On p-Order Cone Relaxation of Sensor Network Localization

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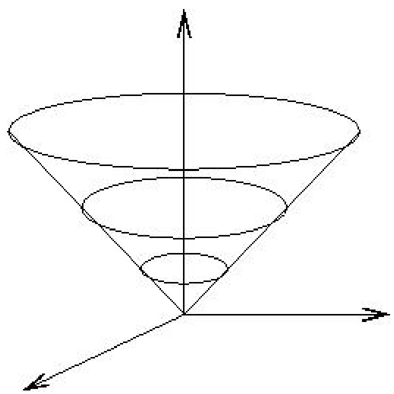
BIRS Workshop, Banff
November 11-16, 2006

Talk Outline

- Problem description
- pOCP relaxations
- Properties of pOCP relaxations
- Performance of SOCP relaxation and efficient solution methods
- Conclusions & Future Directions



Sensor Network Localization

Basic Problem:

- n pts in $\Re^d$ ($d = 1, 2, 3$).

- Know last $n - m$ pts ('anchors') $x_{m+1}, ..., x_n$ and ℓ_p-dist. ($1 < p < \infty$) estimate for pairs of 'neighboring' pts

$$d_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A}$$

with $\mathcal{A} \subseteq \{(i,j) : 1 \leq i < j \leq n\}$.

- Estimate first m pts ('sensors').

History? Graph realization, position estimation in wireless sensor network, determining protein structures, ...

Optimization Problem Formulation

$$v_{\mathrm{opt}} := \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|_p - d_{ij} \right|$$

- Objective function is nonconvex.
- Problem is NP-hard (reduction from PARTITION).
- Use a convex (but not SDP) relaxation.

Other formulations: "$\left|\|x_i - x_j\|_p - d_{ij}\right|^2$" or "$\left|\|x_i - x_j\|_p^p - d_{ij}^p\right|$" or ...

p-Order Cone Program Relaxation

$$
\begin{aligned}
v_{\text{opt}} = \min_{x_1,\ldots,x_m,y_{ij}} \quad & \sum_{(i,j)\in\mathcal{A}} |y_{ij} - d_{ij}| \\
\text{s.t.} \quad & y_{ij} = \|x_i - x_j\|_p \quad \forall (i,j) \in \mathcal{A}
\end{aligned}
$$

Relax "$=$" to "$\geq$" constraint:

$$
\begin{aligned}
v_{\text{socp}} := \min_{x_1,\ldots,x_m,y_{ij}} \quad & \sum_{(i,j)\in\mathcal{A}} |y_{ij} - d_{ij}| \\
\text{s.t.} \quad & y_{ij} \geq \|x_i - x_j\|_p \quad \forall (i,j) \in \mathcal{A}
\end{aligned}
$$

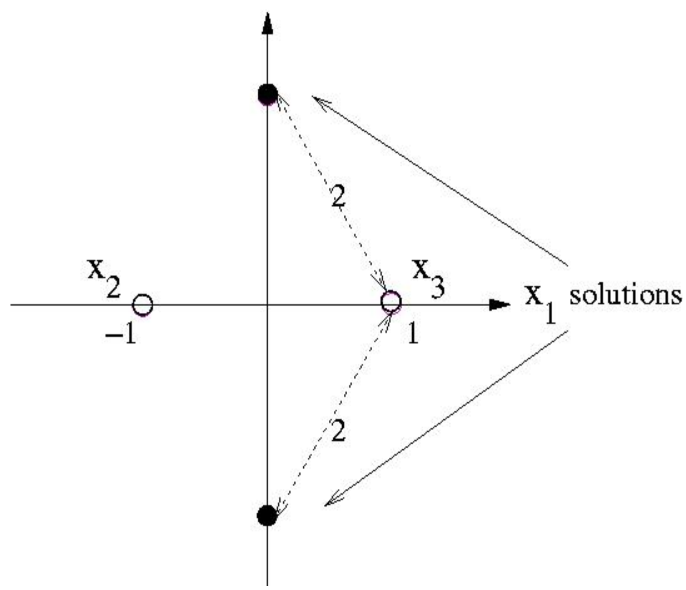
(p=2: Doherty,Pister,El Ghaoui '03)

Properties of pOCP Relaxation

$d = 2$, $n = 3$, $m = 1$, $d_{12} = d_{13} = 2$

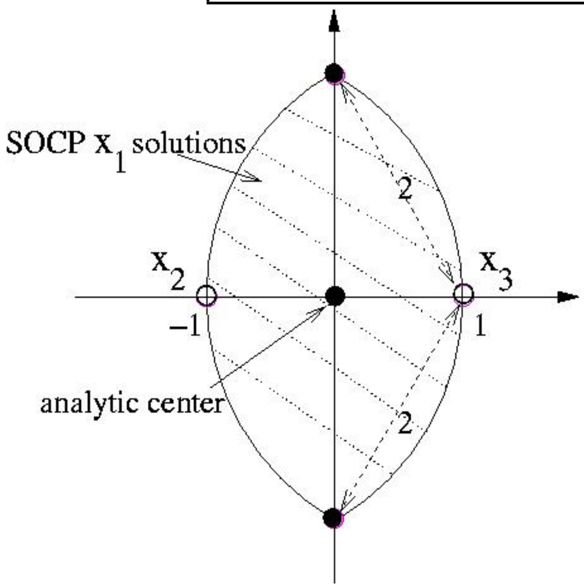
Problem:

$$0 = \min_{x_1 \in \Re^2} \quad |\|x_1 - (0,1)\|_p - 2| + |\|x_1 - (0,-1)\|_p - 2|$$



pOCP Relaxation:

$$
\begin{aligned}
0 = \quad & \min_{\substack{x_1 \in \Re^2 \\ y_{12}, y_{13} \in \Re}} \quad |y_{12} - 2| + |y_{13} - 2| \\
& \text{s.t.} \quad y_{12} \geq \|x_1 - (0,1)\|_p \\
& \qquad\;\; y_{13} \geq \|x_1 - (0,-1)\|_p
\end{aligned}
$$



If solve pOCP by IP method, then likely get analy. center of soln set.

Analytic Center Solution

Define

$$\mathcal{A}_{\text{active}} := \left\{(i,j) \in \mathcal{A} \mid y_{ij} = \|x_i - x_j\|_p \ \forall \text{ solns } x_1, \ldots, x_m, (y_{ij})_{(i,j)\in\mathcal{A}} \text{ of } p\text{OCP}\right\}$$

$$\text{Relative} - \text{interior soln} := \text{soln } x_1, \ldots, x_m, (y_{ij})_{(i,j)\in\mathcal{A}} \text{ of } p\text{OCP with } y_{ij} > \|x_i - x_j\|_p \ \forall\ (i,j) \in \mathcal{A} \setminus \mathcal{A}_{\text{active}}$$

$$\text{Analytic center soln} := \underset{\substack{\text{rel.}-\text{int. soln} \\ x_1,\ldots,x_m,(y_{ij})_{(i,j)\in\mathcal{A}}}}{\arg\min} \sum_{(i,j)\in\mathcal{A}\setminus\mathcal{A}_{\text{active}}} -\log\left(y_{ij}^p - \|x_i - x_j\|_p^p\right)$$

Properties of pOCP Relaxations

Key Property: If $p = 2$ or $d = 2$, then

$$\arg\min_x \sum_{i=1}^{k} \|x - x_i\|_p \in \text{conv}\,\{x_1, ..., x_k\}$$

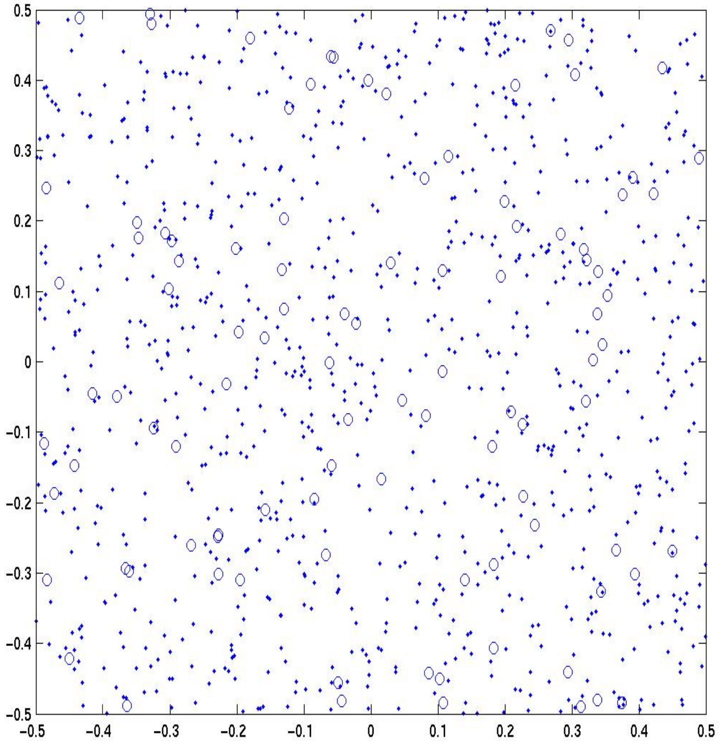
for any k and $x_1, ..., x_k \in \Re^d$. (Remains true if $\|x - x_i\|_p$ is raised to pth power.)

Q: What if $d \geq 3$ and $p \neq 2$? (False if $\|x - x_i\|_p$ is raised to pth power. (S. Zhang '05, C.-K. Sim '03))

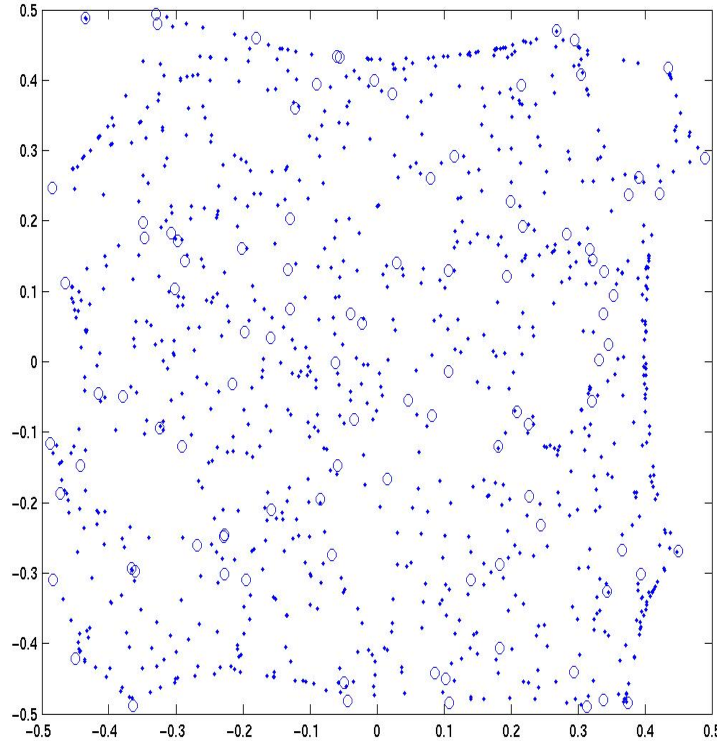
Fact 1: If $(x_1, ..., x_m, y_{ij})_{(i,j)\in\mathcal{A}}$ is the analytic center soln of pOCP, then

$$x_i \in \mathrm{conv}\,\{x_j\}_{j\in\mathcal{N}(i)} \quad \forall i \leq m$$

with $\mathcal{N}(i) := \{j \mid (i,j) \in \mathcal{A}\}$.



True soln ($m = 900$, $n = 1000$, nhbrs if ℓ_2-dist$< .06$)



SOCP soln found by IP method (SeDuMi)

Fact 2: If $(x_1, ..., x_m, y_{ij})_{(i,j)\in\mathcal{A}}$ is a relative-interior pOCP soln (e.g., analytic center), then for each $i \leq m$,

$$\|x_i - x_j\|_p = y_{ij} \quad \text{for some } j \in \mathcal{N}(i) \quad \iff \quad x_i \text{ appears in every } p\text{OCP soln.}$$

Error Bounds

What if distances have errors?

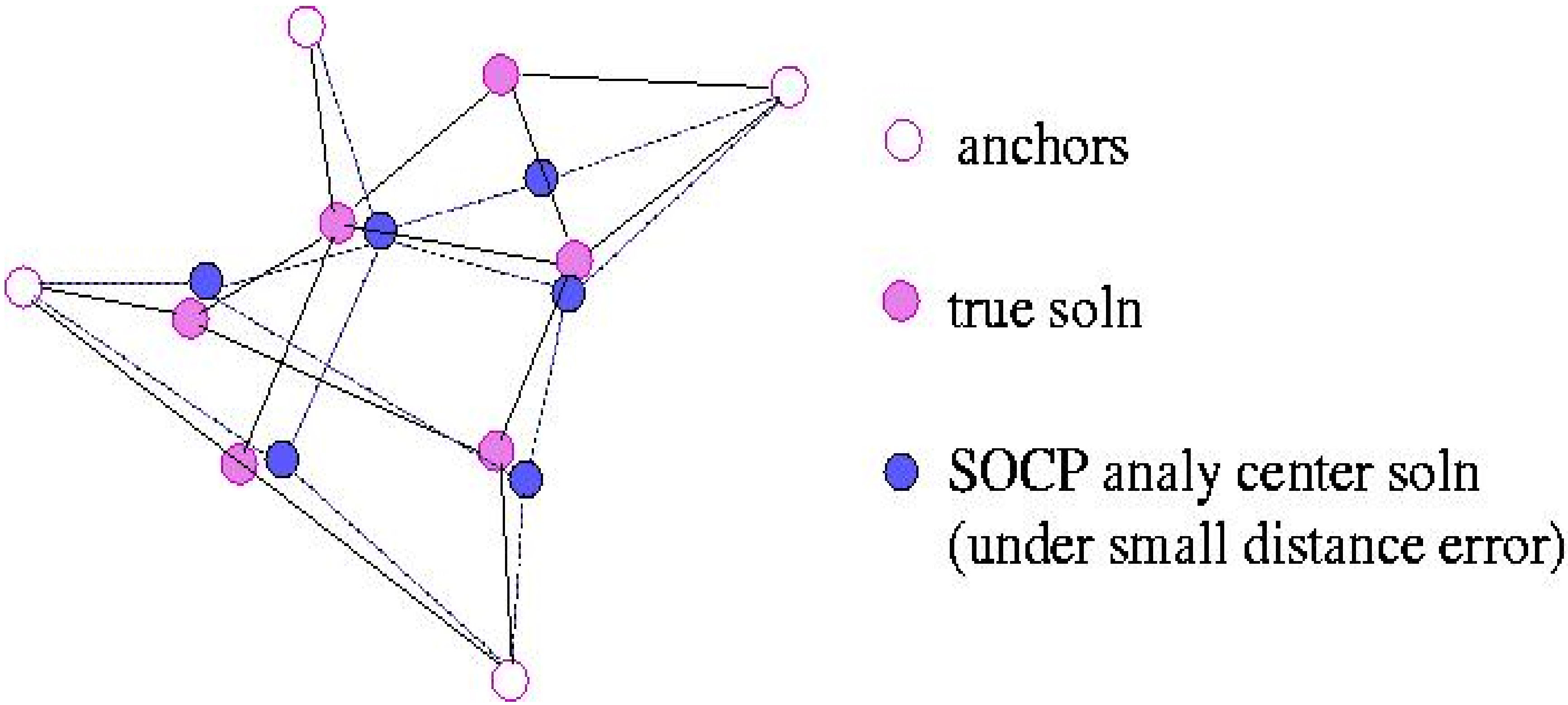
$$d_{ij} = d_{ij}^{\text{true}} + \delta_{ij},$$

where $\delta_{ij} \in \Re$ and $d_{ij}^{\text{true}} := \|x_i^{\text{true}} - x_j^{\text{true}}\|_p$ ($x_i^{\text{true}} = x_i \ \forall \ i > m$).

Fact 3: If $(x_1, ..., x_m, y_{ij})_{(i,j)\in\mathcal{A}}$ is a relative-interior pOCP soln corresp. $(d_{ij})_{(i,j)\in\mathcal{A}}$ and $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \leq \delta$, then for each i,

$$\|x_i - x_j\|_p = y_{ij} \quad \text{for some } j \in \mathcal{N}(i) \quad \Longrightarrow \quad \|x_i - x_i^{\text{true}}\|_p = O\Big(\sqrt[p]{\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}|}\Big).$$

Fact 4: As $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \to 0$, (analytic center pOCP soln corresp. $(d_{ij})_{(i,j)\in\mathcal{A}}$) $\to$ (analytic center pOCP soln corresp. $(d_{ij}^{\text{true}})_{(i,j)\in\mathcal{A}}$).



Error bounds for the analytic center pOCP soln when distances have small errors.

Solving pOCP Relaxation I: IP Method

$$\begin{array}{lll} \min\limits_{x_1,\ldots,x_m,y_{ij}} & \sum\limits_{(i,j)\in\mathcal{A}} |y_{ij}-d_{ij}| & \\ \text{s.t.} & y_{ij} \geq \|x_i - x_j\|_p & \forall (i,j)\in\mathcal{A} \end{array}$$

Put into conic form:

$$\begin{array}{llll} \min & \sum\limits_{(i,j)\in\mathcal{A}} u_{ij} & & \\ \text{s.t.} & x_i - x_j - w_{ij} & = 0 & \forall (i,j)\in\mathcal{A} \\ & y_{ij} - u_{ij} & = d_{ij} & \forall (i,j)\in\mathcal{A} \\ & u_{ij} \geq 0, & (y_{ij}, w_{ij}) \in p\text{−order cone} & \forall (i,j)\in\mathcal{A}. \end{array}$$

Solve by an IP method, e.g., SeDuMi or Mosek for $p = 2$.

Solving pOCP Relaxation II: Smoothing + Coordinate Gradient Descent

$$\min_{y \geq z} |y - d| = \max\{0, z - d\}$$

So pOCP relaxation:

$$\min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \max\{0, \|x_i - x_j\|_p - d_{ij}\}$$

This is an unconstrained nonsmooth convex program.

- Smooth approximation:

$$\max\{0, t\} \approx h_\mu(t) := \frac{(t^2 + \mu^2)^{1/2} + t}{2} \quad (\mu > 0)$$

pOCP approximation:

$$\min \quad f_\mu(x_1, .., x_m) := \sum_{(i,j)\in\mathcal{A}} h_\mu\left(\|x_i - x_j\|_p - d_{ij} + \mu\right)$$

Add a smoothed log-barrier term $-\mu \sum_{(i,j)\in\mathcal{A}} \log\left(\mu + h_\mu\left(d_{ij} - \mu - \|x_i - x_j\|_p\right)\right)$

Solve the smooth approximation using coordinate gradient descent (SCGD):

- If $\|\nabla_{x_i} f_\mu\| = \Omega(\mu)$, then update x_i by moving it along the Newton direction $-[\nabla^2_{x_i x_i} f_\mu]^{-1} \nabla_{x_i} f_\mu$, with Armijo stepsize rule, and re-iterate.
- Decrease μ when $\|\nabla_{x_i} f_\mu\| = O(\mu)\ \forall i$.

$\mu^{\text{init}} = 1e - 5$. $\mu^{\text{end}} = 1e - 9$. Decrease μ by a factor of 10.

Code in Fortran. Computation easily distributes.

Simulation Results for $p = 2$

- Uniformly generate $x_1^{\mathrm{true}}, ..., x_n^{\mathrm{true}}$ in $[0, 1]^2$, $m = .9n$, two pts are nhbrs if dist$< \ radioR$.
  Set
  $$d_{ij} = \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|_2 \cdot |1 + \epsilon_{ij} \cdot nf|,$$
  $\epsilon_{ij} \sim N(0, 1)$ (Biswas, Ye '03)

- Solve SOCP using SeDuMi 1.05 or SCGD.

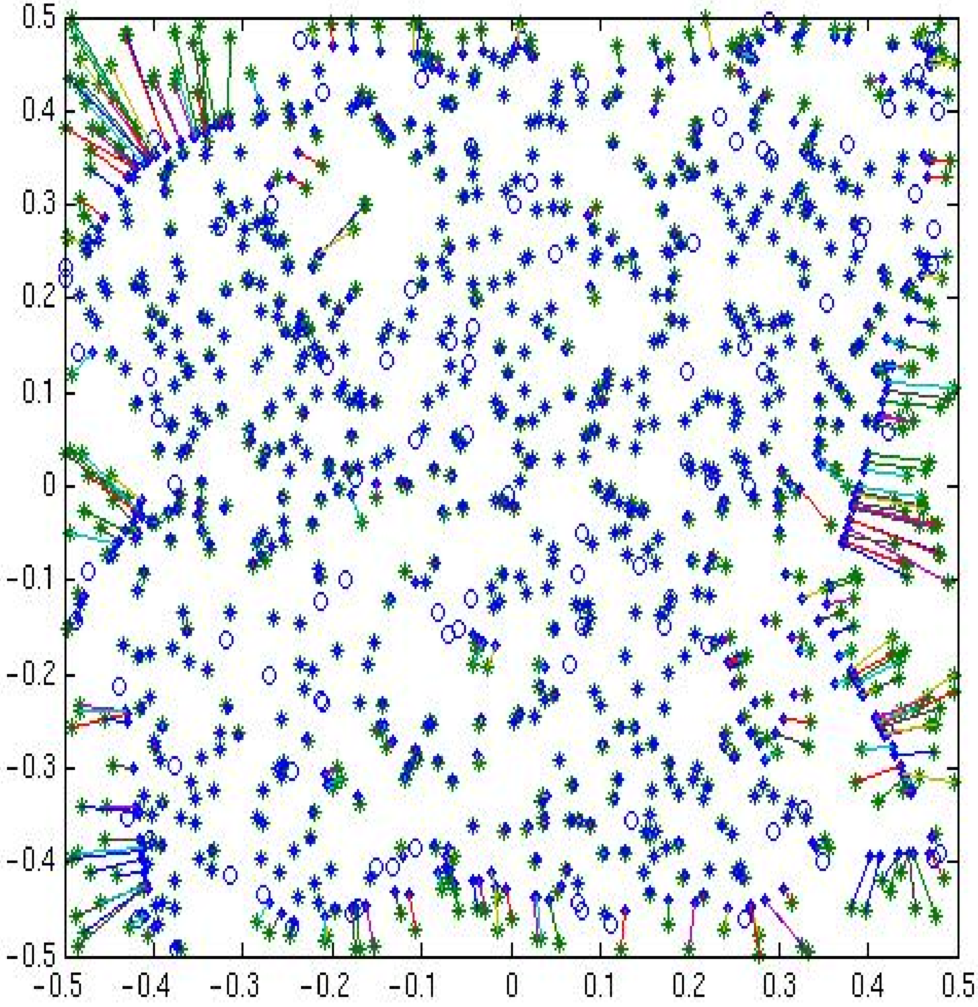
- Sensor i is uniquely positioned if

$$\Big| \|x_i - x_j\|_2 - y_{ij} \Big| \leq 10^{-7} d_{ij} \quad \text{for some } j \in \mathcal{N}(i).$$

n	m	nf	SeDuMi cpu/$m_{\rm up}$/Err$_{\rm up}$	"SCGD" cpu/$m_{\rm up}$/Err$_{\rm up}$
1000	900	0	1.0/327/4.6e-5	.2/357/3.8e-5
1000	900	.001	0.9/438/1.6e-3	.4/442/1.5e-3
1000	900	.01	1.1/555/1.5e-2	1.6/518/1.1e-2
2000	1800	0	33.7/1497/4.3e-4	0.8/1541/3.3e-4
2000	1800	.001	38.0/1465/3.3e-3	1.8/1466/3.6e-3
2000	1800	.01	16.2/1704/6.3e-2	2.9/1707/5.1e-2
4000	3600	0	17.8/2758/3.0e-4	1.6/2844/3.2e-4
4000	3600	.001	20.4/2907/3.2e-3	5.1/2894/3.0e-3
4000	3600	.01	17.2/3023/9.1e-3	6.1/3020/9.1e-3

Table 1: $radioR = .06(.035)$ for $n = 1000, 2000(4000)$

- cpu (minutes) times are on a HP DL360 workstation, running Linux 3.5.
- $m_{\rm up} :=$ number of uniquely positioned sensors.
- Err$_{\rm up} := \max_{i \text{ uniq. pos.}} \|x_i - x_i^{\rm true}\|_2$.



○ = anchors $\ast$ = true positions of sensors
● = SOCP soln found by SCGD ($m = 900$, $n = 1000$, $radioR = .06$, $nf = .01$)

Conclusions & Future Directions

- Finding an analytic center soln of pOCP is key.
- Exploit network structures of pOCP? Distributed computation?
- Additional linear constraints?
- Other objective functions, e.g., $\sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|_p - d_{ij} \right|^2$?
- How to localize sensors not uniquely positioned? Smoothing $+$ local descent on original objective function...
- Extend error bound result to SDP and ESDP relaxations (Biswas, Ye '04, ..., Wang et al. '06)) of $\min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|_2^2 - d_{ij}^2 \right|$?