathcal{K}$. Let $z : \mathrm{Sht}^r_{U(n)} \cong \mathcal{Z}^r_{\mathcal{K}}[\mathcal{K}] \hookrightarrow \mathcal{Z}^r_{\mathcal{K}}$ be the locus of zero maps $\mathcal{K} \to \mathcal{F}_i$ (for varying $\{\mathcal{F}_i\} \in \mathrm{Sht}^r_{U(n)}$). Its complement is the union of $\mathcal{Z}^r_{\mathcal{K}}(a)$ for non-zero $a \in \mathcal{A}_{\mathcal{K}}(k)$ and $\mathcal{Z}^r_{\mathcal{K}}(0)^*$. Note that $\mathcal{Z}^r_{\mathcal{K}}(a)$ is open-closed, and $\mathcal{Z}^r_{\mathcal{K}}(0)^*$ is proper over $\mathrm{Sht}^r_{U(n)}$ by [FYZ21, Proposition 7.5]. Therefore

$\mathcal{Z}^r_{\mathcal{K}}(0)^* \hookrightarrow \mathcal{Z}^r_{\mathcal{K}}(0)$ is open-closed and z is open-closed. The inclusion $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \hookrightarrow \mathcal{Z}^r_{\mathcal{E}}$ is the base change of z along $r_{\mathcal{K}}$, hence also open-closed. □

Corollary 4.2. *The substack $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ \subset \mathcal{Z}^r_{\mathcal{E}}$ is open-closed. In particular, $\mathcal{Z}^{r,\circ}_{\mathcal{E}}$ is open-closed in $\mathcal{Z}^r_{\mathcal{E}}$.*

Proof. Combine Lemma 4.1 with (4.2) □

Thus we have a decomposition of $\mathcal{Z}^r_{\mathcal{E}}$ in open-closed substacks

$$\mathcal{Z}^r_{\mathcal{E}} = \coprod_{\text{sub-bundles } \mathcal{K} \subset\subset \mathcal{E}} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ. \tag{4.4}$$

Remark 4.3. For a sub-bundle $\mathcal{K} \subset\subset \mathcal{E}$, there is an identification over $\mathrm{Sht}^r_{U(n)}$

$$\mathcal{Z}^r_{\mathcal{E}/\mathcal{K}} \xrightarrow{\sim} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \tag{4.5}$$

given by inflating $t_i \colon \mathcal{E}/\mathcal{K} \to \mathcal{F}_i$ along $\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{K}$. It restricts to an isomorphism

$$\mathcal{Z}^{r,\circ}_{\mathcal{E}/\mathcal{K}} \xrightarrow{\sim} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ. \tag{4.6}$$

4.2. **The cycle class $[\mathcal{Z}^{r,\circ}_{\mathcal{E}}]$.** Consider $\mathcal{M}^\circ_{\mathrm{GL}(m)',\mathrm{GL}(n)'}$, which is smooth by Proposition 3.11. Form the stack of Hitchin-shtukas $\mathrm{Sht}^r_{\mathcal{M}^\circ_{\mathrm{GL}(m)',\mathrm{GL}(n)'}}$. In Definition 3.21 we have defined a cycle class

$$[\mathrm{Sht}^r_{\mathcal{M}^\circ_{\mathrm{GL}(m)',\mathrm{GL}(n)'}}] \in \mathrm{Ch}_{r(2n-1-m)}(\mathrm{Sht}^r_{\mathcal{M}^\circ_{\mathrm{GL}(m)',\mathrm{GL}(n)'}}). \tag{4.7}$$

Note we have a decomposition

$$\mathrm{Sht}^r_{\mathcal{M}^\circ_{\mathrm{GL}(m)',\mathrm{GL}(n)'}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} \mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'}. \tag{4.8}$$

We define $[\mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'}] \in \mathrm{Ch}_{r(2n-1-m)}(\mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(n)'})$ to be the projection of $[\mathrm{Sht}^r_{\mathcal{M}^\circ_{\mathrm{GL}(m)',\mathrm{GL}(n)'}}]$ to the summand indexed by $\mathcal{E}$.

We have a Cartesian diagram from Lemma 2.12

$$\begin{tikzcd} \mathcal{Z}^{r,\circ}_{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(m)'} \arrow[d] \\ \mathrm{Sht}^r_{U(n)} \arrow[r, "u"] & \mathrm{Sht}^r_{\mathrm{GL}(n)'} \end{tikzcd}$$

Note that u is a regular local immersion, so that the refined Gysin pullback $u^!$ is defined.

Definition 4.4. We define

$$[\mathcal{Z}^{r,\circ}_{\mathcal{E}}] := u^![\mathcal{Z}^{r,\circ}_{\mathcal{E},\mathrm{GL}(m)'}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^{r,\circ}_{\mathcal{E}}). \tag{4.9}$$

Here we are using the equality

$$r(2n-1-m) - (\dim \mathrm{Sht}^r_{\mathrm{GL}(n)'} - \dim \mathrm{Sht}^r_{U(n)}) = r(n-m) \tag{4.10}$$

to determine the (virtual) dimension of the resulting cycle.

4.3. **Tautological line bundles.** For $i = 1, \cdots, r$ we have a line bundle ℓ_i on $\mathrm{Hk}^r_{U(n)}$ whose fiber at $(\{x'_j\}, \{\mathcal{F}_j, h_j\})$ is the line $\mathcal{F}_i/\mathcal{F}^\flat_{i-1/2}$ (supported at $\sigma(x'_i)$). Recall here that $\mathcal{F}^\flat_{i-1/2} = \mathcal{F}_{i-1} \cap \mathcal{F}_i$. We use the same notation ℓ_i to denote its pullback to $\mathrm{Sht}^r_{U(n)}$. We call them *tautological line bundles* on $\mathrm{Sht}^r_{U(n)}$.

Definition 4.5. Let $\mathcal{K} \subset \mathcal{E}$ be a sub-bundle. In Definition 4.4 we have defined a cycle class $[\mathcal{Z}^{r,\circ}_{\mathcal{E}/\mathcal{K}}] \in \mathrm{Ch}_{r(n-m+m_0)}(\mathcal{Z}^{r,\circ}_{\mathcal{E}/\mathcal{K}})$, where m_0 is the rank of $\mathcal{K}$ if X' is connected, and if $X' = X \sqcup X$ is disconnected then m_0 is the average rank of $\mathcal{K}$ on the two components (note that in this latter case, r must be even for $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ to be non-empty, so rm_0 is an integer). Using (4.6) we view $[\mathcal{Z}^{r,\circ}_{\mathcal{E}/\mathcal{K}}] \in \mathrm{Ch}_{r(n-m+m_0)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ)$ We define

$$[\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ] := (\prod_{i=1}^r c_{\mathrm{top}}(p_i^*\sigma^*\mathcal{K}^* \otimes \ell_i)) \cap [\mathcal{Z}^{r,\circ}_{\mathcal{E}/\mathcal{K}}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ). \tag{4.11}$$

Here $\mathcal{K}^* := \underline{\mathrm{Hom}}(\mathcal{K}, \mathcal{O}_{X'})$ is the linear dual of $\mathcal{K}$, and recall that $p_i : \mathrm{Sht}^r_{U(n)} \to X'$ records the leg x'_i. The notation $c_{\mathrm{top}}(\dots)$ denotes the "top Chern class". In the case when $X' = X \sqcup X$ is disconnected, the rank of $p_i^*\sigma^*\mathcal{K}^* \otimes \ell_i$ is locally constant on $\mathcal{Z}^{r,\circ}_{\mathcal{E}}$ (but not necessarily globally constant), and for each connected component "top" is understood to be the rank of the restriction to that connected component.

Remark 4.6. More generally, if BH_2 is any gerbe of unitary type as in Definition 3.3, then the same formula defines a tautological bundle ℓ_i on $\mathrm{Sht}^r_{H_2}$. We may then define an analogous class $[\mathcal{Z}^r_{\mathcal{E},H_2}[\mathcal{K}]^\circ]$. We will not have much need for this extra generality, so we prefer to focus on the case $BH_2 = BU(n)_{\mathfrak{L}}$ for concreteness. The general unitary gerbe case is only invoked in Example 7.6 and §11.

4.4. **Virtual fundamental classes for special cycles.** Finally we have the definition of the cycle class $[\mathcal{Z}^r_{\mathcal{E}}]$.

Definition 4.7 (Definition of special cycle classes)**.**

1. Under the decomposition (4.13), let $[\mathcal{Z}^r_{\mathcal{E}}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}})$ be the cycle class whose restriction to the open-closed substack $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ is the class $[\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ]$ from Definition 4.5, for all sub-bundles $\mathcal{K}$ of $\mathcal{E}$.
2. Let $a \in \mathcal{A}_{\mathcal{E}}(k)$. Define $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a))$ to be the projection of $[\mathcal{Z}_{\mathcal{E}}]$ to the summand $\mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a))$.

Let $a \in \mathcal{A}_{\mathcal{E}}(k)$. We define substacks of $\mathcal{Z}^r_{\mathcal{E}}(a)$:

$$\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a) := \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \cap \mathcal{Z}^r_{\mathcal{E}}(a), \quad \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ := \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ \cap \mathcal{Z}^r_{\mathcal{E}}(a). \tag{4.12}$$

It is clear that $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)$ is non-empty only when $\mathcal{K} \subset \ker(a)$. For $\mathcal{K} \subset \ker(a)$, a descends to a Hermitian map $\overline{a} : \mathcal{E}/\mathcal{K} \to \sigma^*(\mathcal{E}/\mathcal{K})^\vee \otimes \nu^*\mathfrak{L}$, i.e., $\overline{a} \in \mathcal{A}_{\mathcal{E}/\mathcal{K}}(k)$. Then (4.5) restricts to isomorphisms $\mathcal{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a}) \xrightarrow{\sim} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)$ and $\mathcal{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ \xrightarrow{\sim} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ$.

The open-closed decomposition (4.4) restricts to an open-closed decomposition of $\mathcal{Z}^r_{\mathcal{E}}(a)$,

$$\mathcal{Z}^r_{\mathcal{E}}(a) = \coprod_{\text{sub-bundles } \mathcal{K} \subset\subset \ker(a)} \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ. \tag{4.13}$$

We define $[\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ)$ to be the restriction of $[\mathcal{Z}^r_{\mathcal{E}}(a)]$.

Remark 4.8. Note that a different definition of $[\mathcal{Z}^r_{\mathcal{E}}(a)^\circ]$ has already been given in [FYZ21, §7] when a is non-singular, at least in special cases where $\mathcal{E}$ is a direct sum of line bundles or $\operatorname{rank}\mathcal{E} = n$. We will establish later (Proposition 8.3) that the definitions are consistent.

We denote the natural projection from special cycles to $\mathrm{Sht}^r_{U(n)}$ by

$$\zeta : \mathcal{Z}^r_{\mathcal{E}}(a) \to \mathrm{Sht}^r_{U(n)}\,. \tag{4.14}$$

Recall from [FYZ21, Proposition 7.5] that ζ is finite, the map ζ_* on Chow groups is therefore defined. In particular we have the Chow class

$$\zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n)}) \tag{4.15}$$

for any $a \in \mathcal{A}_{\mathcal{E}}(k)$.

Recall from the decomposition (4.13) that for a singular a, $\mathcal{Z}^r_{\mathcal{E}}(a)$ may have infinitely many components $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ$ indexed by sub-bundles $\mathcal{K} \subset \ker(a)$. The cycle $\zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is still well-defined because ζ is finite on the whole $\mathcal{Z}^r_{\mathcal{E}}(a)$ and not just on each $\mathcal{Z}^r_{\mathcal{E}/\mathcal{K}}(a)^\circ$. Although not logically needed, we give an independent proof of the following fact that assures us that $\zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is a locally finite union of algebraic cycles.

Lemma 4.9. *Fix $(\mathcal{E}, a)$ as above. For each sub-bundle $\mathcal{K} \subset \ker(a)$, let $\mathfrak{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ \subset \mathrm{Sht}^r_{U(n)}$ be the image of $\mathcal{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ \cong \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}](a)^\circ$ under ζ. Then the collection of closed substacks $\{\mathfrak{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ\}_{\mathcal{K}\subset\ker(a)}$ of $\mathrm{Sht}^r_{U(n)}$ is locally finite.*

Proof. For $d \in \mathbf{Q}$, let $\mathrm{Sht}^{r,\le d}_{U(n)}$ be the open substack of those Hermitian Shtukas $\mathcal{F}_\bullet$ such that all slopes of $\mathcal{F}_0$ (as vector bundles over X') are $\le d$. It suffices to show that the intersection $\mathrm{Sht}^{r,\le d}_{U(n)} \cap \mathfrak{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ$ is non-empty only for finitely many sub-bundles $\mathcal{K} \subset \ker(a)$. Now suppose $\mathrm{Sht}^{r,\le d}_{U(n)} \cap \mathfrak{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ \neq \varnothing$, and let $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{K}$. For any vector bundle $\mathcal{V}$ on X' let $\mu_{\max}(\mathcal{V})$ and $\mu_{\min}(\mathcal{V})$ be the maximal and minimal slopes of $\mathcal{V}$. On one hand, a $\overline{k}$-point in $\mathrm{Sht}^{r,\le d}_{U(n)} \cap \mathfrak{Z}^r_{\mathcal{E}/\mathcal{K}}(\overline{a})^\circ$ gives an injective map $\overline{\mathcal{E}}_{\overline{k}} \to \mathcal{F}_0$, which implies $\mu_{\max}(\overline{\mathcal{E}}) \le \mu_{\max}(\mathcal{F}_0) \le d$.

On the other hand, $\overline{\mathcal{E}}$ being a quotient of $\mathcal{E}$ implies that $\mu_{\min}(\overline{\mathcal{E}}) \geq \mu_{\min}(\mathcal{E})$. Thus all slopes of $\overline{\mathcal{E}}$ are within the range $[\mu_{\min}(\mathcal{E}), d]$. This leaves finitely many possibilities for vector bundles $\overline{\mathcal{E}}$ over X' of rank bounded by the rank of $\ker(a)$. □

4.5. **The modularity conjecture.** Let $\mathrm{Bun}_{GU(2m)}$ be the moduli stack of triples $(\mathcal{G}, \mathfrak{M}, h)$ where $\mathcal{G}$ is a vector bundle of rank $2m$ over X', $\mathfrak{M}$ is a line bundle over X, and h is a Hermitian isomorphism $h : \mathcal{G} \xrightarrow{\sim} \sigma^*\mathcal{G}^\vee \otimes \nu^*\mathfrak{M} = \sigma^*\mathcal{G}^* \otimes \nu^*(\omega_X \otimes \mathfrak{M})$. Let $c : \mathrm{Bun}_{GU(2m)} \to \mathrm{Pic}_X$ be the map recording $\omega_X \otimes \mathfrak{M}$. Then for any $\mathfrak{L} \in \mathrm{Pic}_X(k)$, $c^{-1}(\mathfrak{L}) = \mathrm{Bun}_{U(2m),\omega_X^{-1}\otimes\mathfrak{L}}$ as defined in §2.3.

A priori $\mathrm{Bun}_{GU(2m)}(k)$ has a decomposition

$$\mathrm{Bun}_{GU(2m)}(k) = \coprod_{\xi} \widetilde{H}_\xi(F)\backslash \widetilde{H}_\xi(\mathbb{A})/\widetilde{H}_\xi(\widehat{\mathcal{O}}) \tag{4.16}$$

where ξ runs through $2m$-dimensional Hermitian spaces over F' that are locally split at all places, and $\widetilde{H}_\xi$ is the corresponding unitary similitude group. By the Hasse principle for Hermitian spaces [Sch85, Theorem 6.2] ξ must be globally split. Let $\widetilde{H}_m = GU(2m)$ be the unitary similitude group for a fixed split $2m$-dimensional Hermitian spaces over F'. Then

$$\mathrm{Bun}_{GU(2m)}(k) = \widetilde{H}_m(F)\backslash \widetilde{H}_m(\mathbb{A})/\widetilde{H}_m(\widehat{\mathcal{O}}) \tag{4.17}$$

We can similarly define the moduli $\mathrm{Sht}^r_{GU(n)}$ of shtukas for $GU(n)$. It simply adds the similitude line bundle $\mathfrak{L}$ as part of the data which is invariant under Frobenius pullback, and it is the disjoint union

$$\mathrm{Sht}^r_{GU(n)} = \coprod_{\mathfrak{L}\in \mathrm{Pic}_X(k)} \mathrm{Sht}^r_{U(n),\mathfrak{L}}\,. \tag{4.18}$$

Let $\mathrm{Bun}_{\widetilde{P}_m}$ be the moduli stack of quadruples $(\mathcal{G}, \mathfrak{M}, h, \mathcal{E})$ where $(\mathcal{G}, \mathfrak{M}, h) \in \mathrm{Bun}_{GU(2m)}$, and $\mathcal{E} \subset \mathcal{G}$ is a Lagrangian sub-bundle (of rank m). Let $\mathrm{Bun}_{P_m,\mathfrak{M}}$ be the substack with the fixed similitude line bundle $\mathfrak{M}$. We usually omit h and write a point in $\mathrm{Bun}_{P_m,\mathfrak{M}}$ as $(\mathcal{G}, \mathcal{E})$.

The map $\mathrm{Bun}_{\widetilde{P}_m} \to \mathrm{Bun}_{GU(2m)}$ forgetting the Lagrangian sub-bundle is surjective as map of stacks, and it is also surjective on k-points. Indeed, since the generic fiber of any $(\mathcal{G}, \mathfrak{M}, h) \in \mathrm{Bun}_{GU(2m)}(k)$ is a split Hermitian space over F' of dimension $2m$, it has a Lagrangian sub-bundle at the generic point, hence a Lagrangian sub-bundle over X' by saturation. If we write $\widetilde{P}_m \subset \widetilde{H}_m$ for the Siegel parabolic subgroup stabilizing a Lagrangian subspace, then

$$\mathrm{Bun}_{\widetilde{P}_m}(k) = \widetilde{P}_m(F)\backslash \widetilde{H}_m(\mathbb{A})/\widetilde{H}_m(\widehat{\mathcal{O}}). \tag{4.19}$$

Now fix $\mathfrak{L} \in \mathrm{Pic}_X(k)$. For $(\mathcal{G}, \mathcal{E}) \in \mathrm{Bun}_{P_m,\omega_X^{-1}\otimes\mathfrak{L}}(k)$, we have a short exact sequence

$$0 \to \mathcal{E} \to \mathcal{G} \to \sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \nu^*\mathfrak{L}) \to 0 \tag{4.20}$$

which gives an extension class

$$e_{\mathcal{G},\mathcal{E}} \in \mathrm{Ext}^1_{X'}(\sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \nu^*\mathfrak{L}), \mathcal{E}). \tag{4.21}$$

On the other hand, a Hermitian map $a \in \mathcal{A}_{\mathcal{E}}$ can be viewed as an element

$$a \in \mathrm{Hom}_{X'}(\mathcal{E}, \sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \omega_{X'} \otimes \nu^*\mathfrak{L})) = \mathrm{Hom}_{X'}(\mathcal{E}, \sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \nu^*\mathfrak{L}) \otimes \omega_{X'}). \tag{4.22}$$

Serre duality gives a perfect pairing

$$\langle\cdot,\cdot\rangle : \mathrm{Ext}^1_{X'}(\sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \nu^*\mathfrak{L}), \mathcal{E}) \times \mathrm{Hom}_{X'}(\mathcal{E}, \sigma^*\underline{\mathrm{Hom}}(\mathcal{E}, \nu^*\mathfrak{L}) \otimes \omega_{X'}) \to k. \tag{4.23}$$

In particular, $\langle e_{\mathcal{G},\mathcal{E}}, a\rangle \in k$ is defined.

Now we fix a nontrivial character $\psi_0 : k \to \overline{\mathbf{Q}}_\ell^\times$. Finally, recall that $\eta : \mathrm{Pic}_X(k) \to \{\pm 1\}$ is the character with kernel $\mathrm{Nm}(\mathrm{Pic}_{X'}(k))$. Let $\chi : \mathrm{Pic}_{X'}(k) \to \mathbf{C}^\times$ be a character such that $\chi|_{\mathrm{Pic}_X(k)} = \eta^n$.[3]

Definition 4.10. Define a map

$$\begin{array}{rcl} \widetilde{Z}^r_{m,\mathfrak{L}} : \mathrm{Bun}_{P_m,\omega_X^{-1}\otimes\mathfrak{L}}(k) & \to & \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n),\mathfrak{L}}) \\ (\mathcal{G}, \mathcal{E}) & \mapsto & \chi(\det\mathcal{E}) q^{n(\deg\mathcal{E} - \deg\mathfrak{L} - \deg\omega_X)/2} \displaystyle\sum_{a\in\mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)} \psi_0(\langle e_{\mathcal{G},\mathcal{E}}, a\rangle)\zeta_*[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)]. \end{array} \tag{4.24}$$

[3]The existence of such χ is justified in Footnote 1 of [FYZ21].

Taking the union over $\mathfrak{L} \in \mathrm{Pic}_X(k)$, we get a map

$$\widetilde{Z}^r_m : \mathrm{Bun}_{\widetilde{P}_m}(k) \to \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}). \tag{4.25}$$

Remark 4.11. Using (4.19), we may identify $\widetilde{Z}^r_m$ as a function

$$\widetilde{H}_m(\mathbb{A}) \ni g \mapsto \widetilde{Z}^r_m(g) \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}) \tag{4.26}$$

such that

- $\widetilde{Z}^r_m$ is left invariant under the Siegel parabolic $\widetilde{P}_m(F)$ and right invariant under $\widetilde{H}_m(\widehat{\mathcal{O}})$ (everywhere unramified);
- if $g \in \widetilde{H}_m(\mathbb{A})$ has similitude factor $c(g) \in \mathbb{A}^\times$ that projects to the line bundle $\mathfrak{L} \in \mathrm{Pic}_X(k) = F^\times \backslash \mathbb{A}^\times / \widehat{\mathcal{O}}^\times$, then $\widetilde{Z}^r_m(g)$ is supported on $\mathrm{Sht}^r_{U(n),\mathfrak{L}} \subset \mathrm{Sht}^r_{GU(n)}$.

The following is the main conjecture of the paper.

Conjecture 4.12 (Modularity conjecture)**.** *The map $\widetilde{Z}^r_m$ descends to a map*

$$Z^r_m : \mathrm{Bun}_{GU(2m)}(k) \to \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}). \tag{4.27}$$

i.e., the function (4.26) *is left $\widetilde{H}_m(F)$-invariant.*

In other words, the Chow class $\widetilde{Z}^r_m(\mathcal{G}, \mathcal{E}) \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)})$ should depend only on the Hermitian bundle $\mathcal{G}$ and not on its Lagrangian sub-bundle $\mathcal{E}$.

Remark 4.13. When $r = 0$, $\mathrm{Ch}_0(\mathrm{Sht}^0_{GU(n)})$ is simply the space of $\mathbf{Q}$-valued functions on $\mathrm{Bun}_{GU(n)}(k)$. The conjecture in this case follows from the automorphy of the theta series constructed from the Weil representation for the dual pair $(GU(2m), GU(n))$.

Remark 4.14. Suppose $r > 0$ and $n > 1$. We expect based on §10.6.1 that $\mathrm{Ch}_0(\mathrm{Sht}^r_{GU(n)})$ vanishes, making the conjecture vacuous for $m = n$ in this situation. In [FYZ21], for the non-singular terms we constructed cycle classes in the Chow group of *proper* cycles on $\mathrm{Sht}^r_{U(n)}$, and proved a higher Siegel-Weil formula for those terms. It remains an open problem to formulate a more refined version of the generating series where the singular terms also have a meaningful notion of degree.

4.6. **Special cases.** Let $\mathcal{E}$ be a rank m vector bundle on X'. Let $\mathcal{E}' = \sigma^* \underline{\mathrm{Hom}}(\mathcal{E}, \nu^*\mathfrak{L})$. Consider the Hermitian vector bundle $\mathcal{G} = \mathcal{E} \oplus \mathcal{E}'$ with the natural Hermitian form isotropic on each summand and induces the natural pairing between the two summands. In this case, both $(\mathcal{G}, \mathcal{E})$ and $(\mathcal{G}, \mathcal{E}')$ are points of $\mathrm{Bun}_{P_m, \omega_X^{-1} \otimes \mathfrak{L}}(k)$ over $\mathcal{G} \in \mathrm{Bun}_{U(2m), \omega_X^{-1} \otimes \mathfrak{L}}(k)$. Conjecture 4.12 specializes to the following identity.

Conjecture 4.15. *In the above situation, we have an identity in $\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n),\mathfrak{L}})$:*

$$\chi(\det \mathcal{E}) q^{n \deg \mathcal{E}/2} \sum_{a \in \mathcal{A}_{\mathcal{E}}(k)} \zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)] = \chi(\det \mathcal{E}') q^{n \deg \mathcal{E}'/2} \sum_{a' \in \mathcal{A}_{\mathcal{E}'}(k)} \zeta_*[\mathcal{Z}^r_{\mathcal{E}'}(a')]. \tag{4.28}$$

Equivalently,

$$\eta(\mathfrak{L})^{mn} q^{n(\deg \mathcal{E} - m \deg \mathfrak{L})} \sum_{a \in \mathcal{A}_{\mathcal{E}}(k)} \zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)] = \sum_{a' \in \mathcal{A}_{\mathcal{E}'}(k)} \zeta_*[\mathcal{Z}^r_{\mathcal{E}'}(a')]. \tag{4.29}$$

In the equivalent formulation above, we use that

$$\deg \mathcal{E}' = -\deg \mathcal{E} + 2m \deg \mathfrak{L}, \quad \det(\mathcal{E}') \cong \sigma^*(\det \mathcal{E})^{-1} \otimes \nu^* \mathfrak{L}^{\otimes m}. \tag{4.30}$$

We may further specialize to the case where $\mathcal{E}'$ has large slopes, or equivalently $\mathcal{E}$ has small slopes, so that $\mathcal{A}_{\mathcal{E}'}(k)$ only contains the zero Hermitian map.

Conjecture 4.16. *Suppose the maximal slope $\mu_{\max}(\mathcal{E})$ satisfies*

$$\mu_{\max}(\mathcal{E}) < \deg \mathfrak{L} - \deg \omega_X. \tag{4.31}$$

Then we have an identity in $\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{U(n),\mathfrak{L}})$:

$$\eta(\mathfrak{L})^{mn} q^{n(\deg \mathcal{E} - m \deg \mathfrak{L})} \sum_{a \in \mathcal{A}_{\mathcal{E}}(k)} \zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)] = \zeta_*[\mathcal{Z}^r_{\mathcal{E}'}(0)]. \tag{4.32}$$

4.7. **Test intersection numbers.** To give evidence for Conjecture 4.12, we may start with any cycle *with compact support* $\xi \in \mathrm{Ch}_{rm,c}(\mathrm{Sht}^r_{U(n),\mathfrak{L}})$, and form the numerical function by intersecting $\widetilde{Z}^r_m$ with ξ:

$$\widetilde{I}^r_{m,\xi} := \langle \widetilde{Z}^r_m(-), \xi \rangle_{\mathrm{Sht}^r_{U(n),\mathfrak{L}}} : \mathrm{Bun}_{P_m,\omega_X^{-1}\otimes\mathfrak{L}}(k) \to \mathbf{Q}. \tag{4.33}$$

Conjecture 4.12 predicts that $\widetilde{I}^r_{m,\xi}(\mathcal{G},\mathcal{E})$ is independent of $\mathcal{E}$, hence descends to a function on $\mathrm{Bun}_{U(2m),\omega_X^{-1}\otimes\mathfrak{L}}(k)$. We give two example families of compact r-dimensional cycles ξ on $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$, hence giving test grounds for Conjecture 4.12 in the case $m = 1$.

Example 4.17 (Corank $n-1$ special cycles)**.** Let $\mathcal{E}$ be a rank $n-1$ vector bundle over X', and $a \in \mathcal{A}^{\mathrm{ns}}_{\mathcal{E}}(k)$ be a non-singular Hermitian map. Then the special cycle $\mathcal{Z}^r_{\mathcal{E}}(a)$ is proper (we omit the proof here). We have the cycle class $[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_r(\mathcal{Z}^r_{\mathcal{E}}(a))$ by Definition 4.4. Its direct image in $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is then a compact cycle

$$\xi := \zeta_*[\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{r,c}(\mathrm{Sht}^r_{U(n),\mathfrak{L}}). \tag{4.34}$$

Example 4.18 (CM cycles)**.** Let Y be another smooth projective curve over $\mathbf{F}_q$, and $\theta : Y \to X$ be a map of degree n, possibly ramified. Let $\nu_Y : Y' = X' \times_X Y \to Y$, and assume this double covering is nonsplit over each connected component of Y. Let $\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}$ be the moduli stack of rank 1 $\theta^*\mathfrak{L}$-twisted Hermitian shtukas (cf. §3.1.3 for the definition) on Y' (with respect to the double cover ν_Y). Then push-forward along ν_Y gives a map $\Theta : \mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}} \to \mathrm{Sht}^r_{U(n),\mathfrak{L}}$. Now $\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}$ is smooth and proper of pure dimension r, we have the compact cycle class

$$\xi = \Theta_*[\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}] \in \mathrm{Ch}_{r,c}(\mathrm{Sht}^r_{U(n),\mathfrak{L}}). \tag{4.35}$$

The intersection number of the generating series of corank 1 and this cycle will be calculated in §11. In particular, we will verify the modularity of such intersection numbers.

Remark 4.19. It is possible to give a general construction that includes both examples as special cases, but the details will not be included here.

4.8. **The split case.** In the case where $X' = X^{(1)} \coprod X^{(2)}$ is the split double cover of X (so each $X^{(i)} \cong X$), the definition of the cycle classes $[\mathcal{Z}^r_{\mathcal{E}}]$ needs to be modified as follows.

First we spell out some of the definitions more explicitly. In this case, an $\mathfrak{L}$-twisted Hermitian bundle $\mathcal{F}$ on X' identifies with a pair of vector bundles $(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})$, each living on one copy of X, equipped with an isomorphism $\mathcal{F}^{(2)} \cong \mathcal{F}^{(1),\vee} \otimes \mathfrak{L}$. Therefore we have $\mathrm{Bun}_{U(n),\mathfrak{L}} \cong \mathrm{Bun}_{\mathrm{GL}(n)}$ by recording only $\mathcal{F}^{(1)}$. Since every $\mathfrak{L}$ is a norm, without loss of generality we can and will assume $\mathfrak{L} = \mathcal{O}_X$. Then we have a disjoint union

$$\mathrm{Sht}^r_{U(n),\mathfrak{L}} = \coprod_{\mu \in \{\pm 1\}^r} \mathrm{Sht}^{\mu}_{\mathrm{GL}(n)}$$

where the $\mu = (\mu_1, \cdots, \mu_r)$-th component is empty unless $\sum_{i=1}^r \mu_i = 0$; see [FYZ21, §12.3] which also recalled the definition of $\mathrm{Sht}^{\mu}_{\mathrm{GL}(n)}$.

A vector bundle $\mathcal{E}$ on X' of rank m corresponds to two rank m vector bundles $(\mathcal{E}^{(1)}, \mathcal{E}^{(2)})$, each living on one copy of X. Now $\mathcal{A}_{\mathcal{E}}(k) = \mathcal{A}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}(k)$ may be identified with the set of maps $a : \mathcal{E}^{(1)} \to \mathcal{E}^{(2),\vee}$.

We now fix a $\mu = (\mu_1, \cdots, \mu_r)$ such that $\sum_{i=1}^r \mu_i = 0$. The special cycle $\mathcal{Z}^{\mu}_{\mathcal{E}} = \mathcal{Z}^{\mu}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}$ in the split case parametrizes

$$\{(\{x_i\}_{1\le i\le r}, \mathcal{F}_0 \dashrightarrow \ldots \dashrightarrow \mathcal{F}_r \cong {}^\tau\mathcal{F}_0, \mathcal{E}^{(1)} \xrightarrow{t_i^{(1)}} \mathcal{F}_i, \mathcal{E}^{(2)} \xrightarrow{t_i^{(2)}} \mathcal{F}_i^\vee)\} \tag{4.36}$$

where $x_i \in X$, $\mathcal{F}_i$ are vector bundles of rank n on X, the dashed arrow $\mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ is a lower modification of length 1 at x_i if $\mu_i = -1$, and an upper modification of length 1 at x_i if $\mu_i = +1$. The maps $t_i^{(1)}$ and $t_i^{(2)}$ are required to be compatible with the chain of modifications.

The kernel decomposition of $\mathcal{Z}^{\mu}_{\mathcal{E}}$ in this case is indexed by $\mathcal{K} = (\mathcal{K}^{(1)}, \mathcal{K}^{(2)}) \subset (\mathcal{E}^{(1)}, \mathcal{E}^{(2)})$ where we note that the ranks of $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ may be different. We have an open-closed decomposition

$$\mathcal{Z}^{\mu}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}} = \coprod_{\mathcal{K}^{(1)} \subset\subset \mathcal{E}^{(1)}} \coprod_{\mathcal{K}^{(2)} \subset\subset \mathcal{E}^{(2)}} \mathcal{Z}^{\mu}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}[\mathcal{K}^{(1)}, \mathcal{K}^{(2)}]^\circ$$

where $\mathcal{Z}^{\mu}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}[\mathcal{K}^{(1)}, \mathcal{K}^{(2)}]^\circ$ is the substack of those points in (4.36) where $\ker t_i^{(1)} = \mathcal{K}^{(1)}$ and $\ker t_i^{(2)} = \mathcal{K}^{(2)}$ for any (equivalently, all) $0 \le i \le r$. With $\overline{\mathcal{E}}^{(1)} = \mathcal{E}^{(1)}/\mathcal{K}^{(1)}$ and $\overline{\mathcal{E}}^{(2)} = \mathcal{E}^{(2)}/\mathcal{K}^{(2)}$, we have $\mathcal{Z}^{\mu}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}[\mathcal{K}^{(1)}, \mathcal{K}^{(2)}]^\circ \cong$

$\mathcal{Z}^{\mu,\circ}_{\overline{\mathcal{E}}^{(1)},\overline{\mathcal{E}}^{(2)}}$. The virtual classes are then defined by summing over all $(\mathcal{K}^{(1)},\mathcal{K}^{(2)})$ the product of $[\mathcal{Z}^{\mu,\circ}_{\overline{\mathcal{E}}^{(1)},\overline{\mathcal{E}}^{(2)}}]$ with $\prod_{i=1}^r c_{\mathrm{top}}(p_i^*\mathcal{K}^{(?_i)}\otimes \ell_i)$ where $?_i=1$ if $\mu=1$ and $?_i=2$ if $\mu=-1$.

In this case, $\mathrm{Bun}_{U(2m),\omega_X^{-1}}$ from §4.5 is isomorphic to $\mathrm{Bun}_{\mathrm{GL}(2m)}$. Let $P_{(m,m)}$ be the maximal parabolic subgroup of $\mathrm{GL}(2m)$ corresponding to the partition $2m=m+m$. Then we have an isomorphism $\mathrm{Bun}_{P_m,\omega_X^{-1}}\simeq \mathrm{Bun}_{P_{(m,m)}}$, which classifies pairs $(\mathcal{G},\mathcal{E}^{(1)})$ where $\mathcal{E}^{(1)}$ is a rank m sub-bundle of a rank $2m$ vector bundle $\mathcal{G}$ on X. We define another rank m bundle $\mathcal{E}^{(2)}$ by the exact sequence

$$0\to\mathcal{E}^{(1)}\to\mathcal{G}\to\mathcal{E}^{(2),*}\to 0.$$

Given $(\mathcal{E}^{(1)},\mathcal{E}^{(2)})$, the space of such extensions is $\mathrm{Ext}^1(\mathcal{E}^{(2),*},\mathcal{E}^{(1)})$, which is dual to $\mathcal{A}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}(k)=\mathrm{Hom}(\mathcal{E}^{(1)},\mathcal{E}^{(2),\vee})$. The class of $\mathcal{G}$ defines $e_{\mathcal{G},\mathcal{E}^{(1)}}\in\mathrm{Ext}^1(\mathcal{E}^{(2),*},\mathcal{E}^{(1)})$ and we denote $\langle e_{\mathcal{G},\mathcal{E}^{(1)}},-\rangle$ the induced k-linear functional on $\mathcal{A}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}(k)$.

Then $\widetilde{Z}^\mu_m$ as a function on $\mathrm{Bun}_{P_{(m,m)}}(k)$ takes the form

$$(\mathcal{G},\mathcal{E}^{(1)})\mapsto q^{n(\deg\mathcal{E}^{(1)}+\deg\mathcal{E}^{(2)}-\deg\omega_X)/2}\sum_{a\in\mathcal{A}_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}(k)}\psi_0(\langle e_{\mathcal{G},\mathcal{E}^{(1)}},a\rangle)\zeta_*[\mathcal{Z}^\mu_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}(a)]$$

where $\zeta:\mathcal{Z}^\mu_{\mathcal{E}^{(1)},\mathcal{E}^{(2)}}(a)\to\mathrm{Sht}^\mu_{\mathrm{GL}(n)}$ is the natural projection map.

Conjecture 4.12 then says in this case that for each μ, the map $\widetilde{Z}^\mu_m:\mathrm{Bun}_{P_{(m,m)}}(k)\to\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^\mu_{\mathrm{GL}(n)})$ descends to a map

$$Z^\mu_m:\mathrm{Bun}_{\mathrm{GL}(2m)}(k)\longrightarrow\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^\mu_{\mathrm{GL}(n)}).$$

Part 2. Properties of the special cycles

5. Derived Hitchin stacks

5.1. **Overview.** In the next two sections, we explain the special cycle classes of Definition 4.7 from the perspective of derived algebraic geometry. To motivate this, we recall that in [FYZ21], certain "Hitchin stacks" $\mathcal{M}$ were introduced and it was proved that the virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ for *non-singular* a could be obtained from $\mathcal{M}$ by taking the derived intersection of a Hecke correspondence $\mathrm{Hk}^r_{\mathcal{M}}$ for $\mathcal{M}$ with the graph of Frobenius on $\mathcal{M}$. This interpretation was key to the proof of the Higher Siegel-Weil formula [FYZ21, Theorem 1.1].

The restriction to non-singular a can be explained thus: for such a, the intersection involves only the smooth part of the Hitchin stack $\mathcal{M}$. But if we try to repeat such a construction to obtain the cycles indexed by singular a, we necessarily run into loci in $\mathcal{M}$ whose geometry is too poorly behaved to carry it out.

It turns out that these problems can be resolved with *derived algebraic geometry.* In this section we will introduce *derived* Hitchin stacks $\mathscr{M}$ which are always quasi-smooth (the derived analogue of LCI) and with the "correct" virtual dimension, whose classical truncation is $\mathcal{M}$. Derived algebraic geometry then allows associate virtual fundamental classes $[\mathscr{M}]\in\mathrm{Ch}_*(\mathcal{M})$. By taking the derived intersection of derived Hecke correspondences $\mathscr{H}\!k^r_{\mathscr{M}}$ for $\mathscr{M}$ with the graph of Frobenius on $\mathscr{M}$, we then obtain certain virtual fundamental classes in the Chow group of the underlying classical special cycles $\mathcal{Z}^r_{\mathcal{E}}$. On general grounds it is non-trivial to compute these virtual fundamental classes "explicitly" in terms of classical objects. Nonetheless, we will be able to prove that they coincide with the explicit constructions introduced earlier in Definition 4.7. This gives a pleasing derivation of the cycle classes for singular terms, which is on the same conceptual footing as for the non-singular terms.

The fruits of this labor are not merely philosophical: in §7 we use this derived algebraic geometry interpretation of the cycle classes to prove the *linear invariance* property of our special cycles. The number field analogue of this property is a well-known conjectural property of arithmetic theta series [Kud04, Problem 5]. The statement can be formulated in purely classical terms, but *we do not know a proof without derived algebraic geometry.* In turn, §7 will also be used later in §11 numerical evidence for modularity conjecture.

5.2. **Derived stacks.**

5.2.1. *Orientation on derived algebraic geometry.* We give an introductory discussion on derived algebraic geometry, in order to help orient readers not accustomed to this formalism. We confine ourselves to informal and sometimes vague remarks, referring to [Lur04, TV08] for more complete treatments. Some relevant introductory references are [Toe09, Toe10].

Just as Grothendieck's schemes are spaces built locally from "spectra" of commutative rings, *derived* schemes are built locally from "spectra" of *simplicial commutative rings*. Roughly speaking, one can think of simplicial commutative rings as a model for the concept of topological rings.

We use the adjective "classical" (ring, scheme, stack...) to refer to the usual notions of non-derived algebraic geometry. From a formal perspective, the relationship between derived schemes and classical schemes is analogous to the relationship between classical schemes and reduced classical schemes. A derived scheme has an underlying classical scheme, and intuitively one thinks of a derived scheme as an "infinitesimal thickening" of its underlying classical scheme. Formation of the underlying classical scheme (an operation called "classical truncation") defines a functor which is right adjoint to a fully faithful embedding from classical schemes to derived schemes.

$$\begin{tikzcd} \text{Class. Sch.} \arrow[r, hook, bend left] & \text{Der. Sch.} \arrow[l, bend left, "\pi_0"] \end{tikzcd} \quad :: \quad \begin{tikzcd} \text{Reduced Sch.} \arrow[r, hook, bend left] & \text{Class. Sch.} \arrow[l, bend left, "(\cdot)_{\mathrm{red}}"] \end{tikzcd}$$

One advantage of considering non-reduced schemes is that it gives a natural interpretation of the tangent space, as maps from the spectrum of the dual numbers. Analogously, derived algebraic geometry gives a very natural interpretation of the *cotangent complex* (which governs deformation theory), even for a classical scheme. Indeed, the higher cohomology groups of the tangent complex can be viewed in terms of maps from certain "derived infinitesimal schemes", which are derived generalizations of dual numbers. The theory of the cotangent complex plays a crucial technical role in this section.

The passage from classical schemes to classical stacks goes through the "functor of points" perspective: a scheme can be viewed as a functor from commutative rings to sets, and a stack can be viewed as a functor from commutative rings to groupoids. Generalizing this perspective, a derived scheme can be interpreted as a functor from *simplicial* commutative rings to *simplicial* sets. However, when working with simplicial objects, the notion of equivalence should be homotopy-theoretic. Consequently, the test and target categories should be the "non-abelian derived categories" of simplicial commutative rings and of simplicial sets, respectively, which are called the ∞-*category of simplicial rings* and the ∞-*category of simplicial sets*, respectively. We therefore define a *derived stack* to be a functor from the ∞-category of simplicial rings to the ∞-category of simplicial sets, satisfying certain descent conditions. A derived stack has a classical truncation by restricting its domain to classical rings, and this defines a functor which has a fully faithful left adjoint, embedding classical stacks into derived stacks.

$$\begin{tikzcd} \text{Class. Stacks.} \arrow[r, hook, bend left] & \text{Der. Stacks} \arrow[l, bend left, "\pi_0"] \end{tikzcd}$$

For us, derived algebraic geometry will be used in the following way. We wish to attach fundamental classes to our special cycles $\mathcal{Z}^r_{\mathcal{E}}(a)$, but these spaces are poorly behaved in general, so we need to construct *virtual* fundamental cycles. For example, the virtual fundamental cycle of $\mathcal{Z}^r_{\mathcal{E}}(a)$ should have dimension $(n - \operatorname{rank}\mathcal{E})r$, but $\dim \mathcal{Z}^r_{\mathcal{E}}(a)$ may be much larger, e.g when $a = 0$ then $\mathcal{Z}^r_{\mathcal{E}}(a)$ has components of the maximal dimension nr. However, it turns out that there is a natural *derived special cycle* $\mathscr{Z}^r_{\mathcal{E}}(a)$ with the correct *virtual* dimension, which is moreover quasi-smooth (the analogue of LCI in derived algebraic geometry), and whose classical truncation recovers $\mathcal{Z}^r_{\mathcal{E}}(a)$. The property that $\mathscr{Z}^r_{\mathcal{E}}(a)$ is quasi-smooth implies that it has an intrinsic virtual fundamental class $[\mathscr{Z}^r_{\mathcal{E}}(a)]$, and derived invariance of Chow groups (see §6.1.4) allows to view it as an element of $\mathrm{Ch}_{(n-\operatorname{rank}\mathcal{E})r}(\mathcal{Z}^r_{\mathcal{E}}(a))$.

Example 5.1. Even when working with classical schemes, derived structure often shows up implicitly because of derived intersections. Locally, this is based on the "derived tensor product" operation, which can produce a non-classical simplicial commutative ring even when the inputs are classical. From this optic, the virtual fundamental cycles of [FYZ21] come from derived stacks obtained by taking derived intersections in a particular presentation of $\mathcal{Z}_{\mathcal{E}}^r(a)$ as a fibered product of smooth classical stacks. In particular, for non-singular a the derived stacks $\mathscr{Z}_{\mathcal{E}}^r(a)$ is a global complete intersection in the derived sense; more generally, derived algebraic geometry provides an *intrinsic* construction of a virtual fundamental class to any derived stack which *locally* looks like a derived fibered product of smooth classical schemes (this is one formulation of quasi-smoothness). Crucially this is a local property and we do not require any *global* presentation as a derived intersection of smooth stacks, which we do not have in the case of singular coefficients.

5.2.2. *Notational conventions.* We will use script letters such as $\mathscr{X}, \mathscr{Y}$ for derived stacks, and calligraphic letters such as $\mathcal{X}, \mathcal{Y}$ for classical stacks. We will often use $\mathcal{X}$ to denote the classical truncation of $\mathscr{X}$ (defined later in §5.2.6).

5.2.3. *Derived (Artin) stacks.* For the framework of derived stacks, we follow [Kha19b, §1.1]. To summarize, derived stacks are defined as functors from a test category to a target category, satisfying a sheaf condition, where:

- The test category is the ∞-category of simplicial commutative rings. This can be constructed as in [Lur09, Definition 4.1.1]; an intrinsic characterization can be found in [CS19, §5.1]. Following Clausen-Scholze we call it the category of *animated rings*, and use the phrase "animated ring" to indicate an object of this category.
- The target category is the ∞-category of simplicial sets. Similar remarks apply as above. Following Clausen-Scholze we call it the category of *anima*[4], and use the phrase "anima" to indicate an object of this category.

Thus, derived stacks $\mathcal{Y}$ over k are functors from the category of animated rings to the category of anima, denoted $R_\bullet \mapsto \mathcal{Y}(R_\bullet)$, satisfying étale hyperdescent.

We define *n-geometric derived stacks* as in [TV08, §1.3.3][5], and *derived Artin stacks* to be derived stacks which are n-geometric for some n.

5.2.4. *Representable morphisms.* Following [TV08, Definition 1.3.3.1, Definition 1.3.3.7], we say that a morphism of derived stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *n-representable* if for any derived scheme S and any map $S \to \mathcal{Y}$, the fibered product $\mathcal{X} \times_{\mathcal{Y}} S$ is n-geometric. We say f is *representable* if it is n-representable for some n. (Note that this condition is much broader than representability for morphisms of classical Artin stacks, the latter of which is analogous to "(-1)-representable" in our sense.)

By [TV08, Proposition 1.3.3.3], the class of representable morphisms is closed under isomorphisms, (homotopy) pullbacks, and compositions.

5.2.5. *Derived terminology.* We remind the reader that all operations in ∞-categories are "homotopical", so that tensor products of animated rings correspond to "derived tensor products", fiber products of derived stacks correspond to "homotopy fiber products", the "fiber" of a map of complexes $\mathcal{K} \xrightarrow{f} \mathcal{K}'$ in the derived category means the "derived fiber" $\mathrm{Cone}(f)[-1]$, etc. (If we need to refer to a classical fibered product of classical stacks $\mathcal{X}$ and $\mathcal{Y}$ over $\mathcal{T}$, we will denote it by $\mathcal{X} \overset{\mathrm{cl}}{\times}_{\mathcal{T}} \mathcal{S}$.) At some points we include the adjectives "homotopy" or "derived" to emphasize this, but it applies everywhere in this section.

5.2.6. *Classical truncation.* We shall frequently invoke the notion of the "underlying classical stack", i.e. "classical truncation", of a derived stack. Here we recall what this means. If $R_\bullet$ is a simplicial commutative ring, then its "underlying classical ring" is $\pi_0(R_\bullet)$. (In topological terminology this is the "first Postnikov truncation" of $R_\bullet$, which explains the synonymous terminology "classical truncation".) This descends to a functor on animated rings, which is left adjoint to the inclusion of classical (i.e. discrete) commutative rings into animated rings. On the opposite categories, we get a fully faithful functor from affine schemes to derived affine schemes which is left adjoint to the classical truncation.

This operation then glues in the Zariski topology to give a functor $T \mapsto \pi_0(T)$ from derived schemes to classical (discrete) schemes, which is right adjoint to a fully faithful inclusion functor from classical schemes into derived schemes. By abuse of notation we may regard $\pi_0(T)$ as a derived scheme via this inclusion; then the unit of the adjunction is a map $\pi_0(T) \to T$, natural in T.

[4]Also called "∞-groupoid", or "space".

[5]There are differing conventions on n-stacks – for example the above notion differs from the "n-algebraic stacks" of [Toe10, §5.2] – but they all produce the same notion of Artin stack, which is the only one of importance to us.

Finally, if $\mathscr{X}$ is a derived stack, then its underlying classical stack $\mathscr{X}_{\mathrm{cl}}$ is the restriction of $\mathscr{X}$ along the embedding {Classical affine schemes} $\hookrightarrow$ {Derived affine schemes}. The classical truncation functor $\mathscr{X} \mapsto \mathscr{X}_{\mathrm{cl}}$ has a left adjoint, which can be described as the sheafification of the left Kan extension on the underlying prestacks, and is fully faithful [GR17, §I.2.6]. The left adjoint gives an inclusion {Classical stacks} $\hookrightarrow$ {Derived stacks}, and the unit of the adjunction is a map

$$\iota_{\mathscr{X}} : \mathscr{X}_{\mathrm{cl}} \to \mathscr{X}$$

functorial in $\mathscr{X}$, which we call the inclusion of the underlying classical stack.

5.2.7. *Derived mapping stacks.* We give some examples of derived Artin stacks which are of particular relevance to this paper.

Example 5.2. Let X be a proper scheme over k and $\mathscr{Y}$ a derived Artin stack locally of finite presentation over k. The *derived mapping stack* $\mathscr{M}ap(X, \mathscr{Y})$ sends an animated ring $R_\bullet$ to the anima of morphisms

$$X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R_\bullet \to \mathscr{Y}.$$

More generally, in the above situation, if both X and $\mathscr{Y}$ are over a scheme S over k, we can define the derived mapping stack $\mathscr{M}ap_S(X, \mathscr{Y})$ whose $R_\bullet$-points form the anima of morphisms $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R_\bullet \to \mathscr{Y}$ over S. We note that $\mathscr{M}ap_S(X, \mathscr{Y})$ may be expressed as the (homotopy) fiber of $\mathscr{M}ap(X, \mathscr{Y}) \to \mathscr{M}ap(X, S)$, induced by $\mathscr{Y} \to S$, over the given map $X \to S$. By this observation and [Toe14, Corollary 3.3], if X is proper over k and $\mathscr{Y}$ is locally of finite presentation over S, then $\mathscr{M}ap_S(X, \mathscr{Y})$ is a derived Artin stack locally of finite presentation over S.[6]

When $S = X$ we write $\mathscr{S}ect(X, \mathscr{Y})$ for $\mathscr{M}ap_X(X, \mathscr{Y})$.

Example 5.3. Let X be a scheme X over k, and let G be a smooth algebraic group over X. Regard the classical classifying stack $BG = [X/G]$ as a derived stack over X via the embedding discussed above. Then the derived mapping stack $\mathscr{S}ect(X, BG)$ sends $R_\bullet$ to the anima of G-bundles on $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R_\bullet$. When X is a smooth projective curve we will see in Corollary 5.7 that $\mathscr{S}ect(X, BG)$ coincides with its underlying classical stack, which is Bun_G.

Example 5.4. Let X be a proper scheme over k. Let $G \to X$ be an algebraic group scheme and $V \to X$ a vector bundle that is a representation of G. We apply Example 5.2 with $\mathscr{Y} = V/G$ (a classical stack). There is a *derived stack of sections* $\mathscr{S}ect(X, V/G)$ sends an animated ring $R_\bullet$ to the ∞-groupoid of $(\mathcal{E}_G, s)$ where

- $\mathcal{E}_G \xrightarrow{\pi} X$ is a G-bundle on $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, R_\bullet$,
- f is an element of the (homotopy) fibered product $\mathrm{Map}_G(\mathcal{E}_G, V) \overset{h}{\times}_{\mathrm{Map}(\mathcal{E}_G, X)} \{\pi\}$ where the map $\mathrm{Map}_G(\mathcal{E}_G, V) \to \mathrm{Map}(\mathcal{E}_G, X)$ is induced by composition with the tautological map $V \to X$.

The (derived) fiber of the map $\mathscr{S}ect(X, V/G) \to \mathrm{Bun}_G$[7] over a field-valued point $\mathcal{E}_G \in \mathrm{Bun}_G(\kappa)$ is the derived scheme $R\Gamma(X_\kappa, V \times^G \mathcal{E}_G)$; note for contrast that the classical fiber of the map of classical stacks $\mathrm{Sect}(X_\kappa, V/G) \to \mathrm{Bun}_G$ is $H^0(X_\kappa, V \times^G \mathcal{E}_G)$. We spell out how $R\Gamma(X_\kappa, V \times^G \mathcal{E}_G)$ is viewed as a derived scheme:

(1) $R\Gamma(X_\kappa, V \times^G \mathcal{E}_G)$ is a connective (i.e., cohomology groups vanish in negative degrees) perfect cochain complex in the derived category of κ-modules.
(2) Its dual $R\Gamma(X_\kappa, V \times^G \mathcal{E}_G)^*$ is a connective (i.e., homology groups vanish in negative degrees) perfect *chain* complex in the derived category of κ-modules, which by the Dold-Kan correspondence may be viewed as an animated κ-module.
(3) The forgetful functor from animated κ-algebras to animated κ-modules admits a left adjoint, the derived symmetric algebra functor $\mathrm{Sym}^\bullet_\kappa$.
(4) The derived scheme $R\Gamma(X_\kappa, V \times^G \mathcal{E}_G)$ is the spectrum of $\mathrm{Sym}^\bullet_\kappa R\Gamma(X_\kappa, V \times^G \mathcal{E}_G)^*$.

[6] We caution that $\mathscr{M}ap_S(X, \mathscr{Y})$ has a different meaning than the relative mapping stack "$\mathbb{R}\,\mathrm{Map}_{\mathrm{dSt}/S}(X, \mathscr{Y})$" in *loc. cit.*.
[7] Here we are using Example 5.3 to identify $\mathscr{S}ect(X, BG)$, which is a priori a "derived version" of Bun_G, with Bun_G.

5.2.8. *Cotangent complexes.* We refer to [TV08, Toe10] for the theory of the cotangent complex to a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of derived stacks, denoted $\mathbf{L}_f$. The *tangent complex* to f is $\mathbf{T}_f := \underline{\mathrm{RHom}}_{\mathcal{O}_{\mathscr{X}}}(\mathbf{L}_f, \mathcal{O}_{\mathscr{X}})$. Sometimes these will be denoted $\mathbf{L}_{\mathscr{X}/\mathscr{Y}}$ and $\mathbf{T}_{\mathscr{X}/\mathscr{Y}}$ when the map is clear. When f is the structure morphism $f\colon \mathscr{X} \to \operatorname{Spec} k$, we abbreviate $\mathbf{T}_{\mathscr{X}} := \mathbf{T}_{\mathscr{X}/\operatorname{Spec} k}$ and $\mathbf{L}_{\mathscr{X}} := \mathbf{L}_{\mathscr{X}/\operatorname{Spec} k}$.

A useful characterization of the cotangent complex of $f\colon \mathscr{X} \to \operatorname{Spec} k$ is as follows [Toe09, p.37]. Let $A_\bullet$ be an animated k-algebra and recall that for any animated $A_\bullet$-module $M_\bullet$ there is an animated $A_\bullet$-algebra $A_\bullet \oplus M_\bullet$, which on homotopy groups is the square-zero extension of $\pi_*(A_\bullet)$ by $\pi_*(M_\bullet)$. Then for any map $a\colon \operatorname{Spec} A_\bullet \to \mathscr{X}$ and any animated $A_\bullet$-module $M_\bullet$, there is a natural equivalence between $\mathrm{RHom}_{A_\bullet-\mathrm{Mod}}(a^*\mathbf{L}_f, M_\bullet)$ and the homotopy fiber of $\mathscr{X}(A_\bullet \oplus M_\bullet) \to \mathscr{X}(A_\bullet)$ over $a \in \mathscr{X}(A_\bullet)$.

The following fundamental facts will be used frequently:

- For a sequence of morphisms $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$, there is an exact triangle in $\mathrm{QCoh}(\mathscr{X})$:
$$f^*\mathbf{L}_g \to \mathbf{L}_{g\circ f} \to \mathbf{L}_f.$$
- For a Cartesian square
$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Y}' \arrow[r, "g"] & \mathscr{Y} \end{tikzcd}$$
we have $(g')^*\mathbf{L}_f \xrightarrow{\sim} \mathbf{L}_{f'}$. Given compatible maps to a base derived stack $\mathscr{S}$, we then deduce an exact triangle
$$\mathbf{L}_{\mathscr{X}'/\mathscr{S}} \to (g')^*\mathbf{L}_{\mathscr{X}/\mathscr{S}} \oplus (f')^*\mathbf{L}_{\mathscr{Y}'/\mathscr{S}} \to (f \circ g')^*\mathbf{L}_{\mathscr{Y}/\mathscr{S}}.$$

Lemma 5.5. *Let $\mathcal{Y}$ be a noetherian derived Artin stack over k. Suppose that $\mathcal{Y}$ has cotangent complex concentrated in non-negative degrees. Then $\iota_{\mathcal{Y}}\colon (\mathcal{Y})_{\mathrm{cl}} \to \mathcal{Y}$ is an equivalence.*

Proof. Well-known; see [Lur19, Lemma 6.1.2.4] and [TV08, §2.2.2]. □

5.2.9. *Quasi-smoothness.* A key role is played by the notion of *quasi-smooth* derived Artin stacks, and more generally quasi-smooth morphisms. Recall that a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of derived Artin stacks is *quasi-smooth* if it is locally of finite presentation and the relative cotangent complex $\mathbf{L}_f$ is perfect of Tor-amplitude $[-1, \infty)$[8]. Given f locally of finite presentation with $\mathbf{L}_f$ a perfect complex, f is quasi-smooth if and only if the fiber of $\mathbf{L}_f$ at all geometric points is acyclic in (cohomological) degrees < -1 [AG15, §2.1]. This is the derived analogue of being LCI, and for that reason is also sometimes referred to as "derived LCI". In particular, a classical LCI morphism between classical stacks, regarded as derived stacks, is quasi-smooth.

The following facts are immediate from basic properties of the cotangent complex:

- The composition of quasi-smooth morphisms is quasi-smooth.
- The (derived) base change of any quasi-smooth morphism is quasi-smooth. Note that the classical analogue is completely false for classical LCI morphisms!

If $\mathscr{X} \to \operatorname{Spec} k$ is a quasi-smooth morphism, then we simply say that $\mathscr{X}$ *is quasi-smooth* in particular, a classical LCI stack over k is quasi-smooth when regarded as a derived stack.. As we shall see later, quasi-smooth derived Artin stacks are those to which we can naturally associate a virtual fundamental class, which is why this notion is important for us.

We recall for comparison that a morphism of derived stacks $f\colon \mathscr{X} \to \mathscr{Y}$ is *smooth* if it is locally of finite presentation and $\mathbf{L}_f$ is perfect of Tor-amplitude $[0, \infty)$. In particular, this includes smooth morphisms of classical stacks.

5.3. Tangent complexes to derived mapping stacks.

Lemma 5.6. *Let S be a derived stack over k with perfect cotangent complex and X be a smooth proper scheme over k with a map to S. Let $\mathscr{Y}$ be a finite type derived stack over S with perfect relative cotangent complex $\mathbf{L}_{\mathscr{Y}/S}$. Then the cotangent complex $\mathbf{L}_{\mathscr{M}aps_S(X,\mathscr{Y})}$ is perfect, and its pullback to any $R_\bullet$-point $f\colon X_{R_\bullet} \to \mathscr{Y}$, for*

[8]Here we are using *cohomological* grading (as opposed to the homological grading of [Kha19b]), so this means that $H^i(\mathbf{L}_f \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{E})$ vanishes for $i < -1$ for every discrete quasi-coherent sheaf $\mathcal{E}$ on $\mathscr{X}$.

any animated ring $R_\bullet$, is naturally in $R_\bullet$ isomorphic[9] *to* $R\,\mathrm{pr}_*(f^*\mathbf{L}_{\mathscr{Y}/S}\otimes\omega_{X/k})$*, where* pr *is the projection map* $X_{R_\bullet}\to\mathrm{Spec}\,R_\bullet$ *and* $\omega_{X/k}$ *is the dualizing sheaf of* X*.*

In particular, $\mathbf{T}_{\mathscr{M}aps_S(X,\mathscr{Y})}|_f$ *is naturally in* $R_\bullet$ *isomorphic to* $R\,\mathrm{pr}_*(f^*\mathbf{T}_{\mathscr{Y}/S})$*.*

Proof. We apply [HLP14, Proposition 5.1.10], which implies[10] that $\mathbf{L}_{\mathscr{M}aps_S(X,\mathscr{Y})}|_f$ is isomorphic to $\mathrm{pr}_+(f^*\mathbf{L}_{\mathscr{Y}/S})$, where pr_+ is the left adjoint to pr^*. Since X/k is smooth and proper, $\mathrm{pr}^!(-)=\omega_{X/k}\otimes\mathrm{pr}^*(-)$. The left adjoint of $\mathrm{pr}^!$ is pr_*, so the left adjoint of pr^* is $\mathrm{pr}_*(-\otimes\omega_{X/k})$.

The last sentence follows from applying Serre duality to the description of $\mathbf{L}_{\mathscr{M}aps_S(X,\mathscr{Y})}|_f$. □

Corollary 5.7. *Let X be a smooth proper scheme over k. Let $\mathscr{G}$ be a smooth (classical) gerbe over X. Then $\mathscr{S}ect(X,\mathscr{G})$ is isomorphic to its classical truncation* $\mathrm{Sect}(X,\mathscr{G})$*, which is smooth.*

Proof. Because a smooth gerbe is locally in the smooth topology isomorphic to the classifying stack of a group scheme, the relative tangent complex $\mathbf{T}_{\mathscr{G}/X}$ is concentrated in degree -1, hence for any section $f\colon X_{R_\bullet}\to\mathscr{G}$ the cohomology groups of $R\,\mathrm{pr}_*(f^*\mathbf{T}_{\mathscr{G}/X})$ are non-vanishing only in degrees $-1,0$. We conclude by applying Lemma 5.5. □

Example 5.8. Let X be a smooth proper scheme over k and $G\to X$ a smooth group scheme. Then $\mathbf{T}_{BG/X}\cong\mathrm{Lie}(G/X)[1]$. Lemma 5.6 implies that $\mathbf{L}_{\mathscr{S}ect(X,BG)}$ is perfect, and $\mathbf{T}_{\mathscr{S}ect(X,BG)}$ pulled back to $\mathrm{Spec}\,R_\bullet$ via a G-torsor $\mathcal{F}$ over $X_{R_\bullet}$ is isomorphic to $R\,\mathrm{pr}_*(\mathcal{F}\times^G\mathrm{Lie}(G/X))[1]$ naturally in $R_\bullet$.

Corollary 5.9. *Let S be a derived stack over k with perfect cotangent complex. Let X be a smooth proper scheme over k with a map to S. Suppose $\mu\colon\mathscr{Y}\to\mathscr{Y}'$ is a morphism of finite type derived stacks over S such that $\mathbf{L}_\mu$ is perfect. Then the induced map $\mu_*\colon\mathscr{M}aps_S(X,\mathscr{Y})\to\mathscr{M}aps_S(X,\mathscr{Y}')$ has perfect relative cotangent complex, and for any $R_\bullet$-point $f\colon X_{R_\bullet}\to\mathscr{Y}$, $\mathbf{T}_{\mu_*}|_f$ is isomorphic to $R\,\mathrm{pr}_*(f^*\mathbf{T}_\mu)$ naturally in $R_\bullet$.*

In particular, if X is a smooth projective curve and μ is smooth, then μ_* is quasi-smooth.

Proof. The perfectness of the cotangent complex for μ_* and description of the tangent complex follow from the functoriality of Lemma 5.6 with respect to $\mathscr{Y}$. The last sentence follows because in this situation, $R\,\mathrm{pr}_*$ has cohomological amplitude 1 and $\mathbf{T}_\mu$ is concentrated in degree ≤ 0, so $R\,\mathrm{pr}_*(\mathbf{T}_\mu)$ has cohomology concentrated in degrees ≤ 1. □

Example 5.10. Let X be a smooth proper scheme over k. Let G be a smooth group scheme over X and $V\to X$ a vector bundle that is a representation of G over X. Consider an $R_\bullet$-point of $\mathscr{S}ect(X,V/G)$, represented by a G-torsor $\mathcal{F}$ on $X_{R_\bullet}$ and $s\in R\Gamma(X_{R_\bullet},\mathcal{F}\times^G V)$.

We give a more concrete description of various tangent complexes in this situation.

(1) The tangent complex to $\mathscr{S}ect(X,V/G)$ at the $R_\bullet$-point $(\mathcal{F},s)$ is naturally in $R_\bullet$ isomorphic to

$$R\,\mathrm{pr}_*(\underbrace{\mathcal{F}\times^G\mathrm{Lie}(G/X)}_{\deg -1}\xrightarrow{\cdot s}\underbrace{\mathcal{F}\times^G V}_{\deg 0})$$

where the meaning of the differential $\cdot s$ is as in §3.4, and $\mathrm{pr}\colon X_{R_\bullet}\to\mathrm{Spec}\,R_\bullet$ is the projection map.

(2) The map of tangent complexes induced by $\mathscr{S}ect(X,V/G)\xrightarrow{\pi}\mathrm{Bun}_G$ pulled back to $\mathrm{Spec}\,R_\bullet$ via $(\mathcal{F},s)$ is naturally in $R_\bullet$ isomorphic to

$$R\,\mathrm{pr}_*(\underbrace{\mathcal{F}\times^G\mathrm{Lie}(G/X)}_{\deg -1}\xrightarrow{\cdot s}\underbrace{\mathcal{F}\times^G V}_{\deg 0})\xrightarrow{t}R\,\mathrm{pr}_*(\underbrace{\mathcal{F}\times^G\mathrm{Lie}(G/X)}_{\deg -1})\tag{5.1}$$

where the map t is induced by the truncation of complexes.

(3) If X is a smooth projective curve, $\mathscr{S}ect(X,V/G)\xrightarrow{\pi}\mathrm{Bun}_G$ is quasi-smooth, and $\mathscr{S}ect(X,V/G)$ is quasi-smooth.

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[9] By this we mean that there is natural transformation between the two functors; informally speaking, that the isomorphisms base change coherently along $R'_\bullet\to R_\bullet$.

[10] Here we use the presentation of $\mathscr{M}aps_S(X,\mathscr{Y})$ as the homotopy fiber of $\mathscr{M}ap(X,\mathscr{Y})\to\mathscr{M}ap(X,S)$ over the given point of $\mathscr{M}ap_k(X,S)$.

5.4. **(Un)derived** Hk_G^r **and** Sht_G^r**.** We now resume our convention that X is a smooth projective curve over k. For a smooth gerbe $\mathscr{G} \to X$, Corollary 5.7 implies that the derived stack $\mathscr{S}ect(X, \mathscr{G})$ is isomorphic to its classical truncation. Next we define and analyze derived Hecke stacks for gerbes of unitary type and $B\,\mathrm{GL}(n)'$. They will also turn out to be isomorphic to their classical truncations, a non-trivial fact that is needed later to compute cycle classes in explicit terms.

Definition 5.11 (Derived Hecke stacks for unitary type gerbes)**.** Let $\mathscr{G} \cong \prod_{Y_\alpha} R_{Y_\alpha/X} BU(n_\alpha)_{\mathfrak{L}_\alpha}$ be a gerbe of unitary type. We set $Y' := Y \times_X X' \cong \coprod Y'_\alpha$. Then we define $\mathscr{H}k^1_{\mathscr{G}}$ to be the derived stack with $R_\bullet$-points being the anima of:

- $y' \in Y'(R_\bullet)$,
- Hermitian bundles $(\mathcal{F}_0, h_0)$ and $(\mathcal{F}_1, h_1)$, with each $\mathcal{F}_i$ a vector bundle on Y', of rank n_α on Y'_α, and h_i a $\mathfrak{L}_\alpha$-twisted Hermitian structure on $\mathcal{F}_i$, and
- a diagram

$$\begin{tikzcd} & \mathcal{F}^\flat_{1/2} \arrow[dl, hook', "h^{\leftarrow}"'] \arrow[dr, hook, "h^{\rightarrow}"] & \\ \mathcal{F} & & \mathcal{F}_1 \end{tikzcd}$$

where $\mathcal{F}_{1/2}$ is a vector bundle on Y', of rank n_α on Y'_α, and such that $\mathrm{cone}(h^{\leftarrow})$ and $\mathrm{cone}(h^{\rightarrow})$ are supported on $\Gamma_{x'}$ and locally isomorphic to $R_\bullet$ (as $R_\bullet$-modules).

We define $\mathscr{H}k^r_{\mathscr{G}}$ to be the r-fold (derived) fibered product

$$\mathscr{H}k^r_{\mathscr{G}} := \mathscr{H}k^1_{\mathscr{G}} \times_{\mathrm{Bun}_{\mathscr{G}}} \mathscr{H}k^1_{\mathscr{G}} \times_{\mathrm{Bun}_{\mathscr{G}}} \cdots \times_{\mathrm{Bun}_{\mathscr{G}}} \mathscr{H}k^1_{\mathscr{G}} \tag{5.2}$$

where on the i^{th} factor of $\mathscr{H}k^1_{\mathscr{G}}$, parametrizing $\mathcal{F}_{i-1} \leftarrow \mathcal{F}^\flat_{i-1/2} \to \mathcal{F}_i$, the left and right maps to $\mathrm{Bun}_{\mathscr{G}}$ project to $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ respectively. A point of $\mathscr{H}k^r_{\mathscr{G}}$ will be denoted

$$\begin{tikzcd}[column sep=small] & \mathcal{F}^\flat_{1/2} \arrow[dl] \arrow[dr] & & \cdots & & \mathcal{F}^\flat_{r-1/2} \arrow[dl] \arrow[dr] & \\ \mathcal{F}_0 \arrow[rr, dashed, "f_0"] & & \mathcal{F}_1 \arrow[r, dashed] & \cdots \arrow[r, dashed] & \mathcal{F}_{r-1} \arrow[rr, dashed, "f_{r-1}"] & & \mathcal{F}_r \end{tikzcd}$$

and the projection map to $\mathcal{F}_i$ denoted $\mathrm{pr}_i \colon \mathscr{H}k^r_{\mathscr{G}} \to \mathrm{Bun}_{\mathscr{G}}$.

Definition 5.12 (Derived Hecke stacks for $\mathrm{GL}(n)'$)**.** We define $\mathscr{H}k^1_{\mathrm{GL}(n)'}$ to be the derived stack with $R_\bullet$-points being the anima of $x' \in X'(R_\bullet)$, $\mathcal{F}_0, \mathcal{F}_1 \in \mathrm{Bun}_{\mathrm{GL}(n)'}(R_\bullet)$, and a rank n vector bundle $\mathcal{F}^\flat_{1/2}$ on $X'_{R_\bullet}$ plus a diagram

$$\begin{tikzcd} & \mathcal{F}^\flat_{1/2} \arrow[dl, hook', "h^{\leftarrow}"'] \arrow[dr, hook, "h^{\rightarrow}"] & \\ \mathrm{V}(\mathcal{F}_0) \arrow[rr, dashed, "f_0"] & & \mathrm{V}(\mathcal{F}_1) \end{tikzcd}$$

such that $\mathrm{cone}(h^{\leftarrow})$ and $\mathrm{cone}(h^{\rightarrow})$ are supported on $\Gamma_{x'}$ and locally isomorphic to $R_\bullet$ (as $R_\bullet$-modules). Here the associated vector bundles are formed with respect to the standard embedding.

We define $\mathscr{H}k^r_{\mathrm{GL}(n)'}$ to be the r-fold (derived) fibered product

$$\mathscr{H}k^r_{\mathrm{GL}(n)'} := \mathscr{H}k^1_{\mathrm{GL}(n)'} \times_{\mathrm{Bun}_{\mathrm{GL}(n)'}} \mathscr{H}k^1_{\mathrm{GL}(n)'} \times_{\mathrm{Bun}_{\mathrm{GL}(n)'}} \cdots \times_{\mathrm{Bun}_{\mathrm{GL}(n)'}} \mathscr{H}k^1_{\mathrm{GL}(n)'} \tag{5.3}$$

where on the i^{th} factor of $\mathscr{H}k^1_{\mathrm{GL}(n)'}$, parametrizing $\mathrm{V}(\mathcal{F}_{i-1}) \leftarrow \mathcal{F}^\flat_{i-1/2} \to \mathrm{V}(\mathcal{F}_i)$, the left and right maps to $\mathrm{Bun}_{\mathrm{GL}(n)'}$ project to $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ respectively.

In order to make the notation more uniform, we will denote a gerbe of unitary type by $BG \to X$ (even if it does not arise as the classifying stack of a group scheme G). We will write $\mathscr{H}k^r_G$ for $\mathscr{H}k^r_{BG}$ (in the unitary type case) or if $G = \mathrm{GL}(n)'$. It is immediate from the definition that the classical truncation of $\mathscr{H}k^r_{\mathscr{G}}$ is the $\mathrm{Hk}^r_{\mathscr{G}}$ from §3.1.3 (for BG of unitary type) or Definition 2.1 (for $G = \mathrm{GL}(n)'$). We prove below that the canonical map $\mathrm{Hk}^r_G \to \mathscr{H}k^r_G$ is an isomorphism in both cases.

Lemma 5.13. *Let BG be of unitary type or $B\,\mathrm{GL}(n)'$. Then:*

(1) $\iota\colon \mathrm{Hk}_G^r \to \mathscr{H}\!k_G^r$ *is an isomorphism.*
(2) The following diagram of classical stacks is derived Cartesian

$$\begin{tikzcd} \mathrm{Sht}_G^r \arrow[r] \arrow[d] & \mathrm{Hk}_G^r \arrow[d, "(\mathrm{pr}_0, \mathrm{pr}_r)"] \\ \mathrm{Bun}_G \arrow[r, "(\mathrm{Id}, \mathrm{Frob})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{tikzcd}$$

Proof. (1) We argue by induction on r that $(\mathrm{pr}_r, s)\colon \mathscr{H}\!k_G^r \to \mathrm{Bun}_G \times X'$ has perfect cotangent complex concentrated in degree 0. Once this is established, we will know that the cotangent complex of $\mathscr{H}\!k_G^r$ is concentrated in non-negative degrees at all geometric points, hence $\mathscr{H}\!k_G^r$ is isomorphic to its classical truncation by Lemma 5.5.

For the base case, by the same argument as in [FYZ21, Lemma 6.9] (for $G = U(n)$) or Lemma 2.6 (for $G = \mathrm{GL}(n)'$), the map $\mathscr{H}\!k_G^1 \xrightarrow{(\mathrm{pr}_1, s)} \mathscr{M}\!ap(X, BG) \times X$ is a projective space bundle, and $\mathscr{M}\!ap(X, BG) = \mathrm{Bun}_G$ by Corollary 5.7. This shows that $\mathbf{L}_{(\mathrm{pr}_1, s)}$ is perfect and concentrated in degree 0. For the inductive step, we consider the Cartesian square

$$\begin{tikzcd} \mathscr{H}\!k_G^r \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{pr}_r"] & \mathscr{H}\!k_G^{r-1} \arrow[r, "\mathrm{pr}_{r-1}"] \arrow[d, "\mathrm{pr}_{r-2}"] & \mathrm{Bun}_G \\ \mathscr{H}\!k_G^1 \arrow[r, "\mathrm{pr}_1"] & \mathrm{Bun}_G & \end{tikzcd}$$

Using the behavior of cotangent complexes in Cartesian squares, we deduce that the cotangent complex of $\mathscr{H}\!k_G^r \to \mathscr{H}\!k_G^{r-1}$ is also perfect and concentrated in degree 0. Then applying the inductive hypothesis for pr_{r-1} and the distinguished triangle of cotangent complexes for the upper horizontal composition completes the induction.

(2) Since Bun_G is smooth, it suffices to show that the maps $(\mathrm{pr}_0, \mathrm{pr}_r)$ and $(\mathrm{Id}, \mathrm{Frob})$ are transversal. The differential of Frob is zero, so this follows from the smoothness of pr_r ([FYZ21, Lemma 6.9(1)] for BG of unitary type, or Lemma 2.6 for $G = \mathrm{GL}(n)'$). $\square$

Thanks to Lemma 5.13, we may and do write Hk_G^r instead of $\mathscr{H}\!k_G^r$ in the sequel.

5.5. **Derived Hitchin stacks.** We now define derived versions of the Hitchin stacks introduced in §3. Unlike Bun_G and Hk_G^r, these will be genuinely non-classical in general.

Definition 5.14. Let $BH_1 \to B\,\mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be homomorphisms of smooth gerbes over X. We define the *derived Hitchin stack* $\mathscr{M}_{H_1,H_2}$ to be the derived stack taking $R_\bullet$ to the anima of data:

- $\mathcal{E} \in \mathscr{S}\!ect(X, BH_1)(R_\bullet)$,
- $\mathcal{F} \in \mathscr{S}\!ect(X, BH_2)(R_\bullet)$.
- A derived section t of $\underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathrm{V}(\mathcal{F}))$ on $X'_{R_\bullet}$.

As the map $\mathscr{M}_{H_1,H_2} \to \mathscr{S}\!ect(X, BH_1 \times BH_2)$ is evidently representable, this is a derived Artin stack. It is immediate from the definition that the classical truncation of $\mathscr{M}_{H_1,H_2}$ is $\mathcal{M}_{H_1,H_2}$.

We define $\mathscr{M}^\circ_{H_1,H_2} \subset \mathscr{M}_{H_1,H_2}$ to be the open derived substack whose classical truncation is $\mathcal{M}^\circ_{H_1,H_2} \subset \mathcal{M}_{H_1,H_2}$ (cf. [TV08, §2.2.2] for the notion of Zariski open immersion in derived algebraic geometry).

Example 5.15. If BH_1 and BH_2 are the classifying stacks of group schemes H_1 and H_2 over X with homomorphisms $H_1 \to \mathrm{GL}(m)'$ and $H_2 \to \mathrm{GL}(n)'$, then $\mathscr{M}_{H_1,H_2}$ is the instance of $\mathscr{S}\!ect(X, V/G)$ where $G = H_1 \times H_2$ and V is the vector bundle $\nu_* \underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(\mathrm{Std}_{\mathrm{GL}(m)'}, \mathrm{Std}_{\mathrm{GL}(n)'}) \cong \nu_*(\mathcal{O}_{X'}^{\oplus mn})$ over X.

Remark 5.16. We indicate a more concrete interpretation of the derived structure on $\mathscr{M}_{\mathrm{GL}(m)', U(n)}$. Abbreviate $\mathscr{M} := \mathscr{M}_{\mathrm{GL}(m)', U(n)}$.

We have a tautological bundle $\mathcal{H}$ over $\mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)} \times X'$ whose restriction to $\{(\mathcal{E}, \mathcal{F})\} \times X'$ is $\underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F})$. Let $p_{\mathrm{Bun}} : \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)} \times X' \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)}$ be the projection. Then there exists a Zariski cover $\coprod \mathcal{U}_\alpha \to \mathrm{Bun}_{\mathrm{GL}(m)'} \times \mathrm{Bun}_{U(n)}$ on which $(Rp_{\mathrm{Bun},*}\mathcal{H})|_{\mathcal{U}_\alpha}$ can be represented by a two-step perfect complex $\mathcal{K}^0 \xrightarrow{d} \mathcal{K}^1$ over $\mathcal{U}_\alpha$ in degrees 0 and 1. Then $\mathscr{M}|_{\mathcal{U}_\alpha}$ is isomorphic to the *derived* fiber product

$$\begin{tikzcd} \mathscr{M}|_{\mathcal{U}_\alpha} \arrow[r] \arrow[d] & \mathcal{K}^0 \arrow[d, "d"] \\ 0_{\mathcal{U}_\alpha} \arrow[r, hook] & \mathcal{K}^1 \end{tikzcd} \tag{5.4}$$

Here we are viewing $\mathcal{K}^i$ as the total space of the vector bundle $\mathcal{K}^i$, and $0_{\mathcal{U}_\alpha}$ denotes the zero section in $\mathcal{K}^1$.

This leads to another explicit way of describing the derived stack $\mathscr{M}$. As discussed above, the complex $(Rp_{\mathrm{Bun},*}\mathcal{H})$ is Zariski locally represented by a perfect cochain complex $\mathcal{K}^\bullet$ on Bun_G, for $G = \mathrm{GL}(m)' \times U(n)$, which is connective (i.e. its cohomology sheaves are concentrated in non-negative degrees). Then its dual is Zariski locally a connective perfect *chain* complex of quasi-coherent sheaves on Bun_G. Then we may apply the (derived) symmetric algebra functor [Kha19a, §1] to obtain an animated algebra in quasicoherent sheaves on Bun_G, and $\mathscr{M}$ is the relative spectrum.

Corollary 5.17. *Let $BH_1 \to \mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be as in Definition 5.14.*

(1) For any animated ring $R_\bullet$, the tangent complex of the morphism $\mathscr{M}_{H_1,H_2} \xrightarrow{\pi} \mathrm{Bun}_{H_1} \times \mathrm{Bun}_{H_2}$ at $(\mathcal{E}, \mathcal{F}, t \in R\Gamma(X'_{R_\bullet}, \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathrm{V}(\mathcal{F})))) \in \mathscr{M}_{H_1,H_2}(R_\bullet)$ is naturally in $R_\bullet$ isomorphic to

$$R\,\mathrm{pr}_*(\underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathrm{V}(\mathcal{F}))).$$

where $\mathrm{pr}\colon X'_{R_\bullet} \to \mathrm{Spec}\ R_\bullet$ is the projection map. In particular, π is quasi-smooth, hence $\mathscr{M}_{H_1,H_2}$ is quasi-smooth.

(2) If $BH_2 \xrightarrow{=} B\,\mathrm{GL}(n)'$, then $\mathscr{M}^\circ_{H_1,H_2}$ is smooth and the natural map $\iota\colon \mathcal{M}^\circ_{H_1,H_2} \to \mathscr{M}^\circ_{H_1,H_2}$ is an isomorphism.

Proof. (1) is a special case of Corollary 5.9 applied with $\mathscr{Y}' = BH_1 \times_X BH_2$ and $\mathscr{Y} \to \mathscr{Y}'$ equal to pullback of $\underline{\mathrm{Hom}}(\mathrm{Std}_{\mathrm{GL}(m)'}, \mathrm{Std}_{\mathrm{GL}(n)'})/(\mathrm{GL}(m)' \times_X \mathrm{GL}(n)') \to B\,\mathrm{GL}(m)' \times_X B\,\mathrm{GL}(n)'$ along the given map $\mathscr{Y}' \to B\,\mathrm{GL}(m)' \times_X B\,\mathrm{GL}(n)'$, where $\mathrm{Std}_{\mathrm{GL}(m)'}$ is the standard representation of $\mathrm{GL}(m)'$.

For (2), we apply Lemma 5.5 and the calculations of §3.4 and Proposition 3.11(1). ☐

Corollary 5.18. *(1) Let BH_1 be a smooth gerbe over X and $BH_1 \to B\,\mathrm{GL}(m)'$ be any morphism. Then the classical truncation map $\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'} \to \mathscr{M}^\circ_{H_1,\mathrm{GL}(n)'}$ is an isomorphism, and both stacks are smooth.*

(2) Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$. Let $\mathcal{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}} \subset \mathcal{M}_{H_1,U(n),\mathfrak{L}}$ be the preimage of $\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}$ under the Hitchin fibration (see §3.3), and $\mathscr{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}} \subset \mathscr{M}_{H_1,U(n),\mathfrak{L}}$ be the corresponding open derived substack. Then the classical truncation map $\mathcal{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}} \to \mathscr{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}}$ is an isomorphism, and both stacks are smooth.

Proof. Both statements follows from Lemma 5.5 plus the description of the cotangent complexes in Corollary 5.17, which implies that they are perfect with cohomology groups are concentrated in degrees ≤ 0 by the earlier computations in Proposition 3.11. ☐

Remark 5.19. It is important that we restrict to the injective locus for Corollary 5.18(1). The statement would not be true for $\mathscr{M}_{H_1,\mathrm{GL}(n)'}$ in place of $\mathscr{M}^\circ_{H_1,\mathrm{GL}(n)'}$. Furthermore, in part (2), we would not have been able to make the same argument with $\mathscr{M}^\circ_{H_1,U(n),\mathfrak{L}}$ in place of $\mathscr{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}}$.

5.6. **Derived Hecke stack for derived Hitchin spaces.** We shall define derived Hecke stacks $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}}$ and $\mathrm{Hk}^r_{\mathscr{M}^\circ_{H_1,H_2}}$ whose classical truncation recovers the classical stacks $\mathrm{Hk}^r_{\mathcal{M}_{H_1,H_2}}$ and $\mathrm{Hk}^r_{\mathcal{M}^\circ_{H_1,H_2}}$.

Definition 5.20. Let $BH_1 \to \mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be as in Definition 5.14. Assume that BH_2 is unitary type or $B\,\mathrm{GL}(n)'$, so that $\mathrm{Hk}^r_{H_2}$ is defined, and that its morphism to $B\,\mathrm{GL}(n)'$ is the standard map. For $\mathscr{M} = \mathscr{M}_{H_1,H_2}$ or $\mathscr{M}^\circ_{H_1,H_2}$, we define the derived stack $\mathrm{Hk}^1_{\mathscr{M}}$ sending an animated ring $R_\bullet$ to the anima of

$$\begin{tikzcd} & \mathcal{F}^\flat_{1/2} \arrow[dl, hook] \arrow[dr, hook] & \\ \mathrm{V}(\mathcal{F}_0) \arrow[rr, dashed, "f_0"] & & \mathrm{V}(\mathcal{F}_1) \end{tikzcd} \in \mathrm{Hk}^1_{H_2}(R_\bullet)$$

and $t \in \mathscr{S}ect(X_{R_\bullet}, \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^\flat_{1/2}))$. We denote such an $R_\bullet$-point of $\mathrm{Hk}^1_{\mathscr{M}}$ by the diagram

$$\begin{tikzcd}
& \mathrm{V}(\mathcal{E}) \arrow[d, "t^\flat_{1/2}"] \arrow[ddl, bend right, "t_0"'] \arrow[ddr, bend left, "t_1"] & \\
& \mathcal{F}^\flat_{1/2} \arrow[dl, hook'] \arrow[dr, hook] & \\
\mathrm{V}(\mathcal{F}_0) \arrow[rr, dashed, "f_0"] & & \mathrm{V}(\mathcal{F}_1)
\end{tikzcd}$$

Then we define $\mathrm{Hk}^r_{\mathscr{M}}$ as the r-fold derived fibered product

$$\mathrm{Hk}^r_{\mathscr{M}} := \underbrace{\mathrm{Hk}^1_{\mathscr{M}} \times_{\mathscr{M}} \mathrm{Hk}^1_{\mathscr{M}} \times_{\mathscr{M}} \ldots \times_{\mathscr{M}} \mathrm{Hk}^1_{\mathscr{M}}}_{r \text{ times}} \tag{5.5}$$

where the maps are as in Definition 5.11.

Again it is clear that the classical truncation of $\mathrm{Hk}^r_{\mathscr{M}}$ is $\mathrm{Hk}^r_{\mathcal{M}}$ where $\mathcal{M}$ is the classical truncation of $\mathscr{M}$, i.e. $\mathcal{M} = \mathcal{M}_{H_1,H_2}$ or $\mathcal{M}^\circ_{H_1,H_2}$.

Lemma 5.21. *Let BH_1 be a smooth gerbe over X with a map $BH_1 \to B\,\mathrm{GL}(m)'$ and BH_2 a gerbe of unitary type or $B\,\mathrm{GL}(n)'$, equipped with the standard map to $B\,\mathrm{GL}(n)'$. Then the morphism $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}} \xrightarrow{\pi} \mathrm{Bun}_{H_1} \times \mathrm{Hk}^r_{H_2}$ is quasi-smooth. For any animated ring $R_\bullet$, the tangent complex of π at any $(\mathcal{E}, \{x'_i\}, \{\mathcal{F}_i\}, \{t_i\}) \in \mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}}(R_\bullet)$ is naturally in $R_\bullet$ isomorphic to (using notation from Definition 5.20)*

$$R\,\mathrm{pr}_* \left(\underbrace{\bigoplus_{i=1}^{r} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^\flat_{i-1/2})}_{\deg 0} \to \underbrace{\bigoplus_{i=1}^{r-1} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathrm{V}(\mathcal{F}_i))}_{\deg 1} \right)$$

where $\mathrm{pr}\colon X'_{R_\bullet} \to \mathrm{Spec}\ R_\bullet$ is the projection, and the differential is induced by the map

$$\bigoplus_{i=1}^{r} \mathcal{F}^\flat_{i-1/2} \to \bigoplus_{i=1}^{r-1} \mathrm{V}(\mathcal{F}_i) \tag{5.6}$$

sending $(v_{1/2}, \ldots, v_{r-1/2}) \mapsto (v_{1/2} - v_{3/2}, v_{3/2} - v_{5/2}, \ldots, v_{r-3/2} - v_{r-1/2})$, using the given embeddings $\mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathrm{V}(\mathcal{F}_{i-1})$ and $\mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathrm{V}(\mathcal{F}_i)$. In particular, $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}}$ is quasi-smooth.

Proof. The case $r = 1$ is a direct consequence of the definition and Lemma 5.6. Abbreviate $\mathscr{M} = \mathscr{M}_{H_1,H_2}$. Consider an $R_\bullet$-point of $\mathrm{Hk}^r_{\mathscr{M}}$, which is represented by a diagram

$$\begin{tikzcd}[column sep=small]
& & & \mathrm{V}(\mathcal{E}) \arrow[dll, "t_0"'] \arrow[drr, "t_{r-1}"] & & & \\
& \mathcal{F}^\flat_{1/2} \arrow[dl, hook'] \arrow[dr, hook] & & \cdots & & \mathcal{F}^\flat_{r-1/2} \arrow[dl, hook'] \arrow[dr, hook] & \\
\mathrm{V}(\mathcal{F}_0) & & \mathrm{V}(\mathcal{F}_1) & \cdots & \mathrm{V}(\mathcal{F}_{r-1}) & & \mathrm{V}(\mathcal{F}_r)
\end{tikzcd} \tag{5.7}$$

The presentation of $\mathrm{Hk}^r_{\mathscr{M}}$ from (5.5) induces an exact triangle

$$\mathbf{T}_{\mathrm{Hk}^r_{\mathscr{M}}} \longrightarrow \mathbf{T}^{\oplus r}_{\mathrm{Hk}^1_{\mathscr{M}}}|_{\mathrm{Hk}^r_{\mathscr{M}}} \longrightarrow \mathbf{T}^{\oplus(r-1)}_{\mathscr{M}}|_{\mathrm{Hk}^r_{\mathscr{M}}}$$

Similarly, the presentations (5.2),(5.3) induces an exact triangle

$$\mathbf{T}_{\mathrm{Bun}_{H_1} \times \mathrm{Hk}^r_{H_2}} \longrightarrow \mathbf{T}^{\oplus r}_{\mathrm{Bun}_{H_1} \times \mathrm{Hk}^1_{H_2}}|_{\mathrm{Bun}_{H_1} \times \mathrm{Hk}^r_{H_2}} \longrightarrow \mathbf{T}^{\oplus(r-1)}_{\mathrm{Bun}_{H_1} \times \mathrm{Bun}_{H_2}}|_{\mathrm{Bun}_{H_1} \times \mathrm{Hk}^r_{H_2}}$$

So the (derived) fiber of $\mathbf{T}_{\mathrm{Hk}^r_{\mathscr{M}}} \to \mathbf{T}_{\mathrm{Hk}^r_{H_2}}|_{\mathrm{Hk}^r_{\mathscr{M}}}$ is the limit of the diagram

$$\begin{tikzcd}
\mathbf{T}^{\oplus r}_{\mathrm{Hk}^1_{\mathscr{M}}}|_{\mathrm{Hk}^r_{\mathscr{M}}} \arrow[r] \arrow[d] & \mathbf{T}^{\oplus(r-1)}_{\mathscr{M}}|_{\mathrm{Hk}^r_{\mathscr{M}}} \arrow[d] \\
\mathbf{T}^{\oplus r}_{\mathrm{Bun}_{H_1} \times \mathrm{Hk}^1_{H_2}}|_{\mathrm{Hk}^r_{\mathscr{M}}} \arrow[r] & \mathbf{T}^{\oplus(r-1)}_{\mathrm{Bun}_{H_1} \times \mathrm{Bun}_{H_2}}|_{\mathrm{Hk}^r_{\mathscr{M}}}
\end{tikzcd}$$

This limit may be calculated by first forming vertical fibers along the columns, and then taking the horizontal fiber. The vertical fibers restricted to the given $R_\bullet$-point are described by Corollary 5.17:

$$R\,\mathrm{pr}_*\left(\bigoplus_{i=1}^r \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^\flat_{i-1/2})\right) \longrightarrow R\,\mathrm{pr}_*\left(\bigoplus_{i=1}^{r-1} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathrm{V}(\mathcal{F}_i))\right)$$

In particular, as the map (5.6) has torsion cokernel, the fiber of this map is concentrated in degrees ≤ 1. $\square$

Corollary 5.22. *(1) Let BH_1 be a smooth gerbe over X and $BH_1 \to B\,\mathrm{GL}(m)'$ be any morphism. Then the classical truncation map $\mathrm{Hk}^1_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}} \to \mathrm{Hk}^1_{\mathscr{M}^\circ_{H_1,\mathrm{GL}(n)'}}$ is an isomorphism, and both stacks are smooth.*

(2) Let $BH_1 \xrightarrow{=} B\,\mathrm{GL}(m)'$. Let $\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}}} \subset \mathrm{Hk}^1_{\mathcal{M}_{H_1,U(n),\mathfrak{L}}}$ be the preimage of $\mathcal{A}^{\mathrm{ns}}_{H_1,\mathfrak{L}}$ (under the Hitchin fibration composed with pr_1) , and $\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}}} \subset \mathrm{Hk}^1_{\mathscr{M}_{H_1,U(n),\mathfrak{L}}}$ be the corresponding open derived substack. Then the classical truncation map $\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}}} \to \mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}_{H_1,U(n),\mathfrak{L}}}$ is an isomorphism, and both stacks are smooth.

Proof. Repeat the same argument of Corollary 5.18, but using the proofs of Lemmas 3.14 and 3.15 instead of Proposition 3.11. $\square$

5.7. **Derived shtukas for derived Hitchin spaces.** We now introduce derived stacks of shtukas $\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}$ and $\mathrm{Sht}^r_{\mathscr{M}^\circ_{H_1,H_2}}$, whose classical truncation recovers the classical stacks $\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}$ and $\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,H_2}}$ from §3.

Definition 5.23. Let H_1 and H_2 be as in Definition 5.20. Let $\mathscr{M} = \mathscr{M}_{H_1,H_2}$ or $\mathscr{M}^\circ_{H_1,H_2}$. We define $\mathrm{Sht}^r_{\mathscr{M}}$ by the (homotopy) Cartesian diagram

$$\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathscr{M}} \arrow[d, "\mathrm{pr}_0 \times \mathrm{pr}_r"] \\
\mathscr{M} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathscr{M} \times \mathscr{M}
\end{tikzcd} \tag{5.8}$$

We are primarily interested in the case where $BH_2 = BU(n)_{\mathfrak{L}}$ (although we will make some remarks on the unitary type case below). In order to study the tangent complex of $\mathrm{Sht}^r_{\mathscr{M}}$ in this case, we introduce a vector bundle on $\mathrm{Hk}^r_{U(n),\mathfrak{L}}$.

5.7.1. *Excess bundle.* Let $\mathcal{E}$ be a vector bundle on X'. We denote by $\mathcal{V}^r_{\mathcal{E}}$ the rank r tautological vector bundle over $\mathrm{Hk}^r_{U(n),\mathfrak{L}}$ whose fiber at $(\{x'_j\}, \{\mathcal{F}_j, h_j\})$ is the cokernel of the map

$$\bigoplus_{i=1}^r \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^\flat_{i-1/2}) \to \bigoplus_{i=1}^r \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_i) \tag{5.9}$$

given by

$$(v_{1/2}, v_{3/2}, \cdots, v_{r-1/2}) \mapsto (v_{1/2} - v_{3/2}, v_{3/2} - v_{5/2}, \cdots, v_{r-3/2} - v_{r-1/2}, v_{r-1/2}).$$

Here we use the natural inclusions $\mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathcal{F}_{i-1}$ and $\mathcal{F}^\flat_{i-1/2} \hookrightarrow \mathcal{F}_i$. (Note that as (5.9) is injective, the cone coincides with the cokernel, which is a torsion sheaf on X'.)

We use the same notation $\mathcal{V}^r_{\mathcal{E}}$ to denote the pullback of $\mathcal{V}^r_{\mathcal{E}}$ to $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$. In the future we will typically consider the latter object. We define $\mathcal{V}^r$ to be the bundle on $\mathrm{Bun}_{\mathrm{GL}(m)'}(k) \times \mathrm{Sht}^r_{U(n),\mathfrak{L}}$ whose restriction to $\{\mathcal{E}\} \times \mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is $\mathcal{V}^r_{\mathcal{E}}$.

The following lemma is easily checked.

Lemma 5.24. *The bundle $\mathcal{V}^r_{\mathcal{E}}$ on $\mathrm{Hk}^r_{U(n),\mathfrak{L}}$ carries a filtration with associated graded*

$$\bigoplus_{i=1}^{r} p_i^* \sigma^* \mathcal{E}_i^* \otimes \ell_i.$$

Here ℓ_i are the tautological bundles over $\mathrm{Hk}^r_{U(n),\mathfrak{L}}$ introduced in §4.3. In particular, we have

$$\prod_{i=1}^{r} c_m(p_i^* \sigma^* \mathcal{E}^* \otimes \ell_i) = c_{mr}(\mathcal{V}^r_{\mathcal{E}}) \in \mathrm{Ch}^{mr}(\mathrm{Hk}^r_{U(n),\mathfrak{L}}). \tag{5.10}$$

Lemma 5.25. *Let BH_1 be a smooth gerbe over X with a map $BH_1 \to B\,\mathrm{GL}(m)'$ and $BH_2 = BU(n)_{\mathfrak{L}} \to B\,\mathrm{GL}(n)'$ the standard map. Abbreviate $\mathscr{M} := \mathscr{M}_{H_1,H_2}$. Then the relative tangent complex for the map $\mathrm{Sht}^r_{\mathscr{M}} \xrightarrow{\pi} \mathrm{Bun}_{H_1}(k) \times \mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is perfect, and for $\mathcal{E} \in \mathrm{Bun}_{H_1}(k)$ we have*

$$\mathbf{T}_{\pi}|_{\pi^{-1}(\{\mathcal{E}\} \times \mathrm{Sht}^r_{U(n),\mathfrak{L}})} \cong \pi^* \mathcal{V}^r_{\mathrm{V}(\mathcal{E})}[-1].$$

In particular, $\mathrm{Sht}^r_{\mathscr{M}} \to \mathrm{Sht}^r_{U(n),\mathfrak{L}}$ is quasi-smooth, so $\mathrm{Sht}^r_{\mathscr{M}}$ is quasi-smooth.

Proof. We calculate the tangent complex of $\mathrm{Sht}^r_{\mathscr{M}}$ using the presentation (5.8). Consider an $R_\bullet$-point of $\mathrm{Sht}^r_{\mathscr{M}}$, represented by the data $(\underline{x}', \mathcal{E}, \mathcal{F}_0, \mathcal{F}_1, \ldots \mathcal{F}_r)$ and a diagram

$$\begin{tikzcd}
& & & \mathrm{V}(\mathcal{E}) \arrow[dll, "t_0"'] \arrow[drr, "t_{r-1}"] \arrow[rrrr, equal] & & & & \mathrm{V}({}^\tau\mathcal{E}) \arrow[dd, "{}^\tau(i_0 \circ t_0)"] \\
& \mathcal{F}^\flat_{1/2} \arrow[dl, hook', "i_0"'] \arrow[dr, hook] & & \cdots & & \mathcal{F}^\flat_{r-1/2} \arrow[dl, hook'] \arrow[dr, hook] & & \\
\mathcal{F}_0 & & \mathcal{F}_1 & \cdots & \mathcal{F}_{r-1} & & \mathcal{F}_r \arrow[r, "\sim"] & {}^\tau\mathcal{F}_0
\end{tikzcd} \tag{5.11}$$

By the behavior of cotangent complexes in Cartesian squares, we see from Corollary 5.17 and Lemma 5.21 that $\mathbf{L}_\pi$ is perfect, and the tangent complex of $\mathrm{Sht}^r_{\mathscr{M}}$ is the derived fiber (i.e. cone shifted by 1) of the map

$$\mathbf{T}_{\mathrm{Hk}^r_{\mathscr{M}}}|_{\mathrm{Sht}^r_{\mathscr{M}}} \times \mathbf{T}_{\mathscr{M}}|_{\mathrm{Sht}^r_{\mathscr{M}}} \to \mathbf{T}_{\mathscr{M}^2}|_{\mathrm{Sht}^r_{\mathscr{M}}}$$

The Cartesian square (5.8) fits into a commutative diagram where the back and front faces are Cartesian

$$\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}} \arrow[rr] \arrow[d] \arrow[ddr] & & \mathrm{Hk}^r_{\mathscr{M}} \arrow[d, "\mathrm{pr}_0 \times \mathrm{pr}_r"] \arrow[ddr] & \\
\mathscr{M} \arrow[rr, "(\mathrm{Id},\mathrm{Frob})"] \arrow[ddr] & & \mathscr{M}^2 \arrow[ddr] & \\
& \mathrm{Bun}_{H_1}(k) \times \mathrm{Sht}^r_{U(n),\mathfrak{L}} \arrow[rr] \arrow[d] & & \mathrm{Bun}_{H_1} \times \mathrm{Hk}^r_{U(n),\mathfrak{L}} \arrow[d, "(\Delta, \mathrm{pr}_0 \times \mathrm{pr}_r)"] \\
& \mathrm{Bun}_{H_1} \times \mathrm{Bun}_{U(n),\mathfrak{L}} \arrow[rr, "((\mathrm{Id},\mathrm{Frob}),(\mathrm{Id},\mathrm{Frob}))"] & & \mathrm{Bun}^2_{H_1} \times \mathrm{Bun}^2_{U(n),\mathfrak{L}}
\end{tikzcd}$$

To shorten notation, we write $S := \mathrm{Sht}^r_{\mathscr{M}}$. By the same argument as in the proof of Lemma 5.21, $\mathbf{T}_\pi$ is the (homotopy) limit of the diagram

$$\begin{tikzcd}
\mathbf{T}_{\mathrm{Hk}^r_{\mathscr{M}}}|_S \oplus \mathbf{T}_{\mathscr{M}}|_S \arrow[r] \arrow[d] & \mathbf{T}^{\oplus 2}_{\mathscr{M}}|_S \arrow[d] \\
(\mathbf{T}_{\mathrm{Bun}_{H_1}}|_S \oplus \mathbf{T}_{\mathrm{Hk}^r_{U(n),\mathfrak{L}}}|_S) \oplus (\mathbf{T}_{\mathrm{Bun}_{H_1}}|_S \oplus \mathbf{T}_{\mathrm{Bun}_{U(n),\mathfrak{L}}}|_S) \arrow[r] & \mathbf{T}^{\oplus 2}_{\mathrm{Bun}_{H_1}}|_S \oplus \mathbf{T}^{\oplus 2}_{\mathrm{Bun}_{U(n),\mathfrak{L}}}|_S
\end{tikzcd}$$

To compute this we take fiber of the vertical morphisms, using Corollary 5.17 and Lemma 5.21. This says that for any $R_\bullet$-point of S, the pullback of the above diagram to $R_\bullet$ is (naturally in $R_\bullet$) isomorphic to $R\,\mathrm{pr}_*(-)$ applied to the (homotopy) limit of the diagram of complexes on $X'_{R_\bullet}$ below:

$$\begin{tikzcd}
\left(\bigoplus_{i=1}^{r} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^{\flat}_{i-1/2})\right) \oplus \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) \oplus \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_r) \\
\bigoplus_{i=1}^{r-1} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_i) &
\end{tikzcd}$$

where the maps are:

$$\begin{tikzcd}
(v_{1/2}, \cdots, v_{r-1/2}; v_0) \arrow[r] \arrow[d] & (v_0 + v_{1/2}; v_{r-1/2}) \\
(v_{1/2} - v_{3/2}, \cdots, v_{r-3/2} - v_{r-1/2}) &
\end{tikzcd}$$

We may rewrite the homotopy limit as the complex

$$\begin{tikzcd}
\left(\bigoplus_{i=1}^{r} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}^{\flat}_{i-1/2})\right) \oplus \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) \arrow[d] \\
\left(\bigoplus_{i=1}^{r-1} \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_i)\right) \oplus \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0) \oplus \underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_r)
\end{tikzcd}$$

Unraveling down the definitions of the maps, this is seen to be quasi-isomorphic to (5.9) after cancelling the summand $\underline{\mathrm{Hom}}(\mathrm{V}(\mathcal{E}), \mathcal{F}_0)$.

□

Remark 5.26. More generally, for any unitary type gerbe BH_2 we can define the excess bundle $\mathcal{V}^r_{\mathcal{E}}$ on $\mathrm{Sht}^r_{H_2}$ by the same formulas. It carries a filtration with associated graded given by the same formulas as in Lemma 5.24. The analogue of Lemma 5.25 holds, by the same argument.

6. Fundamental classes of derived special cycles

6.1. **Summary of derived intersection theory.** For the framework of intersection on derived stacks, we will use the work [Kha19b] of Khan. In order to make this paper as self-contained as possible, we give a quick summary of the basic facts from [Kha19b, §2,3] that we will need, simplified to our situation of interest.

6.1.1. *Motivic Borel-Moore homology.* The role of Chow groups of a locally finite type derived Artin stack $\mathscr{X}$ will be played by its *motivic Borel-Moore homology* groups $\mathrm{H}^{\mathrm{BM}}_s(\mathscr{X}/\operatorname{Spec} k, \mathbf{Q}(r))$ as defined in [Kha19b, Definition 2.1, Example 2.10]. (Only the case $s = 2r$ will be of interest to us.) According to [Kha19b, Example 2.10], for $\mathcal{X}$ a classical Artin stack locally of finite type over k, $\mathrm{H}^{\mathrm{BM}}_{2r}(\mathcal{X}, \mathbf{Q}(r))$ identifies with the Chow groups (with $\mathbf{Q}$-coefficients) of Joshua [Jos02]; when $\mathcal{X}$ is of finite type they are identified with the Chow groups (with $\mathbf{Q}$-coefficients) of Kresch [Kre99]. We shall see shortly in §6.1.4 that for a locally finite type derived Artin stack $\mathscr{Y}$ over Spec k, $\mathrm{H}^{\mathrm{BM}}_{2r}(\mathscr{X}, \mathbf{Q}(r))$ can be identified with the motivic Borel-Moore homology of the underlying classical stack $\mathcal{X} := \mathscr{X}_{\mathrm{cl}}$, and thereby interpreted in terms of Chow groups.

More generally, if $\mathscr{X} \to \mathscr{S}$ is a locally finite type morphism of derived Artin stacks over k, then there is a theory of *relative motivic Borel-Moore homology* groups $\mathrm{H}^{\mathrm{BM}}_s(\mathscr{X}/\mathscr{S}, \mathbf{Q}(r))$. In this paper we are mainly concerned with the absolute groups; the relative groups play a technical role in some intermediate statements.

We next discuss the basic functorialities enjoyed by $\mathrm{H}^{\mathrm{BM}}(\mathscr{X}/\mathscr{S})$.

6.1.2. *Proper pushforward.* ([Kha19b, §2.2.1]) If $f\colon \mathscr{X} \to \mathscr{Y}$ is a representable (cf. §5.2.4) proper morphism of derived Artin stacks, locally of finite type over $\mathscr{S}$, then there are functorial direct image morphisms

$$f_*\colon \mathrm{H}^{\mathrm{BM}}_s(\mathscr{X}/\mathscr{S}, \mathbf{Q}(r)) \to \mathrm{H}^{\mathrm{BM}}_s(\mathscr{Y}/\mathscr{S}, \mathbf{Q}(r)).$$

6.1.3. *Smooth pullback.* ([Kha19b, §2.2.2]) If $f\colon \mathscr{X} \to \mathscr{Y}$ is a representable smooth morphism of derived Artin stacks, locally of finite type over $\mathscr{S}$, of relative dimension d, then there is a functorial pullback

$$f^!\colon \mathrm{H}^{\mathrm{BM}}_s(\mathscr{Y}/\mathscr{S}, \mathbf{Q}(r)) \to \mathrm{H}^{\mathrm{BM}}_{s+2d}(\mathscr{X}/\mathscr{S}, \mathbf{Q}(r+d)).$$

6.1.4. *Derived invariance.* For any derived Artin stack $\mathscr{X}$ over k, we denote by $i_{\mathscr{X}}\colon \mathcal{X} \to \mathscr{X}$ the inclusion of the underlying classical stack (cf. §5.2.6). According to [Kha19b, Theorem 2.19(ii)], if $\mathscr{X}$ is locally finite type over $\mathscr{S}$ then the direct image

$$(\iota_{\mathscr{X}})_*\colon \mathrm{H}_s^{\mathrm{BM}}(\mathcal{X}/\mathscr{S}, \mathbf{Q}(r)) \to \mathrm{H}_s^{\mathrm{BM}}(\mathscr{X}/\mathscr{S}, \mathbf{Q}(r))$$

is an isomorphism.

6.1.5. *Base change.* Consider a commutative square of derived Artin stacks

$$\begin{tikzcd} \mathscr{Y} \arrow[r] \arrow[d] & \mathscr{T} \arrow[d, "f"] \\ \mathscr{X} \arrow[r] & \mathscr{S} \end{tikzcd}$$

which is Cartesian on the underlying classical stacks. There is a base change homomorphism [Kha19b, §2.2.3]

$$f^*\colon \mathrm{H}_s^{\mathrm{BM}}(\mathscr{X}/\mathscr{S}; \mathbf{Q}(r)) \to \mathrm{H}_s^{\mathrm{BM}}(\mathscr{Y}/\mathscr{T}, \mathbf{Q}(r))$$

Example 6.1. We note that in the special case where $\mathscr{T} = \mathscr{S}$ and f is the identity map, chasing through the definitions reveals f^* to be the isomorphism of derived invariance §6.1.4. In particular, when $\mathscr{Y} = \mathcal{X}$, the underlying classical truncation of $\mathscr{X}$ with its canonical map to $\mathscr{X}$, f^* becomes $(i_{\mathscr{X}})_*^{-1}$.

6.1.6. *Quasi-smooth pullback.* If $f\colon \mathscr{X} \to \mathscr{Y}$ is a quasi-smooth morphism of derived Artin stacks locally finite type over $\mathscr{S}$, then we may define the *relative virtual dimension* of f at $x \in \mathscr{X}$ to be the Euler characteristic of $\mathbf{L}_f$ at x (which could be negative).

Letting d be the relative virtual dimension of $f\colon \mathscr{X} \to \mathscr{Y}$, there is a Gysin map [Kha19b, Construction 3.4]

$$f^!\colon \mathrm{H}_s^{\mathrm{BM}}(\mathscr{Y}/\mathscr{S}, \mathbf{Q}(r)) \to \mathrm{H}_{s+2d}^{\mathrm{BM}}(\mathscr{X}/\mathscr{S}, \mathbf{Q}(r+d))$$

By [Kha19b, §3.3], if $\mathscr{X}$ and $\mathscr{Y}$ are classical and f is representable in (classical) Deligne-Mumford stacks, then the resulting $f^!$ agrees with the Gysin pullback of classical stacks [Man12].

6.1.7. *Compatibility with the refined Gysin homomorphism.* We shall need the following compatibility of the quasi-smooth pullback with the classical refined Gysin homomorphism. Suppose $f\colon \mathcal{S} \to \mathcal{T}$ is a quasi-smooth DM-type morphism between classical Artin stacks of relative dimension d, such that f satisfies the hypotheses of [Man12, Construction 3.6], $\mathscr{Y}$ is a quasi-smooth derived Artin stack, and $g\colon \mathscr{Y} \to \mathcal{T}$ is locally of finite type. Let $\mathcal{Y}$ be the classical truncation of $\mathscr{Y}$, and suppose that the classical fiber product $\mathcal{X} := \mathcal{Y} \overset{\mathrm{cl}}{\times}_{\mathcal{T}} \mathcal{S} \to \mathcal{Y}$ satisfies the hypotheses of [Man12, Construction 3.6]. Note that $\mathcal{X}$ is the classical truncation of $\mathscr{X} := \mathscr{Y} \times_{\mathcal{T}} \mathcal{S}$. Consider the diagram with the bottom square being derived Cartesian and the outer square being Cartesian as classical stacks:

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "\iota_{\mathscr{X}}"] & \mathcal{Y} \arrow[d, "\iota_{\mathscr{Y}}"] \\ \mathscr{X} \arrow[r, "f'"] \arrow[d] & \mathscr{Y} \arrow[d, "g"] \\ \mathcal{S} \arrow[r, "f"] & \mathcal{T} \end{tikzcd}$$

The hypotheses ensure that the refined virtual pullback $f^!_{\mathrm{ref}}\colon \mathrm{Ch}_*(\mathcal{Y}) \to \mathrm{Ch}_{*+d}(\mathcal{X})$ is defined [Man12, Construction 3.6]. On the other hand, we have the identification $\iota_{\mathscr{Y}*}\colon \mathrm{Ch}_s(\mathcal{Y}) \cong \mathrm{H}_{2s}^{\mathrm{BM}}(\mathcal{Y}, \mathbf{Q}(r)) \xrightarrow{\sim} \mathrm{H}_{2s}^{\mathrm{BM}}(\mathscr{Y}, \mathbf{Q}(r))$ from derived invariance.

Lemma 6.2. *Following the notation above, the diagram below commutes.*

$$\begin{tikzcd} \mathrm{Ch}_s(\mathcal{Y}) \arrow[r, "f^!_{\mathrm{ref}}"] \arrow[d, "\sim"', "i_{\mathscr{Y}*}"] & \mathrm{Ch}_{s+d}(\mathcal{X}) \arrow[d, "\sim"', "i_{\mathscr{X}*}"] \\ \mathrm{H}_{2s}^{\mathrm{BM}}(\mathscr{Y}, \mathbf{Q}(r)) \arrow[r, "(f')^!"] & \mathrm{H}_{2s+2d}^{\mathrm{BM}}(\mathscr{X}, \mathbf{Q}(r+d)) \end{tikzcd}$$

Proof. The argument is very similar to that of [Kha19b, §3.3], which handles the case where $\mathscr{Y} \to \mathcal{T}$ is the identity map (and in particular $\mathscr{X}$ is classical). We explain the necessary adjustments in the present situation. Let $\mathrm{C}_{\mathcal{X}/\mathcal{Y}}$ be the intrinsic normal cone for $\mathcal{X} \to \mathcal{Y}$, and $\mathrm{D}_{\mathcal{X}/\mathcal{Y}}$ be Kresch's deformation, so we have a diagram

$$\begin{array}{ccccc} \mathrm{C}_{\mathcal{Y}/\mathcal{X}} & \longrightarrow & \mathrm{D}_{\mathcal{X}/\mathcal{Y}} & \longleftarrow & \mathcal{Y} \times \mathbf{G}_m \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ 0 & \hookrightarrow & \mathbf{A}^1 & \hookleftarrow & \mathbf{G}_m \end{array}$$

Since f is quasi-smooth, it has a normal bundle stack N_f, which is the stack associated to the (co-connective) two-term complex $\mathbf{T}_f[1]$ [Kha19b, §1.3]. The intrinsic normal cone $\mathrm{C}_{\mathcal{X}/\mathcal{Y}}$ admits an embedding into $f^*\mathrm{N}_f \cong \iota_{\mathscr{X}}^*\mathrm{N}_{f'}$, which fits into a commutative diagram

$$\begin{array}{ccccc} \mathrm{C}_{\mathcal{Y}/\mathcal{X}} & \longrightarrow & \mathrm{D}_{\mathcal{X}/\mathcal{Y}} & \longleftarrow & \mathcal{Y} \times \mathbf{G}_m \\ \big\downarrow{\scriptstyle a} & & \big\downarrow & & \big\downarrow \\ f^*\mathrm{N}_f & \longrightarrow & \mathrm{D}_{\mathscr{X}/\mathscr{Y}} & \longleftarrow & \mathscr{Y} \times \mathbf{G}_m \end{array}$$

where $\mathrm{D}_{\mathscr{X}/\mathscr{Y}}$ is the deformation to the normal bundle stack for the quasi-smooth morphism $f' : \mathscr{X} \to \mathscr{Y}$ [Kha19b, §1.4]. The rest of the argument concludes as in [Kha19b, §3.3]. □

6.1.8. *Top Chern class.* ([Kha19b, §2.2.4]) If $\mathcal{E}$ is a finite locally free sheaf of rank r on a derived Artin stack $\mathscr{X}$ of finite type over k, then there is a *top Chern class* $c_r(\mathcal{E}) \in \mathrm{H}^{\mathrm{BM}}_{-2r}(\mathscr{X}/\mathscr{X}, \mathbf{Q}(-r))$.[11]

Next we will discuss some operations on these motivic Borel-Moore homology groups.

6.1.9. *Composition product.* ([Kha19b, §2.2.5]) Given a derived Artin stack $\mathscr{T}$ locally of finite type over $\mathscr{S}$, and a derived Artin stack $\mathscr{X}$ locally of finite type over $\mathscr{T}$, there is a composition product

$$\circ \colon \mathrm{H}^{\mathrm{BM}}_s(\mathscr{X}/\mathscr{T}, \mathbf{Q}(r)) \otimes \mathrm{H}^{\mathrm{BM}}_{s'}(\mathscr{T}/\mathscr{S}, \mathbf{Q}(r')) \to \mathrm{H}^{\mathrm{BM}}_{s+s'}(\mathscr{X}/\mathscr{S}, \mathbf{Q}(r+r')). \tag{6.1}$$

6.1.10. *Virtual fundamental classes.* We next discuss one of the key features provided by derived algebraic geometry, namely the intrinsic construction of virtual fundamental classes.

Let $f \colon \mathscr{X} \to \mathscr{S}$ be a quasi-smooth morphism of derived Artin stacks, of relative virtual dimension d. Write $1_{\mathscr{S}}$ for the unit of $\mathrm{H}^{\mathrm{BM}}_0(\mathscr{S}/\mathscr{S}; \mathbf{Q}(0))$. Then the *relative fundamental class* of f is [Kha19b, Construction 3.6]

$$[\mathscr{X}/\mathscr{S}] := f^!(1_{\mathscr{S}}) \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathscr{X}/\mathscr{S}; \mathbf{Q}(d)).$$

Of particular importance is the case $\mathscr{S} = \mathrm{Spec}\ k$, in which case we write $[\mathscr{X}] := [\mathscr{X}/\mathrm{Spec}\ k]$ and call it the *virtual fundamental class of* $\mathscr{X}$. Note that by §6.1.4, we may view $[\mathscr{X}] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathcal{X}; \mathbf{Q}(d)) \cong \mathrm{Ch}_d(\mathcal{X})$ where $\mathcal{X}$ is the underlying classical stack of $\mathscr{X}$, and we will frequently do so.

When $\iota_{\mathscr{X}} : \mathcal{X} \to \mathscr{X}$ is an isomorphism and $\mathcal{X}$ is smooth, then $[\mathscr{X}]$ is the usual fundamental class $[\mathcal{X}]^{\mathrm{naive}}$.

We next establish some basic properties of these virtual fundamental classes.

6.1.11. *Intersection product of virtual fundamental classes.* Let $\mathscr{X}, \mathscr{Y}$ and $\mathscr{Y}'$ be derived Artin stacks locally finite type and equidimensional k, and suppose furthermore that $\mathscr{X}$ is smooth and $\mathscr{Y}, \mathscr{Y}'$ are quasi-smoooth over k. Suppose we have maps (not necessarily quasi-smooth) $f \colon \mathscr{Y} \to \mathscr{X}$ and $f' \colon \mathscr{Y}' \to \mathscr{X}$. Consider the Cartesian square

$$\begin{array}{ccc} \mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}' & \xrightarrow{\Delta'_{\mathscr{X}}} & \mathscr{Y} \times \mathscr{Y}' \\ \big\downarrow & & \big\downarrow \\ \mathscr{X} & \xrightarrow{\Delta_{\mathscr{X}}} & \mathscr{X} \times \mathscr{X} \end{array}$$

The morphism $\Delta'_{\mathscr{X}}$ is quasi-smooth as it is the base change of the quasi-smooth morphism $\Delta_{\mathscr{X}}$. In particular $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}'$ is also quasi-smooth over k of virtual dimension $r = \dim \mathscr{Y} + \dim \mathscr{Y}' - \dim \mathscr{X}$. We write

$$[\mathscr{Y}] \cdot_{\mathscr{X}} [\mathscr{Y}'] := (\Delta'_{\mathscr{X}})^!([\mathscr{Y} \times \mathscr{Y}']) \in \mathrm{H}^{\mathrm{BM}}_{2r}(\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}', \mathbf{Q}(r)).$$

[11] To compare this with the usual formulation of Chern classes, we observe that $\mathrm{H}^{\mathrm{BM}}_{-s}(\mathscr{X}/\mathscr{X}, \mathbf{Q}(-r))$ is naturally isomorphic to the motivic *cohomology* groups $\mathrm{H}^s(\mathscr{X}, \mathbf{Q}(r))$.

Lemma 6.3. *In the situation above, we have*

$$[\mathscr{Y}] \cdot_{\mathscr{X}} [\mathscr{Y}'] = [\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}']. \tag{6.2}$$

Proof. By definition $[\mathscr{Y} \times \mathscr{Y}'] = \mathrm{pr}^![\mathrm{Spec}\ k]$ where $\mathrm{pr}\colon \mathscr{Y} \times \mathscr{Y}' \to \mathrm{Spec}\ k$ is the structure map, and $\mathrm{pr}^!$ is defined because $\mathscr{Y} \times \mathscr{Y}'$ is quasi-smooth. Hence we have

$$[\mathscr{Y}] \cdot_{\mathscr{X}} [\mathscr{Y}'] = (\Delta'_{\mathscr{X}})^! \,\mathrm{pr}^![\mathrm{Spec}\ k] = (\mathrm{pr} \circ \Delta'_{\mathscr{X}})^![\mathrm{Spec}\ k] = [\mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}'].$$

□

6.1.12. *Excess intersection formula.* We shall make crucial use of the following *excess intersection formula*, which is [Kha19b, Proposition 3.15][12]. Suppose we have a commutative (but *not* necessarily Cartesian) square of derived Artin stacks over k,

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g"] \arrow[d, "p"] & \mathscr{Y}' \arrow[d, "q"] \\ \mathscr{X} \arrow[r, "f"] & \mathscr{Y} \end{tikzcd} \tag{6.3}$$

where f and g are quasi-smooth, and equidimensional. We say that (6.3) is an *excess intersection square* if it is Cartesian on underlying classical stacks, and the homotopy fiber of the canonical map $p^*\mathbf{L}_{\mathscr{X}/\mathscr{Y}}[-1] \to \mathbf{L}_{\mathscr{X}'/\mathscr{Y}'}[-1]$ is a locally free $\mathcal{O}_{\mathscr{X}'}$-module of finite rank r, whose dual we call the *excess bundle* $\mathcal{E}$. Then we have the top Chern class $c_r(\mathcal{E}) \in \mathrm{H}^{\mathrm{BM}}_{-2r}(\mathscr{X}'/\mathscr{X}', \mathbf{Q}(-r))$. The excess intersection formula asserts that

$$q^*[\mathscr{X}/\mathscr{Y}] = c_r(\mathcal{E}) \circ [\mathscr{X}'/\mathscr{Y}'] \in \mathrm{H}^{\mathrm{BM}}_{2d}(\mathscr{X}'/\mathscr{Y}', \mathbf{Q}(d)),$$

where d is the virtual dimension of f and q^* is the base change map of §6.1.5.

Lemma 6.4. *Let $p : \mathscr{X}' \to \mathscr{X}$ be a map of quasi-smooth derived Artin stacks locally finite type over k that induces an isomorphism on their classical truncations $p_{\mathrm{cl}} : \mathcal{X}' \xrightarrow{\sim} \mathcal{X}$. Assume $\mathbf{L}_p[-2]$ is a locally free $\mathcal{O}_{\mathscr{X}'}$-module of finite rank r. Then*

$$[\mathscr{X}] = c_r(\mathbf{T}_p[2]) \circ [\mathscr{X}'] \in \mathrm{Ch}_d(\mathcal{X}),$$

where d is the virtual dimension of $\mathscr{X}$ (note here $\mathbf{T}_p[2]$ is a locally free $\mathcal{O}_{\mathscr{X}'}$-module of finite rank r).

Proof. Apply the excess intersection formula to the square

$$\begin{tikzcd} \mathscr{X}' \arrow[r] \arrow[d, "p"] & \mathrm{Spec}\ k \arrow[d, equal] \\ \mathscr{X} \arrow[r] & \mathrm{Spec}\ k \end{tikzcd}$$

□

6.2. **Calculation of virtual fundamental classes.** We now return to the (derived) Hitchin stacks. Fix $m \le n$ and let $\mathscr{M} = \mathscr{M}_{\mathrm{GL}(m)',U(n),\mathfrak{L}}$ and $\mathcal{M} = \mathcal{M}_{\mathrm{GL}(m)',U(n),\mathfrak{L}}$, which is the classical truncation of $\mathscr{M}$. In the future we will suppress $\mathfrak{L}$ for notational simplicity. As $\mathrm{Sht}^r_{\mathscr{M}}$ is quasi-smooth by Lemma 5.25, the virtual fundamental class $[\mathrm{Sht}^r_{\mathscr{M}}] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}})$ is defined by §6.1.10. We can now confirm that the virtual fundamental classes of $\mathcal{Z}^r_{\mathcal{E}}$ constructed earlier in §4.4 agree with the components of $[\mathrm{Sht}^r_{\mathscr{M}}]$.

Theorem 6.5. *Recall $\mathscr{M} = \mathscr{M}_{\mathrm{GL}(m)',U(n)}$ and $\mathcal{M} = \mathcal{M}_{\mathrm{GL}(m)',U(n)}$. We have*

$$[\mathrm{Sht}^r_{\mathscr{M}}]|_{\mathcal{Z}^r_{\mathcal{E}}} = [\mathcal{Z}^r_{\mathcal{E}}] \in \mathrm{Ch}_{(n-m)r}(\mathcal{Z}^r_{\mathcal{E}}),$$

where the latter is as in Definition 4.7(1). Here, the notation $(\cdot)|_{\mathcal{Z}^r_{\mathcal{E}}}$ means projection to the summand indexed by the union of connected components of $\mathrm{Sht}^r_{\mathcal{M}}$ corresponding to $\mathcal{Z}^r_{\mathcal{E}}$.

The rest of this subsection is devoted to the proof of Theorem 6.5. Recall the open-closed decomposition (4.4) of $\mathcal{Z}^r_{\mathcal{E}}$ into $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ according to the kernel of $t_i : \mathcal{E} \to \mathcal{F}_i$. We will prove the above theorem first for the non-degenerate term $\mathcal{Z}^{r,\circ}_{\mathcal{E}}$, then for the most degenerate term $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{E}]$, and finally for general terms by reducing them to the two extreme cases.

[12]We note that the conventions of [Kha19b] are off from ours by a dualization, e.g. the virtual fundamental class of a self-intersection in [Kha19b, Corollary 3.17] is the top Chern class of what is called the "conormal bundle" in *loc. cit.*, whereas we would call it the normal bundle.

6.2.1. *Kernel decomposition.* Recall that $\mathrm{Sht}^r_{\mathcal{M}}$ is the disjoint union of open-closed substacks $\mathcal{Z}^r_{\mathcal{E}}$ for $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)$, and each $\mathcal{Z}^r_{\mathcal{E}}$ is the disjoint union of open-closed substacks $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ indexed by sub-bundles $\mathcal{K} \subset\subset \mathcal{E}$ (see §4.1 and Corollary 4.2).

Since $\mathrm{Sht}^r_{\mathscr{M}}$ has underlying classical stack $\mathrm{Sht}^r_{\mathcal{M}}$, it similarly decomposes into open-closed derived substacks $\mathscr{Z}^r_{\mathcal{E}}$ (whose underlying classical stack is $\mathcal{Z}^r_{\mathcal{E}}$), which further decompose into open-closed derived substacks $\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$ (whose underlying classical stack is $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ$). Similarly, we have the open-closed derived substack $\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}] \subset \mathscr{Z}^r_{\mathcal{E}}$ whose classical truncation is $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]$.

To summarize, we have a decomposition into open-closed derived substacks

$$\mathrm{Sht}^r_{\mathscr{M}} = \coprod_{\substack{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k), \\ \mathcal{K} \subset\subset \mathcal{E}}} \mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ. \tag{6.4}$$

The virtual fundamental classes $[\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ)$ and $[\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}])$ are defined as the restriction of the virtual fundamental class $[\mathrm{Sht}^r_{\mathscr{M}}]$.

6.2.2. *Non-degenerate terms.* We consider $\mathscr{Z}^{r,\circ}_{\mathcal{E}} := \mathscr{Z}^r_{\mathcal{E}}[0]^\circ$ whose underlying classical stack is $\mathcal{Z}^{r,\circ}_{\mathcal{E}}$. We will show:

Lemma 6.6. *We have*

$$[\mathscr{Z}^{r,\circ}_{\mathcal{E}}] = [\mathcal{Z}^{r,\circ}_{\mathcal{E}}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^{r,\circ}_{\mathcal{E}}). \tag{6.5}$$

Let $H_1 \to \mathrm{GL}(m)'$ be any homomorphism of smooth group schemes over X (although we shall only need the case where this map is the identity).

Lemma 6.7.

(1) The following square is (derived) Cartesian:

$$\begin{tikzcd} \mathscr{M}_{H_1,U(n)} \arrow[r] \arrow[d] & \mathscr{M}_{H_1,\mathrm{GL}(n)'} \arrow[d] \\ \mathrm{Bun}_{U(n)} \arrow[r] & \mathrm{Bun}_{\mathrm{GL}(n)'} \end{tikzcd}$$

(2) The following square is (derived) Cartesian:

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathscr{M}_{H_1,U(n)}} \arrow[r] \arrow[d] & \mathrm{Sht}^r_{\mathscr{M}_{H_1,\mathrm{GL}(n)'}} \arrow[d] \\ \mathrm{Sht}^r_{U(n)} \arrow[r] & \mathrm{Sht}^r_{\mathrm{GL}(n)'} \end{tikzcd}$$

Proof. Immediate from the definitions. □

Applying Lemma 6.3, we obtain:

Corollary 6.8. *We have* $[\mathscr{M}_{H_1,U(n)}] = [\mathscr{M}_{H_1,\mathrm{GL}(n)'}] \cdot_{\mathrm{Bun}_{\mathrm{GL}(n)'}} [\mathrm{Bun}_{U(n)}] \in \mathrm{Ch}_*(\mathscr{M}_{H_1,U(n)})$.

Corollary 6.9. *We have*

$$[\mathrm{Sht}^r_{\mathscr{M}^\circ_{H_1,\mathrm{GL}(n)'}}] = [\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}}] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}})$$

where the right side is defined in Definition 3.21.

Proof. We abbreviate $\mathscr{M}'^\circ := \mathscr{M}^\circ_{H_1,\mathrm{GL}(n)'}$ and $\mathcal{M}'^\circ := \mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}$, which is the classical truncation of $\mathscr{M}'^\circ$. Consider the Cartesian square

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathscr{M}'^\circ} \arrow[r] \arrow[d] & (\mathrm{Hk}^1_{\mathscr{M}'^\circ})^r \arrow[d, "q"] \\ (\mathscr{M}'^\circ)^r \arrow[r, "\Phi"] & (\mathscr{M}'^\circ)^{2r} \end{tikzcd}$$

By Lemma 6.3, we have

$$[\mathrm{Sht}^r_{\mathscr{M}'^\circ}] = [(\mathrm{Hk}^1_{\mathscr{M}'^\circ})^r] \cdot_{(\mathscr{M}'^\circ)^{2r}} [(\mathscr{M}'^\circ)^r]. \tag{6.6}$$

According to Corollaries 5.18(1) and 5.22(1), the three corners of the above diagram (except $\mathrm{Sht}^r_{\mathscr{M}'^\circ}$) are smooth and isomorphic to their classical truncations. By §6.1.10, we then have

$$[(\mathrm{Hk}^1_{\mathscr{M}'^\circ})^r] = [(\mathrm{Hk}^1_{\mathcal{M}'^\circ})^r]^{\mathrm{naive}} \in \mathrm{Ch}_*(\mathrm{Hk}^r_{\mathcal{M}'^\circ})$$

and

$$[(\mathscr{M}'^\circ)^r] = [(\mathcal{M}'^\circ)^r]^{\mathrm{naive}} \in \mathrm{Ch}_*((\mathcal{M}'^\circ)^r).$$

Inserting these into (6.6) gives

$$[(\mathrm{Hk}_{\mathscr{M}'^\circ})^r] \cdot_{(\mathscr{M}'^\circ)^{2r}} [(\mathscr{M}'^\circ)^r] = [(\mathrm{Hk}^1_{\mathcal{M}'^\circ})^r]^{\mathrm{naive}} \cdot_{(\mathcal{M}'^\circ)^{2r}} [(\mathcal{M}'^\circ)^r]^{\mathrm{naive}} \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}'^\circ}).$$

The right hand side is precisely the definition of $[\mathrm{Sht}^r_{\mathcal{M}'^\circ}]$ in Definition 3.21. □

Proof of Lemma 6.6. Note that $\mathrm{Sht}^r_{\mathscr{M}^\circ_{H_1,U(n)}} = {}_m\mathrm{Sht}^r_{\mathscr{M}_{H_1,U(n)}}$. Applying Lemma 6.7 and Lemma 6.3 , we have

$$[\mathrm{Sht}^r_{\mathscr{M}^\circ_{H_1,U(n)}}] = [\mathrm{Sht}^r_{U(n)}] \cdot_{\mathrm{Sht}^r_{\mathrm{GL}(n)'}} [\mathrm{Sht}^r_{\mathscr{M}^\circ_{H_1,\mathrm{GL}(n)'}}] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,U(n)}}).$$

By Corollary 6.9, the RHS above identifies with $[\mathrm{Sht}^r_{U(n)}] \cdot_{\mathrm{Sht}^r_{\mathrm{GL}(n)'}} [\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,\mathrm{GL}(n)'}}] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}^\circ_{H_1,U(n)}})$. Now specializing to the case $H_1 = \mathrm{GL}(m')$, and decomposing both sides of the resulting equality according to $\mathcal{E} \in \mathrm{Bun}_{H_1}(k)$ yields (6.5). □

6.2.3. *The most degenerate term.* We will next handle the most degenerate term $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{E}]$. Let ${}_0\mathrm{Sht}^r_{\mathcal{M}}$ be the substack of $\mathrm{Sht}^r_{\mathcal{M}}$ where $t_i = 0$. Then ${}_0\mathrm{Sht}^r_{\mathcal{M}}$ is the disjoint union of $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{E}]$ over $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)$, hence open-closed in $\mathrm{Sht}^r_{\mathcal{M}}$. Let ${}_0\mathrm{Sht}^r_{\mathscr{M}} = \coprod \mathscr{Z}^r_{\mathcal{E}}[\mathcal{E}] \subset \mathrm{Sht}^r_{\mathscr{M}}$ be the corresponding open-closed derived substack. Note that the underlying classical stack of ${}_0\mathrm{Sht}^r_{\mathscr{M}}$ is

$$({}_0\mathrm{Sht}^r_{\mathscr{M}})_{\mathrm{cl}} = {}_0\mathrm{Sht}^r_{\mathcal{M}} \cong \mathrm{Bun}_{\mathrm{GL}(m)'}(k) \times \mathrm{Sht}^r_{U(n)}\,. \tag{6.7}$$

In §5.7.1 we defined a bundle $\mathcal{V}^r$ on $\mathrm{Bun}_{\mathrm{GL}(m)'}(k) \times \mathrm{Sht}^r_{U(n)}$. Below we write $\mathcal{V}^r|_{{}_0\mathrm{Sht}^r_{\mathscr{M}}}$ to denote the restriction of $\mathcal{V}^r$ to ${}_0\mathrm{Sht}^r_{\mathscr{M}}$ via the natural map ${}_0\mathrm{Sht}^r_{\mathscr{M}} \to \mathrm{Bun}_{\mathrm{GL}(m)'}(k) \times \mathrm{Sht}^r_{U(n)}$.

Lemma 6.10. *We have*

$$[{}_0\mathrm{Sht}^r_{\mathscr{M}}] = c_{mr}(\mathcal{V}^r|_{{}_0\mathrm{Sht}^r_{\mathscr{M}}}) \cdot [{}_0\mathrm{Sht}^r_{\mathcal{M}}]^{\mathrm{naive}} \in \mathrm{Ch}_{(n-m)r}({}_0\mathrm{Sht}^r_{\mathcal{M}}). \tag{6.8}$$

Proof. We apply Lemma 6.4 to the map $\iota : {}_0\mathrm{Sht}^r_{\mathcal{M}} \to {}_0\mathrm{Sht}^r_{\mathscr{M}}$. Note that ${}_0\mathrm{Sht}^r_{\mathscr{M}}$ is quasi-smooth by Lemma 5.25, and ${}_0\mathrm{Sht}^r_{\mathcal{M}}$ is smooth by (6.7) (using [FYZ21, Lemma 6.9]). To apply the excess intersection formula, we claim that $\mathbf{L}_\iota$ is concentrated in degree -2, and $H^{-2}\mathbf{L}_\iota \cong (\mathcal{V}^r)^*|_{{}_0\mathrm{Sht}^r_{\mathscr{M}}}$.

We have an exact triangle

$$\iota^*\mathbf{L}_{{}_0\mathrm{Sht}^r_{\mathscr{M}}} \to \mathbf{L}_{{}_0\mathrm{Sht}^r_{\mathcal{M}}} \to \mathbf{L}_\iota. \tag{6.9}$$

Consider the composition

$${}_0\mathrm{Sht}^r_{\mathcal{M}} \xrightarrow{\iota} {}_0\mathrm{Sht}^r_{\mathscr{M}} \xrightarrow{\pi} \mathrm{Bun}_{\mathrm{GL}(m)'}(k) \times \mathrm{Sht}^r_{U(n)}\,.$$

This induces an exact triangle

$$\iota^*\mathbf{L}_\pi \to \mathbf{L}_{\pi\circ\iota} \to \mathbf{L}_\iota.$$

Note that $\pi \circ \iota = \mathrm{Id}$, so that $\mathbf{L}_{\pi\circ\iota} = 0$. Hence $\mathbf{L}_\iota = \iota^*\mathbf{L}_\pi[1]$, which is $(\mathcal{V}^r)^*[2]|_{{}_0\mathrm{Sht}^r_{\mathscr{M}}}$ by Lemma 5.25. □

6.2.4. *Intermediate terms.* In order to simplify notation, we will conflate $\mathrm{GL}(m)$ torsors with rank m vector bundles in this section. Also, for ease of language we will give the argument in the case where X' is connected. At the end in Remark 6.15, we will summarize the adjustments that need to be made if X' is disconnected.

Lemma 6.11. *For any sub-bundle $\mathcal{K} \subset \mathcal{E}$ with quotient $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{K}$ of rank i, we have*

$$[\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]] = c_{(m-i)r}(\mathcal{V}^r_{\mathcal{K}}|_{\mathscr{Z}^r_{\overline{\mathcal{E}}}}) \cdot [\mathscr{Z}^r_{\overline{\mathcal{E}}}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]). \tag{6.10}$$

Here $\mathcal{V}^r_{\mathcal{K}}|_{\mathscr{Z}^r_{\overline{\mathcal{E}}}}$ denotes the pullback of $\mathcal{V}^r_{\mathcal{K}}$ along $\mathscr{Z}^r_{\overline{\mathcal{E}}} \to \mathrm{Sht}^r_{U(n)}$, and $[\mathscr{Z}^r_{\overline{\mathcal{E}}}] \in \mathrm{Ch}_{r(n-i)}(\mathcal{Z}^r_{\overline{\mathcal{E}}})$ is viewed as an element in $\mathrm{Ch}_{r(n-i)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}])$ via the isomorphism $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}] \cong \mathcal{Z}^r_{\overline{\mathcal{E}}}$.

Proof of Theorem 6.5 assuming Lemma 6.11. Restricting (6.10) to the open-closed $\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ \cong \mathcal{Z}^{r,\circ}_{\overline{\mathcal{E}}}$, we get

$$[\mathrm{Sht}^r_{\mathscr{M}}]|_{\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ} = c_{(m-i)r}(\mathcal{V}^r_{\mathcal{K}}|_{\mathscr{Z}^r_{\overline{\mathcal{E}}}}) \cdot [\mathscr{Z}^{r,\circ}_{\overline{\mathcal{E}}}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]^\circ). \tag{6.11}$$

By Lemma 6.6 we have $[\mathscr{Z}^{r,\circ}_{\overline{\mathcal{E}}}] = [\mathcal{Z}^{r,\circ}_{\overline{\mathcal{E}}}]$. Inserting this into (6.11), we get exactly the expression for $[\mathcal{Z}^r_{\mathcal{E}}]$ from Definition 4.7. □

It remains to establish Lemma 6.11. Suppose $0 \leq i \leq m \leq n$. We abbreviate $\mathscr{M}(m,n) = \mathscr{M}_{\mathrm{GL}(m)',U(n)}$. We define two auxiliary variants of derived Hitchin stacks.

- $\mathscr{M}'$ classifies $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(R_\bullet)$, $\mathcal{F} \in \mathrm{Bun}_{U(n)}(R_\bullet)$, a vector sub-bundle $\mathcal{K} \subset \mathcal{E}$ of rank $m-i$ (so $\mathcal{E}/\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(i)'}$) and a derived section $t \in R\Gamma(X'_{R_\bullet}, \underline{\mathrm{Hom}}(\mathcal{E}/\mathcal{K}, \mathcal{F}))$. Projecting such data to $(\mathcal{E}/\mathcal{K}, \mathcal{F}, t)$ induces a map $\mathscr{M}' \to \mathscr{M}(i,n)$.
- $\mathscr{M}''$ classifies $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(R_\bullet)$,$\mathcal{F} \in \mathrm{Bun}_{U(n)}(R_\bullet)$, a vector sub-bundle $\mathcal{K} \subset \mathcal{E}$ of rank $m-i$ (so $\mathcal{E}/\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(i)'}$) and a derived section $t \in R\Gamma(X'_{R_\bullet}, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F}))$. Projecting such data to $(\mathcal{E}, \mathcal{F}, t)$ induces a map $\mathscr{M}'' \to \mathscr{M}(m,n)$, while sending it to $(\mathcal{K}, \mathcal{F}, t|_{\mathcal{K}} \in R\Gamma(X'_{R_\bullet}, \underline{\mathrm{Hom}}(\mathcal{K}, \mathcal{F})))$ induces a map $\mathscr{M}'' \to \mathscr{M}(m-i,n)$.

From the constructions we get a canonical map $\mathscr{M}' \to \mathscr{M}''$ sending $(\mathcal{K} \subset \mathcal{E}, \mathcal{F}, t)$ to $(\mathcal{K} \subset \mathcal{E}, \mathcal{F}, t')$ where t' is the image of t under the natural map $R\Gamma(X'_{R_\bullet}, \underline{\mathrm{Hom}}(\mathcal{E}/\mathcal{K}, \mathcal{F})) \to R\Gamma(X'_{R_\bullet}, \underline{\mathrm{Hom}}(\mathcal{E}, \mathcal{F}))$. So we have a diagram

$$\begin{tikzcd} \mathscr{M}' \arrow[r] \arrow[d] & \mathscr{M}'' \arrow[r] \arrow[d] & \mathscr{M}(m,n) \\ \mathscr{M}(i,n) & \mathscr{M}(m-i,n) & \end{tikzcd} \tag{6.12}$$

We define $\mathrm{Hk}^r_{\mathscr{M}'} := \mathrm{Hk}^r_{\mathscr{M}(i,n)} \times_{\mathscr{M}(i,n)} \mathscr{M}'$ and $\mathrm{Sht}^r_{\mathscr{M}'}$ by the Cartesian square

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathscr{M}'} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathscr{M}'} \arrow[d, "\mathrm{pr}_0 \times \mathrm{pr}_r"] \\ \mathscr{M}' \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathscr{M}' \times \mathscr{M}' \end{tikzcd}$$

We have an open-closed decomposition

$$\mathrm{Sht}^r_{\mathscr{M}(i,n)} = \coprod_{\overline{\mathcal{E}} \in \mathrm{Bun}_{\mathrm{GL}(i)'}(k)} \mathscr{Z}^r_{\overline{\mathcal{E}}}$$

and an open-closed decomposition of $\mathrm{Sht}^r_{\mathscr{M}'}$ according to the discrete data $(\mathcal{K} \subset \mathcal{E})$, or equivalently according to $\mathcal{E}$ and $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{K}$:

$$\mathrm{Sht}^r_{\mathscr{M}'} = \coprod_{\substack{\overline{\mathcal{E}} \in \mathrm{Bun}_{\mathrm{GL}(i)'}(k), \\ \mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k), \\ \mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}}} \mathrm{Sht}^r_{\mathscr{M}'}(\mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}). \tag{6.13}$$

Let $\mathscr{A}(i,n)$ be the Hitchin base for $\mathscr{M}(i,n)$, classifying $\overline{\mathcal{E}} \in \mathrm{Bun}_{\mathrm{GL}(i)'}$ and a derived section a of $\underline{\mathrm{Hom}}(\overline{\mathcal{E}}, \sigma^* \overline{\mathcal{E}}^\vee \otimes \nu^* \mathfrak{L})$ such that $\sigma^* a^\vee = a$. Let $\mathscr{A}'$ be the Hitchin base for $\mathscr{M}'$, classifying $\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}$, a vector sub-bundle $\mathcal{K} \subset \mathcal{E}$ of rank $m-i$, with quotient bundle $\overline{\mathcal{E}}$ of rank i, and a derived section a of $\underline{\mathrm{Hom}}(\overline{\mathcal{E}}, \sigma^* \overline{\mathcal{E}}^\vee \otimes \nu^* \mathfrak{L})$ such that $\sigma^* a^\vee = a$.

Lemma 6.12. *Let $\mathscr{Y}$ be a locally finite type derived stack over k. Then the diagram*

$$\begin{tikzcd} \mathscr{Y}(k) \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d, "(\mathrm{Id},\mathrm{Frob})"] \\ \mathscr{Y} \arrow[r, "\Delta"] & \mathscr{Y} \times \mathscr{Y} \end{tikzcd}$$

is derived Cartesian.

Proof. Let $\mathscr{Y}^{h\,\mathrm{Frob}}$ be the derived fibered product

$$\begin{tikzcd} \mathscr{Y}^{h\,\mathrm{Frob}} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d, "(\mathrm{Id},\mathrm{Frob})"] \\ \mathscr{Y} \arrow[r, "\Delta"] & \mathscr{Y} \times \mathscr{Y} \end{tikzcd}$$

Clearly $\mathscr{Y}^{h\,\mathrm{Frob}}$ receives a canonical map from $\mathscr{Y}(k)$ (regarded as a constant stack), and we want to show that this map is an isomorphism. It suffices to show that $\mathscr{Y}^{h\,\mathrm{Frob}}$ is isomorphic to its classical truncation, in which case it follows from the analogous statement for finite type classical stacks. To this end, let us examine the tangent complex of the derived fibered product: since Frob induces the zero map on tangent complexes, it is the derived fibered product of the diagram of complexes

$$\begin{tikzcd}
 & \mathbf{T}_{\mathscr{Y}}|_{\mathscr{Y}^{h\,\mathrm{Frob}}} \arrow[d, "(\mathrm{Id},0)"] \\
\mathbf{T}_{\mathscr{Y}}|_{\mathscr{Y}^{h\,\mathrm{Frob}}} \arrow[r, "(\mathrm{Id},\mathrm{Id})"] & \mathbf{T}_{\mathscr{Y}}|_{\mathscr{Y}^{h\,\mathrm{Frob}}} \oplus \mathbf{T}_{\mathscr{Y}}|_{\mathscr{Y}^{h\,\mathrm{Frob}}}
\end{tikzcd}$$

which is evidently zero. Then we conclude using Lemma 5.5. □

Lemma 6.13. *For any choice of $\mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}$ as in* (6.13)*, the natural map $\mathrm{Sht}^r_{\mathscr{M}'}(\mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}) \to \mathrm{Sht}^r_{\mathscr{M}(i,n)}(\overline{\mathcal{E}}) = \mathscr{Z}^r_{\overline{\mathcal{E}}}$ is an isomorphism.*

Proof. We have isomorphisms $\mathscr{M}' \to \mathscr{A}' \times_{\mathscr{A}(i,n)} \mathscr{M}(i,n)$ and $\mathrm{Hk}^r_{\mathscr{M}'} \to \mathscr{A}' \times_{\mathscr{A}(i,n)} \mathrm{Hk}^r_{\mathscr{M}(i,n)}$, which induce $\mathrm{Sht}^r_{\mathscr{M}'} \xrightarrow{\sim} \mathscr{A}'(k) \times_{\mathscr{A}(i,n)(k)} \mathrm{Sht}^r_{\mathscr{M}(i,n)}$ by the diagram below (where we have used Lemma 6.12).

$$\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}'} \arrow[r] \arrow[d] \arrow[ddrr] & \mathrm{Hk}^r_{\mathscr{M}'} \arrow[d] \arrow[ddrr] & & \\
\mathscr{M}' \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] \arrow[ddrr] & \mathscr{M}' \times \mathscr{M}' \arrow[ddrr] & & \\
 & & \mathscr{A}'(k) \times_{\mathscr{A}(i,n)(k)} \mathrm{Sht}^r_{\mathscr{M}(i,n)} \arrow[r] \arrow[d] & \mathscr{A}' \times_{\mathscr{A}(i,n)} \mathrm{Hk}^r_{\mathscr{M}(i,n)} \arrow[d] \\
 & & \mathscr{A}' \times_{\mathscr{A}(i,n)} \mathscr{M}(i,n) \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & (\mathscr{A}' \times_{\mathscr{A}(i,n)} \mathscr{M}(i,n)) \times (\mathscr{A}' \times_{\mathscr{A}(i,n)} \mathscr{M}(i,n))
\end{tikzcd}$$

Decomposing this last isomorphism over $\mathcal{A}'(k)$ gives the result. □

Similarly, we define $\mathrm{Hk}_{\mathscr{M}''} := \mathrm{Hk}^r_{\mathscr{M}(m,n)} \times_{\mathscr{M}(m,n)} \mathscr{M}''$ and $\mathrm{Sht}_{\mathscr{M}''}$ by the Cartesian square

$$\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}''} \arrow[r] \arrow[d] & \mathrm{Hk}^r_{\mathscr{M}''} \arrow[d, "\mathrm{pr}_0 \times \mathrm{pr}_r"] \\
\mathscr{M}'' \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathscr{M}'' \times \mathscr{M}''
\end{tikzcd}$$

We have open-closed decompositions

$$\mathrm{Sht}^r_{\mathscr{M}(m,n)} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} \mathscr{Z}^r_{\mathcal{E}},$$

$$\mathrm{Sht}^r_{\mathscr{M}(m-i,n)} = \coprod_{\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(m-i)'}(k)} \mathscr{Z}^r_{\mathcal{K}},$$

and

$$\mathrm{Sht}^r_{\mathscr{M}''} = \coprod_{\substack{\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(m-i)'}(k), \\ \mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(m)'}(k), \\ \mathcal{K} \subset\subset \mathcal{E}}} \mathrm{Sht}^r_{\mathscr{M}''}(\mathcal{K} \subset \mathcal{E}).$$

We remind that the notation $\mathcal{K} \subset\subset \mathcal{E}$ means that $\mathcal{K} \subset \mathcal{E}$ is a vector sub-bundle of $\mathcal{E}$, i.e. $\mathcal{E}/\mathcal{K}$ is a vector bundle (as opposed to merely a sub coherent sheaf).

Lemma 6.14. *For any rank m vector bundle $\mathcal{E}$ over X' and any vector sub-bundle $\mathcal{K} \subset \mathcal{E}$ of rank $m-i$, the map $\mathrm{Sht}^r_{\mathscr{M}''} \to \mathrm{Sht}^r_{\mathscr{M}(m,n)}$ restricts to an isomorphism $\mathrm{Sht}^r_{\mathscr{M}''}(\mathcal{K} \subset \mathcal{E}) \xrightarrow{\sim} \mathscr{Z}^r_{\mathcal{E}}$.*

Proof. The argument is similar to that for Lemma 6.13. □

We have a map $z : \mathrm{Bun}_{\mathrm{GL}(m-i)'} \times \mathrm{Bun}_{U(n)} \to \mathscr{M}(m-i,n)$ sending $(\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(m-i)'}, \mathcal{F} \in \mathrm{Bun}_{U(n)})$ to $(\mathcal{K}, \mathcal{F}, \mathcal{K} \xrightarrow{0} \mathcal{F}) \in \mathscr{M}(m-i,n)$. This map fits into a Cartesian square

$$\begin{tikzcd} \mathscr{M}' \arrow[r] \arrow[d] & \mathscr{M}'' \arrow[d] \\ \mathrm{Bun}_{\mathrm{GL}(m-i)'} \times \mathrm{Bun}_{U(n)} \arrow[r, "z"] & \mathscr{M}(m-i,n) \end{tikzcd}$$

This in turn induces a Cartesian square

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathscr{M}'} \arrow[r] \arrow[d] & \mathrm{Sht}^r_{\mathscr{M}''} \arrow[d] \\ \mathrm{Bun}_{\mathrm{GL}(m-i)'}(k) \times \mathrm{Sht}^r_{U(n)} \arrow[r] & \mathrm{Sht}^r_{\mathscr{M}(m-i,n)} \end{tikzcd} \tag{6.14}$$

Proof of Lemma 6.11. Thanks to Lemma 6.14 and Lemma 6.13, we have open-closed decompositions

$$\mathrm{Sht}^r_{\mathscr{M}(m-i,n)} = \coprod_{\mathcal{K}} \mathscr{Z}^r_{\mathcal{K}}, \quad \mathrm{Sht}^r_{\mathscr{M}''} = \coprod_{\mathcal{K} \subset\subset \mathcal{E}} \mathscr{Z}^r_{\mathscr{E}}, \quad \mathrm{Sht}^r_{\mathscr{M}'} = \coprod_{\mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}} \mathscr{Z}^r_{\overline{\mathcal{E}}}.$$

Inserting these decompositions into (6.14) and then restricting to the open-closed $\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]$, we obtain a Cartesian square (where $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{K}$)

$$\begin{tikzcd} \mathscr{Z}^r_{\overline{\mathcal{E}}} \arrow[r, "\zeta"] \arrow[d, "\pi_{\overline{\mathcal{E}}}"] & \mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}] \arrow[d] \\ \mathrm{Sht}^r_{U(n)} \arrow[r, "\iota"] & \mathscr{Z}^r_{\mathcal{K}} \end{tikzcd} \tag{6.15}$$

Note that the classical truncation of the top arrow ζ is the canonical isomorphism $\mathcal{Z}^r_{\overline{\mathcal{E}}} \cong \mathcal{Z}^r_{\mathcal{E}}[\mathcal{K}]$. We then apply Lemma 6.4 to ζ. Note that both $\mathscr{Z}^r_{\overline{\mathcal{E}}}$ and $\mathscr{Z}^r_{\mathcal{E}}[\mathcal{K}]$ are quasi-smooth by Lemma 5.25 because they are open-closed in $\mathrm{Sht}^r_{\mathscr{M}(i,n)}$ and $\mathrm{Sht}^r_{\mathscr{M}(m,n)}$ respectively. By the proof of Lemma 6.10, $\mathbf{L}_\iota \cong (\mathcal{V}^r_{\mathcal{K}})^*[2]$. By the base change property of cotangent complexes, $\mathbf{L}_\zeta \cong \pi^*_{\overline{\mathcal{E}}}(\mathcal{V}^r_{\mathcal{K}})^*[2]$, so $\mathbf{T}_\zeta[2] \cong \pi^*_{\overline{\mathcal{E}}}\mathcal{V}^r_{\mathcal{K}}$. Now the formula (6.10) follows from Lemma 6.4. □

Remark 6.15. In the case where X' is disconnected, the sub-bundles $\mathcal{K} \subset \mathcal{E}$ occurring in the "decomposition according to the kernel" need not have the same rank on the two components of $X' = X \sqcup X$. Hence, in that case one needs to replace the unions over $\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(m-i)'}(k)$ above by unions over all sub-bundles $\mathcal{K} \subset\subset \mathcal{E}$, and similarly replace the unions over quotients $\overline{\mathcal{E}} = \mathcal{E}/\mathcal{K} \in \mathrm{Bun}_{\mathrm{GL}(i)'}(k)$ by unions over all quotients $\mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}$. With these adjustments, the proof goes through exactly as above.

7. Linear Invariance

In this section we prove various "functoriality" results for the virtual fundamental cycles $[\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}]$, regarding their compatibility with respect to morphisms induced by gerbe maps $BH_1 \to BH_1'$ and $BH_2 \to BH_2'$.

Throughout this section we fix a line bundle $\mathfrak{L}$ on X and all Hermitian bundles will be $\mathfrak{L}$-twisted, all unitary gerbes will be twisted by $\mathfrak{L}$, etc. For conciseness we suppress this from the notation.

For example, we will prove the following property, which resolves the function field analogue of [Kud04, Problem 5].

Theorem 7.1 (Linear Invariance for special cycles)**.** *Let $\mathcal{E}$ be a rank m vector bundle on X' admitting a decomposition $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \ldots \oplus \mathcal{E}_j$ with $\mathcal{E}_i$ having rank m_i. Let $a_i \in \mathcal{A}_{\mathcal{E}_i}(k)$, and $a \in \mathcal{A}_{\mathcal{E}}(k)$ whose restriction to $\mathcal{E}_i$ is a_i, i.e.,*

$$\text{the composition } \mathcal{E}_i \to \mathcal{E} \xrightarrow{a} \sigma^*\mathcal{E}^\vee \to \sigma^*\mathcal{E}_i^\vee \text{ is } a_i \text{ for each } 1 \leq i \leq j. \tag{7.1}$$

Then we have an equality of cycle classes in $\mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a))$,

$$\Big([\mathcal{Z}^r_{\mathcal{E}_1}(a_1)] \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{E}_2}(a_2)] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{E}_j}(a_j)]\Big)|_{\mathcal{Z}^r_{\mathcal{E}}(a)} = [\mathcal{Z}^r_{\mathcal{E}}(a)]$$

where $(\cdot)|_{\mathcal{Z}^r_{\mathcal{E}}(a)}$ *denotes the projection to the corresponding component of the open-closed decomposition*

$$\mathcal{Z}^r_{\mathcal{E}_1}(a_1) \overset{\mathrm{cl}}{\times}_{\mathrm{Sht}^r_{U(n)}} \mathcal{Z}^r_{\mathcal{E}_2}(a_2) \overset{\mathrm{cl}}{\times}_{\mathrm{Sht}^r_{U(n)}} \ldots \overset{\mathrm{cl}}{\times}_{\mathrm{Sht}^r_{U(n)}} \mathcal{Z}^r_{\mathcal{E}_j}(a_j) = \coprod_{a \text{ satisfying } (7.1)} \mathcal{Z}^r_{\mathcal{E}}(a).$$

We call this property "Linear Invariance" in analogy to a result of Howard [How19], which could be viewed as a mixed characteristic local analogue of the special case where $\mathcal{E}_1, \ldots, \mathcal{E}_j$ are all line bundles. It is closely related to functoriality in the "H_1-variable" (cf. §7.1 below). Functoriality in the "H_2-variable", explained in §7.2 below, will also be used in the next part in order to compute numerical evidence for modularity.

7.1. **Functoriality in H_1.** Let $BH_1 \xrightarrow{\phi} BH_1' \to B\,\mathrm{GL}(m)'$ and $BH_2 \to B\,\mathrm{GL}(n)'$ be maps of smooth gerbes over X. Then we have a map of the corresponding derived Hitchin stacks $\mathscr{M}_{H_1,H_2} \to \mathscr{M}_{H_1',H_2}$, whose classical truncation is $\mathcal{M}_{H_1,H_2} \to \mathcal{M}_{H_1',H_2}$. Assume further that BH_2 is unitary type or $B\,\mathrm{GL}(n)'$, with the standard map to $B\,\mathrm{GL}(n)'$. Then we get induced maps $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}} \to \mathrm{Hk}^r_{\mathscr{M}_{H_1',H_2}}$, $\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}} \to \mathrm{Sht}^r_{\mathscr{M}_{H_1',H_2}}$, etc.

Lemma 7.2. *In the situation above, ϕ induces isomorphisms (where fibered products are derived)*

(1) $\mathscr{M}_{H_1,H_2} \cong \mathscr{M}_{H_1',H_2} \times_{\mathrm{Bun}_{H_1'}} \mathrm{Bun}_{H_1}$, *and*
(2) $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}} \cong \mathrm{Hk}^r_{\mathscr{M}_{H_1',H_2}} \times_{\mathrm{Bun}_{H_1'}} \mathrm{Bun}_{H_1}$.

Proof. Immediate from the definitions. □

Proposition 7.3. *In the situation above, ϕ induces an isomorphism of derived stacks*

$$\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}} \cong \mathrm{Sht}^r_{\mathscr{M}_{H_1',H_2}} \times_{\mathrm{Bun}_{H_1'}(k)} \mathrm{Bun}_{H_1}(k),$$

so that

$$[\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}] = [\mathrm{Sht}^r_{\mathscr{M}_{H_1',H_2}}] \cdot_{\mathrm{Bun}_{H_1'}(k)} [\mathrm{Bun}_{H_1}(k)] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}).$$

Proof. Abbreviate $\mathscr{M} := \mathscr{M}_{H_1,H_2}$ and $\mathscr{M}' := \mathscr{M}_{H_1',H_2}$. Consider the commutative diagram below.

$$\begin{tikzcd}
\mathscr{M}' \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] \arrow[d] & \mathscr{M}' \times \mathscr{M}' \arrow[d] & \mathrm{Hk}^r_{\mathscr{M}'} \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)"'] \arrow[d] \\
\mathrm{Bun}_{H_1'} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathrm{Bun}_{H_1'} \times \mathrm{Bun}_{H_1'} & \mathrm{Bun}_{H_1'} \arrow[l, "\Delta"'] \\
\mathrm{Bun}_{H_1} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] \arrow[u] & \mathrm{Bun}_{H_1} \times \mathrm{Bun}_{H_1} \arrow[u] & \mathrm{Bun}_{H_1} \arrow[l, "\Delta"'] \arrow[u]
\end{tikzcd} \tag{7.2}$$

The derived fibered products along the rows of (7.2) are (using Lemma 6.12)

$$\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}'} \arrow[d] \\
\mathrm{Bun}_{H_1'}(k) \\
\mathrm{Bun}_{H_1}(k) \arrow[u]
\end{tikzcd} \tag{7.3}$$

Each term is quasi-smooth by Lemma 5.25, and moreover $\mathrm{Bun}_{H_1'}(k)$ and $\mathrm{Bun}_{H_1}(k)$ are smooth.

Using Lemma 7.4, we compute that the derived fibered products along the columns of (7.2) are

$$\mathscr{M} \xrightarrow{(\mathrm{Id},\mathrm{Frob})} \mathscr{M} \times \mathscr{M} \xleftarrow{(\mathrm{pr}_0,\mathrm{pr}_r)} \mathrm{Hk}^r_{\mathscr{M}} \tag{7.4}$$

The same proof as for [YZ17, Lemma A.9] gives canonical isomorphisms of derived stacks between the derived fibered products of (7.3) and (7.4). The derived fibered product of (7.4) is $\mathrm{Sht}^r_{\mathscr{M}}$. We then conclude by applying (6.2) to (7.3). □

7.2. **Functoriality in H_2.** Let $BH_1 \to B\,\mathrm{GL}(m)'$, and $BH_2 \xrightarrow{\phi} BH_2' \to B\,\mathrm{GL}(n)'$ be maps of smooth gerbes over X. Then we have a map of the corresponding derived Hitchin stacks $\mathscr{M}_{H_1,H_2} \to \mathscr{M}_{H_1,H_2'}$, whose classical truncation is $\mathcal{M}_{H_1,H_2} \to \mathcal{M}_{H_1,H_2'}$. Assume further that BH_2 and BH_2' are unitary type or $B\,\mathrm{GL}(n)'$, with the standard map to $B\,\mathrm{GL}(n)'$. Then these induce $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}} \to \mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2'}}$, $\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}} \to \mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2'}}$, etc.

Lemma 7.4. *In the situation above, ϕ induces isomorphisms (where fibered products are derived)*

(1) $\mathscr{M}_{H_1,H_2} \cong \mathscr{M}_{H_1,H_2'} \times_{\mathrm{Bun}_{H_2'}} \mathrm{Bun}_{H_2}$, *and*

(2) $\mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2}} \cong \mathrm{Hk}^r_{\mathscr{M}_{H_1,H_2'}} \times_{\mathrm{Hk}^r_{H_2'}} \mathrm{Hk}^r_{H_2}$.

Proof. Immediate from the definitions. □

Proposition 7.5. *Then ϕ induces an isomorphism of derived stacks*

$$\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}} \cong \mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2'}} \times_{\mathrm{Sht}^r_{H_2'}} \mathrm{Sht}^r_{H_2},$$

(with the RHS a derived fibered product), so that

$$[\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}] = [\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2'}}] \cdot_{\mathrm{Sht}^r_{H_2'}} [\mathrm{Sht}^r_{H_2}] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}).$$

Proof. Abbreviate $\mathscr{M} := \mathscr{M}_{H_1,H_2}$ and $\mathscr{M}' := \mathscr{M}_{H_1,H_2'}$. Consider the commutative diagram below.

$$\begin{tikzcd} \mathscr{M}' \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] \arrow[d] & \mathscr{M}' \times \mathscr{M}' \arrow[d] & \mathrm{Hk}^r_{\mathscr{M}'} \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)"'] \arrow[d] \\ \mathrm{Bun}_{H_2'} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathrm{Bun}_{H_2'} \times \mathrm{Bun}_{H_2'} & \mathrm{Hk}^r_{H_2'} \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)"'] \\ \mathrm{Bun}_{H_2} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] \arrow[u] & \mathrm{Bun}_{H_2} \times \mathrm{Bun}_{H_2} \arrow[u] & \mathrm{Hk}^r_{H_2} \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)"'] \arrow[u] \end{tikzcd} \tag{7.5}$$

The derived fibered products along the rows of (7.5) are

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathscr{M}'} \arrow[d] \\ \mathrm{Sht}^r_{H_2'} \\ \mathrm{Sht}^r_{H_2} \arrow[u] \end{tikzcd} \tag{7.6}$$

Each term is quasi-smooth by Lemma 5.25, and moreover $\mathrm{Sht}^r_{H_2'}$ and $\mathrm{Sht}^r_{H_2}$ are smooth.

Using Lemma 7.2, we compute that the derived fibered products along the columns of (7.5) are

$$\mathscr{M} \xrightarrow{(\mathrm{Id},\mathrm{Frob})} \mathscr{M} \times \mathscr{M} \xleftarrow{(\mathrm{pr}_0,\mathrm{pr}_r)} \mathrm{Hk}^r_{\mathscr{M}} \tag{7.7}$$

The same proof as for [YZ17, Lemma A.9] gives canonical isomorphisms of derived stacks between the derived fibered products of (7.6) and (7.7). The derived fibered product of (7.7) is $\mathrm{Sht}^r_{\mathscr{M}}$. We then conclude by applying (6.2) to (7.6). □

Example 7.6. Consider the situation of Example 4.18, with Y be another smooth projective curve over $\mathbf{F}_q$, and $\theta : Y \to X$ be a map of degree n, possibly ramified. Let $\theta' : Y' \to X'$ (resp. $\nu' : Y' \to Y$) be the base change of θ (resp. ν).

We define the moduli of shtukas[13] $\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}$ to be $\mathrm{Sht}^r_{H_2}$ for $BH_2 = BR_{Y/X}U(1)_{\theta^*\mathfrak{L}}$ (defined in §3.1.3). Take $BH_1 \xrightarrow{=} \mathrm{GL}(m)'$, $BH_2 \to BH_2' = U(n)_{\mathfrak{L}}$ the canonical map. Proposition 7.5 implies that

$$[\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}] = [\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2'}}] \cdot_{\mathrm{Sht}^r_{H_2'}} [\mathrm{Sht}^r_{H_2}] \in \mathrm{Ch}_*(\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}}).$$

[13]Since Y may be disconnected, this is not covered by our previous definitions.

The right hand side is, by Theorem 6.5,

$$\bigoplus_{\mathcal{E}\in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} [\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}] \cdot_{\mathrm{Sht}^r_{U(n),\mathfrak{L}}} [\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}]$$

and the left hand side is

$$\bigoplus_{\mathcal{E}\in \mathrm{Bun}_{\mathrm{GL}(m)'}(k)} [\mathcal{Z}^r_{\theta'^*\mathcal{E},\theta^*\mathfrak{L}}],$$

where the summands are the special cycles defined relative to Y'/Y. In particular, projecting to the component indexed by $(\mathcal{E},a)$ (where $a\in\mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)$) yields

$$\bigoplus_{\substack{\widetilde{a}\in\mathcal{A}_{\theta'^*\mathcal{E},\theta^*\mathfrak{L}}(k)\\ \mathrm{tr}(\widetilde{a})=a}} [\mathcal{Z}^r_{\theta'^*\mathcal{E},\theta^*\mathfrak{L}}(\widetilde{a})] = [\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)] \cdot_{\mathrm{Sht}^r_{U(n),\mathfrak{L}}} [\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}].$$

Here the trace map is defined as follows. Recall that $\mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)$ is the set of Hermitian maps $a:\mathcal{E}\to\sigma^*\mathcal{E}^\vee\otimes\nu^*\mathfrak{L}$, i.e., σ-invariant elements in $\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E},\sigma^*\mathcal{E}^\vee\otimes\nu^*\mathfrak{L})$. Having defined $(\theta'^*\mathcal{E})^\vee=(\theta'^*\mathcal{E})^*\otimes\omega_{Y'}$, we have natural isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_{Y'}}(\theta'^*\mathcal{E},\sigma^*(\theta'^*\mathcal{E})^\vee\otimes\nu'^*(\theta^*\mathfrak{L}))\\
\cong\ &\mathrm{Hom}_{\mathcal{O}_{Y'}}(\theta'^*\mathcal{E},\theta'^*(\sigma^*\mathcal{E}^*\otimes\nu^*\mathfrak{L})\otimes\omega_{Y'})\\
\cong\ &\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E},\sigma^*\mathcal{E}^*\otimes\nu^*\mathfrak{L}\otimes\theta'_*\omega_{Y'}) \quad \text{(By adjunction).}
\end{aligned}$$

Post-composition with the trace map

$$\mathrm{tr}_{Y'/X'}\colon \theta_*\omega_{Y'}\to\omega_{X'}$$

defines a map

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E},\sigma^*\mathcal{E}^*\otimes\nu^*\mathfrak{L}\otimes\theta'_*\omega_{Y'})\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E},\sigma^*\mathcal{E}^*\otimes\nu^*\mathfrak{L}\otimes\omega_{X'})$$

and hence a trace map

$$\mathrm{tr}\colon\ \mathrm{Hom}_{\mathcal{O}_{Y'}}(\theta'^*\mathcal{E},\sigma^*(\theta'^*\mathcal{E})^\vee\otimes\nu'^*(\theta^*\mathfrak{L}))\longrightarrow \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E},\sigma^*\mathcal{E}^\vee\otimes\nu^*\mathfrak{L}) \tag{7.8}$$

It is easy to see that the map (7.8) preserves Hermitian elements and therefore defines the desired trace map

$$\mathrm{tr}\colon \mathcal{A}_{\theta'^*\mathcal{E},\theta^*\mathfrak{L}}(k)\longrightarrow\mathcal{A}_{\mathcal{E},\mathfrak{L}}(k)\ . \tag{7.9}$$

7.3. **Proof of Linear Invariance.** We will work up to the proof of Theorem 7.1 with two intermediate steps. Below, we repeatedly use without further comment that the intersection product of §6.1.11 is compatible with that of [YZ17, §A.1.4], by Lemma 6.2.

Lemma 7.7. *Fix $m\le n$ and $m=m_1+\ldots+m_j$. Let $BH_1^{(1)}=B\,\mathrm{GL}(m_1)', BH_1^{(2)}=B\,\mathrm{GL}(m_2)',\ldots,BH_1^{(j)}=B\,\mathrm{GL}(m_j)'$, $BH_1=BH_1^{(1)}\times\ldots\times BH_1^{(j)}$, and $BH_2=BU(n)$. Define the derived Hitchin stacks $\mathscr{M}_{H_1^{(i)},H_2}$ using the identity map $BH_1^{(i)}\xrightarrow{=}B\,\mathrm{GL}(m_i)'$ and the standard map $BH_2\to B\,\mathrm{GL}(n)'$, and $\mathscr{M}_{H_1,H_2}$ using the standard block diagonal map $BH_1\to B\,\mathrm{GL}(m)'$ and the standard map $BH_2\to B\,\mathrm{GL}(n)'$.*

Then we have the following equality in $\mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{\mathcal{M}_{H_1,H_2}})$:

$$[\mathrm{Sht}^r_{\mathscr{M}_{H_1^{(1)},H_2}}]\cdot_{\mathrm{Sht}^r_{U(n)}}\ldots\cdot_{\mathrm{Sht}^r_{U(n)}}[\mathrm{Sht}^r_{\mathscr{M}_{H_1^{(j)},H_2}}]=[\mathrm{Sht}^r_{\mathscr{M}_{H_1,H_2}}]. \tag{7.10}$$

Proof. For $1\le i\le j$ we abbreviate $\mathscr{M}^{(i)}:=\mathscr{M}_{H_1^{(i)},H_2}$. Consider the diagram below.

$$\begin{tikzcd}
\mathscr{M}^{(1)}\times\ldots\times\mathscr{M}^{(j)} \arrow[r, "(\mathrm{Id},\mathrm{Frob})^j"] \arrow[d] & (\mathscr{M}^{(1)}\times\mathscr{M}^{(1)})\times\ldots\times(\mathscr{M}^{(j)}\times\mathscr{M}^{(j)}) \arrow[d] & \mathrm{Hk}^r_{\mathscr{M}^{(1)}}\times\ldots\times\mathrm{Hk}^r_{\mathscr{M}^{(j)}} \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)^j"'] \arrow[d] \\
(\mathrm{Bun}_{U(n)})^j \arrow[r, "(\mathrm{Id},\mathrm{Frob})^j"] & (\mathrm{Bun}_{U(n)}\times\mathrm{Bun}_{U(n)})^j & (\mathrm{Hk}^r_{U(n)})^j \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)^j"'] \\
\mathrm{Bun}_{U(n)} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] \arrow[u, "\Delta_{\mathrm{Bun}_{U(n)}}"] & \mathrm{Bun}_{U(n)}\times\mathrm{Bun}_{U(n)} \arrow[u, "\Delta_{\mathrm{Bun}_{U(n)}\times\mathrm{Bun}_{U(n)}}"] & \mathrm{Hk}^r_{U(n)} \arrow[l, "(\mathrm{pr}_0,\mathrm{pr}_r)"'] \arrow[u, "\Delta"']
\end{tikzcd} \tag{7.11}$$

Using Lemma 5.13, we see that the derived fibered products along the rows of (7.11) are

$$
\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}^{(1)}} \times \ldots \times \mathrm{Sht}^r_{\mathscr{M}^{(j)}} \arrow[d] \\
(\mathrm{Sht}^r_{U(n)})^j \\
\mathrm{Sht}^r_{U(n)} \arrow[u, "\Delta"]
\end{tikzcd}
\tag{7.12}
$$

Each term is quasi-smooth by Lemma 5.25, and $(\mathrm{Sht}^r_{U(n)})^j$ and $\mathrm{Sht}^r_{U(n)}$ are smooth by [FYZ21, Lemma 6.9(2)].

The derived fibered products along the columns of (7.11) are

$$\mathscr{M} \xrightarrow{(\mathrm{Id},\mathrm{Frob})} \mathscr{M} \times \mathscr{M} \xleftarrow{(\mathrm{pr}_0,\mathrm{pr}_r)} \mathrm{Hk}^r_{\mathscr{M}} \tag{7.13}$$

where we abbreviated $\mathscr{M} := \mathscr{M}_{H_1,H_2}$.

The same proof as for [YZ17, Lemma A.9] gives canonical isomorphisms of derived stacks between the derived fibered products of (7.12) and (7.13). The derived fibered product of (7.12) is $\mathrm{Sht}^r_{\mathscr{M}}$, which is quasi-smooth by Lemma 5.25. We then conclude by applying (6.2) to (7.12). $\square$

Proof of Theorem 7.1. We have by definition

$$[\mathrm{Sht}^r_{\mathscr{M}_{\mathrm{GL}(m)',U(n)}}] = \bigoplus_{\mathcal{E}\in\mathrm{Bun}_{\mathrm{GL}(m)'}(k)} [\mathscr{Z}^r_{\mathcal{E}}] \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{\mathcal{M}_{\mathrm{GL}(m)',U(n)}}), \tag{7.14}$$

where the virtual fundamental classes are defined because $\mathrm{Sht}^r_{\mathscr{M}_{\mathrm{GL}(m)',U(n)}}$ is quasi-smooth.

Similarly, we have for each $i = 1, \ldots, j$ that

$$[\mathrm{Sht}^r_{\mathscr{M}_{\mathrm{GL}(m_i)',U(n)}}] = \bigoplus_{\mathcal{E}_i\in\mathrm{Bun}_{\mathrm{GL}(m_i)'}(k)} [\mathscr{Z}^r_{\mathcal{E}_i}]. \tag{7.15}$$

Let H_1 be the subgroup $\mathrm{GL}(m_1)' \times \cdots \times \mathrm{GL}(m_j)'$ of $\mathrm{GL}(m)'$ as in the hypotheses of Lemma 7.7. By Lemma 7.7, we then have

$$[\mathrm{Sht}^r_{\mathscr{M}_{H_1,U(n)}}] = \bigoplus_{\mathcal{E}_i\in\mathrm{Bun}_{\mathrm{GL}(m_i)'}(k), i=1,\ldots,j} [\mathscr{Z}^r_{\mathcal{E}_1}] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathscr{Z}^r_{\mathcal{E}_j}]. \tag{7.16}$$

Applying Proposition 7.3 to the inclusion $H_1 \hookrightarrow H_1' = \mathrm{GL}(m')$ we get

$$[\mathrm{Sht}^r_{\mathscr{M}_{H_1,U(n)}}] = [\mathrm{Sht}^r_{\mathscr{M}_{\mathrm{GL}(m)',U(n)}}] \cdot_{\mathrm{Bun}_{\mathrm{GL}(m)'}(k)} [\mathrm{Bun}_{H_1}(k)] \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{\mathcal{M}_{H_1,U(n)}}). \tag{7.17}$$

Projecting the above equality to the component indexed by $(\mathcal{E}_1, \ldots, \mathcal{E}_j) \in \mathrm{Bun}_{H_1}(k)$ and using (7.14) and (7.16) yields

$$[\mathscr{Z}^r_{\mathcal{E}_1}] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathscr{Z}^r_{\mathcal{E}_j}] = [\mathscr{Z}^r_{\mathcal{E}}]. \tag{7.18}$$

By Theorem 6.5, we have

$$[\mathcal{Z}^r_{\mathcal{E}}] = [\mathscr{Z}^r_{\mathcal{E}}] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}),$$

and similarly (using the compatibility of §6.1.11 and the refined intersection product) we have

$$[\mathcal{Z}^r_{\mathcal{E}_1}] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{E}_j}] = [\mathscr{Z}^r_{\mathcal{E}_1}] \cdot_{\mathrm{Sht}^r_{U(n)}} \cdots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathscr{Z}^r_{\mathcal{E}_j}].$$

Putting these equalities into (7.18) and then projecting to the component indexed by a gives the result. $\square$

8. Compatibility with the cycle classes of [FYZ21]

Let $\mathcal{E}$ be a rank m vector bundle on X' and $a \in \mathcal{A}^{\mathrm{ns}}_{\mathcal{E}}(k)$ (i.e., a is *non-singular*). In [FYZ21] we gave a different definition of the virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ in the case either $\mathcal{E}$ is a direct sum of line bundles and or $\operatorname{rank}\mathcal{E} = n$. The goal of this section is to prove that these cycles defined in [FYZ21] agree with the same-named cycles defined in Definition 4.4. Although [FYZ21] was written with the twisting line bundle $\mathfrak{L}$ being trivial, a completely analogous construction applies with any $\mathfrak{L}$. We shall fix a choice of $\mathfrak{L}$ throughout this section and suppress it from the notation.

8.1. **Corank one special cycles.** In this subsection, we show that the injective part of any corank one special cycle, either for $BH_2 = BU(n)_{\mathfrak{L}}$ or $BH_2 = B\,\mathrm{GL}(n)'$, is LCI.

Throughout this subsection we abbreviate $\mathcal{M}' := \mathcal{M}^{\circ}_{\mathrm{GL}(1)',\mathrm{GL}(n)'}$.

Proposition 8.1. *(1) The stack $\mathrm{Sht}^r_{\mathcal{M}'}$ is LCI of pure dimension $r(2n-2)$. In particular, for any line bundle $\mathcal{L}$ on X', $\mathcal{Z}^{r,\circ}_{\mathcal{L},\mathrm{GL}(n)'}$ is LCI of pure dimension $r(2n-2)$.*

(2) The class $[\mathrm{Sht}^r_{\mathcal{M}'}] \in \mathrm{Ch}_{r(2n-2)}(\mathrm{Sht}^r_{\mathcal{M}'})$ from Definition 3.16 agrees with

$$\sum_{\mathcal{L} \in \mathrm{Bun}_{\mathrm{GL}(1)'}(k)} [\mathcal{Z}^{r,\circ}_{\mathcal{L}}]^{\mathrm{naive}} \in \mathrm{Ch}_{r(2n-2)}(\mathrm{Sht}^r_{\mathcal{M}'}),$$

where $[\mathcal{Z}^{r,\circ}_{\mathcal{L}}]^{\mathrm{naive}} \in \mathrm{Ch}_{r(2n-2)}(\mathrm{Sht}^r_{\mathcal{M}'})$ is the fundamental class of that component.

Proof. (1) We may write $\mathrm{Sht}^r_{\mathcal{M}'}$ by the Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^r_{\mathcal{M}'} \arrow[r] \arrow[d] & (\mathrm{Hk}^1_{\mathcal{M}'})^r \arrow[d, "(p_0,p_1)^r"] \\ \mathcal{M}'^r \arrow[r, "\Phi^r_{\mathcal{M}'}"] & \mathcal{M}'^{2r} \end{tikzcd} \tag{8.1}$$

By the smoothness of $\mathcal{M}'$ and $\mathrm{Hk}^1_{\mathcal{M}'}$ and the relative dimension calculations in Proposition 3.11 and Lemma 3.14, we see that $\mathrm{Sht}^r_{\mathcal{M}'}$ has local dimension $\geq r(2n-2)$ everywhere.

On the other hand, we will show in Proposition 8.6(3) and Corollary 8.5 that $\dim \mathcal{Z}^{r,\circ}_{\mathcal{L},\mathrm{GL}(n)'} \leq r(2n-2)$ for any line bundle $\mathcal{L}$, hence $\dim \mathrm{Sht}^r_{\mathcal{M}'} \leq r(2n-2)$. Combining this with the lower bound of local dimension given above, and the fact that $\mathrm{Sht}^r_{\mathcal{M}'}$ is a fibered product of smooth stacks, we conclude that $\mathrm{Sht}^r_{\mathcal{M}'}$ is LCI of pure dimension $r(2n-2)$.

(2) We have seen that $\mathrm{Sht}^r_{\mathcal{M}'}$ is LCI and the fibered product in (8.1) exhibits it as a proper intersection, so the claim follows from [Ful98, Proposition 7.1][14] □

We may now establish a result that was promised in [FYZ21, Remark 7.10].

Corollary 8.2. *The Cartesian square*

$$\begin{tikzcd} \mathcal{Z}^{r,\circ}_{\mathcal{L}} \arrow[r] \arrow[d] & \mathcal{Z}^{r,\circ}_{\mathcal{L},\mathrm{GL}(n)'} \arrow[d] \\ \mathrm{Sht}^r_{U(n)} \arrow[r] & \mathrm{Sht}^r_{\mathrm{GL}(n)'} \end{tikzcd}$$

is a proper intersection. Hence for all $a \in \mathcal{A}_{\mathcal{L}}(k)$ [15] we have that $\mathcal{Z}^r_{\mathcal{L}}(a)^{\circ}$ is LCI of dimension $r(n-1)$.

In particular, $[\mathcal{Z}^r_{\mathcal{L}}(a)^{\circ}]^{\mathrm{naive}} = [\mathcal{Z}^r_{\mathcal{L}}(a)^{\circ}] \in \mathrm{Ch}_*(\mathcal{Z}^{r,\circ}_{\mathcal{L}})$, the latter being as in Definition 4.4.

Proof. Lemma 2.12 implies that $\coprod_{a \in \mathcal{A}(k)} \mathcal{Z}^r_{\mathcal{L}}(a)^{\circ}$ is the fibered product of $\mathcal{Z}^{r,\circ}_{\mathcal{L},\mathrm{GL}(n)'}$ and $\mathrm{Sht}^r_{U(n)}$ over $\mathrm{Sht}^r_{\mathrm{GL}(n)'}$. These have dimensions $r(2n-2)$, rn, and $r(2n-1)$ respectively, as established in Proposition 8.1, [FYZ21, Lemma 6.9(2)], and Lemma 2.6 respectively. Since $\mathrm{Sht}^r_{U(n)} \to \mathrm{Sht}^r_{\mathrm{GL}(n)'}$ is a regular local immersion of smooth Deligne-Mumford stacks, this implies that the fibered product has dimension $\geq r(n-1)$.

[14] Strictly speaking, the statement in *loc. cit.* is for schemes, so we apply it after adding truncating and adding sufficient level structure, and then taking a limit over truncations. When adding level structure along a finite subscheme $D \subset X$, we ask that the leg maps avoid D, so this lies over an open substack of $\mathrm{Sht}^r_{U(n)}$. As D varies, these substacks form an open cover as D varies. The equality in question can be checked on this open cover because it is an equality of *top-dimensional* cycles.

[15] Note our $\mathcal{A}_{\mathcal{L}}(k)$ is denoted $\mathcal{A}^{\mathrm{all}}_{\mathcal{L}}(k)$ in [FYZ21, §7], i.e., it includes singular a.

On the other hand, it was already established in [FYZ21, Proposition 9.1, 9.5] that $\dim \mathcal{Z}^r_{\mathcal{L}}(a)^\circ \le r(n-1)$, so equality holds. As this presentation realizes $\mathcal{Z}^{r,\circ}_{\mathcal{L}}$ as the pullback of the LCI Deligne-Mumford stack $\mathcal{Z}^{r,\circ}_{\mathcal{L},\mathrm{GL}(n)'}$ against a regular local immersion, we conclude that $\mathcal{Z}^{r,\circ}_{\mathcal{L}}$ is also LCI.

For the last statement, we use that $\mathrm{Sht}^r_{U(n)} \to \mathrm{Sht}^r_{\mathrm{GL}(n)'}$ is a regular local immersion (as both are smooth) and Proposition 8.1.[16] □

8.2. **Agreement of definitions.** Let $\mathcal{E}$ be a rank m vector bundle on X'. In [FYZ21] we gave a different definition of the virtual fundamental class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ when $a \in \mathcal{A}^{\mathrm{ns}}_{\mathcal{E}}(k)$ (i.e., a is *non-singular*), in the following cases:

(1) $\mathcal{E} \approx \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_m$ is a direct sum of line bundles on X' [FYZ21, §7.8], or
(2) $\operatorname{rank} \mathcal{E} = n$ [FYZ21, §7.9].

We denote the class defined in [FYZ21] by $[\mathcal{Z}^r_{\mathcal{E}}(a)]^{\mathrm{old}}$. In this section we prove that $[\mathcal{Z}^r_{\mathcal{E}}(a)]^{\mathrm{old}}$ agrees with the class $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ defined in Definition 4.4.

Proposition 8.3. *For $\mathcal{E}$ as in Cases (1) or (2) above and $a \in \mathcal{A}^{\mathrm{ns}}_{\mathcal{E}}(k)$, we have*

$$[\mathcal{Z}^r_{\mathcal{E}}(a)]^{\mathrm{old}} = [\mathcal{Z}^r_{\mathcal{E}}(a)] \in \mathrm{Ch}_{r(n-m)}(\mathcal{Z}^r_{\mathcal{E}}(a)).$$

Proof. (1) If $\mathcal{E} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_m$, [FYZ21, §7.8] defined $[\mathcal{Z}^r_{\mathcal{E}}(a)]^{\mathrm{old}}$ to be the projection of $[\mathcal{Z}^r_{\mathcal{L}_1}(a_1)^\circ]^{\mathrm{naive}} \cdot_{\mathrm{Sht}^r_{U(n)}} \ldots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{L}_m}(a_m)^\circ]^{\mathrm{naive}}$ to the components indexed by a (where $a_i \in \mathcal{A}_{\mathcal{L}_i(k)}$ is the restriction of a to $\mathcal{L}_i$). By Theorem 7.1, $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is described in the same way with respect to $[\mathcal{Z}^r_{\mathcal{L}_1}(a_1)] \cdot_{\mathrm{Sht}^r_{U(n)}} \ldots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{L}_m}(a_m)]$. Moreover, since a is non-singular, $\mathcal{Z}^r_{\mathcal{E}}(a)$ is contained in the fiber product of $\mathcal{Z}^r_{\mathcal{L}_i}(a_i)^\circ$ over $\mathrm{Sht}^r_{U(n)}$, hence $[\mathcal{Z}^r_{\mathcal{E}}(a)]$ is the projection of $[\mathcal{Z}^r_{\mathcal{L}_1}(a_1)^\circ] \cdot_{\mathrm{Sht}^r_{U(n)}} \ldots \cdot_{\mathrm{Sht}^r_{U(n)}} [\mathcal{Z}^r_{\mathcal{L}_m}(a_m)^\circ]$ to the components indexed by a. So we are reduced to showing that $[\mathcal{Z}_{\mathcal{L}_i}(a_i)^\circ]^{\mathrm{naive}} = [\mathcal{Z}^r_{\mathcal{L}_i}(a_i)^\circ]$ (here we allow $a_i = 0$). This follows from Corollary 8.2.

(2) Suppose $\operatorname{rank} \mathcal{E} = n$. Take $BH_1 = B\,\mathrm{GL}(n)'$, $BH_2 = BU(n)_{\mathfrak{L}}$, and form $\mathscr{M}^{\mathrm{ns}} := \mathscr{M}^\circ_{H_1,H_2}|_{\mathscr{A}^{\mathrm{ns}}_{H_1}}$ with respect to the identity map $H_1 \xrightarrow{=} \mathrm{GL}(n)'$ and the standard map $BH_2 \to B\,\mathrm{GL}(n)'$. Let $\mathcal{M}^{\mathrm{ns}}$ be the classical truncation of $\mathscr{M}^{\mathrm{ns}}$.

In terms of the open-closed decomposition

$$\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{ns}}} = \coprod_{\mathcal{E} \in \mathrm{Bun}_{\mathrm{GL}(n)'}(k)} \coprod_{a \in \mathcal{A}^{\mathrm{ns}}_{\mathcal{E}}(k)} \mathcal{Z}^r_{\mathcal{E}}(a),$$

[FYZ21, Theorem 10.1] establishes that $[\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{ns}}}]$, defined as in [FYZ21, Definition 8.16] using the (classical) Gysin pullback, is the direct sum of $[\mathcal{Z}^r_{\mathcal{E}}(a)]^{\mathrm{old}}$ over all $(\mathcal{E}, a)$ as above. Hence it suffices to show that $[\mathrm{Sht}^r_{\mathscr{M}^{\mathrm{ns}}}] = [\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{ns}}}] \in \mathrm{Ch}_0(\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{ns}}})$.

Rewriting $\mathrm{Hk}^r_{\mathscr{M}^{\mathrm{ns}}}$ as the derived fibered product $(\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}})^r \times_{\mathscr{M}^{\mathrm{ns},2r}} \mathscr{M}^{\mathrm{ns},r}$, we see that $\mathrm{Sht}^r_{\mathscr{M}^{\mathrm{ns}}}$ may be rewritten as the fibered product below, where $\Phi^r_{\mathscr{M}^{\mathrm{ns}}}$ is as in Definition 3.20.

$$\begin{tikzcd}
\mathrm{Sht}^r_{\mathscr{M}^{\mathrm{ns}}} \arrow[r] \arrow[d] & (\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}})^r \arrow[d] \\
\mathscr{M}^{\mathrm{ns},r} \arrow[r, "\Phi^r_{\mathscr{M}^{\mathrm{ns}}}"] & \mathscr{M}^{\mathrm{ns},2r}
\end{tikzcd} \tag{8.2}$$

By Corollaries 5.18(2) and 5.22(2), the canonical maps $\mathcal{M}^{\mathrm{ns}} := (\mathscr{M}^{\mathrm{ns}})_{\mathrm{cl}} \to \mathscr{M}^{\mathrm{ns}}$ and $\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}} \to \mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}}$ are isomorphisms of smooth stacks, so in particular $[\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}}] = [\mathrm{Hk}^1_{\mathcal{M}^{\mathrm{ns}}}]^{\mathrm{naive}}$. Lemma 6.3 then implies that $[\mathrm{Sht}^r_{\mathscr{M}^{\mathrm{ns}}}] = [(\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}})^r] \cdot_{\mathscr{M}^{\mathrm{ns},2r}} [\mathscr{M}^{\mathrm{ns},r}]$, which is the same as $(\Phi^r_{\mathscr{M}^{\mathrm{ns}}})^![(\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}})^r]$. By Lemma 6.2, $(\Phi^r_{\mathscr{M}^{\mathrm{ns}}})^![(\mathrm{Hk}^1_{\mathscr{M}^{\mathrm{ns}}})^r]$ agrees with $[\mathrm{Sht}^r_{\mathcal{M}^{\mathrm{ns}}}]$.

□

8.3. **Stratification of shtukas for $\mathcal{M}_{\mathrm{GL}(1)',\mathrm{GL}(n)'}$.** The goal of this subsection is to show that $\dim \mathcal{Z}^r_{\mathcal{L},\mathrm{GL}(n)'} \le r(2n-2)$ for line bundles $\mathcal{L}$ on X', as has been used in the proof of Proposition 8.1. The idea is similar to that of [FYZ21, §9].

[16] Strictly speaking, we need to apply the same considerations here as in Footnote 14.

Fix a line bundle $\mathcal{L}$ on X'. We define $\mathcal{Z}' := (\mathcal{Z}^r_{\mathcal{L},\mathrm{GL}(n)'})_{\overline{k}}$. Let $I_0 \sqcup I_+ \sqcup I_- \sqcup I_\pm$ be a partition of $\{1, 2, \cdots, r\}$. We denote this partition simply by $I_\bullet$. For any $N \in \mathbf{Z}_{\geq 0}$, define $\mathfrak{D}(N; I_\bullet)$ to be the moduli space of sequences of effective divisors $(D_i)_{0\leq i\leq r}$ on $X'_{\overline{k}}$ such that

(1) $\deg(D_0) \leq N$.
(2) For $? \in \{0, +, -, \pm\}$, and $i \in I_?$, the pair (D_{i-1}, D_i) belongs to the corresponding Case (?) beow
  (0) $D_i = D_{i-1}$;
  (+) $D_i = D_{i-1} + \sigma x'_i$ for some $x'_i \in X'_{\overline{k}}$;
  (-) $D_i = D_{i-1} - x'_i$ for some $x'_i \in X'_{\overline{k}}$;
  (±) $D_i = D_{i-1} - x'_i + \sigma x'_i$ for some $x'_i \in X'_{\overline{k}}$.
(3) $D_r = {}^\tau D_0$.

For $? = +, -$ or $\pm$, (D_{i-1}, D_i) determines a point $x'_i \in X_{\overline{k}}$. This gives a map.

$$(\pi_+, \pi_-, \pi_\pm)\colon \mathfrak{D}(N, I_\bullet) \to (X'_{\overline{k}})^{I_+\sqcup I_-\sqcup I_\pm}.$$

Lemma 8.4. *The map* $\pi_+ : \mathfrak{D}(N, I_\bullet) \to (X'_{\overline{k}})^{I_+}$ *is quasi-finite. In particular,* $\dim \mathfrak{D}(N, I_\bullet) \leq |I_+|$.

Proof. Let $D_\bullet \in \mathfrak{D}(N; I_\bullet)$ and $\overline{D}_i = \nu(D_i)$ be the image of D_i in $X_{\overline{k}}$. Let $x_i = \nu(x'_i)$, then $\overline{D}_i = \overline{D}_{i-1}$ if $i \in I_0 \sqcup I_\pm$, and $\overline{D}_i = \overline{D}_{i-1} + x_i$ if $i \in I_+$, $\overline{D}_i = \overline{D}_{i-1} - x_i$ if $i \in I_-$. By condition (3) above, $\overline{D}_0$ satisfies the equation

$$\overline{D}_0 + \sum_{i\in I_+} x_i = {}^\tau\overline{D}_0 + \sum_{j\in I_-} x_j. \tag{8.3}$$

By [FYZ21, Lemma 9.4], for fixed $\{x_i\}_{i\in I_+}$, there are only finitely many $\overline{D}_\bullet$ satisfying (8.3) and $\deg D_0 \leq N$. If $\overline{D}_\bullet$ is fixed then $D_\bullet$ has finitely many choices. We conclude that there are finitely many $\overline{k}$-points in $\mathfrak{D}(N; I_\bullet)$ with fixed image in $(X'_{\overline{k}})^{I_+}$. □

For a partition $I_\bullet = (I_0, I_+, I_-, I_\pm)$ of $\{1, 2, \ldots, r\}$, let $\mathcal{Z}'[N, I_\bullet]$ be the stack classifying $(\{D_i\}_{0\leq i\leq r}, \{x'_i\}_{1\leq i\leq r}, \{\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i\})$ where $(\{x'_i\}_{1\leq i\leq r}, \{\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i\}) \in \mathcal{Z}'(S)$, and $\{D_i\} \in \mathfrak{D}(N; I_\bullet)$ with image $\{x'_i\}_{i\in I_?}$ under $\pi_?$ $(? = +, -, \pm)$, and t_i extends to a saturated embedding $\mathcal{L}(D_i) \hookrightarrow \mathcal{F}_i$. Since D_i is determined by t_i, the natural map $\mathcal{Z}'[N, I_\bullet] \hookrightarrow \mathcal{Z}'$ is a locally closed immersion. As in [FYZ21, §9.2.2], we define the map

$$\pi'[N; I_\bullet]\colon \mathcal{Z}'[N, I_\bullet] \to (X'_{\overline{k}})^{I_0} \times \mathfrak{D}(N; I_\bullet).$$

Corollary 8.5 (of Lemma 8.4)**.** *When* $n = 1$, $\dim \mathcal{Z}'[N; I_\bullet] = 0$.

Proof. When $n = 1$, $\mathcal{Z}'[N; I_\bullet]$ classifies $(\{D_i\}_{0\leq i\leq r}, \{x'_i\}_{1\leq i\leq r}, \{\mathcal{L} \xrightarrow{t_i} \mathcal{F}_i\})$ such that t_i extends to an isomorphism $\mathcal{L}(D_i) \cong \mathcal{F}_i$. This implies that $I_\pm = \{1, 2, \cdots, r\}$, and the forgetful map $\mathcal{Z}'[N; I_\bullet] \to \mathfrak{D}(N; I_\bullet)$ is an isomorphism. By Lemma 8.4, $\dim \mathcal{Z}'[N; I_\bullet] = \dim \mathfrak{D}(N; I_\bullet) = 0$. □

Proposition 8.6. *Assume* $n \geq 2$.

(1) For varying $N \in \mathbf{Z}_{\geq 0}$ *and partitions* $I_\bullet$ *of* $\{1, 2, \cdots, r\}$ *with* $|I_+| = |I_-|$*, the substacks* $\mathcal{Z}'[N; I_\bullet]$ *give a partition of* $\mathcal{Z}'$.
(2) The fiber dimension of the map $\pi'[N; I_\bullet]$ *is* $\leq (2n-3)|I_0| + (2n-2)|I_+| + (n-2)|I_-| + (n-1)|I_\pm|$.
(3) We have $\dim \mathcal{Z}'[N; I_\bullet] \leq r(2n-2)$*. Moreover, the equality is achieved only when* $I_0 = \{1, 2, \cdots, r\}$*, i.e., all* D_i *are equal to the same divisor of* X' *defined over* $\mathbf{F}_q$.

Proof. (1) is clear (note that $|I_+| = |I_-|$ is implied by the assumption that $D_r = {}^\tau D_0$).

(2) The analysis is similar to that of [FYZ21, Proposition 9.1], although the cases behave differently, so let us explain how they play out.

Fix $D_\bullet \in \mathfrak{D}(N; I_\bullet)(\overline{k})$, let $\mathcal{Z}'[D_\bullet]$ be the fiber of the projection $\mathcal{Z}'[N; I_\bullet] \to \mathfrak{D}(N; I_\bullet)$ over $D_\bullet$. Let $\mathcal{M}' = \mathcal{M}^\circ_{\mathrm{GL}(1)',\mathrm{GL}(n)'}$.

Let $\mathcal{H}'[D_\bullet]$ be the substack of $\mathrm{Hk}^r_{\mathcal{M}',\overline{k}}$ classifying data $(x'_\bullet, \mathcal{L} \xrightarrow{t_i} \mathcal{F}_i)$ such that t_i extends to a map $t'_i : \mathcal{L}(D_i) \to \mathcal{F}_i$. Note that for $i \notin I_0$, the x'_i are determined by $D_\bullet$. Let $\mathcal{M}'[D_i]$ be the substack of $\mathcal{M}'_{\overline{k}}$ classifying maps $t : \mathcal{L} \to \mathcal{F}$ that extend to a saturated map $t' : \mathcal{L}(D_i) \to \mathcal{F}$. Then we have a Cartesian diagram of stacks over $\overline{k}$

$$\begin{tikzcd} \mathcal{Z}'[D_\bullet] \arrow[r] \arrow[d] & \mathcal{H}'[D_\bullet] \arrow[d, "(p_0,p_r)"] \\ \mathcal{M}'[D_0] \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathcal{M}'[D_0]\times\mathcal{M}'[D_r] \end{tikzcd} \tag{8.4}$$

Note since $D_r = {}^\tau D_0$, the Frobenius morphism sends $\mathcal{M}'[D_0]$ to $\mathcal{M}'[D_r]$. We claim that the map

$$\Pi[D_\bullet] : \mathcal{H}'[D_\bullet] \to \mathcal{M}'[D_r] \times X'^{I_0}_{\overline{k}} \tag{8.5}$$

is smooth of relative dimension $(2n-3)|I_0| + (2n-2)|I_+| + (n-2)|I_-| + (n-1)|I_\pm|$. Then by Lemma [FYZ21, Lemma 9.3], the fibers of $\mathcal{Z}'[D_\bullet] \to (X'_{\overline{k}})^{I_0}$, which are fibers of $\pi'[N; I_\bullet]$, have dimension $\le (2n-3)|I_0| + (2n-2)|I_+| + (n-2)|I_-| + (n-1)|I_\pm|$.

For $0 \le j \le r$, let $\mathcal{H}'_{\ge j}$ be the moduli stack defined similarly to $\mathcal{H}'[D_\bullet]$ but classifying only $x'_\bullet \in X'^{I_0}_{\overline{k}}$ and saturated maps $\{t_i : \mathcal{L}(D_i) \to \mathcal{F}_i\}_{j\le i\le r}$ (and $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are still related to each other by elementary modifications at x'_{i+1} for $j \le i < r$). We can factorize $\Pi[D_\bullet]$ as

$$\Pi[D_\bullet] : \mathcal{H}'[D_\bullet] = \mathcal{H}'_{\ge 0} \xrightarrow{\Pi_1} \mathcal{H}'_{\ge 1} \xrightarrow{\Pi_2} \cdots \xrightarrow{\Pi_r} \mathcal{H}'_{\ge r} = \mathcal{M}'[D_r] \times X'^{I_0}_{\overline{k}}. \tag{8.6}$$

The smoothness claim follows after we establish the following four statements:

- $(H0)$ If $i \in I_0$, then Π_i exhibits $\mathcal{H}'_{\ge i-1}$ as an open substack in a $\mathbf{P}^{n-1}$-bundle over a $\mathbf{P}^{n-2}$-bundle over $\mathcal{H}'_{\ge i}$.
- $(H+)$ If $i \in I_+$, then Π_i exhibits $\mathcal{H}'_{\ge i-1}$ as an open substack in a $\mathbf{P}^{n-1}$-bundle over a $\mathbf{P}^{n-1}$-bundle over $\mathcal{H}'_{\ge i}$.
- $(H-)$ If $i \in I_-$, then Π_i exhibits $\mathcal{H}'_{\ge i-1}$ as an open substack in a $\mathbf{P}^{n-2}$-bundle over $\mathcal{H}'_{\ge i}$.
- $(H\pm)$ If $i \in I_\pm$, then Π_i exhibits $\mathcal{H}'_{\ge i-1}$ as an open substack in a $\mathbf{P}^{n-1}$-bundle over $\mathcal{H}'_{\ge i}$.

Proof of $(H0)$. When $i \in I_0$, $D_{i-1} = D_i$. We write the modification $\mathcal{F}_{i-1} \dashrightarrow \mathcal{F}_i$ as

$$\mathcal{F}_{i-1} \xleftarrow{x'_i} \mathcal{F}^\flat_{i-1/2} \xrightarrow{\sigma x'_i} \mathcal{F}_i \tag{8.7}$$

Here both arrows have cokernel of length one supported at the labelled points. Such modification $\mathcal{F}^\flat_{i-1/2}$ of $\mathcal{F}_i$ are parametrized by a hyperplane H in the fiber $\mathcal{F}_i|_{\sigma x'_i}$ and a line L in the fiber $\mathcal{F}_i|_{x'_i}$. The lower modifications of $\mathcal{F}_i$ at $\sigma x'_i$ allowed in this case are those for which the map $t_i : \mathcal{L}(D_i) \to \mathcal{F}_i$ factors through $\mathcal{F}^\flat_{i-1/2}$, which is parametrized by the closed subset of hyperplanes $H \subset \mathcal{F}_i|_{\sigma x'_i}$ containing the line given by the image of $\mathcal{L}(D_i)|_{\sigma x'_i}$. The space of choices for H thus form a copy of $\mathbf{P}^{n-2}$. The upper modifications of $\mathcal{F}_i$ at x'_i allowed in this case are those for which the map $t_{i-1} : \mathcal{L}(D_i) \to \mathcal{F}^\flat_{i-1/2} \to \mathcal{F}_{i-1}$ is saturated, which is parametrized by the open subset of those lines $L \subset \mathcal{F}_i|_{x'_i}$ not equal to the image of $t_i(x'_i)$. The space of such choices of L thus form a copy of $\mathbf{P}^{n-1} - \{\mathrm{pt}\}$.

This argument globalizes in the evident way as $(\{\mathcal{L}(D_j) \xrightarrow{t_j} \mathcal{F}_j\}_{i\le j\le r}, \{x'_i\}_{i\in I_0})$ moves over $\mathcal{H}'_{\ge i}$, exhibiting that Π_i as an open substack in a $\mathbf{P}^{n-2} \times \mathbf{P}^{n-1}$-bundle. This applies similarly for the analogous arguments below for the other cases, so we focus on analyzing the fibers.

Proof of $(H+)$. When $i \in I_+$, we have $D_{i-1} = D_i - \sigma x'_i$. We use the same notation $(H, L) \in \mathbf{P}^\vee(\mathcal{F}_i|_{\sigma x'_i}) \times \mathbf{P}(\mathcal{F}_i|_{x'_i})$ as in the $(H0)$ case. This time the allowable lower modifications of $\mathcal{F}_i$ at $\sigma x'_i$ are parametrized by the open subset of $H \subset \mathbf{P}(\mathcal{F}_i|_{\sigma(x'_i)})$ that do not contain the image of $\mathcal{L}(D_i)|_{\sigma(x'_i)}$. This forms a copy of $\mathbf{P}^{n-1} - \mathbf{P}^{n-2}$. The allowable upper modifications of $\mathcal{F}_i$ at x'_i are again parametrized by those L not equal to the image of $t_i(x'_i)$. This is a copy of $\mathbf{P}^{n-1} - \{\mathrm{pt}\}$. So the fibers of Π_i in this case are isomorphic to $(\mathbf{P}^{n-1} - \mathbf{P}^{n-2}) \times (\mathbf{P}^{n-1} - \{\mathrm{pt}\})$.

Proof of $(H-)$. When $i \in I_-$, we have $D_{i-1} = D_i + x'_i$. This time the allowable lower modifications of $\mathcal{F}_i$ at $\sigma x'_i$ are parametrized by the closed subset of $H \subset \mathbf{P}(\mathcal{F}_i|_{\sigma(x'_i)})$ that contain the image of $\mathcal{L}(D_i)|_{\sigma(x'_i)}$. This forms a copy of $\mathbf{P}^{n-2}$. The allowable upper modifications of $\mathcal{F}_i$ at x'_i are parametrized by a single point where L is equal to the image of $t_i(x'_i)$. So the fibers of Π_i in this case are isomorphic to $\mathbf{P}^{n-2}$.

Proof of $(H\pm)$. When $i \in I_\pm$, we have $D_i + x'_i = D_{i-1} + \sigma x'_i$. This time the allowable lower modifications of $\mathcal{F}_i$ at $\sigma x'_i$ are parametrized by the open subset of $H \subset \mathbf{P}(\mathcal{F}_i|_{\sigma(x'_i)})$ that do not contain the image of $\mathcal{L}(D_i)|_{\sigma(x'_i)}$. This forms a copy of $\mathbf{P}^{n-1} - \mathbf{P}^{n-2}$. The allowable upper modifications of $\mathcal{F}_i$ at x'_i are parametrized by a single point where L is equal to the image of $t_i(x_i')$. So the fibers of Π_i in this case are isomorphic to $\mathbf{P}^{n-1} - \mathbf{P}^{n-2}$.

(3) By (2) and Lemma 8.4 we have

$$\begin{aligned} \dim \mathcal{Z}'[N, I_\bullet] &\leq |I_0| + \dim \mathfrak{D}(N; I_\bullet) + (2n-3)|I_0| + (2n-2)|I_+| + (n-2)|I_-| + (n-1)|I_\pm| \\ &\leq |I_0| + |I_+| + (2n-3)|I_0| + (2n-2)|I_+| + (n-2)|I_-| + (n-1)|I_\pm| \\ &= (2n-2)|I_0| + \frac{3n-3}{2}|I_+| + \frac{3n-3}{2}|I_-| + (n-1)|I_\pm|. \end{aligned}$$

Here we use that $|I_+| = |I_-|$. Since $(3n-3)/2 \leq 2n-2$ and $n-1 \leq 2n-2$, we conclude that the last quantity above is $\leq (2n-2)(|I_0| + |I_+| + |I_-| + |I_\pm|) = (2n-2)r$. Moreover, if equality holds, then we must have $|I_+| = |I_-| = |I_\pm| = 0$, i.e., $I_0 = \{1, 2, \cdots, r\}$. □

Part 3. Evidence

For the whole of Part 3, we assume X'/X is a geometrically nontrivial double cover.

9. Nonsingular Fourier coefficients for unitary similitude groups

In this section we extend the main result of [FYZ21] to the case of unitary similitude groups. One advantage of doing this is that we get central derivative formulas for the Siegel-Eisenstein series when the sign of the functional equation is -1 (when n is odd).

9.1. Siegel–Eisenstein series on unitary groups with similitudes. We extend the result from [FYZ21, §2] to the case of unitary groups with similitudes. For any one-dimensional F-vector space L, let $\mathrm{Herm}_n(F, L)$ be the F-vector space of F'/F-Hermitian forms $h : F'^n \times F'^n \to L \otimes_F F'$ (with respect to the involution $1 \otimes \sigma$ on $L \otimes_F F'$). For any F-algebra R, $\mathrm{Herm}_n(R, L) := \mathrm{Herm}_n(F, L) \otimes_F R$ is the set of $L \otimes_F R'$-valued R'/R-Hermitian forms on R'^n, where $R' = R \otimes_F F'$. When $L = F$ we write $\mathrm{Herm}_n(F) = \mathrm{Herm}_n(F, F)$ and $\mathrm{Herm}_n(R) = \mathrm{Herm}_n(F) \otimes_F R$ for any F-algebra R.

Let W be the standard split F'/F-skew-Hermitian space of dimension $2n$. Let $H_n = U(W)$ be the unitary group, and let $\widetilde{H}_n = GU(W)$ be the unitary group with similitudes, both as algebraic groups over F. Let $c : \widetilde{H}_n \to \mathbb{G}_m$ denote the similitude character. Let $P_n(\mathbb{A}) = M_n(\mathbb{A})N_n(\mathbb{A})$ be the standard Siegel parabolic subgroup of $H_n(\mathbb{A})$, where

$$M_n(\mathbb{A}) = \left\{ m(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\bar{\alpha}^{-1} \end{pmatrix} : \alpha \in \mathrm{GL}_n(\mathbb{A}_{F'}) \right\},$$

$$N_n(\mathbb{A}) = \left\{ n(\beta) = \begin{pmatrix} 1_n & \beta \\ 0 & 1_n \end{pmatrix} : \beta \in \mathrm{Herm}_n(\mathbb{A}) \right\}.$$

Similarly, let $\widetilde{P}_n(\mathbb{A}) = \widetilde{M}_n(\mathbb{A})N_n(\mathbb{A})$ be the standard Siegel parabolic subgroup of $\widetilde{H}_n(\mathbb{A})$, where

$$\widetilde{M}_n(\mathbb{A}) = \left\{ m(\alpha, c) = \begin{pmatrix} \alpha & 0 \\ 0 & c\, {}^t\bar{\alpha}^{-1} \end{pmatrix} : c \in \mathbb{A}^\times, \alpha \in \mathrm{GL}_n(\mathbb{A}_{F'}) \right\} \cong M_n(\mathbb{A}) \times \mathbb{A}^\times.$$

Let $\eta : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be the quadratic character associated to F'/F by class field theory. Since X'/X is étale, the character descends to $\eta : \mathrm{Pic}_X(k)/\mathrm{Pic}_{X'}(k) \to \{\pm 1\}$. Fix $\chi : \mathbb{A}_{F'}^\times / F'^\times \to \mathbb{C}^\times$ a character such that $\chi|_{\mathbb{A}_F^\times} = \eta^n$. We may view χ as a character on $M_n(\mathbb{A}) \simeq \mathrm{GL}_n(\mathbb{A}_{F'})$ by $\chi(\alpha) = \chi(\det(\alpha))$ and extend it to $P_n(\mathbb{A})$ trivially on $N_n(\mathbb{A})$. Fix a character $\chi_0 : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$. Define the *degenerate principal series* of $\widetilde{H}_n(\mathbb{A})$ to be the unnormalized smooth induction

$$I_n(s, (\chi, \chi_0)) = \mathrm{Ind}_{\widetilde{P}_n(\mathbb{A})}^{\widetilde{H}_n(\mathbb{A})}(\chi|\det|_{F'}^{s+n/2}, \chi_0|\cdot|_F^{-n(s+n/2)}), \quad s \in \mathbb{C}.$$

In other words, its sections $\Phi(-, s)$ satisfy

$$\Phi(m(\alpha, c)n(\beta)g, s) = \chi(\alpha)\chi_0(c)|c|_F^{-n(s+n/2)}|\det\alpha|_{F'}^{s+n/2}\Phi(g, s).$$

For a standard section $\Phi(-, s) \in I_n(s, \chi)$, define the associated *Siegel–Eisenstein series*

$$E(g, s, \Phi) = \sum_{\gamma \in P_n(F)\backslash H_n(F)} \Phi(\gamma g, s), \quad g \in \widetilde{H}_n(\mathbb{A}),$$

which converges for $\Re(s) \gg 0$ and admits meromorphic continuation to $s \in \mathbb{C}$. Here we have used $P_n(F)\backslash H_n(F) \simeq \widetilde{P}_n(F)\backslash \widetilde{H}_n(F)$.

Notice that $E(g,s,\Phi)$ depends on the choice of χ. In this paper, we will assume both χ and χ_0 are unramified everywhere. Eventually it will be convenient to take $\chi_0 = \eta^n$ but we do not make this assumption until §10.6. Then $I_n(s,(\chi,\chi_0))$ is unramified and we fix $\Phi(-,s) \in I_n(s,(\chi,\chi_0))$ as the unique $K = \widetilde{H}_n(\widehat{\mathcal{O}})$-invariant section normalized by

$$\Phi(1_{2n}, s) = 1. \tag{9.1}$$

Similarly we normalize $\Phi_v \in I_n(s,(\chi_v,\chi_0))$ for every $v \in |X|$ and we then have a factorization $\Phi = \bigotimes_{v\in|X|} \Phi_v$.

9.2. **Fourier expansion.** Let ω_F be the generic fiber of the canonical bundle of X, and $\mathbb{A}_{\omega_F} = \mathbb{A} \otimes_F \omega_F$. The residue pairing $\mathrm{Res} : \mathbb{A}_{\omega_F} \times \mathbb{A} \to k$ induces a pairing

$$\langle\cdot,\cdot\rangle : \mathrm{Herm}_n(\mathbb{A},\omega_F) \times \mathrm{Herm}_n(\mathbb{A}) \to k$$

given by $\langle T,b\rangle = \mathrm{Res}(-\mathrm{Tr}(Tb))$. Composing this pairing with the fixed nontrivial additive character $\psi_0 : k \to \mathbb{C}^\times$ exhibits $\mathrm{Herm}_n(\mathbb{A},\omega_F)$ as the Pontryagin dual of $\mathrm{Herm}_n(\mathbb{A})$. Moreover, it exhibits $\mathrm{Herm}_n(F,\omega_F)$ as the Pontryagin dual of $\mathrm{Herm}_n(F)\backslash \mathrm{Herm}_n(\mathbb{A}) = N_n(F)\backslash N_n(\mathbb{A})$. The global residue pairing is the sum of local residue pairings $\langle\cdot,\cdot\rangle_v : \mathrm{Herm}_n(F_v,\omega_{F_v}) \times \mathrm{Herm}_n(F_v) \to k$ defined by $\langle T,b\rangle_v = \mathrm{tr}_{k_v/k}\mathrm{Res}_v(-\mathrm{Tr}(Tb))$.

We have a Fourier expansion

$$E(g,s,\Phi) = \sum_{T\in\mathrm{Herm}_n(F,\omega_F)} E_T(g,s,\Phi), \quad g \in \widetilde{H}_n(\mathbb{A}),$$

where

$$E_T(g,s,\Phi) = \int_{N_n(F)\backslash N_n(\mathbb{A})} E(n(b)g,s,\Phi)\psi_0(\langle T,b\rangle)\,\mathrm{d}n(b),$$

and the Haar measure $\mathrm{d}n(b)$ is normalized such that $N_n(F)\backslash N_n(\mathbb{A})$ has volume 1.

When T is nonsingular, for a factorizable $\Phi = \bigotimes_{v\in|X|}\Phi_v$ we have a factorization of the Fourier coefficient into a product (cf. [Kud97, §4])

$$E_T(g,s,\Phi) = |\omega_X|_F^{-n^2/2} \prod_v W_{T,v}(g_v,s,\Phi_v), \tag{9.2}$$

where the *local (generalized) Whittaker function* is defined by

$$W_{T,v}(g_v,s,\Phi_v) = \int_{N_n(F_v)} \Phi_v(w_n^{-1}n(b)g_v,s)\psi_0(\langle T,b\rangle_v)\,\mathrm{d}_v n(b), \quad w_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

and has an analytic continuation to $s \in \mathbb{C}$. Here the local Haar measure $\mathrm{d}_v n(b)$ is normalized so that the volume of $N_n(\mathcal{O}_v)$ is 1. The factor $|\omega_X|_F^{-n^2/2}$ is the ratio between the global measure $\mathrm{d}n$ and the product of the local measures $\prod_v \mathrm{d}_v n$.

Note that for $m(\alpha,c) \in \widetilde{M}_n(F_v)$,

$$W_{T,v}(m(\alpha,c),s,\Phi_v) = \chi(\alpha)(\chi_0\eta^n)(c)|c|_{F_v}^{-n(-s+n/2)}|\det(\alpha)|_{F'_v}^{-s+n/2} W_{c^{-1}\,{}^t\bar{\alpha}T\alpha,v}(1,s,\Phi_v). \tag{9.3}$$

We define the *regular* part of the Eisenstein series to be

$$E^{\mathrm{reg}}(g,s,\Phi) = \sum_{\substack{T\in\mathrm{Herm}_n(F,\omega_F)\\ \mathrm{rank}\,T=n}} E_T(g,s,\Phi), \quad g \in \widetilde{H}_n(\mathbb{A}). \tag{9.4}$$

Analogous to [FYZ21, §2.6] we view E^{reg} as a function on

$$\widetilde{M}_n(F)\backslash \widetilde{M}_n(\mathbb{A})/\widetilde{M}_n(\widehat{\mathcal{O}}) \simeq \mathrm{Bun}_{\mathrm{GL}(n)'}(k) \times \mathrm{Pic}_X(k).$$

For $(\mathcal{E},\mathfrak{L}) \in \mathrm{Bun}_{\mathrm{GL}(n)'}(k) \times \mathrm{Pic}_X(k)$ and $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{L}$, we can define the a-th Fourier coefficient $E_a(m(\mathcal{E},\mathfrak{L}),s,\Phi)$ (similar to what is done in [FYZ21, §2.6]).

Theorem 9.1. *Let $\mathcal{E}$ be a vector bundle over X' of rank n. Then*

$$E^{\mathrm{reg}}(m(\mathcal{E},\mathfrak{L}),s,\Phi)=\sum_{a}E_a(m(\mathcal{E},\mathfrak{L}),s,\Phi) \tag{9.5}$$

where the sum runs over all injective Hermitian maps $a:\mathcal{E}\to\sigma^*\mathcal{E}^\vee\otimes\nu^*\mathfrak{L}$, and

$$\begin{aligned}E_a(m(\mathcal{E},\mathfrak{L}),s,\Phi)=&(\chi_0\eta^n)(\mathfrak{L})\chi(\det\mathcal{E})q^{-(\deg(\mathcal{E})-n\deg(\mathfrak{L}))(s-n/2)-\frac{1}{2}n^2\deg(\omega_X)}\\&\times\mathscr{L}_n(s)^{-1}\operatorname{Den}(q^{-2s},\operatorname{coker}(a)).\end{aligned}$$

Here

$$\mathscr{L}_n(s)=\prod_{i=1}^{n}L(i+2s,\eta^i). \tag{9.6}$$

The density function $\operatorname{Den}(q^{-2s},\operatorname{coker}(a))$ *(see* [FYZ21, §2.6, §5.1]*) is a polynomial in q^{-s} of degree*

$$\operatorname{len}(\operatorname{coker}(a))=\deg(\sigma^*\mathcal{E}^\vee\otimes\mathfrak{L})-\deg(\mathcal{E})=2n(\deg\mathfrak{L}+\deg\omega_X)-2\deg(\mathcal{E}).$$

Proof. By (9.2) and (9.3) we have

$$\begin{aligned}E_T(m(\alpha,c),s,\Phi_v)=&\chi(\alpha)(\chi_0\eta^n)(c)|c|_F^{-n(-s+n/2)}|\det(\alpha)|_{F'}^{-s+n/2}|\omega_X|^{-\frac{1}{2}n^2}\\&\times\prod_{v\in|X|}W_{c^{-1}\,{}^t\bar{\alpha}T\alpha,v}(1,s,\Phi_v).\end{aligned}$$

Note

$$|\det(\alpha)|_{F'}=q^{\deg(\mathcal{E})},\quad |c|_F=q^{\deg(\mathfrak{L})}.$$

The rest is the same as (the proof of) [FYZ21, Thm. 2.7, Thm. 5.1]. □

9.3. **Normalized Eisenstein series.** There is an intertwining operator

$$M(s):I(s,(\chi,\chi_0))\to I(-s,(\chi,\chi_0\eta^n))=I(-s,(\chi,\chi_0))\otimes(\eta^n\circ c)$$

where $c:\widetilde{H}_n(\mathbb{A})\to\mathbb{A}^\times$ is the similitude factor. The image of the unramified section is

$$M(s)\Phi(s,g)=q^{-\frac{n^2}{2}\deg\omega_X}\frac{\mathscr{L}_n(s-1/2)}{\mathscr{L}_n(s)}\eta^n(c(g))\Phi(-s,g),$$

Our result does not rely on these facts; later in §10.1 we will recall the well-known computation when $n=1$.

We define a normalized Eisenstein series

$$\widetilde{E}(g,s,\Phi)=q^{n\deg\omega_X s}\mathscr{L}_n(s)E(g,s,\Phi). \tag{9.7}$$

Then it satisfies a functional equation

$$\widetilde{E}(g,s,\Phi)=\eta^n(c(g))\widetilde{E}(g,-s,\Phi),\quad g\in\widetilde{H}_n(\mathbb{A}). \tag{9.8}$$

Note that when n is odd and $\eta(c(g))=-1$, the sign of the functional equation is -1.

By Theorem 9.1, for injective $a:\mathcal{E}\to\sigma^*\mathcal{E}^\vee\otimes\nu^*\mathfrak{L}$, the a-th Fourier coefficient (expanded at $g=m(\mathcal{E},\mathfrak{L})$) then has a very simple form

$$\widetilde{E}_a(m(\mathcal{E},\mathfrak{L}),s,\Phi)=(\chi_0\eta^n)(\mathfrak{L})\chi(\det\mathcal{E})q^{d(s-n/2)}\operatorname{Den}(q^{-2s},\operatorname{coker}(a)), \tag{9.9}$$

where

$$d=n(\deg\mathfrak{L}+\deg\omega_X)-\deg\mathcal{E} \tag{9.10}$$

is the half of the degree of $\operatorname{Den}(q^{-2s},\operatorname{coker}(a))$ (as a polynomial in q^{-s}). Note that d depends on $\mathcal{E}$ via its degree. This normalization differs from [FYZ21] in that here we do not absorb the trivial terms.

9.4. **Non-singular terms with similitudes.** Now we can state a generalization of the main result of [FYZ21] to Hermitian shtukas with similitudes.

Theorem 9.2. *Let $\mathcal{E}$ be a vector bundle of rank n on X', and let $d = -\deg(\mathcal{E}) + n(\deg \mathfrak{L} + \deg \omega_X)$. Let $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{L}$ be an injective (i.e. non-singular) Hermitian map. Then*

$$\deg[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)] = \frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(q^{ds}\operatorname{Den}(q^{-2s},\operatorname{coker}(a))\right).$$

Remark 9.3. Here we note that $\operatorname{Den}(q^{-2s},\operatorname{coker}(a))$ is a polynomial in q^{-s} of degree $2d = -2\deg\mathcal{E} + 2(\deg\omega_X + \deg\mathfrak{L})$. The right hand side of the above formula is symmetric up to the sign $\eta^n(\mathfrak{L})$ with respect to the substitution $s \mapsto -s$. Therefore the right side vanishes if $(-1)^r \neq \eta^n(\mathfrak{L})$.

On the other hand, by Lemma 2.10, $\operatorname{Sht}^r_{U(n),\mathfrak{L}}$ is empty when $(-1)^r \neq \eta^n(\mathfrak{L})$, so in that case the identity in the theorem holds trivially. The theorem is nontrivial only when $(-1)^r = \eta^n(\mathfrak{L})$.

Proof. The proof is similar to [FYZ21, Thm. 12.1]. We introduce a generalization of the moduli stack of torsion Hermitian sheaves $\operatorname{Herm}_{2d}(X'/X,\mathfrak{L})$ that classifies $(\mathcal{Q},h)$ where $\mathcal{Q}$ is a torsion coherent sheaf on X' of length $2d$, and h is an isomorphism $\mathcal{Q} \xrightarrow{\sim} \sigma^*\mathcal{Q}^\vee \otimes \nu^*\mathfrak{L}$ such that $\sigma^*h^\vee = h$.

The arguments in *loc. cit.* also show

(1) there is a graded virtual perverse sheaf on $\operatorname{Herm}_{2d}(X'/X,\mathfrak{L})$

$$\mathcal{K}^{\mathrm{Eis}}_d = \bigoplus_{i=0}^d \mathcal{K}^{\mathrm{Eis}}_{d,i}$$

such that

$$\operatorname{Den}(q^{-2s},\mathcal{Q}) = \sum_{i=0}^d \operatorname{Tr}(\operatorname{Frob}_{\mathcal{Q}},(\mathcal{K}^{\mathrm{Eis}}_{d,i})_{\mathcal{Q}})q^{-2is}, \quad \text{for } \mathcal{Q} \in \operatorname{Herm}_{2d}(X'/X,\mathfrak{L})(k). \tag{9.11}$$

(2) there is a graded virtual perverse sheaf on $\operatorname{Herm}_{2d}(X'/X,\mathfrak{L})$

$$\mathcal{K}^{\mathrm{Int}}_d = \bigoplus_{i=0}^d \mathcal{K}^{\mathrm{Int}}_{d,i}$$

such that

$$\deg[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)] = \sum_{i=0}^d \operatorname{Tr}(\operatorname{Frob}_{\mathcal{Q}},(\mathcal{K}^{\mathrm{Int}}_{d,i})_{\mathcal{Q}})\cdot(d-2i)^r. \tag{9.12}$$

Here $\mathcal{Q} = \operatorname{coker}(a) \in \operatorname{Herm}_{2d}(X'/X,\mathfrak{L})(k)$.

By the same proof of *loc. cit.*, $\mathcal{K}^{\mathrm{Eis}}_d$ and $\mathcal{K}^{\mathrm{Int}}_d$ are virtual linear combinations of isotypic summands of the Hermitian-Springer sheaf $\operatorname{Spr}^{\mathrm{Herm}}_{2d}$ on $\operatorname{Herm}_{2d}(X'/X,\mathfrak{L})$ under the action of $W_d = (\mathbf{Z}/2\mathbf{Z})^d \rtimes S_d$. The same proof of [FYZ21, Prop.12.3] again shows

$$\mathcal{K}^{\mathrm{Eis}}_d \cong \mathcal{K}^{\mathrm{Int}}_d \tag{9.13}$$

as graded virtual perverse sheaves on $\operatorname{Herm}_{2d}(X'/X,\mathfrak{L})$, and the proof is complete. □

Remark 9.4. When $\eta^n(\mathfrak{L}) = -1$ (so n is necessarily odd), $q^{ds}\operatorname{Den}(q^{-2s},\operatorname{coker}(a))$ is an odd function in s. Theorem 9.2 then gives a geometric interpretation of odd order central derivative of nonsingular Fourier coefficients of the normalized Eisenstein series in terms of degrees of special cycles. This complements the even derivative case treated in [FYZ21].

9.5. **A refinement of non-singular coefficients.** In certain cases, the special cycles $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$ can be further decomposed into the union of two open-closed parts. We will prove a refinement of Theorem 9.2 that calculates the degree of the 0-cycle on each part.

Below we consider the case $\eta^n(\mathfrak{L}) = 1$. In this case, $\operatorname{Sht}^r_{U(n),\mathfrak{L}} = \varnothing$ unless r is even, by Lemma 2.10. So we also assume r is even.

Let $\mathfrak{L}^{(n)} := \omega^{\otimes n-1} \otimes \mathfrak{L}^{\otimes n}$. Taking determinant induces a map

$$\det : \operatorname{Sht}^r_{U(n),\mathfrak{L}} \to \operatorname{Sht}^r_{U(1),\mathfrak{L}^{(n)}}. \tag{9.14}$$

Let $\mathfrak{N} = \omega_X^{\otimes n} \otimes \mathfrak{L}^{\otimes n}$. Since $\mathfrak{N}$ is a norm (as $\eta(\mathfrak{L}^{\otimes n}) = 1$ by assumption and ω_X is known to be a norm), Lemma 2.16 implies that the set $\mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$ of irreducible components of $\mathrm{Prym}_{\mathfrak{N}}$ (defined over k) is $\mathbf{Z}/2\mathbf{Z}$-torsor. For $\epsilon \in \mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$, let $\mathrm{Prym}_{\mathfrak{N}}^{\epsilon}$ be the corresponding component. Let $p : \mathrm{Sht}^r_{U(1),\mathfrak{L}^{(n)}} \to \mathrm{Prym}_{\mathfrak{N}}$ be the map recording $\mathcal{F}_0$. Let $\mathrm{Sht}^{r,\epsilon}_{U(1),\mathfrak{L}^{(n)}}$ be the preimage of $\mathrm{Prym}^{\epsilon}_{\mathfrak{N}}$ under p, and let $\mathrm{Sht}^{r,\epsilon}_{U(n),\mathfrak{L}}$ be the further preimage under det. Define $\mathcal{Z}^{r,\epsilon}_{\mathcal{E},\mathfrak{L}}(a)$ to be the preimage of $\mathrm{Sht}^{r,\epsilon}_{U(n),\mathfrak{L}}$ under the map $\zeta : \mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a) \to \mathrm{Sht}^r_{U(n),\mathfrak{L}}$.

Theorem 9.5. *Assume that $\eta^n(\mathfrak{L}) = 1$ and r even and $r > 0$. Let $\mathcal{E}$ be a vector bundle of rank n on X', and $d = -\deg(\mathcal{E}) + n(\deg\mathfrak{L} + \deg\omega_X)$. Let $a : \mathcal{E} \to \sigma^*\mathcal{E}^\vee \otimes \nu^*\mathfrak{L}$ be an injective (i.e. non-singular) Hermitian map. Then for any $\epsilon \in \mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$ we have:*

$$\deg[\mathcal{Z}^{r,\epsilon}_{\mathcal{E},\mathfrak{L}}(a)] = \frac{1}{2}\deg[\mathcal{Z}^{r}_{\mathcal{E},\mathfrak{L}}(a)] = \frac{1}{2(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(q^{ds}\,\mathrm{Den}(q^{-2s}, \mathrm{coker}(a))\right).$$

Proof. By Theorem 9.2, it suffices to show that $\deg[\mathcal{Z}^{r,\epsilon}_{\mathcal{E},\mathfrak{L}}(a)] = \frac{1}{2}\deg[\mathcal{Z}^{r}_{\mathcal{E},\mathfrak{L}}(a)]$.

Define X_d to be the d^{th} symmetric power of X, X'_d similarly, and $\nu_d \colon X'_d \to X_d$ to be the map induced by ν. Consider the moduli stack $\mathcal{P}_d$ that classifies $(D, \mathcal{F}, \iota)$ where D is an effective divisor on X of degree d, $\mathcal{F} \in \mathrm{Pic}_{X'}$ and ι is an isomorphism $\mathrm{Nm}(\mathcal{F}) \cong \mathcal{O}_X(D)$. The map $p : \mathcal{P}_d \to X_d$ given by forgetting $\mathcal{F}$ is a torsor for Prym. Let $\mathcal{P}_d \to X^\sharp_d \xrightarrow{\mu_d} X_d$ be the Stein factorization of p. Since Prym has two geometrically connected components, $\mu_d : X^\sharp_d \to X_d$ is an étale double cover. Consider the map $\alpha : X'_d \to \mathcal{P}_d$ (over X_d) sending $D' \in X'_d$ to $(\nu_d(D'), \mathcal{O}_{X'}(D'), \iota)$ where ι is the canonical isomorphism $\mathrm{Nm}(\mathcal{O}_{X'}(D')) \cong \mathcal{O}_X(D)$. It induces a map $\nu^\sharp_d : X'_d \to X^\sharp_d$ such that $\nu_d : X'_d \to X_d$ factorizes as

$$\nu_d : X'_d \xrightarrow{\nu^\sharp_d} X^\sharp_d \xrightarrow{\mu_d} X_d. \tag{9.15}$$

If we base change μ_d along the symmetrization map $X^d \to X_d$ we get an étale double cover $X^{d,\sharp}$ of X^d. We claim that the étale double cover $X^{d,\sharp} \to X^d$ is given by the homomorphism

$$\pi_1(X^d) \to (\mathbf{Z}/2\mathbf{Z})^d \xrightarrow{\mathrm{sum}} \mathbf{Z}/2\mathbf{Z} \tag{9.16}$$

where the first map classifies the $(\mathbf{Z}/2\mathbf{Z})^d$-torsor X'^d/X^d. Indeed we may consider the map $\widetilde{\alpha} : X'^d \to X'_d \xrightarrow{\alpha} \mathcal{P}_d$; it sends $(x'_1, \cdots, x'_d)$ to $(\nu(x'_1) + \cdots + \nu(x'_d), \mathcal{O}(x'_1 + \cdots + x'_d))$. Each time we change x'_i to $\sigma(x'_i)$ the resulting point under $\widetilde{\alpha}$ moves to a different component of the corresponding fiber of $\mathcal{P}_d \to X_d$, hence the resulting map $X'^d \to X^\sharp_d$ factors through the quotient of X'^d by the subgroup $\ker(\mathrm{sum} : (\mathbf{Z}/2\mathbf{Z})^d \to \mathbf{Z}/2\mathbf{Z})$.

To summarize, $\mu_d : X^\sharp_d \to X_d$ is the double cover attached to the local system η_d on X_d (see [FYZ21, §11.4]).

Let $\mathcal{M}^{\mathrm{ns}}_d$ be the open-closed substack of $\mathcal{M}^{\mathrm{ns}}_{\mathrm{GL}(n)',U(n),\mathfrak{L}} = \mathcal{M}^{\circ}_{\mathrm{GL}(n)',U(n),\mathfrak{L}}$ where $\deg\mathcal{E} = n(\deg\mathfrak{L} + \deg\omega_X) - d$. Let $\mathcal{A}^{\mathrm{ns}}_d$ be the corresponding Hitchin base. Write $\mathrm{Lagr}_{2d} := \mathrm{Lagr}_{2d}(X'/X, \mathfrak{L})$ and $\mathrm{Herm}_{2d} := \mathrm{Herm}_{2d}(X'/X, \mathfrak{L})$. Recall from [FYZ21, Lemma 8.8] we have a commutative diagram where the left side square is Cartesian

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{ns}}_d \arrow[r, "\epsilon'_d"] \arrow[d, "f_d"] & \mathrm{Lagr}_{2d} \arrow[r] \arrow[d, "\upsilon_{2d}"] & X'_d \arrow[d, "\nu_d"] \\
\mathcal{A}^{\mathrm{ns}}_d \arrow[r, "\epsilon_d"] & \mathrm{Herm}_{2d} \arrow[r] & X_d
\end{tikzcd}$$

The map $\mathrm{Herm}_{2d} \to X_d$ is the descent of the divisor of the Hermitian torsion sheaf, and $\mathrm{Lagr}_{2d} \to X'_d$ records the divisor of the Lagrangian subsheaf.

Let $\mu^{\mathrm{Herm}}_d : \mathrm{Herm}^\sharp_{2d} \to \mathrm{Herm}_{2d}$ and $\mu^{\mathcal{A}}_d : \mathcal{A}^{\mathrm{ns},\sharp}_d \to \mathcal{A}^{\mathrm{ns}}_d$ be the base changes of the double cover $\mu_d : X^\sharp_d \to$

X_d. Then we have a diagram where all squares are Cartesian

$$\begin{tikzcd} \mathcal{M}_d^{\mathrm{ns}} \arrow[r, "\epsilon'_d"] \arrow[d, "f_d^\sharp"] & \mathrm{Lagr}_{2d} \arrow[d, "\upsilon_{2d}^\sharp"] \\ \mathcal{A}_d^{\mathrm{ns},\sharp} \arrow[r, "\epsilon_d^\sharp"] \arrow[d, "\mu_d^{\mathcal{A}}"] & \mathrm{Herm}_{2d}^\sharp \arrow[d, "\mu_d^{\mathrm{Herm}}"] \\ \mathcal{A}_d^{\mathrm{ns}} \arrow[r, "\epsilon_d"] & \mathrm{Herm}_{2d} \end{tikzcd}$$

We claim that the double cover $\mu_d^{\mathcal{A}} : \mathcal{A}_d^{\mathrm{ns},\sharp} \to \mathcal{A}_d^{\mathrm{ns}}$ is trivial. Indeed, by assumption, $\mathfrak{L}^{\otimes n}$ is a norm, hence $(\omega_X \otimes \mathfrak{L})^{\otimes n}$ is also a norm since ω_X is known to be a norm. Say $(\omega_X \otimes \mathfrak{L})^{\otimes -n} = \mathrm{Nm}(\mathfrak{M})$ for some $\mathfrak{M} \in \mathrm{Pic}_{X'}(k)$. This means $\mathfrak{M}$ carries a Hermitian form $a_{\mathfrak{M}} : \mathfrak{M} \xrightarrow{\sim} \sigma^*\mathfrak{M}^{\otimes -1} \otimes \nu^*(\omega \otimes \mathfrak{L})^{\otimes -n}$. We define a map $\beta_{\mathfrak{M}} : \mathcal{A}_d^{\mathrm{ns}} \to \mathcal{P}_d$ sending $(\mathcal{E}, a)$ to $(\mathrm{div}(a), (\det \mathcal{E})^{\otimes -1} \otimes \mathfrak{M}^{\otimes -1}, \iota)$. Here $\iota : \mathrm{Nm}((\det \mathcal{E})^{\otimes -1} \otimes \mathfrak{M}^{\otimes -1}) \cong \mathcal{O}(\mathrm{div}(a))$ is defined as follows. Note that $\det(a)$ is a Hermitian map $\det \mathcal{E} \to \sigma^*(\det \mathcal{E})^{\otimes -1} \otimes \nu^*(\omega_X \otimes \mathfrak{L})^{\otimes n}$. Then $\det(a) \otimes a_{\mathfrak{M}}$ gives a Hermitian map $\det \mathcal{E} \otimes \mathfrak{M} \to \sigma^*(\det \mathcal{E} \otimes \mathfrak{M})^{\otimes -1}$ whose divisor descends to $\mathrm{div}(a) \in X_d$. Therefore it induces a canonical Hermitian isomorphism $\det \mathcal{E} \otimes \mathfrak{M} \xrightarrow{\sim} \sigma^*(\det \mathcal{E} \otimes \mathfrak{M})^{\otimes -1} \otimes \nu^*\mathcal{O}(-\mathrm{div}(a))$, which gives the desired isomorphism ι. The map $\beta_{\mathfrak{M}}$ then induces a map $\mathcal{A}_d^{\mathrm{ns}} \to X_d^\sharp$ over X_d, which shows that $\mathcal{A}_d^{\mathrm{ns},\sharp} \to \mathcal{A}_d^{\mathrm{ns}}$ is split (but not canonically, since the splitting depends on the choice of $\mathfrak{M}$).

Let $(\mathcal{E}, a_1^\sharp)$ and $(\mathcal{E}, a_2^\sharp) \in \mathcal{A}_d^{\mathrm{ns},\sharp}(k)$ be the two preimages of $(\mathcal{E}, a)$. We have a canonical map $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a) \to \mathcal{M}_d^{\mathrm{ns}}$ by recording $\mathcal{E} \xrightarrow{t_0} (\mathcal{F}_0, h_0)$. We have a further map $\mathcal{M}_d^{\mathrm{ns}} \to \mathcal{P}_d$ mapping $(\mathcal{E} \xrightarrow{t} (\mathcal{F}, h))$ to $(\mathrm{div}(a), \det \mathcal{F} \otimes (\det \mathcal{E})^{\otimes -1})$ where $a = \sigma^* t^\vee \circ h \circ t$. By construction, for each $\epsilon \in \mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$, $\mathcal{Z}^{r,\epsilon}_{\mathcal{E},\mathfrak{L}}(a)$ maps exactly to the one (out of two) components of the fiber $p^{-1}(\mathrm{div}(a)) \subset \mathcal{P}_d$ over $\mathrm{div}(a) \in X_d(k)$. Therefore, the decomposition of $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)$ according to ϵ is the same as the decomposition given by the map $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a) \to f_d^{-1}(\mathcal{E}, a) \to \mu_d^{\mathcal{A},-1}(\mathcal{E}, a) = \{(\mathcal{E}, a_1^\sharp), (\mathcal{E}, a_2^\sharp)\}$. Let $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a_i^\sharp)$ be the fiber over $(\mathcal{E}, a_i^\sharp)$. We thus reduce to show

$$\deg[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a_1^\sharp)] = \deg[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a_2^\sharp)]. \tag{9.17}$$

Here $[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a_i^\sharp)]$ is defined as the restriction of $[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a)]$ to the open-closed substack $\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a_i^\sharp)$, $i = 1, 2$.

Let $\mathcal{Q}_i^\sharp \in \mathrm{Herm}_{2d}^\sharp(k)$ be the image of $(\mathcal{E}, a_i^\sharp)$. Define $\mathcal{Z}^r_{\mathcal{Q}_i^\sharp}$ and its fundamental class using the Shtuka construction for Lagr_{2d} (see [FYZ21, §11.1]). Then the smoothness of ϵ_d implies $\deg[\mathcal{Z}^r_{\mathcal{E},\mathfrak{L}}(a_i^\sharp)] = \deg[\mathcal{Z}^r_{\mathcal{Q}_i^\sharp}]$. Recall the self-correspondence $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ of Lagr_{2d} over Herm_{2d}. Applying the Lefschetz trace formula [FYZ21, Prop. 11.8] to the same situation as [FYZ21, Corollary 11.9] except that we are now working over the base $\mathrm{Herm}_{2d}^\sharp$ rather than Herm_{2d}, we get

$$\deg[\mathcal{Z}^r_{\mathcal{Q}_i^\sharp}] = \mathrm{Tr}\left([\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]^r \circ \mathrm{Frob}_{\mathcal{Q}_i^\sharp}, (R(\upsilon_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell)_{\mathcal{Q}_i^\sharp}\right), \quad i = 1, 2. \tag{9.18}$$

The only caveat here is that the self-correspondence $\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}$ of Lagr_{2d} is not over $\mathrm{Herm}_{2d}^\sharp$ but only over Herm_{2d}: the two compositions $\upsilon_{2d}^\sharp \circ \mathrm{pr}_0$ and $\upsilon_{2d}^\sharp \circ \mathrm{pr}_1 : \mathrm{Hk}^1_{\mathrm{Lagr}_{2d}} \to \mathrm{Herm}_{2d}^\sharp$ differ by the involution σ_d of $\mathrm{Herm}_{2d}^\sharp / \mathrm{Herm}_{2d}$ (base changed from the involution of $X_d^\sharp / X_d$). Therefore $[\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]$ induces a map $R(\upsilon_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell \to \sigma_d^* R(\upsilon_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell$. Since r is even, $[\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]^r$ induces an endomorphism of $R(\upsilon_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell$. The proof of the Lefschetz trace formula in *loc.cit* adapts to this twisted situation.

Let $W_d = (\mathbf{Z}/2\mathbf{Z})^d \rtimes S_d$ and $\chi_d : W_d \to \mathbf{Z}/2\mathbf{Z}$ be trivial on S_d and nontrivial on each factor of $\mathbf{Z}/2\mathbf{Z}$. Let $W'_d = \ker(\chi_d)$. Recall the map $\pi_{2d}^{\mathrm{Herm}} : \widetilde{\mathrm{Herm}}_{2d} \to \mathrm{Herm}_{2d}$ ([FYZ21, §4.2]), and it factors through $\pi_{2d}^\sharp : \widetilde{\mathrm{Herm}}_{2d} \to \mathrm{Herm}_{2d}^\sharp$. This is a small map that is generically a W'_d-torsor. Therefore $\mathrm{Spr}_{2d}^\sharp := R(\pi_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell$ is a middle extension perverse sheaf that carries an action of W'_d. For a representation ρ of W'_d, let $\mathrm{Spr}_{2d}^\sharp[\rho] = \mathrm{Hom}_{W'_d}(\rho, \mathrm{Spr}_{2d}^\sharp) = (\rho^\vee \otimes R(\pi_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell)^{W'_d}$. Then, analogous to [FYZ21, Lemma 11.4, Corollary 11.5], $R(\upsilon_{2d}^\sharp)_*\overline{\mathbf{Q}}_\ell \cong \mathrm{Spr}_{2d}^\sharp[\mathrm{Ind}_{S_d}^{W'_d} \mathbf{1}]$. Recall $\{\rho_i\}_{0 \le i \le d}$ are irreducible direct summands of $\mathrm{Ind}_{S_d}^{W_d} \mathbf{1}$ (as

W_d-representation) defined in [FYZ21, §11.2]. We have a corresponding decomposition of W'_d-representations

$$\mathrm{Ind}_{S_d}^{W'_d}\mathbf{1}=\bigoplus_{0\le i\le[d/2]}\rho'_i$$

where $\rho'_i\in\mathrm{Irr}(W'_d)$ is characterized by:

$$\mathrm{Ind}_{W'_d}^{W_d}\rho'_i\cong\begin{cases}\rho_i\oplus\rho_{d-i}, & i<d/2;\\ \rho_{d/2}, & i=d/2.\end{cases}\tag{9.19}$$

Let $\mathcal{K}^\sharp_{d,i}=\mathrm{Spr}^\sharp_{2d}[\rho'_i]$ for $0\le i\le[d/2]$, then

$$R(\upsilon^\sharp_{2d})_*\overline{\mathbf{Q}}_\ell\cong\bigoplus_{0\le i\le[d/2]}\mathcal{K}^\sharp_{d,i}.$$

The action of $[\mathrm{Hk}^1_{\mathrm{Lagr}_{2d}}]^r$ preserves each $\mathcal{K}^\sharp_{d,i}$ and acts on it by the scalar $(d-2i)^r$. Therefore (9.18) implies

$$\deg[\mathcal{Z}^r_{\mathcal{Q}^\sharp_1}]=\sum_{0\le i<d/2}(d-2i)^r\,\mathrm{Tr}(\mathrm{Frob}_{\mathcal{Q}^\sharp_1},(\mathcal{K}^\sharp_{d,i})_{\mathcal{Q}^\sharp_1}).$$

Here we have omitted the term $i=d/2$ because its coefficient $(d-2i)^r=0$ (using that $r>0$).

The same formula is true when $\mathcal{Q}^\sharp_1$ is replaced by $\mathcal{Q}^\sharp_2$. Therefore to show (9.17) it suffices to show that $\mathcal{K}^\sharp_{d,i}$ has the same Frobenius trace at conjugate points under σ_d. Better, we will show that $\mathcal{K}^\sharp_{d,i}$ descends to Herm_{2d}. Indeed, (9.19) implies $\rho_i|_{W'_d}\cong\rho'_i$ for $0\le i<d/2$, hence $\mathrm{Spr}^\sharp_{2d}[\rho_i|_{W'_d}]\cong\mathrm{Spr}^\sharp_{2d}[\rho'_i]$. Note that $\mathrm{Spr}^\sharp_{2d}[\rho_i|_{W'_d}]\cong\mu_d^{\mathrm{Herm},*}\mathcal{K}^{\mathrm{Int}}_{d,i}$, hence $\mathcal{K}^\sharp_{d,i}$ is the pullback of $\mathcal{K}^{\mathrm{Int}}_{d,i}$. The proof is complete. □

Remark 9.6. Interestingly, the statement of Theorem 9.5 does not hold for $r=0$ in general, because the sheaf $\mathcal{K}^\sharp_{d,d/2}$ occurring in the proof does not descend to Herm_{2d}, so its Frobenius trace cannot be the same at all conjugate rational points for all $\mathbf{F}_q$.

10. Modularity: the case of $U(1)$

In this section we prove the Modularity Conjecture 4.12 for $n=m=1$, which we show in Corollary 10.10 follows from the modularity after taking the degrees of special cycles (on each connected component, if there are multiple). The degrees of nonzero terms in the generating series in this case are taken care of by Theorem 9.5. The bulk of this section is devoted to the calculation of the degree of the 0-th term in the generating series, which we relate to the higher derivatives of an L-function, completing the higher Siegel-Weil formula in this case.

10.1. **The constant term of the Eisenstein series.** We will switch the notation $\mathcal{E}$ to $\mathcal{L}$ to indicate a line bundle on X'. We compute the constant Fourier coefficient of the Siegel-Eisenstein series for $\widetilde{H}_1=GU(2)$. We use notations from §9.

By definition, the constant term is equal to

$$E_0(g,s,\Phi)=\Phi(g,s)+M(s)\Phi(g,s),$$

where $M(s)$ is the intertwining operator

$$M(s):I(s,(\chi,\chi_0))\longrightarrow I(-s,(\chi,\chi_0\eta^n))$$

defined by

$$M(s)\Phi(g,s):=\int_{N(\mathbb{A})}\Phi(w^{-1}n(b)g,s)\,\mathrm{d}n(b).$$

Since our section Φ is unramified (see (9.1)), so is $M(s)\Phi(g,s)$. Therefore it suffices to determine the value of $M(s)\Phi(g,s)$ at $g=1$. By [Sha10, Lem. 4.3.2], translating into the current context and noting that $\mathrm{vol}(\widehat{\mathcal{O}})=q^{-\frac12\deg\omega_X}$ for the self-dual measure, we obtain

$$M(s)\Phi(1,s)=q^{-\frac12\deg\omega_X}\frac{L(2s,\eta)}{L(2s+1,\eta)}$$

and hence

$$M(s)\Phi(g,s) = \eta(c(g))q^{-\frac{1}{2}\deg\omega_X}\frac{L(2s,\eta)}{L(2s+1,\eta)}\Phi(g,-s).$$

Therefore

$$E_0(g,s,\Phi) = \Phi(g,s) + \eta(c(g))q^{-\frac{1}{2}\deg\omega_X}\frac{L(2s,\eta)}{L(2s+1,\eta)}\Phi(g,-s).$$

Remark 10.1. Note that the formula for the constant term is consistent with the functional equation

$$E(g,s,\Phi) = E(g,-s,M(s)\Phi),$$

or equivalently $M(-s)M(s)\Phi = \Phi$. In fact, by the above formula on $M(s)\Phi$, we have

$$M(-s)M(s)\Phi = q^{-\frac{1}{2}\deg\omega_X}\frac{L(-2s,\eta)}{L(-2s+1,\eta)}\cdot q^{-\frac{1}{2}\deg\omega_X}\frac{L(2s,\eta)}{L(2s+1,\eta)}\Phi.$$

Then $M(-s)M(s)\Phi = \Phi$ follows from the functional equation

$$q^{\frac{1}{2}\deg\omega_X s}L(s,\eta) = q^{\frac{1}{2}\deg\omega_X(1-s)}L(1-s,\eta), \tag{10.1}$$

where we note that $L(s,\eta)$ is a polynomial in q^{-s} of degree $\deg\omega_X$.

Now we evaluate the constant term at $g = m(\mathcal{L},\mathfrak{L})$ for line bundles $\mathcal{L}\in\mathrm{Pic}_{X'}(k), \mathfrak{L}\in\mathrm{Pic}_X(k)$,

$$\begin{aligned} E_0(m(\mathcal{L},\mathfrak{L}),s,\Phi) =& \chi(\mathcal{L})\chi_0(\mathfrak{L})q^{(\deg\mathcal{L}-\deg\mathfrak{L})(s+1/2)} \\ &+\eta(\mathfrak{L})\chi(\mathcal{L})\chi_0(\mathfrak{L})q^{-\frac{1}{2}\deg\omega_X}\frac{L(2s,\eta)}{L(2s+1,\eta)}q^{(\deg\mathcal{L}-\deg\mathfrak{L})(-s+1/2)}. \end{aligned} \tag{10.2}$$

The normalized Eisenstein series (9.7), specialized to the case $n=1$, gives

$$\widetilde{E}(g,s,\Phi) = q^{\deg\omega_X s}L(2s+1,\eta)E(g,s,\Phi)$$

and (9.10) becomes

$$d = \deg\mathfrak{L} + \deg\omega_X - \deg\mathcal{L}. \tag{10.3}$$

By (10.2), when $a=0$,

$$\begin{aligned} &\widetilde{E}_0(m(\mathcal{L},\mathfrak{L}),s,\Phi) \\ =&(\chi_0\eta)(\mathfrak{L})\chi(\mathcal{L})q^{d(s-1/2)}L(2s,\eta) + \chi_0(\mathfrak{L})\chi(\mathcal{L})q^{-d(s+1/2)}q^{\deg\omega_X(1/2+2s)}L(2s+1,\eta). \end{aligned} \tag{10.4}$$

By the functional equation (10.1), the two summands in (10.4) are switched (up to the sign $\eta(\mathfrak{L})$) with respect to the substitution $s\mapsto -s$.

10.2. **The constant term of the generating series.** Fix a line bundle $\mathfrak{L}\in\mathrm{Pic}_X(k)$. For $1\le i\le r$, let ℓ_i be the line bundle on $\mathrm{Sht}^r_{U(1),\mathfrak{L}}$ whose fiber at $(\{x_i\},\{\mathcal{F}_i\})$ is the fiber $\mathcal{F}_i|_{\sigma x_i}$. According to Definition 4.5 and Definition 4.7, the proposed constant term for the generating series is a sum of two terms. When $n=1$, one of them vanishes and hence we have

$$[\mathcal{Z}^r_{\mathcal{L}}(0)] = [\mathcal{Z}^r_{\mathcal{L}}[\mathcal{L}](0)] := \prod_{i=1}^r c_1(p_i^*\sigma^*\mathcal{L}^{-1}\otimes\ell_i)\in\mathrm{Ch}^r(\mathrm{Sht}^r_{U(1),\mathfrak{L}}) = \mathrm{Ch}_0(\mathrm{Sht}^r_{U(1),\mathfrak{L}}). \tag{10.5}$$

Note that on the left hand side we have suppressed the dependence on $\mathfrak{L}$, for brevity.

The goal now is to calculate the degree of $[\mathcal{Z}^r_{\mathcal{L}}(0)]$ in terms of higher derivatives of the L-function $L(s,\eta)$. We have $L(s,\eta) = \zeta_{X'}(s)/\zeta_X(s)$, and it is a polynomial in q^{-s} (because of our assumption that X'/X is non-split) of degree $2g-2$.

Theorem 10.2. *Let $r\in\mathbb{Z}_{\ge 0}$ be such that $(-1)^r = \eta(\mathfrak{L})$. Then we have*

$$\deg[\mathcal{Z}^r_{\mathcal{L}}(0)] = 2(\log q)^{-r}\frac{d^r}{ds^r}\Big|_{s=0}\left(q^{ds}L(2s,\eta)\right) \tag{10.6}$$

where d is defined in (10.3).

For the brevity of notation, we will denote

$$\mathrm{Sht}^r_{U(1)} := \mathrm{Sht}^r_{U(1),\mathfrak{L}}, \quad \mathfrak{N} := \mathfrak{L} \otimes \omega_X,$$

Then $d = \deg \mathfrak{N} - \deg \mathcal{L}$ and Theorem 10.2 is equivalent to

$$\deg[\mathcal{Z}^r_{\mathcal{L}}(0)] = 2(\log q)^{-r} \frac{d^r}{ds^r}\Big|_{s=0} \left(q^{ds} L(2s, \eta)\right). \tag{10.7}$$

10.3. **Calculation of the Chern classes.** Recall the Cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}^r_{U(1)} \arrow[r, "p"] \arrow[d, "p_{[1,r]}:=(p_1,\cdots,p_r)"'] & \mathrm{Prym}_{\mathfrak{N}} \arrow[d, "\mathrm{Lang}"] \\ X'^r \arrow[r, "\mathrm{AJ}^r"] & \mathrm{Prym}^{\epsilon(\mathfrak{N})} \end{tikzcd} \tag{10.8}$$

Let $p_{ab} = (p_a, p_b) : \mathrm{Sht}^r_{U(1)} \to X' \times X'$. Let $\Delta \subset X' \times X'$ be the diagonal and $\Delta^- \subset X' \times X'$ be the anti-diagonal consisting of points $(x, \sigma x)$.

Let $\mathcal{P}$ be the Poincaré line bundle over $X' \times \mathrm{Prym}_{\mathfrak{N}}$.

In the following, all Chern classes lie in ℓ-adic cohomology groups. Also, when we write $\mathrm{H}^i(Z, \overline{\mathbf{Q}}_\ell)$ or $\mathrm{H}^i(Z)$ for a stack Z over k we mean $\mathrm{H}^i(Z_{\overline{k}}, \overline{\mathbf{Q}}_\ell)$.

Lemma 10.3. *For $1 \le a \le r$ we have an equality in $\mathrm{H}^2(\mathrm{Sht}^r_{U(1)}, \overline{\mathbf{Q}}_\ell(1))$:*

$$c_1(p_a^*\sigma^*\mathcal{L}^{-1} \otimes \ell_a) = (\sigma p_a, p)^* c_1(\mathcal{P}) + \sum_{b<a} p_{ba}^* c_1(\mathcal{O}(\Delta - \Delta^-)) - p_a^* c_1(\mathcal{L} \otimes \omega_{X'}). \tag{10.9}$$

Proof. We have

$$\ell_a|_{(\{x_i\},\{\mathcal{F}_i\})} = \mathcal{F}_a/\mathcal{F}^\flat_{a-1/2} = (\mathcal{F}_a)_{\sigma x_a} = \mathcal{F}_0(\sigma x_1 + \cdots + \sigma x_a - x_1 - \cdots - x_a)|_{\sigma x_a}.$$

Therefore

$$\ell_a \cong (\sigma p_a, p)^*\mathcal{P} \otimes \left(\otimes_{1\le b<a} p_{ba}^* \mathcal{O}(\Delta - \Delta^-)\right) \otimes p_a^*(\mathcal{O}(\Delta - \Delta^-)|_\Delta).$$

Since $\mathcal{O}(\Delta)|_\Delta \cong \omega_{X'}^{-1}$, $\mathcal{O}(\Delta^-)|_\Delta \cong \mathcal{O}_{X'}$ and $c_1(\sigma^*\mathcal{L}) = c_1(\mathcal{L})$, we obtain the desired formula. □

Denote $V = \mathrm{H}^1(X', \overline{\mathbf{Q}}_\ell)^{\sigma=-1}$ as a Frob-module. Denote the action of Frob on V by ϕ. Let $\xi \in \mathrm{H}^2(X', \overline{\mathbf{Q}}_\ell(1))$ be the fundamental class of any closed point on $X'_{\overline{k}}$ and use it to identify $\mathrm{H}^2(X', \overline{\mathbf{Q}}_\ell(1)) \simeq \overline{\mathbf{Q}}_\ell$. Let

$$\langle \cdot, \cdot \rangle : V \times V \to \overline{\mathbf{Q}}_\ell(-1) \tag{10.10}$$

be the symplectic pairing on V induced from the cup product, i.e.,

$$v \cup v' = \langle v, v' \rangle \xi \in \mathrm{H}^2(X', \overline{\mathbf{Q}}_\ell), \quad v, v' \in V. \tag{10.11}$$

For dual bases $\{v_i\}$ and $\{v^i\}$ under this symplectic pairing, i.e., $\sum_i \langle v_i, \alpha \rangle v^i = \alpha$ for all $\alpha \in V$. Let

$$\beta = \sum_i v_i \otimes v^i \in \wedge^2(V) \subset V \otimes V. \tag{10.12}$$

Lemma 10.4. *We have $c_1(\mathcal{O}(\Delta - \Delta^-)) = -2\beta \in V \otimes V \subset \mathrm{H}^2(X' \times X')$.*

Proof. Note that the group $\mathrm{Aut}(X'/X) \times \mathrm{Aut}(X'/X)$ acts on $\mathrm{H}^2(X' \times X')$ and $V \otimes V$ is exactly the isotypic subspace for the character $\chi : \mathrm{Aut}(X'/X) \times \mathrm{Aut}(X'/X) \to \{\pm 1\}$ that is nontrivial on both factors. By inspection we see that $c_1(\mathcal{O}(\Delta - \Delta^-))$ must be in this isotypic subspace, therefore $c_1(\mathcal{O}(\Delta - \Delta^-)) \in V \otimes V$.

For any class $\gamma \in \mathrm{H}^2(X' \times X')$, let $\gamma^\heartsuit$ be the projection of γ to the χ-isotypic subspace $V \otimes V \subset \mathrm{H}^2(X' \times X')$. We *claim* that $c_1(\mathcal{O}(\Delta))^\heartsuit = -\beta$. Note that $c_1(\mathcal{O}(\Delta))$ is the cycle class $cl(\Delta)$ of the diagonal Δ in $X' \times X'$, so we need to show that $cl(\Delta)^\heartsuit = -\beta$. Under the Kunneth decomposition and the Poincaré duality, $cl(\Delta)$ corresponds to the identity endomorphism of $\mathrm{H}^*(X')$. In particular,

$$cl(\Delta)^\heartsuit \cup (\alpha \otimes 1) = \xi \otimes \alpha \in \mathrm{H}^2(X') \otimes V, \quad \forall \alpha \in V. \tag{10.13}$$

(Here $1 \in H^0(X')$ is the fundamental class of X'.) This property characterizes $cl(\Delta)^\heartsuit$. To show $cl(\Delta)^\heartsuit = -\beta$, it suffices to check that

$$-\beta \cup (\alpha \otimes 1) = \xi \otimes \alpha, \quad \forall \alpha \in V. \tag{10.14}$$

This holds because $-(\sum_i v_i\otimes v^i)\cup(\alpha\otimes 1)=\sum_i(v_i\cup\alpha)\otimes v^i$ (by Koszul sign convention), which is $\sum_i\langle v_i,\alpha\rangle\xi\otimes v^i=\xi\otimes(\sum_i\langle v_i,\alpha\rangle v^i)=\xi\otimes\alpha$.

Finally, by $\mathcal{O}(\Delta^-)\simeq(\sigma,1)^*\mathcal{O}(\Delta)$ we know that $c_1(\mathcal{O}(\Delta^-))^\heartsuit=-c_1(\mathcal{O}(\Delta))^\heartsuit=\beta$. Therefore $c_1(\mathcal{O}(\Delta-\Delta^-))=c_1(\mathcal{O}(\Delta-\Delta^-))^\heartsuit=-2\beta$. □

The Abel-Jacobi map

$$\begin{aligned}\mathrm{AJ}_1: X' &\to \mathrm{Prym}^1\\ x&\mapsto \mathcal{O}(\sigma x-x)\end{aligned}\tag{10.15}$$

induces an injective map AJ_1^* on H^1 and identifies the image:

$$\mathrm{H}^1(\mathrm{Prym}^1)\xrightarrow{\sim}\mathrm{H}^1(X')^{\sigma=-1}=V.\tag{10.16}$$

We claim that $\mathrm{H}^1(\mathrm{Prym}^0)$ as well as $\mathrm{H}^1(\mathrm{Prym}^\epsilon_{\mathfrak{N}})$ for any $\mathfrak{N}\in\mathrm{Pic}_X(k)$ are *canonically* identified with V. Indeed, if A is a (geometrically) connected group scheme over k and A_1 is any an A-torsor over k, then any choice of $b\in A_1(\overline{k})$ identifies $A_{\overline{k}}$ with $A_{1,\overline{k}}$ hence gives an isomorphism $\mathrm{H}^*(A)\cong\mathrm{H}^*(A_1)$. Different choices of b give the same isomorphism because b varies in A_1 which is geometrically connected. Applying this principle to $A=\mathrm{Prym}^0$ and A-torsors $\mathrm{Prym}^\epsilon_{\mathfrak{N}}$ and Prym^1, we see that there are canonical isomorphisms

$$\mathrm{H}^1(\mathrm{Prym}^\epsilon_{\mathfrak{N}})\cong\mathrm{H}^1(\mathrm{Prym}^0)\cong\mathrm{H}^1(\mathrm{Prym}^1)\cong V.\tag{10.17}$$

Lemma 10.5. *For $\epsilon\in\mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$, we have*

$$c_1(\mathcal{P})|_{X'\times\mathrm{Prym}^\epsilon_{\mathfrak{N}}}=2\beta+\deg_X\mathfrak{N}\,(\xi\otimes 1)$$

where $\beta\in V\otimes V\subset\mathrm{H}^1(X')\otimes\mathrm{H}^1(\mathrm{Prym}^\epsilon_{\mathfrak{N}})\subset\mathrm{H}^2(X'\times\mathrm{Prym}^\epsilon_{\mathfrak{N}})$ and $\xi\otimes 1\in\mathrm{H}^2(X')\otimes\mathrm{H}^0(\mathrm{Prym}^\epsilon_{\mathfrak{N}})\subset\mathrm{H}^2(X'\times\mathrm{Prym}^\epsilon_{\mathfrak{N}})$.

Proof. Choose $\mathfrak{N}'\in\mathrm{Pic}_{X'}(\bar{k})$ such that $\mathrm{Nm}(\mathfrak{N}')\cong\mathfrak{N}$. Pulling back by $(\mathrm{Id}_{X'},\mathrm{AJ}_1)$ to $X'\times X'$, $\mathcal{P}$ becomes $\mathcal{O}(\Delta^--\Delta)\otimes\mathrm{pr}_1^*\mathfrak{N}'$ where $\mathrm{pr}_1: X'\times X'\to X'$ is the projection to the first factor. Then note that $c_1(\mathrm{pr}_1^*\mathfrak{N}')=\deg_{X'}\mathfrak{N}'\,(\xi\otimes 1)=\deg_X\mathfrak{N}\,(\xi\otimes 1)\in\mathrm{H}^2(X')\otimes\mathrm{H}^0(X')$. Now the lemma follows from Lemma 10.4. □

Lemma 10.6. *For $1\le a\le r$, we have*

$$\begin{aligned}-\frac{1}{2}(\sigma p_a,p)^*c_1(\mathcal{P})=&-\frac{1}{2}\deg_X\mathfrak{N}\cdot\ p_a^*\xi+\mathrm{Tr}((\phi-1)^{-1}|V)p_a^*\xi\\ &+\sum_{b<a}p_{ba}^*(((\phi-1)^{-1}\otimes 1)\beta)+\sum_{b>a}p_{ab}^*((1\otimes(\phi-1)^{-1})\beta).\end{aligned}$$

Proof. We first have

$$(\sigma p_a,p)^*(\xi\otimes 1)=p_a^*\xi.\tag{10.18}$$

Next we use the commutative diagram

$$\begin{tikzcd}\mathrm{Sht}^r_{U(1)} \arrow[r, "{(p_a,p)}"] \arrow[d, "p_{[1,r]}"'] & X'\times\mathrm{Prym}_{\mathfrak{N}} \arrow[d, "\mathrm{Id}\times\mathrm{Lang}"] \\ X'^r \arrow[r, "{(\mathrm{pr}_a,\mathrm{AJ}^r)}"] & X'\times\mathrm{Prym}^{\epsilon(\mathfrak{N})}\end{tikzcd}\tag{10.19}$$

Here $\mathrm{pr}_a: X'^r\to X'$ is the a-th projection. The pullback along the Lang map $\mathrm{H}^1(\mathrm{Prym}^{\epsilon(\mathfrak{N})})\to\mathrm{H}^1(\mathrm{Prym}^\epsilon_{\mathfrak{N}})$ is the isomorphism $\phi-1$ of V under the isomorphisms (10.17), for each component $\epsilon\in\mathrm{Irr}(\mathrm{Prym}_{\mathfrak{N}})$. Therefore,

$$\beta=(\mathrm{Id}_{X'}\times\mathrm{Lang})^*(1\otimes(\phi-1)^{-1})\beta.\tag{10.20}$$

Here we view $(1\otimes(\phi-1)^{-1})\beta\in V\otimes V$ as an element of $\mathrm{H}^2(X'\times\mathrm{Prym}^{\epsilon(\mathfrak{N})})$. Hence by the above commutative diagram

$$\begin{aligned}(p_a,p)^*\beta=&(p_a,p)^*(\mathrm{Id}_{X'}\times\mathrm{Lang})^*(1\otimes(\phi-1)^{-1})\beta\\ =&p_{[1,r]}^*(\mathrm{pr}_a,\mathrm{AJ}^r)^*(1\otimes(\phi-1)^{-1})\beta.\end{aligned}\tag{10.21}$$

Since AJ^r can be decomposed into a composition $X'^r \xrightarrow{\mathrm{AJ}_1^r} (\mathrm{Prym}^1)^r \xrightarrow{m} \mathrm{Prym}^{\epsilon(\mathfrak{N})}$ (the map m is multiplication), and

$$m^*(v) = 1 \otimes \cdots \otimes 1 \otimes v + \cdots + v \otimes 1 \otimes \cdots \otimes 1$$

for $v \in V = \mathrm{H}^1(\mathrm{Prym}^{\epsilon(\mathfrak{N})})$, we get $\mathrm{AJ}^{r*} v = \sum_{1 \le b \le r} \mathrm{pr}_b^* v \in \mathrm{H}^1(X'^r)$. Hence

$$p_{[1,r]}^*(\mathrm{pr}_a, \mathrm{AJ}^r)^*(1 \otimes (\phi - 1)^{-1})\beta = \sum_i p_a^* v_i \otimes \left(\sum_{b=1}^r p_b^* (\phi - 1)^{-1} v^i \right). \tag{10.22}$$

Note that when $b < a$, the term $\mathrm{pr}_a^* v_i \otimes \mathrm{pr}_b^*(\phi-1)^{-1} v^i$ is $-p_{ba}^*((\phi-1)^{-1} v^i \otimes v_i)$; but after summing over i, using that $\sum_i v^i \otimes v_i = -\beta$, we obtain

$$-p_{ba}^*((\phi - 1)^{-1} \otimes 1)(-\beta) = p_{ba}^*((\phi - 1)^{-1} \otimes 1)(\beta).$$

When $b > a$, the corresponding term in (10.22) is $p_{ab}^*(1 \otimes (\phi - 1)^{-1})(\beta)$. When $b = a$, the corresponding term in (10.22) is

$$\sum_i p_a^*(v_i \cup (\phi - 1)^{-1} v^i) = \mathrm{Tr}((\phi - 1)^{-1}|V) p_a^* \xi.$$

Here we are using that the bases $\{v_i\}$ and $\{v^j\}$ satisfy $v_i \cup v^j = \delta_{ij}\xi$. Combining these with (10.21) and (10.22) we get

$$(p_a, p)^*\beta = \mathrm{Tr}((\phi-1)^{-1}|V) p_a^*\xi + \sum_{b<a} p_{ba}^*(((\phi - 1)^{-1} \otimes 1)\beta) + \sum_{b>a} p_{ab}^*((1 \otimes (\phi - 1)^{-1})\beta). \tag{10.23}$$

Since the action of σ on V is by -1, we have

$$(\sigma p_a, p)^*(\beta) = (p_a, p)^*(\sigma, \mathrm{Id})^*\beta = -(p_a, p)^*(\beta). \tag{10.24}$$

Combining this with (10.23), (10.18) and Lemma 10.5, we get the desired identity. $\square$

10.4. **Taylor expansion of $L(s, \eta)$.**

Lemma 10.7. *Let $\alpha \in \mathbf{C}$. Write the n-th derivative of $\log(1 - \alpha q^{-s})$ as*

$$(-1)^{n-1} (\log q)^n \frac{f_n(\alpha q^{-s})}{(1 - \alpha q^{-s})^n}, \quad n = 1, 2, \cdots. \tag{10.25}$$

where $f_n(x)$ is a polynomial in x. Then

$$f_n(x) = \sum_{c \in C_n} x^{\delta(c)}. \tag{10.26}$$

Here C_n is the set of cyclic permutations on $\{1, 2, \cdots, n\}$; for $c \in C_n$, $\delta(c)$ is the number of $1 \le i \le n$ such that $c(i) \le i$ (when $n = 1$, $\delta(c) = 1$).

Proof. From the definition we get a recursive relation:

$$f_{n+1}(x) = n x f_n(x) + x(1 - x) f_n'(x). \tag{10.27}$$

Also $f_1(x) = x$. From this it is easy to see that $\deg f_n(x) = n$ and $f_n(0) = 0$. Write $f_n(x) = a_1^{(n)} x + a_2^{(n)} x^2 + \cdots + a_n^{(n)} x^n$. Then

$$a_i^{(n+1)} = i a_i^{(n)} + (n + 1 - i) a_{i-1}^{(n)}, \quad i = 1, \cdots, n. \tag{10.28}$$

On the other hand, let $C_{n,i}$ be the set of $c \in C_n$ such that $\delta(c) = i$. We must show that $|C_{n,i}| = a_i^{(n)}$. We do this by checking that $|C_{n,i}|$ satisfies the same recursive relation (10.28).

For $c \in C_{n+1}$, let $1 \le i_c, j_c \le n$ be defined by $c(i_c) = n + 1$ and $c(n + 1) = j_c$. We have a map $\pi : C_{n+1} \to C_n$ sending $c \in C_{n+1}$ to $c' \in C_n$ defined by $c'(i) = c(i)$ if $i \ne i_c$ and $c'(i_c) = j_c$. We decompose $C_{n+1,i} = C'_{n+1,i} \sqcup C''_{n+1,i}$, where $C'_{n+1,i}$ is the set of $c \in C_{n,i}$ such that $i_c > j_c$. Then π restricts to a i to 1 map $\pi' : C'_{n+1,i} \to C_{n,i}$ (the preimage of c' are in bijection with i such that $c'(i) < i$) and an $(n + 1 - i)$ to 1 map $\pi'' : C''_{n+1,i} \to C_{n,i-1}$ (the preimage of $c' \in C_{n,i-1}$ are in bijection with i such that $c'(i) > i$). This shows

$$|C_{n+1,i}| = |C'_{n+1,i}| + |C''_{n+1,i}| = i|C_{n,i}| + (n + 1 - i)|C_{n,i-1}|. \tag{10.29}$$

This shows that $|C_{n,i}|$ satisfies the same recursive relation (10.28) as $(a_i^{(n)})$. Since the initial values match $|C_{1,1}| = 1 = a_1^{(1)}$, the lemma follows. $\square$

Corollary 10.8. *The Taylor expansion of* $\log L(s,\eta)$ *at* $s=0$ *is:*

$$\log L(s,\eta)=\log L(0,\eta)-\sum_{\ell\geq 1}\sum_{c\in C_\ell}\mathrm{Tr}\left(\frac{\phi^{\delta(c)}}{(1-\phi)^\ell}\Big|V\right)\frac{(\log q)^\ell(-s)^\ell}{\ell!}. \tag{10.30}$$

Proof. Let $\{\alpha_i\}$ be the multiset of eigenvalues of ϕ acting on V. By Lemma 10.7 evaluated at $s=0$, we get

$$\log(1-\alpha_i q^{-s})=\log(1-\alpha_i)-\sum_{\ell\geq 1}\sum_{c\in C_\ell}\frac{\alpha_i^{\delta(c)}}{(1-\alpha_i)^\ell}\frac{(\log q)^\ell(-s)^\ell}{\ell!}. \tag{10.31}$$

Taking sum over α_i, noting that $L(s,\eta)=\prod_i(1-\alpha_i q^{-s})$ and $\sum_i\alpha_i^{\delta(c)}/(1-\alpha_i)^\ell=\mathrm{Tr}(\frac{\phi^{\delta(c)}}{(1-\phi)^\ell}|V)$, we get the desired formula. $\square$

10.5. **Proof of Theorem 10.2.** Combining Lemma 10.3, Lemma 10.4 and Lemma 10.6, we get

$$\begin{aligned}&-\frac{1}{2}c_1(p_a^*\sigma^*\mathcal{L}^{-1}\otimes\ell_a)\\ =\quad& p_{[1,r]}^*\left(\sum_{b<a}\mathrm{pr}_{ba}^*((\phi(\phi-1)^{-1}\otimes 1)\beta)+\sum_{b>a}\mathrm{pr}_{ab}^*((1\otimes(\phi-1)^{-1})\beta)\right)\\ +\quad& p_{[1,r]}^*\left(\left(\mathrm{Tr}(\phi(\phi-1)^{-1}|V)-d/2\right)\mathrm{pr}_a^*\xi\right).\end{aligned} \tag{10.32}$$

Here we are using (10.3) and

$$-c_1(\mathcal{L}\otimes\omega_{X'})-2\,\mathrm{Tr}((\phi-1)^{-1}|V)\xi=-2\left(\frac{\deg\mathcal{L}}{2}+\frac{\deg\omega_{X'}}{2}+\mathrm{Tr}((\phi-1)^{-1}|V)\right)\xi.$$

Note that $\frac{\deg\omega_{X'}}{2}=\deg\omega_X=\dim V$, hence the last two terms combine to give $\mathrm{Tr}(1+(\phi-1)^{-1}|V)=\mathrm{Tr}(\phi(\phi-1)^{-1}|V)$.

Taking the product of $-\frac{1}{2}c_1(p_a^*\sigma^*\mathcal{L}^{-1}\otimes\ell_a)$ over all $1\leq a\leq r$, using (10.32) and extracting the coefficient of $p_{[1,r]}^*\xi^r$ we get

$$\prod_{a=1}^r-\frac{1}{2}c_1(p_a^*\sigma^*\mathcal{L}^{-1}\otimes\ell_a)=(\sum_{g\in S_r}A_g)p_{[1,r]}^*(\xi^{\otimes r}) \tag{10.33}$$

where $A_g\in\overline{\mathbf{Q}}_\ell$ is defined as

$$\begin{aligned}A_g\,\xi^{\otimes r}=&\prod_{g(a)<a}(\phi(\phi-1)^{-1}\otimes 1)\beta_{g(a)a}\prod_{g(a)>a}(1\otimes(\phi-1)^{-1})\beta_{ag(a)}\\ &\times\prod_{g(a)=a}\left(\mathrm{Tr}(\phi(\phi-1)^{-1}|V)-d/2\right)\xi_a.\end{aligned} \tag{10.34}$$

Here we use the abbreviations $(-)_{ba}=\mathrm{pr}_{ba}^*(-),(-)_a=\mathrm{pr}_a^*(-)$. When $r=0$ we understand the sum $\sum_{g\in S_r}A_g$ as 1.

We form the generating series of A_g for $g\in S_r$ for all $r\geq 0$. Our aim is to show

$$\sum_{r\geq 0,g\in S_r}A_g\frac{(\log q)^r(-s)^r}{r!}=q^{ds/2}L(s,\eta)L(0,\eta)^{-1}. \tag{10.35}$$

Indeed if this holds, then making a change of variables $s\mapsto 2s$ and extracting the coefficient of s^r we get

$$(-2\log q)^r\sum_{g\in S_r}A_g=L(0,\eta)^{-1}\frac{d^r}{ds^r}\left(q^{ds}L(2s,\eta)\right)\Big|_{s=0}. \tag{10.36}$$

Taking the degrees of both sides of (10.33) we get

$$\deg[\mathcal{Z}_\mathcal{L}^r(0)]=(-2)^r|\operatorname{Prym}(\mathbf{F}_q)|\sum_{g\in S_r}A_g. \tag{10.37}$$

Here the factor $|\operatorname{Prym}(\mathbf{F}_q)|$ is the degree of $p_{[1,r]} : \operatorname{Sht}^r_{U(1)} \to X'^r$. Using (10.37) and the fact that $|\operatorname{Prym}(\mathbf{F}_q)| = 2L(0,\eta)$ we get

$$\deg[\mathcal{Z}^r_{\mathcal{L}}(0)] = 2L(0,\eta)\cdot(-2)^r \sum_{g\in S_r} A_g = 2(\log q)^{-r}\frac{d^r}{ds^r}\left(q^{ds}L(2s,\eta)\right)\Big|_{s=0}. \tag{10.38}$$

This is exactly (10.7) and hence Theorem 10.2 is proved.

Now it remains to prove (10.35). Let $C(g)$ be the set of cycles of g. For each $c\in C(g)$, let $\ell(c)$ be its length and recall $\delta(c)$ is the number of $1\le i\le n$ such that $c(i)$ is defined and $c(i)\le i$. We claim that we can write $A_g = \prod_{c\in C(g)} A_{g,c}$ where

$$A_{g,c} = \begin{cases} -\operatorname{Tr}(\phi(1-\phi)^{-1}|V) - d/2, & \ell(c)=1,\\ -\operatorname{Tr}(\phi^{\delta(c)}(1-\phi)^{-\ell(c)}|V), & \ell(c)>1.\end{cases} \tag{10.39}$$

Indeed, suppose a cycle $c=(a,g(a),\cdots,g^{\ell-1}(a))$ has length $\ell=\ell(c)$. If $\ell=1$ then a is a fixed point of g and the factor corresponding to such a fixed point can be directly read from the definition of A_g in (10.34). If $\ell>1$, write $a_s = g^{s-1}(a)$ for $s=1,2,\cdots,\ell$. We assume that a is the largest element in the cycle. If $a_s > a_{s+1}$, the corresponding factor $(\phi(\phi-1)^{-1}\otimes 1)\beta_{a_{s+1}a_s} = \sum_i \phi(\phi-1)^{-1}v_{i,a_{s+1}}\otimes v^i_{a_s}$ (recall that $v^i_{a_s}$ means v^i put in the a_s-th factor of $\mathrm{H}^*(X')^{\otimes r}$). We rewrite it as

$$\sum_i v^i_{a_s}\otimes \phi(1-\phi)^{-1}v_{i,a_{s+1}} \tag{10.40}$$

where switching the terms produces a minus sign which cancels with the change from $\phi-1$ to $1-\phi$. Similarly, if $a_s<a_{s+1}$, writing $\beta = -\sum_i v^i\otimes v_i$, the corresponding factor $(1\otimes(\phi-1)^{-1})\beta_{a_sa_{s+1}}$ is

$$\sum_i v^i_{a_s}\otimes (1-\phi)^{-1}v_{i,a_{s+1}}. \tag{10.41}$$

Now take the product of the terms (10.40) or (10.41) for $s=1,\cdots,\ell$ and take the cup product of terms that are placed in the same factor of $\mathrm{H}^*(X')^{\otimes r}$. For any endomorphism T of V, we have $Tv_i\cup v^j = T_{ji}\xi$ where T_{ji} is the (i,j)-entry of the matrix of T under the basis $\{v_i\}$. The product above is then a multiple of $\xi^{\otimes r}$, and the multiple is the trace of the product of $\phi(1-\phi)^{-1}$ (for those s such that $a_s>a_{s+1}$, totalling $\delta(c)$ of them) and $(1-\phi)^{-1}$ (for the rest of s), except for a sign that appears in the cup product at the a-th factor, $v^i_a\cup(1-\phi)^{-1}v_{j,a} = -(1-\phi)^{-1}_{ij}\xi$. This proves (10.39).

The formula (10.39) depends only on the cyclic permutation c on an *ordered* set. We write $A_{g,c}$ as A_c with the understanding that the ordered set on which c operates is a subset of $\mathbf{N}$. Now we re-organize the sum over $g\in S_r$ by grouping first according to the partitions of the set $\{1,2,\cdots,r\}$ and then according to the conjugacy classes. We have surjections

$$\pi : S_r \xrightarrow{\pi_1} \Pi_r \xrightarrow{\pi_2} P_r \tag{10.42}$$

where Π_r is the set of partitions of the set $\{1,2,\cdots,r\}$, and P_r is the set of partitions of r. The map π_1 takes $g\in S_r$ to its cycles, and π_2 takes the lengths of the cycles. For $I_\bullet\in\Pi_r$, corresponding to a partition $\{I_\alpha\}$ of $\{1,\cdots,r\}$, the contribution of $\pi_1^{-1}(I_\bullet)$ to $\sum_{g\in S_r} A_g x^r/r!$ is

$$\Sigma_{I_\bullet} := \frac{\prod_\alpha |I_\alpha|!}{r!}\prod_\alpha\left(\sum_{c\in C(I_\alpha)} A_c\frac{x^{|I_\alpha|}}{|I_\alpha|!}\right) \tag{10.43}$$

where the sum is over the set $C(I_\alpha)$ of cyclic permutations of I_α. Clearly the sum $\sum_{c\in C(I_\alpha)} A_c$ depends only on the cardinality $|I_\alpha|$ and not on the ordering of I_α. Denote

$$\Gamma_\ell := \sum_{c\in C_\ell} A_c. \tag{10.44}$$

(Here recall from Lemma 10.7 that C_ℓ is the set of cyclic permutations on $\{1,2,\cdots,\ell\}$.) Write $\lambda := \pi_2(I_\bullet)\in P_r$ as $\lambda_1^{m_1}\cdots\lambda_t^{m_t}$, where $\lambda_1>\cdots>\lambda_t$, and m_i is the multiplicity of λ_i, then

$$\Sigma_{I_\bullet} = \frac{\prod(\lambda_i!)^{m_i}}{r!}\prod_i\left(\Gamma_{\lambda_i}\frac{x^{\lambda_i}}{\lambda_i!}\right)^{m_i}. \tag{10.45}$$

In particular, $\Sigma_{I_\bullet}$ depends only on the partition $\pi_2(I_\bullet) \in P_r$. Therefore the contribution of $\lambda \in P_r$ to $\sum_{g\in S_r} A_g x^r/r!$ is

$$|\pi_2^{-1}(\lambda)| \frac{\prod(\lambda_i!)^{m_i}}{r!} \prod_i \left(\Gamma_{\lambda_i} \frac{x^{\lambda_i}}{\lambda_i!}\right)^{m_i} = \prod_i \frac{1}{m_i!} \left(\Gamma_{\lambda_i} \frac{x^{\lambda_i}}{\lambda_i!}\right)^{m_i}. \tag{10.46}$$

Here we are using $|\pi_2^{-1}(\lambda)| = |O_\lambda|/|\pi_1^{-1}(I_\bullet)|$ (where $O_\lambda \subset S_r$ is the conjugacy class corresponding to λ), $|O_\lambda| = r!/(\prod \lambda_i^{m_i} m_i!)$ and $|\pi_1^{-1}(I_\bullet)| = \prod_i((\lambda_i - 1)!)^{m_i}$.

Summing over all partitions of r and then over all $r \geq 0$, we get

$$\sum_{r\geq 0, g\in S_r} A_g \frac{x^r}{r!} = \prod_{\ell\geq 1} \sum_{m\geq 0} \frac{1}{m!} \left(\Gamma_\ell \frac{x^\ell}{\ell!}\right)^m = \exp\left(\sum_{\ell\geq 1} \Gamma_\ell \frac{x^\ell}{\ell!}\right). \tag{10.47}$$

Using the formula (10.39) for $A_{g,c}$ we have

$$\Gamma_\ell = \begin{cases} -\operatorname{Tr}\left(\frac{\phi}{1-\phi}\Big|V\right) - d/2, & \ell = 1 \\ \sum_{c\in C_\ell} -\operatorname{Tr}\left(\frac{\phi^{\delta(c)}}{(1-\phi)^\ell}\Big|V\right), & \ell > 1. \end{cases} \tag{10.48}$$

Plugging into (10.47) we get

$$\sum_{r\geq 0, g\in S_r} A_g \frac{x^r}{r!} = \exp\left(\left(-\operatorname{Tr}\left(\frac{\phi}{1-\phi}\Big|V\right) - d/2\right) x - \sum_{\ell\geq 2}\sum_{c\in C_\ell} \operatorname{Tr}\left(\frac{\phi^{\delta(c)}}{(1-\phi)^\ell}|V\right) \frac{x^\ell}{\ell!}\right). \tag{10.49}$$

By Corollary 10.8, and letting $x = -(\log q)s$, we have

$$\begin{aligned} &\left(-\operatorname{Tr}\left(\frac{\phi}{1-\phi}\Big|V\right) - d/2\right)(\log q)(-s) - \sum_{\ell\geq 2}\sum_{c\in C_\ell} \operatorname{Tr}\left(\frac{\phi^{\delta(c)}}{(1-\phi)^\ell}\Big|V\right) \frac{(\log q)^\ell(-s)^\ell}{\ell!} \\ = \quad & \log L(s,\eta) - \log L(0,\eta) + d/2\,(\log q)s. \end{aligned}$$

Taking the exponential, and plugging into (10.49) we get

$$\sum_{r\geq 0, g\in S_r} A_g \frac{(\log q)^r(-s)^r}{r!} = q^{ds/2} L(s,\eta)\cdot L(0,\eta)^{-1},$$

which is exactly (10.35). □

10.6. **The complete comparison.** We now take the definition of Eisenstein series in §9.1. We make the following choices of characters:

- $\chi_0 = \eta$;
- χ is any character on $\mathrm{Pic}_{X'}(k)$ such that $\chi|_{\mathrm{Pic}_X(k)} = \eta$.

Recall from Definition 4.10 and (4.26) the generating series

$$\widetilde{Z}_1^r : \mathrm{Bun}_{\widetilde{P}_1}(k) = \widetilde{P}_1(F)\backslash \widetilde{H}_1(\mathbb{A})/\widetilde{H}_1(\widehat{\mathcal{O}}) \to \mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{GU(1)}).$$

Note $\widetilde{Z}_1^r(g)$ is compactly supported because its support is contained in $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ (where $\mathfrak{L} = c(g)$) which is proper.

Theorem 10.9. *We have for all $g \in \widetilde{H}_1(\mathbb{A})$,*

$$\frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\left(\widetilde{E}(g,s,\Phi)\right) = \deg \widetilde{Z}_1^r(g). \tag{10.50}$$

Proof. Since both sides are $\widetilde{H}_1(\widehat{\mathcal{O}})$-invariant, it suffices to show the Fourier expansions at $g = m(\mathcal{L},\mathfrak{L})$ match term-wise:

$$\frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0} \widetilde{E}_a(m(\mathcal{L},\mathfrak{L}),s,\Phi) = \chi(\mathcal{L})q^{-d/2}\deg[\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)] \tag{10.51}$$

for every $\mathcal{L} \in \mathrm{Pic}_{X'}(k), \mathfrak{L} \in \mathrm{Pic}_X(k)$ and $a \in \mathcal{A}_{\mathcal{L},\mathfrak{L}}(k)$. Here d is as in (10.3). We may further assume that $(-1)^r = \eta(\mathfrak{L})$, since otherwise both sides vanish.

When $a \neq 0$, by (9.9) specialized to $\chi_0 = \eta$, we have

$$\widetilde{E}_a(m(\mathcal{L}, \mathfrak{L}), s, \Phi) = \chi(\mathcal{L}) q^{-d/2} q^{ds} \operatorname{Den}(q^{-2s}, \operatorname{coker}(a)).$$

Then (10.51) follows from Theorem 9.2 specialized to $n = 1$, which relates the degree of $\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)$ to the local density.

It remains to consider the case $a = 0$. By (10.4) specialized to $\chi_0 = \eta$ and the symmetry with respect to $s \mapsto -s$, we have

$$\left(\frac{d}{ds}\right)^r \Big|_{s=0} \widetilde{E}_0(m(\mathcal{L}, \mathfrak{L}), s, \Phi) = 2\chi(\mathcal{L}) q^{-d/2} \left(\frac{d}{ds}\right)^r \Big|_{s=0} q^{ds} L(2s, \eta). \tag{10.52}$$

On the geometric side, since $\mathcal{L}$ is a line bundle, there are two terms in the decomposition (4.13):

$$\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(0) = \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}[\mathcal{L}](0) \coprod \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(0)^\circ,$$

where the first term is isomorphic to $\operatorname{Sht}^r_{U(1),\mathfrak{L}}$. Correspondingly, in Definition 4.7, there are two terms in $[\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(0)]$ in this case. Since the rank of $\mathcal{F}_i$ is $n = 1$, an injective $\mathcal{L} \to \mathcal{F}_i$ must give rise to non-zero $a : \mathcal{L} \to \sigma^*\mathcal{L}^\vee \otimes \nu^*\mathfrak{L}$. It follows that the stack $\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(0)^\circ$ is empty. Hence there is only one term left, i.e., $[\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(0)] = [\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}[\mathcal{L}](0)]$, which is defined by the Chern classes of the tautological line bundles. This term has the desired degree by Theorem 10.2 and (10.52). This completes the proof. □

Corollary 10.10. *The generating series $g \in \widetilde{P}_1(F)\backslash \widetilde{H}_1(\mathbb{A})/\widetilde{H}_1(\widehat{\mathcal{O}}) \mapsto \widetilde{Z}^r_1(g)$ is automorphic, i.e., it is left $\widetilde{H}_1(F)$-invariant and hence descends to a map*

$$Z^r_1 : \operatorname{Bun}_{GU(2)}(k) \to \operatorname{Ch}_0(\operatorname{Sht}^r_{GU(1)}).$$

In other words, Conjecture 4.12 holds for $n = m = 1$.

Proof. The case $r = 0$ is classical and follows from the modularity of theta functions (proved by Poisson summation).

Now consider the case $r > 0$. Let $g \in \widetilde{H}_1(\mathbb{A})$ with similitude $c(g) \in \mathbb{A}^\times$ corresponding to $\mathfrak{L} \in \operatorname{Pic}_X(k)$. Then $\widetilde{Z}^r_1(g) \in \operatorname{Ch}_0(\operatorname{Sht}^r_{U(1),\mathfrak{L}})$. By Corollary 10.14 below, the (component-wise) degree map induces an isomorphism $\operatorname{Ch}_0(\operatorname{Sht}^r_{U(1),\mathfrak{L}}) \xrightarrow{\sim} \mathbf{Q}^{\pi_0(\operatorname{Sht}^r_{U(1),\mathfrak{L}})}$. Hence it suffices to show that Z^r_1 is automorphic after composing with component-wise degree.

Assume that $\operatorname{Sht}^r_{U(1),\mathfrak{L}}$ is non-empty (otherwise the statement is vacuously true). According to Lemma 2.16, $\operatorname{Sht}^r_{U(1),\mathfrak{L}}$ has two connected components if r is even (and positive) and one connected component when r is odd. When r is odd, Theorem 10.9 implies that $\deg \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)$ is equal to $\frac{1}{(\log q)^r} \left(\frac{d}{ds}\right)^r \Big|_{s=0} \Big(\widetilde{E}(g, s, \Phi)\Big)$, which is automorphic in g.

When $r > 0$ is even, we claim that $\deg \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a) = \frac{1}{2} \frac{1}{(\log q)^r} \left(\frac{d}{ds}\right)^r \Big|_{s=0} \Big(\widetilde{E}(g, s, \Phi)\Big)$ on both components of $\operatorname{Sht}^r_{U(1),\mathfrak{L}}$, hence is also automorphic. For $a \neq 0$, it follows from Theorem 9.5. For $a = 0$, it is immediate from the calculation of Theorem 10.2 that the degrees of $\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(0)$ on both components of $\operatorname{Sht}^r_{U(1),\mathfrak{L}}$ are the same. □

10.6.1. *Chow groups of zero-cycles.* For a stack $\mathcal{Y}$ over k, we denote by $\operatorname{Ch}_0(\mathcal{Y})^{\deg 0}$ the subgroup of $\operatorname{Ch}_0(\mathcal{Y})$ whose degree on each *proper* connected component of $\mathcal{Y}$ vanishes. We will show that $\operatorname{Ch}_0(\mathcal{Y})^{\deg 0}$ vanishes for DM stacks satisfying mild conditions.

Lemma 10.11. *Let Y be a quasi-compact connected scheme of finite type over a field. Then any zero-cycle on Y lies on a connected (but possibly reducible) curve contained in Y.*

Proof. If Y is quasi-projective, then the result follows from [CP16, Corollary 1.9]. In general, we may cover Y by a finite number of affine varieties $U_1, \ldots, U_s$. Without loss of generality, we may enlarge our zero-cycle D so that whenever $U_i \cap U_j$ is non-empty, then $D \cap (U_i \cap U_j)$ is also non-empty. By the quasi-projective case, for each i we may find a connected curve C_i containing $D \cap U_i$. Then $\bigcup C_i$ is a connected curve containing D. □

Lemma 10.12. *Let Y be a quasi-compact separated scheme of finite type over $\mathbf{F}_q$. Then* $\operatorname{Ch}_0(Y)^{\deg 0} = 0$.

Proof. We immediately reduce to the case where Y is connected. Next we will reduce to the proper connected case. A compactification $\overline{Y}$ of Y exists, by Nagata's Theorem. Then the map $\mathrm{Ch}_0(\overline{Y})^{\deg 0} \to \mathrm{Ch}_0(Y)^{\deg 0}$ is surjective, since to a zero-cycle on Y we may add an appropriate (rational) multiple of any closed point on the boundary point of $\overline{Y}$ so that the sum has degree 0 on $\overline{Y}$. Hence it suffices to show that $\mathrm{Ch}_0(\overline{Y})^{\deg 0} = 0$.

So we may and do assume that Y is proper and connected. Let $D \in \mathrm{Ch}_0(Y)^{\deg 0}$. By Lemma 10.11, we may find a connected curve C in Y containing D. Since Y is proper we may furthermore assume that C is proper by replacing it with its closure if necessary.

We next reduce to the case where C is irreducible. Indeed, suppose that $C = \bigcup C_i$ is the union of irreducible components and that $\mathrm{Ch}_0(C_i)^{\deg 0} = 0$ for each i. Then any zero cycle in $\mathrm{Ch}_0(C_i)$ is equivalent to one concentrated at a single point (with $\mathbf{Q}$-coefficients); applying this repeatedly, any zero-divisor on C is equivalent to one supported on a single C_i.

So we may assume that C is proper and irreducible, and let $\widetilde{C} \to C$ be its normalization. Any $D \in \mathrm{Ch}_0(C)^{\deg 0}$ is the image of $\widetilde{D} \in \mathrm{Ch}_0(\widetilde{C})^{\deg 0} = \mathrm{Pic}^0_{\widetilde{C}}(\mathbf{F}_q) \otimes_{\mathbf{Z}} \mathbf{Q}$, which vanishes by the finiteness of $\mathrm{Pic}^0_{\widetilde{C}}(\mathbf{F}_q)$. $\square$

Corollary 10.13. *Suppose $\mathcal{Y}$ is a finite type separated Deligne-Mumford stack over a field, admitting a Zariski cover by open substacks that each have a finite flat atlas from a quasi-projective scheme. Then* $\mathrm{Ch}_0(\mathcal{Y})^{\deg 0} = 0$.

Proof. By the Keel-Mori Theorem [KM97] (as explained in [Con, Theorem 1.1]), $\mathcal{Y}$ has a coarse moduli space Y. The hypothesis implies that the conditions in [Con, §3] hold. In particular, Y is a scheme and $\mathcal{Y} \to Y$ is a proper universal homeomorphism [Con, Theorem 3.1], so it induces a bijection of connected components that matches proper components with proper components. Applying [Gil84, Theorem 6.8] to each of the connected components of $\mathcal{Y}$, we obtain $\mathrm{Ch}_0(\mathcal{Y}) \xrightarrow{\sim} \mathrm{Ch}_0(Y)$. As the proper components are also in bijection, this isomorphism takes $\mathrm{Ch}_0(\mathcal{Y})^{\deg 0} \xrightarrow{\sim} \mathrm{Ch}_0(Y)^{\deg 0}$, which vanishes by Lemma 10.12. $\square$

Corollary 10.14. *We have* $\mathrm{Ch}_0(\mathrm{Sht}^r_{U(1),\mathfrak{L}})^{\deg 0} = 0$.

Proof. The hypotheses of Corollary 10.13 are satisfied by (a variant with identical proof of) [Var04, Proposition 2.16]. $\square$

11. The corank one case: testing against CM cycles

We provide further evidence for the modularity in the corank one case, by intersecting against a certain class of CM cycles constructed in Example 4.18. In the number field case, an analogous problem was studied by Howard [How12].

11.1. **Setup.** Let X be a smooth projective curve over k, with function field F, and X'/X an unramified cover of degree 2. Let Y be another smooth projective curve and $\theta : Y \to X$ be a map of degree n, and let $Y' = X' \times_X Y$ their fiber product:

$$\begin{tikzcd}
 & Y' \arrow[dl, "\nu'"', "2"] \arrow[dd, "\theta'"', "n"] \\
Y \arrow[dd, "\theta"', "n"] & \\
 & X' \arrow[dl, "\nu"', "2"] \\
X &
\end{tikzcd}$$

Abusing notation, we will let σ denote the nontrivial involution on both Y'/Y and X'/X. We allow Y to be disconnected and ramified over X; but we will assume that the cover Y'/Y remains geometrically non-split over every component (i.e., for every connected component Y_α of Y, $Y_\alpha \times_X X'$ is geometrically connected).

For a line bundle $\mathfrak{L}$ over X, let $\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}$ be the moduli stack constructed in Example 4.18 (see also Example 7.6). The non-split hypothesis ensures that $\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}$ is proper. Taking direct image $\mathcal{F}_\bullet \mapsto \theta'_*\mathcal{F}_\bullet$ along the map $\theta' : Y' \to X'$ induces a finite unramified morphism

$$\Theta \colon \mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}} \longrightarrow \mathrm{Sht}^r_{U(n),\mathfrak{L}} . \tag{11.1}$$

This map defines a class

$$\Theta_*[\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}] \in \mathrm{Ch}_{r,c}(\mathrm{Sht}^r_{U(n),\mathfrak{L}})$$

in the Chow group of proper cycles on $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$.

11.2. **Pullback formula.** Let $\mathcal{L}$ be a line bundle on X'. Recall that $\mathcal{A}_{\mathcal{L},\mathfrak{L}}(k)$ is the set of Hermitian maps $a : \mathcal{L} \to \sigma^*\mathcal{L}^\vee \otimes \mathfrak{L}$, where $\mathcal{L}^\vee$ denotes the Serre dual. Previously in (7.9) we have defined a trace map

$$\mathrm{tr}\colon \mathcal{A}_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(k) \longrightarrow \mathcal{A}_{\mathcal{L},\mathfrak{L}}(k)\ .$$

Proposition 11.1. *Let $\mathcal{L}$ be a line bundle on X' and let $a \in \mathcal{A}_{\mathcal{L},\mathfrak{L}}(k)$. Then there is a natural decomposition into open-closed substacks:*

$$\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}} \times_{\mathrm{Sht}^r_{U(n),\mathfrak{L}}} \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a) \xrightarrow{\sim} \coprod_{\widetilde{a}} \mathcal{Z}^r_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(\widetilde{a}) \tag{11.2}$$

where $\widetilde{a}$ runs over all elements in $\mathcal{A}_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(k)$ such that $\mathrm{tr}(\widetilde{a}) = a$, and the virtual fundamental classes satisfy

$$\Theta^![\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)]\Big|_{\mathcal{Z}^{\theta^*\mathfrak{L},r}_{\theta^*\mathcal{L}}(\widetilde{a})} = [\mathcal{Z}^r_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(\widetilde{a})]. \tag{11.3}$$

Proof. This follows from Example 7.6. □

It follows immediately that, under the intersection pairing

$$\langle -,-\rangle\colon\ \mathrm{Ch}^r(\mathrm{Sht}^r_{U(n),\mathfrak{L}}) \times \mathrm{Ch}_{r,c}(\mathrm{Sht}^r_{U(n),\mathfrak{L}}) \longrightarrow \mathbb{Q}, \tag{11.4}$$

we have the following pullback formula:

$$\left\langle \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a), \Theta_*[\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}]\right\rangle = \sum_{\substack{\widetilde{a}\in\mathcal{A}_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(k),\\ \mathrm{tr}(\widetilde{a})=a}} \deg[\mathcal{Z}^r_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(\widetilde{a})]. \tag{11.5}$$

Remark 11.2. If we assume Y is connected, the pullback relation (11.5) can be proved without using the derived methods behind Example 7.6. We sketch a direct argument. Since Y is connected, a map $\theta'^*_*\mathcal{L} \to \mathcal{F}_\bullet$ is injective if and only if the induced map $\mathcal{L} \to \theta'_*\mathcal{F}_\bullet$ is injective. Therefore (11.2) restricts to the following *Cartesian* diagram for the circle loci of the special cycles (see Definition 2.9):

$$\begin{tikzcd}
\coprod_{\widetilde{a}} \mathcal{Z}^r_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(\widetilde{a})^\circ \arrow[r] \arrow[d] & \mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)^\circ \arrow[d] \\
\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}} \arrow[r, "\Theta"] & \mathrm{Sht}^r_{U(n),\mathfrak{L}}
\end{tikzcd}$$

Note that $\mathcal{Z}^r_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}(0)^\circ$ is empty. By Corollary 8.2, all terms in the disjoint union have the expected dimension (i.e., every $\mathcal{Z}^r_{\theta^*\mathcal{L},\theta^*\mathfrak{L}}(\widetilde{a})^\circ$ has dimension zero, and $\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)^\circ$ has dimension $r(n-1)$). Since the bottom map is a LCI morphism, and $\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)^\circ$ is LCI by Corollary 8.2[17], the Gysin pullbacks along the map Θ of the fundamental classes $[\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}(a)^\circ]$ are represented by the naive fundamental classes. This almost proves the relation (11.3), except for the most degenerate term $\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}[\mathcal{L}](0)$ corresponding to the Chern classes given by Definition 4.5. It remains to show

$$\Theta^![\mathcal{Z}^r_{\mathcal{L},\mathfrak{L}}[\mathcal{L}](0)] = [\mathcal{Z}^r_{\theta'^*\mathcal{L},\theta^*\mathfrak{L}}[\theta'^*\mathcal{L}](0)].$$

Since the tautological line bundles ℓ_i on $\mathrm{Sht}^r_{U(n),\mathfrak{L}}$ pullback to the tautological line bundles on $\mathrm{Sht}^r_{U(1)/Y,\theta'^*\mathfrak{L}}$ via Θ, this identity is easy to check directly.

[17] Corollary 8.2 is much easier to prove in this special case $n = 1$.

11.3. **Evidence for modularity in the corank one case.** Recall the generating series $\widetilde{Z}^r_m$ of corank m special cycles from Definition 4.10 and (4.26)

$$\widetilde{H}_m(\mathbb{A}) \ni g \mapsto \widetilde{Z}^r_m(g) \in \mathrm{Ch}_{r(n-m)}(\mathrm{Sht}^r_{GU(n)}).$$

We now specialize it to the corank one case, i.e., $m=1$. We will denote $\widetilde{Z}^r_{X'/X} := \widetilde{Z}^r_{m=1}$ or $\widetilde{Z}^r$ if X'/X is self-evident. We want to intersect the cycle class $\widetilde{Z}^r(g) \in \mathrm{Ch}_{r(n-1)}(\mathrm{Sht}^r_{GU(n)})$ for g satisfying $c(g) = \mathfrak{L}$ with the cycle $\Theta_*[\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}] \in \mathrm{Ch}_{r,c}(\mathrm{Sht}^r_{U(n),\mathfrak{L}})$. To make the statement more concise, we now introduce $GU(1)_{Y/X}$ to be the subgroup scheme of $GU(1)_Y$ with similitude line bundle in Pic_X (so that $GU(1)_{Y/X}$-torsors are the same as $(\mathcal{F}, \mathfrak{L}, h) \in \mathrm{Pic}_{Y'} \times \mathrm{Pic}_X$ where h is a Hermitian isomorphism $h : \mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}^\vee \otimes \nu'^*\theta^*\mathfrak{L}$), and define $\mathrm{Sht}^r_{GU(1)_{Y/X}}$ accordingly. Then $\mathrm{Sht}^r_{GU(1)_{Y/X}} = \coprod_{\mathfrak{L}\in\mathrm{Pic}_X(k)} \mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}$. Then we have a finite morphism

$$\Theta \colon \mathrm{Sht}^r_{GU(1)_{Y/X}} \longrightarrow \mathrm{Sht}^r_{GU(n)} ,$$

which is the union of components (11.1) indexed by $\mathfrak{L}$.

Suppose that $Y = \coprod_{\alpha\in\mathrm{Irr}(Y)} Y_\alpha$ is the decomposition of Y into connected components. Let $\widetilde{H}_1(\mathbb{A}_{Y_\alpha})$ denote the adelic similitude unitary group $GU(2)$ over (the function field of) Y_α. Let $\widetilde{H}_1(\mathbb{A}_Y) := \prod_\alpha \widetilde{H}_1(\mathbb{A}_{Y_\alpha})$.

By Example 7.6, we have an open-closed partition

$$\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}} = \coprod_{\underline{r}} \mathrm{Sht}^{\underline{r}}_{U(1)/Y,\theta^*\mathfrak{L}}, \tag{11.6}$$

where $\underline{r} = (r_\alpha)_\alpha \in \mathbf{Z}^{\mathrm{Irr}(Y)}$ satisfies $|\underline{r}| := \sum_\alpha r_\alpha = r$, and $\mathrm{Sht}^{\underline{r}}_{U(1)/Y,\theta^*\mathfrak{L}} := \prod_\alpha \mathrm{Sht}^{r_\alpha}_{U(1)/Y,\theta^*\mathfrak{L}}$. Note that by our definition the generating series $\widetilde{Z}_{Y'/Y}$ is a function

$$\widetilde{Z}_{Y'/Y} : \widetilde{H}_1(\mathbb{A}_Y) = \prod_\alpha \widetilde{H}_1(\mathbb{A}_{Y_\alpha}) \to \mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{GU(1)_Y}) = \bigoplus_{\mathfrak{N}\in\mathrm{Pic}_Y(k)} \mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{U(1)/Y,\mathfrak{N}}) \tag{11.7}$$

Viewing $\widetilde{H}_1(\mathbb{A})$ as a subgroup of $\widetilde{H}_1(\mathbb{A}_Y)$ via the diagonal embedding, the restriction $Z^r_{Y'/Y}|_{\widetilde{H}_1(\mathbb{A})}$ takes values in $\mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{GU(1)_{Y/X}}) = \bigoplus_{\mathfrak{L}\in\mathrm{Pic}_X(k)} \mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}})$.

On the analytic side, we denote by $\widetilde{E}(g_\alpha, s, \Phi_{Y_\alpha})$ the normalized Eisenstein series (9.7) in §9 for $n=1$, the covering Y'_α/Y_α, and the spherical section Φ_{Y_α}. It is an automorphic form on $\widetilde{H}_1(\mathbb{A}_{Y_\alpha})$. Let $\Phi_Y = \otimes\Phi_{Y_\alpha}$. We define for $g = (g_\alpha) \in \widetilde{H}_1(\mathbb{A}_Y)$,

$$\widetilde{E}(g, s, \Phi_Y) = \prod_{\alpha\in\mathrm{Irr}(Y)} \widetilde{E}(g_\alpha, s, \Phi_{Y_\alpha}),$$

which is an automorphic form on $\widetilde{H}_1(\mathbb{A}_Y)$.

We have the following result, which provides evidence for the Modularity Conjecture 4.12 in the corank $m=1$ case.

Theorem 11.3. *(1) We have an equality*

$$\Theta^! Z^r_{X'/X} = Z^r_{Y'/Y}|_{\widetilde{H}_1(\mathbb{A})}$$

of functions on $\widetilde{H}_1(\mathbb{A})$ with values in $\mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{GU(1)_{Y/X}})$.

(2) For every $g \in \widetilde{H}_1(\mathbb{A})$, we have

$$\left\langle Z^r_{X'/X}(g), \Theta_*[\mathrm{Sht}^r_{GU(1)_{Y/X}}]\right\rangle = \frac{1}{(\log q)^r}\left(\frac{d}{ds}\right)^r\Big|_{s=0}\widetilde{E}(g, s, \Phi_Y). \tag{11.8}$$

In particular, the function $\widetilde{H}_1(\mathbb{A}) \ni g \mapsto \left\langle Z^r_{X'/X}(g), \Theta_*[\mathrm{Sht}^r_{GU(1)_{Y/X}}]\right\rangle$ defines an automorphic form on $\widetilde{H}_1(\mathbb{A})$.

Proof. To show the first statement, suppose that $g \in \widetilde{H}_1(\mathbb{A})$ has similitude factor $c(g) = \mathfrak{L}$. Then both sides take values in $\mathrm{Ch}_{0,c}(\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}})$ and the equality follows from the pull back relation (11.3).

To show the second statement, for $\underline{r} = (r_\alpha)_\alpha$ satisfying $\sum_\alpha r_\alpha = r$, we have

$$\deg(Z^r_{Y'/Y}(g)|_{\mathrm{Sht}^{\underline{r}}_{GU(1)/Y}}) = \prod_{\alpha \in \mathrm{Irr}(Y)} \deg Z^{r_\alpha}_{Y'_\alpha/Y_\alpha}(g_\alpha),$$

for $g = (g_\alpha) \in \widetilde{H}_1(\mathbb{A}_Y)$. Similarly there is a decomposition of the analytic side, by Lebniz's rule,

$$\left(\frac{d}{ds}\right)^r \Big|_{s=0} \widetilde{E}(g, s, \Phi_Y) = \sum_{\underline{r} \in \mathbf{Z}^{\mathrm{Irr}(Y)}, |\underline{r}|=r} \prod_\alpha \left(\frac{d}{ds}\right)^{r_\alpha} \Big|_{s=0} \widetilde{E}(g_\alpha, s, \Phi_{Y_\alpha}).$$

By the case of modularity when $n = 1$, i.e., Theorem 10.9, we have

$$\deg Z^{r_\alpha}_{Y'_\alpha/Y_\alpha}(g_\alpha) = \frac{1}{(\log q)^{r_\alpha}} \left(\frac{d}{ds}\right)^{r_\alpha} \Big|_{s=0} \widetilde{E}(g_\alpha, s, \Phi_{Y_\alpha})$$

for $g_\alpha \in \widetilde{H}_1(\mathbb{A}_{Y_\alpha})$. The assertion follows by combining these equalities. $\square$

Remark 11.4. In view of (11.6), the proof above shows a refinement of (11.8), i.e., for any $\underline{r} \in \mathbf{Z}^{\mathrm{Irr}(Y)}_{\geq 0}$ such that $|\underline{r}| = r$, we have

$$\left\langle Z^r_{X'/X}(g), \Theta_*[\mathrm{Sht}^{\underline{r}}_{GU(1)_{Y/X}}] \right\rangle = \frac{1}{(\log q)^r} \prod_\alpha \left(\frac{d}{ds}\right)^{r_\alpha} \Big|_{s=0} \widetilde{E}(g, s, \Phi_{Y_\alpha})$$

as a function of $g \in \widetilde{H}_1(\mathbb{A})$.

Remark 11.5. In the number field case, the theorem of Howard [How12] is analogous to our case where Y is connected and $r = 1$. It seems that the analog of the case of disconnected Y in the number field case has not been treated.

Remark 11.6. Since Y is allowed to be ramified over X, there are infinitely many such covers. We may form the subspace of $\mathrm{H}^{2(n-1)r}_c(\mathrm{Sht}^r_{U(n),\mathfrak{L}})$ spanned by the cycle classes $\Theta_*[\mathrm{Sht}^r_{U(1)/Y,\theta^*\mathfrak{L}}]$ for varying coverings Y/X of degree n. It is an interesting question how large this subspace is.

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