Some Convex Programs Without a Duality Gap [1]

(in honor of Alfred Auslender on his 65th birthday–for his many contributions to mathematical programming, including duality)

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Abstract

An important issue in convex programming concerns duality gap. Various conditions have been developed over the years that guarantee no duality gap, including one developed by Rockafellar [22] involving separable objective function and affine constraints. We show that this sufficient condition can be further relaxed to allow the constraint functions to be separable. We also refine a sufficient condition involving weakly analytic functions by allowing them to be extended-real-valued.

Key words. separable convex program, recession direction, duality gap, Hoffman's error bound, weakly analytic function.

1 Introduction

Duality plays an essential role in linear programming and, more generally, convex programming. For a given convex program ("primal problem")

$$v_{\mathrm{P}} = \min \; f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0, \quad i = 1, ..., m, \tag{P}$$

where each $f_i : \Re^n \to (-\infty, \infty]$ is a proper convex lsc (lower semicontinuous) function, the dual problem is

$$v_{\mathrm{D}} = \max_{\mu \ge 0} \; q(\mu), \tag{D}$$

where $q(\mu) = \inf_x \{f_0(x) + \sum_{i=1}^m \mu_i f_i(x)\}$. We assume throughout that the feasible set of (P):

$$F = \{x \in \mathrm{dom} f_0 \mid f_i(x) \le 0, \; i = 1, ..., m\}$$

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is nonempty (i.e., $\upsilon_{\rm P} < \infty$). By weak duality, $\upsilon_{\rm P} \geq \upsilon_{\rm D}$. Notice that the optimal solution set of (P) may be empty or unbounded, and the same is true for (D).

The dual problem can offer new insights, and dual methods or primal-dual methods are often efficient at finding an optimal primal solution or determine that no such solution exists; see [4, 5, 6, 7, 13, 17, 19, 21, 22, 23] and references therein. In order for dual methods and primal-dual methods to be effective at solving the primal problem, it is essential that there is no duality gap, i.e., $\upsilon_{\rm P} = \upsilon_{\rm D}$. It is well known that this is not true in general, as is illustrated by Duffin's 2-variable example [8, p. 46]:

$$n = 2,\ m = 1, \quad f_0(x) = x_2, \quad f_1(x) = \sqrt{x_1^2 + x_2^2} - x_1, \tag{1}$$

for which $\upsilon_{\rm P} = 0$ while $\upsilon_{\rm D} = -\infty$. Thus, much study has focussed on identifying important classes of convex programs for which there is no duality gap. If $\upsilon_{\rm P} = -\infty$, then $\upsilon_{\rm D} = -\infty$ by weak duality and there is no duality gap. Thus the interesting case is when $\upsilon_{\rm P}$ is finite.

It is known that there is no duality gap under any one of the following assumptions:

A1.

$$\mathrm{dom} f_0 \subseteq \mathrm{dom} f_i \quad \text{and} \quad \mathrm{ri}(\mathrm{dom} f_0) \subseteq \mathrm{ri}(\mathrm{dom} f_i), \quad i = 1, ..., m, \tag{2}$$

and the optimal solution set of (P) is nonempty and bounded [18, Thm. 30.4].

A2. (2) and there exists a $\hat{x} \in \mathrm{ri}(\mathrm{dom} f_0) \cap F$ satisfying $f_i(\hat{x}) < 0$ whenever $i \neq 0$ and f_i is not affine [18, Thms. 28.2 and 30.4].

A3. $f_0, f_1, ..., f_m$ are real-valued and there exists a $\hat{x} \in F$ satisfying $f_i(\hat{x}) < 0$ whenever $i \neq 0$ and f_i is not weakly analytic[3] [16, Thm. 4.2], [2, Thm. 5.4.2] (also see [19, 20] for earlier works).

A4. f_0 is separable and $f_1, ..., f_m$ are affine [22, Sec. 11D].

A5. $f_0, f_1, ..., f_m$ are asymptotically level stable (als) and have the same domain [1, Thm. 2], [2, Thm. 5.4.1].[4]

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[3]A function $f : \Re^n \to (-\infty, \infty]$ is weakly analytic if f is constant on a line in $\Re^n$ whenever it is constant on some open segment of the line.

[4]A proper convex lsc function $f : \Re^n \to (-\infty, \infty]$ is als if, for any $\alpha > 0$, any $d \in \Re^n$, any sequence $x^k \in \Re^n$, and any convergent sequence $\epsilon_k \in \Re$, $k = 1, 2, ...$, satisfying

$$f(x^k) \leq \epsilon_k \ \forall k, \qquad \|x^k\| \to \infty, \qquad \frac{x^k}{\|x^k\|} \to d, \qquad f_\infty(d) = 0,$$

there exists $\bar{k}$ such that $f(x^k - \alpha d) \leq \epsilon_k$ for all $k \geq \bar{k}$. Here f_∞ denotes the recession function of f [18, p. 66], also called horizon function or asymptotic function [2, p. 48].

Notice that (2) holds automatically under A3 or A4 or A5. Other variants of the above results are found in, for example, [7, Sec. 6.5.2], [8, p. 90], [10, Prop. 3.2], [15, Thm. 4.5]. A unified treatment of A3, A5, and related assumptions is given in [9].

For Duffin's example, f_0 and f_1 are real-valued (so (2) holds) and the optimal solution set of (P) is nonempty but unbounded. Not surprisingly in view of A1–A5, f_1 is neither weakly analytic nor affine nor als, and $f_1(x) < 0$ has no solution. Moreover, f_1 is not separable. In fact, all known examples of duality gap involve non-separable functions [7, p. 375], [11, p. 23], [18, p. 318]. This raises a natural question of whether the assumption of f_i being affine in A4 can be relaxed to f_i being separable. In other words, does $f_0, f_1, ..., f_m$ being separable imply no duality gap?

Our main result is that the answer is 'yes' (assuming also that $\text{dom} f_0 \subseteq \text{dom} f_i$ for all i). Separable programs form an important class of nonlinear programs [3, Sec. 11.3], [22], for which efficient dual methods and primal-dual methods can be developed [4, 5, 6, 13, 17, 21, 22, 23]. For example, they can be solved efficiently by primal-dual interior-point methods [17] since the objective function Hessian $\nabla^2 f_0$ (assuming f_0 is twice differentiable) is diagonal and each constraint function gradient ∇f_i (assuming f_i is differentiable) has a fixed sparsity pattern, like in linear programs. We also give a refinement of [2, Thm. 5.4.2] that allows $f_0, f_1, ..., f_m$ to be extended-real-valued in A3; see Sec. 5.

Throughout, for a function $f : \Re^n \to (-\infty, \infty]$, $\text{dom} f = \{x | f(x) < \infty\}$. For a convex set $C \subset \Re^n$, $\text{cl}C$ and $\text{ri}C$ denote the closure and relative interior of C, respectively. We abbreviate "convex hull" and "affine hull" as "conv" and "aff", respectively. For any $\alpha \in \Re$, $\alpha^+ = \max\{0, \alpha\}$. For any $x \in \Re^n$, x_j denotes the jth component of x.

2 Main Result

Define the primal function

$$p(u) = \inf_{f_i(x) \leq u_i, i=1,...,m} f_0(x). \tag{3}$$

Then, $p(0) = \upsilon_{\text{P}}$. It is well known that, under the assumption $\text{dom} f_0 \subseteq \cap_{i=1}^m \text{dom} f_i$, we have

$$\liminf_{u \to 0} p(u) = \upsilon_{\text{D}} \tag{4}$$

(see, e.g., [2, Prop. 5.3.1], [7, Prop. 6.5.2], [8, Cor. 4.3.6], [18, Cor. 30.2.2]) and $\upsilon_{\text{P}} = \upsilon_{\text{D}}$ is equivalent to p being lsc at 0.

We state our main result below. Its proof is long and is spread over this and the next two sections. The proof makes essential use of an error bound result of Hoffman [12], namely, the solution set of a linear equation/inequality system changes in a Lipschitzian manner with the right-hand side.

Theorem 1 *Assume* $f_0, f_1, ..., f_m$ *are separable and* $\mathrm{dom} f_0 \subseteq \cap_{i=1}^m \mathrm{dom} f_i$. *Then* p *is lsc at* 0, *so that* $\upsilon_{\mathrm{P}} = \upsilon_{\mathrm{D}}$.

Proof. Since $f_0, f_1, ..., f_m$ are separable, as well as being proper convex lsc, we have

$$f_i(x) \quad = \quad \sum_{j=1}^{n} f_{ij}(x_j), \quad i = 0, 1, ..., m,$$

where each $f_{ij} : \Re \to (-\infty, \infty]$ is a proper convex lsc function. Moreover,

$$\mathrm{dom} f_{0j} \subseteq \mathrm{dom} f_{ij} \quad \forall i, j. \tag{5}$$

By (3) and (4), there exists a sequence $x^k \in \Re^n$, $k = 1, 2, ...$, such that $f_0(x^k) \to \upsilon_{\mathrm{D}}$ and $f_i(x^k)^+ \to 0$, $i = 1, ..., m$. We construct below a sequence $y^k \in \Re^n$, $k = 1, 2, ...$, such that

$$\lim_{k\to\infty} \inf f_0(y^k) \leq \upsilon_{\mathrm{D}}, \qquad f_i(y^k) \leq 0,\ i = 1, ..., m, \quad \forall k \gg 0.$$

(We write "$k \gg 0$" as short for "k sufficiently large".) This shows that $\upsilon_{\mathrm{P}} \leq \upsilon_{\mathrm{D}}$ which, together with $\upsilon_{\mathrm{P}} \geq \upsilon_{\mathrm{D}}$, yields $\upsilon_{\mathrm{P}} = \upsilon_{\mathrm{D}}$.

By assumption, $F \neq \emptyset$. Also, F is convex (not necessarily closed). Let

$$I = \{i \in \{1, ..., m\} \mid f_i(x) = 0\ \forall x \in F\}.^{5} \tag{6}$$

By convexity of f_i, there exists $\bar{x} \in F$ such that $f_i(\bar{x}) < 0$ for all $i \in \{1, ..., m\} \setminus I$. By passing to a subsequence, we can assume that, for each j, either $x_j^k \to \infty$ or $x_j^k \to -\infty$ or $\{x_j^k\}$ converges to some $x_j^\infty \in \Re$. Let

$$J_- = \{j \mid x_j^k \to -\infty\}, \quad J_+ = \{j \mid x_j^k \to \infty\}, \quad J_b = \{j \mid x_j^k \to x_j^\infty\}.$$

Lemma 1 *If* $\upsilon_{\mathrm{P}} \neq -\infty$, *then* $f_{0j}(x_j^k)$ *is bounded above for all* $j \in J_b$.

[5] Notice that (2) may not be satisfied, such as when $\mathrm{dom} f_0$ is an endpoint of $\mathrm{dom} f_i \neq \mathrm{dom} f_0$ for some $i \in \{1, ..., m\}$. Thus, even in the case of $I = \emptyset$, assumption A2 may not be satisfied and Thm. 1 does not follow from existing results.

The proof of Lemma 1 is given in Sec. 3. (The proof uses a conformal decomposition of x^k into components $z^{k,1}, z^{k,2}, \ldots$ that have recession properties with respect to each f_i, $i \neq 0$, in a lexico-graphical sense.) If $\upsilon_{\mathrm{P}} = -\infty$, then the desired $\{y^k\}$ sequence automatically exists. Suppose instead that $\upsilon_{\mathrm{P}} > -\infty$. By Lemma 1 and (5), we have that

$$x_j^{\infty} \in \mathrm{dom} f_{0j} \subseteq \mathrm{dom} f_{ij} \quad \forall i, \ \forall j \in J_b.$$

For each $j \in \{1, \ldots, n\}$, let

$$a_j = \inf_{x \in F} x_j, \qquad b_j = \sup_{x \in F} x_j.$$

Then, for any $j \in \{1, \ldots, n\}$ and any $\epsilon > 0$ sufficiently small, there exists $a^{\epsilon} \in F$ (depending on j) such that $a_j \leq a_j^{\epsilon} \leq a_j + \epsilon$ if $a_j > -\infty$ and $a_j^{\epsilon} \leq -1/\epsilon$ if $a_j = -\infty$. Similarly, there exists $b^{\epsilon} \in F$ (depending on j) such that $b_j \geq b_j^{\epsilon} \geq b_j - \epsilon$ if $b_j < \infty$ and $b_j^{\epsilon} \geq 1/\epsilon$ if $b_j = \infty$. Since $(1-\alpha)a^{\epsilon} + \alpha b^{\epsilon} \in F$ for all $\alpha \in [0,1]$, we have for each $i \in I$ that

$$f_{i\ell}((1-\alpha)a_{\ell}^{\epsilon} + \alpha b_{\ell}^{\epsilon}) \leq (1-\alpha) f_{i\ell}(a_{\ell}^{\epsilon}) + \alpha f_{i\ell}(b_{\ell}^{\epsilon}), \quad \ell = 1, \ldots, n. \tag{7}$$

This yields

$$\begin{aligned} 0 &= f_i((1-\alpha)a^{\epsilon} + \alpha b^{\epsilon}) \\ &= \sum_{\ell=1}^{n} f_{i\ell}((1-\alpha)a_{\ell}^{\epsilon} + \alpha b_{\ell}^{\epsilon}) \\ &\leq \sum_{\ell=1}^{n} (1-\alpha) f_{i\ell}(a_{\ell}^{\epsilon}) + \alpha f_{i\ell}(b_{\ell}^{\epsilon}) \\ &= (1-\alpha) f_i(a^{\epsilon}) + \alpha f_i(b^{\epsilon}) \\ &= 0. \end{aligned}$$

Thus, we must have equalities in (7), implying in particular that

$$f_{ij}((1-\alpha)a_j^{\epsilon} + \alpha b_j^{\epsilon}) = (1-\alpha) f_{ij}(a_j^{\epsilon}) + \alpha f_{ij}(b_j^{\epsilon}) \quad \forall \alpha \in [0,1].$$

Thus f_{ij} is an *affine* function on the interval $[a_j^{\epsilon}, b_j^{\epsilon}]$. Since this is true for any $\epsilon > 0$ sufficiently small and the interval $[a_j^{\epsilon}, b_j^{\epsilon}]$ is nested with respect to decreasing ϵ, we obtain that f_{ij} is affine on $[a_j, b_j]$. (The interval is closed since f_{ij} is lsc.) Since $\bar{x}_j \in [a_j, b_j]$, for each $i \in I$ and j, there exists $A_{ij} \in \Re$ such that

$$f_{ij}(x_j) = A_{ij}(x_j - \bar{x}_j) + f_{ij}(\bar{x}_j) \quad \forall x_j \in [a_j, b_j]. \tag{8}$$

Specifically, if $a_j < b_j$, then A_{ij} is unique, finite, and given by

$$A_{ij} = \frac{f_{ij}(b_j) - f_{ij}(a_j)}{b_j - a_j},$$

and $x_j \mapsto A_{ij}(x_j - \bar{x}_j) + f_{ij}(\bar{x}_j)$ supports f_{ij}. If $a_j = b_j$, then $\bar{x}_j = a_j$ and we set

$$A_{ij} = \begin{cases} 0 & \text{if } j \in J_= \\ \dfrac{f_{ij}(\hat{x}_j) - f_{ij}(\bar{x}_j)}{\hat{x}_j - \bar{x}_j} & \text{if } j \notin J_= \end{cases} \qquad \forall i \in I,$$

where

$$J_= = \{j \in J_b \mid a_j = b_j,\ x_j^\infty = \bar{x}_j\},$$

$$\hat{x}_j = \begin{cases} \bar{x}_j - 1 & \text{if } j \in J_-, \\ \bar{x}_j + 1 & \text{if } j \in J_+, \\ \frac{1}{2}(x_j^\infty + \bar{x}_j) & \text{if } j \in J_b \setminus J_=. \end{cases}$$

(Our choice of A_{ij} ensures that, for $j \notin J_=$, $x_j \mapsto A_{ij}(x_j - \bar{x}_j) + f_{ij}(\bar{x}_j)$ lies below f_{ij} in a neighborhood of x_j^k when $k \gg 0$. This is used to prove Lemma 2 below. If $a_j = b_j$ and $\partial f_{ij}(\bar{x}_j) \neq \emptyset$, we can alternatively set A_{ij} to be any element of $\partial f_{ij}(\bar{x}_j)$.) For $j \notin J_=$, let L_{ij} be the largest interval containing $[a_j, b_j]$ over which f_{ij} has the form (8), for $i \in I$. For $j \in J_=$, let $L_{ij} = \{\bar{x}_j\}$ for $i \in I$. L_{ij} is closed since f_{ij} is lsc. Let $A_i = [A_{i1} \ \cdots \ A_{in}]$, $F_i = L_{i1} \times \cdots \times L_{in}$. By (8) and $f_i(\bar{x}) = 0$ for all $i \in I$, we have

$$f_i(x) = A_i(x - \bar{x}) \quad \forall x \in F_i,\ \forall i \in I. \tag{9}$$

Notice that $F \subseteq [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq F_i$.

By (9), for each $i \in I$, we have

$$0 = f_i(x) = A_i(x - \bar{x}) \quad \forall x \in F \subseteq F_i.$$

Thus, $A_i^T \in S^\perp$ for $i \in I$, where ^{T} denotes transpose, and $S = \text{aff}(F - \bar{x})$. This implies

$$0 = A_i(x - \bar{x}) \quad \forall x \in S + \bar{x},\ \forall i \in I. \tag{10}$$

This and (9) imply

$$f_i(x) = A_i(x - \bar{x}) = 0 \quad \forall x \in (S + \bar{x}) \cap F_i.$$

Thus, if $x^k \in (S + \bar{x}) \cap (\cap_{i \in I} F_i)$, we have $f_i(x^k) = 0\ \forall i \in I$.

For $k = 1, 2, \ldots$, let

$$z_{ij}^k = \arg\min_{\xi \in L_{ij}} |\xi - x_j^k|, \qquad \delta_{ij}^k = |x_j^k - z_{ij}^k|, \qquad \delta^k = \max_{i \in I, j=1,\ldots,n} \delta_{ij}^k.$$

Lemma 2 $\delta^k \to 0$.

The proof of Lemma 2 is given in Sec. 4. (The proof is by contradiction, showing that if $\delta^k \not\to 0$, then there exist $d^k \in S$ and $\bar{x}^k \in \text{conv}\{x^k - d^k, \bar{x}\}$ such that $\bar{x}^k$ has a cluster point $\bar{x}^\infty$ with $\bar{x}^\infty - \bar{x} \notin S$ and $(1-\alpha)\bar{x} + \alpha\bar{x}^\infty \in F$ for $\alpha > 0$ sufficiently small. This contradicts $S = \text{aff}(F - \bar{x})$.) The interval

$$L_j = \bigcap_{i \in I} L_{ij}$$

is nonempty for all j (since $\bar{x}_j \in L_j$). Then, defining

$$z^k = (z_j^k)_{j=1}^n \quad \text{with} \quad z_j^k = \arg\min_{\xi \in L_j} |\xi - x_j^k|,$$

we have from Lemma 2 that $x^k - z^k \to 0$.[6]

For each $i \in I$ and j, since f_{ij} is convex lsc and f_{ij} is affine on L_j, we have that f_{ij} is uniformly continuous on the interval $(L_j + [-\epsilon, \epsilon]) \cap \text{dom} f_{ij}$ for a sufficiently small $\epsilon > 0$. Since $x_j^k - z_j^k \to 0$ and $z_j^k \in L_j$ and $x_j^k \in \text{dom} f_{ij}$, we see that z_j^k and x_j^k are both in this interval for $k \gg 0$, implying $f_{ij}(x_j^k) - f_{ij}(z_j^k) \to 0$. Thus

$$f_i(x^k) - f_i(z^k) \to 0 \quad \forall i \in I.$$

Since $f_i(x^k)^+ \to 0$ for all $i \in I$ and $[\cdot]^+$ is a Lipschitz continous mapping, this implies

$$f_i(z^k)^+ \to 0 \quad \forall i \in I.$$

By passing to a subsequence, we can assume that, for each $i \in I$, either $f_i(z^k) \to 0$ or $f_i(z^k) \leq -\delta$ for all k, where $\delta > 0$. Let

$$I_0 = \{i \in I \mid f_i(z^k) \to 0\}. \tag{11}$$

Since $z_j^k \in L_j \subseteq L_{ij}$ for all $i \in I$ and j, we also have from (9) that $f_i(z^k) = A_i(z^k - \bar{x})$ for all $i \in I$ and k. Hence

$$A_i(z^k - \bar{x}) \to 0 \;\; \forall i \in I_0, \qquad A_i(z^k - \bar{x}) \leq -\delta \;\; \forall k, \; \forall i \in I \setminus I_0.$$

Also, for each j and each $k \gg 0$, either $z_j^k = x_j^k$ or else z_j^k is an endpoint of L_j and $x_j^k \notin L_j$. In the first case, since x_j^k is in $\text{dom} f_{ij}$ for all i, then so is z_j^k. In the second

[6]More precisely, for each j and k, either $z_{ij}^k = x_j^k$ for all $i \in I$ or else z_{ij}^k is an endpoint of L_{ij} for some $i \in I$. In the first case, $z_j^k = x_j^k$; in the second case, z_j^k is one of these endpoints. In either case, $|x_j^k - z_j^k| = \max_{i \in I} \delta_{ij}^k$.

case, since $\bar{x}_j \in L_j$, then z_j^k lies between x_j^k and $\bar{x}_j$. Since x_j^k and $\bar{x}_j$ are both in $\mathrm{dom} f_{ij}$ for all i, then so is z_j^k. Thus $z^k \in \mathrm{dom} f_i$ for all $i = 0, 1, ..., m$.

For each k, consider the linear system

$$\begin{array}{ll} A_i z = A_i z^k & \forall i \in I_0, \\ A_i z \leq A_i \bar{x} - \delta & \forall i \in I \setminus I_0, \end{array} \quad \begin{array}{ll} z_j \leq \bar{x}_j & \forall j \in J_-, \\ z_j \geq \bar{x}_j & \forall j \in J_+, \\ z_j = z_j^k & \forall j \in J_b. \end{array} \tag{12}$$

This linear system is consistent for all $k \gg 0$ (z^k is a solution). Since $x^k - z^k \to 0$ and $A_i(z^k - \bar{x}) \to 0$ for all $i \in I_0$, the right-hand side is bounded as $k \to \infty$. By Hoffman's result [12], it has a solution $\tilde{z}^k$ that is bounded as $k \to \infty$. Since $A_i z^k \to A_i \bar{x}$ for all $i \in I_0$, any cluster point z^∞ of $\{\tilde{z}^k\}$ is a solution of

$$\begin{array}{ll} A_i z = A_i \bar{x} & \forall i \in I_0, \\ A_i z \leq A_i \bar{x} - \delta & \forall i \in I \setminus I_0, \end{array} \quad \begin{array}{ll} z_j \leq \bar{x}_j & \forall j \in J_-, \\ z_j \geq \bar{x}_j & \forall j \in J_+, \\ z_j = x_j^\infty & \forall j \in J_b. \end{array}$$

Since, $z_j^k \to -\infty$ for all $j \in J_-$ and $z_j^k \to \infty$ for all $j \in J_+$, then z^∞ in fact satisfies

$$\begin{array}{ll} A_i z = A_i \bar{x} & \forall i \in I_0, \\ A_i z \leq A_i \bar{x} - \delta & \forall i \in I \setminus I_0, \end{array} \quad \begin{array}{ll} z_j^k \leq z_j \leq \bar{x}_j & \forall j \in J_-, \\ z_j^k \geq z_j \geq \bar{x}_j & \forall j \in J_+, \\ z_j = x_j^\infty & \forall j \in J_b \end{array} \tag{13}$$

for all $k \gg 0$.

For each $k \gg 0$, since z^k satisfies

$$\begin{array}{ll} A_i z = A_i z^k & \forall i \in I_0, \\ A_i z \leq A_i \bar{x} - \delta & \forall i \in I \setminus I_0, \end{array} \quad \begin{array}{ll} z_j^k \leq z_j \leq \bar{x}_j & \forall j \in J_-, \\ z_j^k \geq z_j \geq \bar{x}_j & \forall j \in J_+, \\ z_j = z_j^k & \forall j \in J_b, \end{array}$$

and (13) is consistent and differs from this system only in the right-hand side, Hoffman's result [12] implies (13) has a solution $\bar{z}^k$ such that

$$\|\bar{z}^k - z^k\| = O\left(\sum_{i \in I_0} |A_i(z^k - \bar{x})| + \sum_{j \in J_b} |z_j^k - x_j^\infty| \right),$$

Since $\sum_{i \in I_0} |A_i(z^k - \bar{x})| = \sum_{i \in I_0} |f_i(z^k)| \to 0$ (see (11)) and $z_j^k \to x_j^\infty$ for all $j \in J_b$, we obtain that $\bar{z}^k - z^k \to 0$. This together with $x^k - z^k \to 0$ yields

$$\bar{z}^k - x^k \to 0. \tag{14}$$

For $j \in J_- \cup J_+$, since $z_j^k \in L_j$ and $\bar{x}_j \in L_j$, then $\operatorname{conv}\{\bar{x}_j, z_j^k\} \subseteq L_j$. For $j \in J_b$, since $z_j^k \to x_j^\infty$ and L_j is closed, then $x_j^\infty \in L_j$. Since $\bar{z}^k$ is a solution of (13), this together with (9) implies $\bar{z}^k \in L_1 \times \cdots \times L_n$ and

$$f_i(\bar{z}^k) = 0 \;\; \forall i \in I_0, \qquad f_i(\bar{z}^k) \le -\delta \;\; \forall i \in I \setminus I_0 \quad \forall k \gg 0. \tag{15}$$

For each $j \in J_-$, we have from $x_j^k - z_j^k \to 0$ that $z_j^k \to -\infty$ so the left endpoint of L_j is $-\infty$, implying $x_j^k = z_j^k \in L_j$ for all $k \gg 0$. Hence $\bar{x}_j \ge \bar{z}_j^k \ge x_j^k$ for all $k \gg 0$. Similarly, for each $j \in J_+$, we have $\bar{x}_j \le \bar{z}_j^k \le x_j^k$ for all $k \gg 0$. Thus, for each $j \in J_- \cup J_+$ and $k \gg 0$,

$$\text{either} \quad \bar{z}_j^k = x_j^k \quad \text{or} \quad \bar{z}_j^k \in \operatorname{conv}\{\bar{x}_j, x_j^k\}.$$

By further passing to a subsequence, we assume that for each $i \notin I$ and $j \in J_- \cup J_+$, either $f_{ij}(x_j^k) \to \infty$ or $\{f_{ij}(x_j^k)\}$ is bounded above. If $f_{ij}(x_j^k) \to \infty$, we have from the above relation that

$$\text{either} \quad f_{ij}(\bar{z}_j^k) = f_{ij}(x_j^k) \quad \text{or} \quad f_{ij}(\bar{z}_j^k) \le \max\{f_{ij}(\bar{x}_j), f_{ij}(x_j^k)\} = f_{ij}(x_j^k)$$

for all $k \gg 0$. If $\{f_{ij}(x_j^k)\}$ is bounded above, then the convexity of f_{ij} implies f_{ij} is uniformly continuous on the interval $(-\infty, \bar{x}_j]$ (respectively, $[\bar{x}_j, \infty)$) if $j \in J_-$ (respectively, $j \in J_+$). Since both x_j^k and $\bar{z}_j^k$ lie in this interval, (14) yields

$$f_{ij}(\bar{z}_j^k) - f_{ij}(x_j^k) \to 0.$$

For each $i \notin I$ and $j \in J_b$, since f_{ij} is continuous on the interval $\operatorname{dom} f_{ij}$ and both x_j^k and $\bar{z}_j^k = x_j^\infty$ lie in this interval, (14) yields

$$f_{ij}(\bar{z}_j^k) - f_{ij}(x_j^k) \to 0.$$

Thus, we have

$$\lim_{k\to\infty} \sup\{f_{ij}(\bar{z}_j^k) - f_{ij}(x_j^k)\} \le 0 \;\; \forall i \notin I, \; \forall j.$$

Then, upon summing over all j,

$$\lim_{k\to\infty} \sup\{f_i(\bar{z}^k) - f_i(x^k)\} \le 0 \;\; \forall i \notin I.$$

Since $f_i(x^k)^+ \to 0$ for all $i \ne 0$ and $f_0(x^k) \to \upsilon_{\mathrm{D}}$, this implies

$$f_i(\bar{z}^k)^+ \to 0 \;\; \forall i \in \{1, ..., m\} \setminus I, \qquad \lim_{k\to\infty} \inf f_0(\bar{z}^k) \le \upsilon_{\mathrm{D}}. \tag{16}$$

Since $f_i(\bar{x}) < 0$ for all $i \in \{1, ..., m\} \setminus I$, then for

$$y^k = (1 - \alpha^k)\bar{z}^k + \alpha^k \bar{x}, \qquad \alpha^k = \max_{i \in \{1,\ldots,m\} \setminus I} \left\{ \frac{f_i(\bar{z}^k)^+}{f_i(\bar{z}^k)^+ - f_i(\bar{x})} \right\},$$

we have $\alpha^k \in [0, 1)$ and $f_i(y^k) \le (1-\alpha^k) f_i(\bar{z}^k)^+ + \alpha^k f_i(\bar{x}) \le 0$ for all $i \in \{1, ..., m\} \setminus I$. We have similarly from (15) that $f_i(y^k) \le 0$ for all $i \in I$. Thus $y^k \in F$. Also, we have from (16) that $\alpha^k \to 0$ and hence

$$\lim_{k\to\infty} \inf f_0(y^k) \;\le\; \lim_{k\to\infty} \inf\, (1 - \alpha^k) f_0(\bar{z}^k) + \alpha^k f_0(\bar{x}) \;=\; \lim_{k\to\infty} \inf f_0(\bar{z}^k) \;\le\; \upsilon_{\mathrm{D}}.$$

Thus, y^k is the desired sequence. ∎

Note 1: Our proof is complex since x_j^k and $f_{ij}(x_j^k)$ can tend to ∞ or $-\infty$ at different rates for different i and j. Also, f_{ij} may not be uniformly continuous. As an example, for

$$\min\ e^{-x_1} \quad \text{s.t.} \quad x_1^2 + x_2 \le 0,$$

we have $\upsilon_{\mathrm{P}} = \upsilon_{\mathrm{D}} = 0$, and any sequence $x^k \in \Re^2$, $k = 1, 2, ...,$ satisfying $f_0(x^k) \to \upsilon_{\mathrm{D}}$ and $f_1(x^k)^+ \to 0$ must have $x_1^k \to \infty$ and $x_2^k \to -\infty$ quadratically in x_1^k. On the other hand, by Lemma 1, $f_{0j}(x_j^k)$ *cannot* tend to ∞ for any j (since $\upsilon_{\mathrm{P}} \ne -\infty$). Thus, the constraint functions have different behaviors from the cost function along x^k, which must be taken into account in the proof.

Note 2: By adding the indicator function for $\cap_{i=1}^m \mathrm{cl}(\mathrm{dom} f_i)$ to f_0 (which does not affect p nor q), we can always assume that $\mathrm{dom} f_0 \subseteq \mathrm{cl}(\mathrm{dom} f_i)$ for all i. This raises the question of whether (5) can be relaxed in Thm. 1. The difficulty lies in that a limit point x_j^∞, $j \in J_b$, can then lie outside of $\mathrm{dom} f_{ij}$ for some i. An example is

$$\min\ x_1 \quad \text{s.t.} \quad -\ln(x_1) + x_2 \le 0.$$

Thus, if (5) can be relaxed, a more complicated proof would be needed.

Note 3: The assumption $F \ne \emptyset$ is needed for (4) to hold. As an example, for the linear program

$$\min\ x \quad \text{s.t.} \quad 0 \cdot x + 1 \le 0,$$

we have $F = \emptyset$, so that $\upsilon_{\mathrm{P}} = \infty$. Moreover,

$$p(u) = \min_x \{x \mid 0 \cdot x + 1 \le u\} = \begin{cases} \infty & \text{if } u < 1 \\ -\infty & \text{if } u \ge 1 \end{cases},$$

so that $\lim_{u\to 0} \inf p(u) = \infty$. On the other hand,

$$q(\mu) = \min_x \{x + \mu(0 \cdot x + 1)\} = -\infty \quad \forall \mu \ge 0,$$

so $\upsilon_{\mathrm{D}} = \sup_{\mu \ge 0} q(\mu) = -\infty$. All other assumptions are satisfied.

3 Proof of Lemma 1

In this section we prove Lemma 1. Suppose that, for some $\bar{j} \in J_b$, $f_{0\bar{j}}(x_{\bar{j}}^k)$ is not bounded above. We show that $\upsilon_{\mathrm{P}} = -\infty$. Since $f_{0\bar{j}}$ is convex and $x_{\bar{j}}^k \to x_{\bar{j}}^\infty$, we have that $f_{0\bar{j}}(x_{\bar{j}}^k) \to \infty$. Since $f_{0j}(x_j^k)$ is bounded below for all $j \in J_b$, this implies $\sum_{j\in J_b} f_{0j}(x_j^k) \to \infty$. Since $f_0(x^k) \to \upsilon_{\mathrm{D}} < \infty$, this in turn implies $J_- \cup J_+ \neq \emptyset$ and

$$\sum_{j\in J_-\cup J_+} f_{0j}(x_j^k) \to -\infty. \tag{17}$$

Let $M_1 = \{0, 1, ..., m\}$. Let

$$I_\infty = \{i \in M_1 \mid f_{ij}(x_j^k) \to \infty \text{ for some } j \in J_b\}.$$

Then $0 \in I_\infty$ (since $f_{0\bar{j}}(x_{\bar{j}}^k) \to \infty$). Denote the left and right derivatives [22]:

$$f_{ij}^-(\xi) = \lim_{\zeta\to\xi^-} \frac{f_{ij}(\zeta) - f_{ij}(\xi)}{\zeta - \xi}, \quad f_{ij}^+(\xi) = \lim_{\zeta\to\xi^+} \frac{f_{ij}(\zeta) - f_{ij}(\xi)}{\zeta - \xi}.$$

It is known that f_{ij}^- and f_{ij}^+ are nondecreasing [22, Sec. 8A]. Let

$$c_{ij} = \lim_{\xi\to-\infty} f_{ij}^-(\xi), \qquad d_{ij} = \lim_{\xi\to\infty} f_{ij}^+(\xi) \tag{18}$$

(possibly $c_{ij} = -\infty$ or $d_{ij} = \infty$). If $c_{ij} = -\infty$, then $\lim_{\xi\to-\infty} f_{ij}(\xi)/|\xi| = \infty$. If $d_{ij} = \infty$, then $\lim_{\xi\to\infty} f_{ij}(\xi)/|\xi| = \infty$. If $c_{ij} > -\infty$, then $f_{ij}(\xi) - c_{ij}\xi$ is monotonically decreasing as $\xi \to -\infty$ and $\lim_{\xi\to-\infty}(f_{ij}(\xi) - c_{ij}\xi)/|\xi| = 0$. If $d_{ij} < \infty$, then $f_{ij}(\xi) - d_{ij}\xi$ is monotonically decreasing as $\xi \to \infty$ and $\lim_{\xi\to\infty}(f_{ij}(\xi) - d_{ij}\xi)/|\xi| = 0$.[7] For each $i \in M_1$, denote

$$J_i^- = \{j \in J_- \mid c_{ij} > -\infty\}, \quad J_i^+ = \{j \in J_+ \mid d_{ij} < \infty\},$$

$$K_i^- = \{j \in J_i^- \mid \lim_{\xi\to-\infty} f_{ij}(\xi) - c_{ij}\xi = -\infty\}, \quad K_i^+ = \{j \in J_i^+ \mid \lim_{\xi\to\infty} f_{ij}(\xi) - d_{ij}\xi = -\infty\}.$$

Since $f_i(x^k)$ is bounded above, we must have $J_i^- \cup J_i^+ \neq \emptyset$ for all i. ($K_i^- \cup K_i^+$ may be empty.) Define

$$\eta_i^k = \sum_{j\in J_i^-} c_{ij} x_j^k + \sum_{j\in J_i^+} d_{ij} x_j^k \quad \forall i \in M_1.$$

[7] For example, if $f_{ij}(\xi) = 1/\xi + \xi$ for $\xi > 0$ (otherwise $f_{ij}(\xi) = \infty$), then $c_{ij} = -\infty$, $d_{ij} = 1$, and $\lim_{\xi\to\infty} f_{ij}(\xi) - d_{ij}\xi = 0$. If we change $1/\xi$ to $-\log\xi$, then we still have $d_{ij} = 1$, but $\lim_{\xi\to\infty} f_{ij}(\xi) - d_{ij}\xi = -\infty$. Our analysis will need to distinguish between these two asymptotically linear and sublinear cases.

Notice that $|\eta_i^k| = O(\|x^k\|)$ for all $i \in M_1$. By passing to a subsequence if necessary, we can assume that $\frac{\eta_i^k}{\|x^k\|}$, $i \in M_1$, and $\frac{x^k}{\|x^k\|}$ both converge. Let

$$I_1 = \left\{ i \in M_1 \;\middle|\; \lim_{k\to\infty} \frac{\eta_i^k}{\|x^k\|} \neq 0 \right\}, \quad J_1 = \left\{ j \in \{1, ..., n\} \;\middle|\; \lim_{k\to\infty} \frac{x_j^k}{\|x^k\|} \neq 0 \right\}.$$

(I_1 may be empty. $J_1 \neq \emptyset$. $J_1 \cap J_b = \emptyset$.) Let

$$\gamma_1^k = \max_{i \in M_1 \setminus I_1} |\eta_i^k| + \max_{j \notin J_1} |x_j^k|.$$

For each k, consider the linear system

$$\sum_{j\in J_i^-} c_{ij} x_j + \sum_{j \in J_i^+} d_{ij} x_j = \eta_i^k \;\forall i \in M_1 \setminus I_1, \qquad \begin{array}{ll} x_j \leq \bar{x}_j - \gamma_1^k & \forall j \in J_- \cap J_1, \\ x_j \geq \bar{x}_j + \gamma_1^k & \forall j \in J_+ \cap J_1, \\ x_j = x_j^k & \forall j \notin J_1. \end{array} \tag{19}$$

This system is consistent for all $k \gg 0$ (x^k is a solution). By Hoffman's result [12], it has a solution $x^{k,1}$ such that $\|x^{k,1}\| = O\left(\gamma_1^k\right) = o(\|x^k\|)$. Let

$$z^{k,1} = x^k - x^{k,1}.$$

Then

$$\sum_{j \in J_i^-} c_{ij} z_j^{k,1} + \sum_{j \in J_i^+} d_{ij} z_j^{k,1} = 0 \;\; \forall i \in M_1 \setminus I_1, \tag{20}$$

$$\frac{1}{\eta_i^k} \left(\sum_{j \in J_i^-} c_{ij} z_j^{k,1} + \sum_{j \in J_i^+} d_{ij} z_j^{k,1} \right) \to 1 \;\; \forall i \in I_1, \tag{21}$$

$$z_j^{k,1} = 0 \;\forall j \notin J_1, \qquad \frac{z_j^{k,1}}{x_j^k} \to 1 \;\; \forall j \in J_1. \tag{22}$$

Also, $\{z^{k,1}/\|x^k\|\}$ and $\{x^k/\|x^k\|\}$ have the same cluster points. Since $\{f_i(x^k)\}$ is bounded above for each $i \in M_1$, any cluster point z of $\{z^{k,1}/\|x^k\|\}$ is a recession direction for f_i [2, 8, 18]. Thus z satisfies

$$\sum_{j \in J_i^-} c_{ij} z_j + \sum_{j \in J_i^+} d_{ij} z_j \leq 0 \;\forall i \in M_1, \qquad z_j = 0 \;\forall j \notin \left(\bigcap_{i \in M_1} J_i^- \right) \cup \left(\bigcap_{i \in M_1} J_i^+ \right).$$

(This follows from the limiting properties of $f_{ij}(\cdot)/|\cdot|$ discussed after (18).) Since $\lim_{k\to\infty} \frac{\eta_i^k}{\|x^k\|} \neq 0$ for all $i \in I_1$, the left relation implies that

$$\eta_i^k < 0 \quad \forall k \gg 0, \;\forall i \in I_1. \tag{23}$$

Since $\lim_{k\to\infty} \frac{x_j^k}{\|x^k\|} \neq 0$ for all $i \in J_1$, the right relation implies that $J_1 \subseteq \left(\cap_{i\in M_1} J_i^-\right) \cup \left(\cap_{i\in M_1} J_i^+\right)$ and hence

$$J_1 \subseteq (J_i^- \cup J_i^+) \quad \forall i \in M_1. \tag{24}$$

Combining (23), (24) with (20), (21), (22) yields that, for each $k \gg 0$,

$$z^{k,1} \text{ is a recession direction for } f_i \quad \forall i \in M_1.$$

Let

$$\bar{I}_1 \;=\; I_1 \cup \left\{ i \in M_1 \setminus I_1 \mid (K_i^- \cup K_i^+) \cap J_1 \neq \emptyset \right\}.$$

($\bar{I}_1$ may be empty.) For each $i \in I_1$, we have from (23) that $\eta_i^k < 0$ for all $k \gg 0$ and hence (21) yields

$$\sum_{j\in J_i^-} c_{ij} z_j^{k,1} + \sum_{j\in J_i^+} d_{ij} z_j^{k,1} < 0 \quad \forall k \gg 0.$$

This together with (22) and $\lim_{\xi\to-\infty} f_{ij}(\xi) - c_{ij}\xi < \infty$ for all $j \in J_i^-$ and $\lim_{\xi\to\infty} f_{ij}(\xi) - d_{ij}\xi < \infty$ for all $j \in J_i^+$ implies that, for any $x \in \mathrm{dom} f_i$, $f_i(x + tz^{k,1}) \to -\infty$ as $t \to \infty$. For each $i \in M_1 \setminus I_1$ with $(K_i^- \cup K_i^+) \cap J_1 \neq \emptyset$, we have similarly from (20) and (21) that, for any $x \in \mathrm{dom} f_i$, $f_i(x + tz^{k,1}) \to -\infty$ as $t \to \infty$. Thus, $z^{k,1}$ is a strict recession direction for f_i for all $i \in \bar{I}_1$.[8]

For each $i \in M_1 \setminus \bar{I}_1$, we have $(K_i^- \cup K_i^+) \cap J_1 = \emptyset$, which together with (24) implies

$$\lim_{k\to\infty} \left\{ \sum_{j\in J_1} f_{ij}(x_j^k) - \left(\sum_{j\in J_i^-\cap J_1} c_{ij} x_j^k + \sum_{j\in J_i^+\cap J_1} d_{ij} x_j^k \right) \right\} \quad \text{is finite.}$$

Since $x^k = x^{k,1} + z^{k,1}$ and, by $I_1 \subseteq \bar{I}_1$ and (20), (22), we have

$$\sum_{j\in J_i^-\cap J_1} c_{ij} z_j^{k,1} + \sum_{j\in J_i^+\cap J_1} d_{ij} z_j^{k,1} = 0 \quad \forall k \gg 0,$$

this yields

$$\lim_{k\to\infty} \left\{ \sum_{j\in J_1} f_{ij}(x_j^k) - \left(\sum_{j\in J_i^-\cap J_1} c_{ij} x_j^{k,1} + \sum_{j\in J_i^+\cap J_1} d_{ij} x_j^{k,1} \right) \right\} \quad \text{is finite.}$$

[8] We say $d \in \Re^n$ is a *strict recession direction* for f_i if, for any $x \in \mathrm{dom} f_i$, $f_i(x + td) \to -\infty$ as $t \to \infty$.

Since $(K_i^- \cup K_i^+) \cap J_1 = \emptyset$ and (24) imply

$$\lim_{k\to\infty} \left\{ \sum_{j\in J_1} f_{ij}(x_j^{k,1}) - \left(\sum_{j\in J_i^- \cap J_1} c_{ij} x_j^{k,1} + \sum_{j\in J_i^+ \cap J_1} d_{ij} x_j^{k,1} \right) \right\} \quad \text{is finite,}$$

this yields

$$\lim_{k\to\infty} \left\{ \sum_{j\in J_1} f_{ij}(x_j^k) - \sum_{j\in J_1} f_{ij}(x_j^{k,1}) \right\} \quad \text{is finite.}$$

Since $x_j^k = x_j^{k,1}$ for $j \notin J_1$ (see (22)) so that $f_{ij}(x_j^k) = f_{ij}(x_j^{k,1})$, we obtain upon summing over all j that

$$\lim_{k\to\infty} \left\{ f_i(x^k) - f_i(x^{k,1}) \right\} \quad \text{is finite.}$$

Since $\{f_i(x^k)\}$ is bounded above, then $\{f_i(x^{k,1})\}$ is bounded above.

Since $x^{k,1}$ satisfies (19) and $I_1 \subseteq \bar{I}_1$, then $|\eta_i^k| = O(\|x^{k,1}\|)$ for all $i \in M_2$, where we define

$$M_2 = M_1 \setminus \bar{I}_1.$$

By passing to a subsequence if necessary, we can assume that $\dfrac{\eta_i^k}{\|x^{k,1}\|}$, $i \in M_2$, and $\dfrac{x^{k,1}}{\|x^{k,1}\|}$ both converge. Let

$$I_2 = \left\{ i \in M_2 \;\middle|\; \lim_{k\to\infty} \frac{\eta_i^k}{\|x^{k,1}\|} \neq 0 \right\}, \quad J_2 = \left\{ j \in \{1, ..., n\} \;\middle|\; \lim_{k\to\infty} \frac{x_j^{k,1}}{\|x^{k,1}\|} \neq 0 \right\}.$$

(I_2 may be empty. $J_2 \neq \emptyset$.) Since $x^{k,1}$ satisfies (19), we have

$$x_j^{k,1} \leq \bar{x}_j - \gamma_1^k \quad \forall j \in J_- \cap J_1, \qquad x_j^{k,1} \geq \bar{x}_j + \gamma_1^k \quad \forall j \in J_+ \cap J_1.$$

Since $\|x^{k,1}\| = O(\gamma_1^k)$ and $J_1 \cap J_b = \emptyset$, this shows that $J_1 \subseteq J_2$. By the definition of γ_1^k, either $\bar{I}_1 \neq I_1$ or $I_2 \neq \emptyset$ or $J_1 \neq J_2$. Let

$$\gamma_2^k = \max_{i\in M_2 \setminus I_2} |\eta_i^k| + \max_{j\notin J_2} |x_j^{k,1}|.$$

Then, we proceed exactly as before, but with $M_1, I_1, J_1, x^k, \gamma_1^k$ replaced by M_2, I_2, J_2, $x^{k,1}, \gamma_2^k$, to obtain $x^{k,2}$, $z^{k,2}$, $\bar{I}_2$. In particular, for each $k \gg 0$,

$$z^{k,2} \text{ is a recession direction for } f_i \quad \forall i \in M_2,$$

and $z^{k,2}$ is a strict recession direction for f_i for $i \in \bar{I}_2$. In this manner, we obtain $M_\ell, I_\ell, J_\ell, x^{k,\ell}, z^{k,\ell}, \bar{I}_\ell$, $\ell = 1, 2, \ldots$ We terminate this construction when either (i) $J_\ell \not\subseteq (J_- \cup J_+)$ or (ii) $0 \in \bar{I}_\ell$. The construction must terminate finitely since, for each $\ell = 1, 2, \ldots$, we have $M_\ell \supseteq M_{\ell+1}$, $J_\ell \subseteq J_{\ell+1}$, and either $\bar{I}_\ell \neq I_\ell$ (so that $M_\ell \neq M_{\ell+1}$) or $I_{\ell+1} \neq \emptyset$ (so that $M_{\ell+1} \neq M_{\ell+2}$) or $J_\ell \neq J_{\ell+1}$.

Case (i): We terminate when $J_\ell \not\subseteq (J_- \cup J_+)$ for some ℓ. Then we must have $J_{\ell-1} \subseteq (J_- \cup J_+)$, so $J_\ell \setminus J_{\ell-1} \subseteq J_b$. Since, by construction, we have

$$x_j^{k,\ell-1} = \cdots = x_j^{k,1} = x_j^k \quad \forall j \notin J_{\ell-1},\ \forall k \gg 0, \qquad \lim_{k\to\infty} \frac{x_j^{k,\ell-1}}{\|x^{k,\ell-1}\|} \neq 0 \quad \forall j \in J_\ell,$$

then $\{x^{k,\ell-1}\}$ is bounded. Also, we have $0 \in M_\ell$ (since $0 \notin \bar{I}_1 \cup \cdots \cup \bar{I}_{\ell-1}$) and, by construction, $f_0(x^{k,\ell-1})$ is bounded above. However, since $f_{0\bar{j}}(x_{\bar{j}}^{k,\ell-1}) = f_{0\bar{j}}(x_{\bar{j}}^k) \to \infty$ for some $\bar{j} \in J_b$, we must have $f_{0j}(x_j^{k,\ell-1}) \to -\infty$ and hence $|x_j^{k,\ell-1}| \to \infty$ for some $j \notin J_b$. This contradicts the boundedness of $\{x^{k,\ell-1}\}$.

Case (ii): We terminate when $0 \in \bar{I}_\ell$ for some ℓ. By construction,

$$\left.\begin{array}{lll} z^{k,s} \text{ is a recession direction for } f_i & \forall i \in M_s,\ \forall k \gg 0, \\ z^{k,s} \text{ is a strict recession direction for } f_i & \forall i \in \bar{I}_s,\ \forall k \gg 0, \\ M_{s+1} = M_s \setminus \bar{I}_s, \qquad J_s \subseteq (J_- \cup J_+), & \end{array}\right. \qquad s = 1, \cdots, \ell.$$

Moreover,

$$\|z^{k,s}\| \to \infty, \quad z_j^{k,s} = 0\ \forall j \notin J_s, \quad \lim_{k\to\infty} \frac{z_j^{k,s}}{\|z^{k,s}\|} \begin{cases} < 0 & \forall j \in J_- \cap J_s, \\ > 0 & \forall j \in J_+ \cap J_s, \end{cases} \quad s = 1, \cdots, \ell. \tag{25}$$

Fix any $k \gg 0$. Fix any $\upsilon \in \Re$. For any $\beta_\ell > 0$, since $f_i(\bar{x}) \leq 0$ for all $i \in M_\ell \setminus \{0\}$, we have

$$f_i(\bar{x} + \beta_\ell z^{k,\ell}) \leq 0 \quad \forall i \in M_\ell \setminus \{0\}. \tag{26}$$

Since $0 \in \bar{I}_\ell$ so that $z^{k,\ell}$ is a strict recession direction for f_0, by taking $\beta_\ell > 0$ sufficiently large and using $f_0(\bar{x}) < \infty$, we have

$$f_0(\bar{x} + \beta_\ell z^{k,\ell}) \leq \upsilon. \tag{27}$$

Since $(-\infty, \bar{x}_j] \subseteq \mathrm{dom} f_{ij}$ for all $j \in J_-$ and $[\bar{x}_j, \infty) \subseteq \mathrm{dom} f_{ij}$ for all $j \in J_+$, we also have from (25) that

$$f_i(\bar{x} + \beta_\ell z^{k,\ell}) < \infty \quad \forall i \notin M_\ell. \tag{28}$$

Since $M_\ell = M_{\ell-1} \setminus \bar{I}_{\ell-1}$, then for any $\beta_{\ell-1} > 0$, we have from (26) and (27) that

$$f_0(\bar{x} + \beta_\ell z^{k,\ell} + \beta_{\ell-1} z^{k,\ell-1}) \leq \upsilon, \quad f_i(\bar{x} + \beta_\ell z^{k,\ell} + \beta_{\ell-1} z^{k,\ell-1}) \leq 0 \quad \forall i \in M_\ell \setminus \{0\}.$$

Since $z^{k,\ell-1}$ is a strict recession direction for f_i for $i \in \bar{I}_{\ell-1}$, by taking $\beta_{\ell-1}$ sufficiently large, we have from (28) that

$$f_i(\bar{x} + \beta_\ell z^{k,\ell} + \beta_{\ell-1} z^{k,\ell-1}) \le 0 \quad \forall i \in \bar{I}_{\ell-1}.$$

Since $(-\infty, \bar{x}_j] \subseteq \mathrm{dom} f_{ij}$ for all $j \in J_-$ and $[\bar{x}_j, \infty) \subseteq \mathrm{dom} f_{ij}$ for all $j \in J_+$, we also have from (25) that

$$f_i(\bar{x} + \beta_\ell z^{k,\ell} + \beta_{\ell-1} z^{k,\ell-1}) < \infty \quad \forall i \notin M_{\ell-1}.$$

Continuing in this manner, we obtain

$$f_0(\bar{x} + \sum_{s=1}^{\ell} \beta_s z^{k,s}) \le \upsilon, \quad f_i(\bar{x} + \sum_{s=1}^{\ell} \beta_s z^{k,s}) \le 0 \quad \forall i \in M_1 \setminus \{0\}.$$

Since $M_1 \setminus \{0\} = \{1, ..., m\}$, this shows that $\bar{x} + \sum_{s=1}^{\ell} \beta_s z^{k,s}$ is a feasible solution with cost below υ. Since υ is arbitrary, this shows that $\upsilon_{\mathrm{P}} = -\infty$. ■

Note 4: In the last paragraph, instead of fixing k and add to $\bar{x}$ a suitable positive combination of $z^{k,\ell}, ..., z^{k,1}$, we can instead add to $\bar{x}$ the sum $z^{k_\ell,\ell} + \cdots + z^{k_1,1}$, with $k_\ell, ..., k_1$ suitably chosen ($k_{\ell-1}$ depends on k_ℓ, etc.)

Note 5: Our proof makes essential use of the conformal decomposition of x^k in terms of $z^{k,1}, z^{k,2}, ...$, with $(z^{k,1}, z^{k,2}, ...)$ having recession properties with respect to each f_i in a lexico-graphical sense. Working with, say, recession directions obtained by taking cluster points of $\{x^k/\|x^k\|\}$ does *not* seem to work.

4 Proof of Lemma 2

In this section we prove Lemma 2. We argue by contradiction. Suppose there exists $\delta > 0$ such that $\delta^k \ge \delta$ for an infinite number of k. By passing to a subsequence if necessary, we can assume that for each $i \in I$ and j, either $\delta_{ij}^k \ge \delta$ for all k or $\delta_{ij}^k \to 0$. For each $i \in I$, denote

$$J_i = \{j \in \{1, ..., n\} \mid \delta_{ij}^k \ge \delta \ \forall k\}.$$

Then $J_i \ne \emptyset$ for some $i \in I$. For each j with $a_j < b_j$, since $x_j \mapsto A_{ij}(x_j - \bar{x}_j) + f_{ij}(\bar{x}_j)$ supports f_{ij} for each $i \in I$, we have

$$f_{ij}(x_j^k) \ge A_{ij}(x_j^k - \bar{x}_j) + f_{ij}(\bar{x}_j) \quad \forall k. \tag{29}$$

For each j with $a_j = b_j$ and $j \notin J_=$, we have that $\hat{x}_j \in \text{conv}\{x_j^k, \bar{x}_j\}$ for all $k \gg 0$, so the convexity of f_{ij} and the definition of A_{ij} imply (29) for all $i \in I$ [22, Sec. 8A]. For each $j \in J_=$, we have $z_{ij}^k = \bar{x}_j$ which, together with $x_j^k \to x_j^\infty = \bar{x}_j$, implies $\delta_{ij}^k \to 0$, i.e., $j \notin J_i$ for all $i \in I$. Moreover, for each $i \in I$ and $j \in J_i$, there exists $\rho_{ij} > 0$ such that

$$f_{ij}(x_j^k) \geq A_{ij}(x_j^k - \bar{x}_j) + f_{ij}(\bar{x}_j) + \rho_{ij}\delta_{ij}^k \quad \forall k. \tag{30}$$

This is because $j \notin J_=$, so that f_{ij} is affine on L_{ij}, and is convex but non-affine on any larger interval containing L_{ij}.[9] Let $\rho = \min_{i\in I, j\in J_i} \rho_{ij}$. Then, for each $i \in I$, we have upon using (29), (30), and $A_{ij} = 0$ for all $j \in J_=$ that

$$\begin{aligned} f_i(x^k) &= \sum_{j=1}^n f_{ij}(x_j^k) \\ &\geq \sum_{j=1}^n \Big(A_{ij}(x_j^k - \bar{x}_j) + f_{ij}(\bar{x}_j)\Big) + \sum_{j\in J_=} (f_{ij}(x_j^k) - f_{ij}(\bar{x}_j)) + \rho \sum_{j\in J_i} \delta_{ij}^k \\ &= A_i(x^k - \bar{x}) + \sum_{j\in J_=} (f_{ij}(x_j^k) - f_{ij}(\bar{x}_j)) + \rho \sum_{j\in J_i} \delta_{ij}^k, \end{aligned}$$

where the last equality also uses $f_i(\bar{x}) = 0$. We can decompose x^k uniquely as

$$x^k = u^k + v^k, \tag{31}$$

where $u^k \in S + \bar{x}$ and $v^k \in S^\perp$. Then (10) yields that $A_i(u^k - \bar{x}) = 0$ and hence

$$f_i(x^k) \geq A_i v^k + \sum_{j\in J_=} (f_{ij}(x_j^k) - f_{ij}(\bar{x}_j)) + \rho \sum_{j\in J_i} \delta_{ij}^k \quad \forall i \in I.$$

Since $f_i(x^k)^+ \to 0$ and $x_j^k \to \bar{x}_j$ so that $f_{ij}(x_j^k) \to f_{ij}(\bar{x}_j)$ for all $j \in J_=$, this implies $\rho \sum_{j\in J_i} \delta_{ij}^k \leq -A_i v^k + o(1)$ for all $i \in I$. Thus, if $J_i \neq \emptyset$, then $\frac{\rho}{2} \sum_{j\in J_i} \delta_{ij}^k \leq -A_i v^k$ for all $k \gg 0$. This yields

$$\delta^k \leq \max_{\substack{i\in I \\ J_i\neq\emptyset}} \sum_{j\in J_i} \delta_{ij}^k \leq \frac{2}{\rho} \max_{\substack{i\in I \\ J_i\neq\emptyset}} (-A_i v^k) \leq \frac{2}{\rho} \max_{\substack{i\in I \\ J_i\neq\emptyset}} \|A_i^T\| \|v^k\| = O(\|v^k\|) \quad \forall k \gg 0.$$

By further passing to a subsequence, we can assume that, for each j, either $x_j^k/\|v^k\| \to \infty$ or $x_j^k/\|v^k\| \to -\infty$ or $|x_j^k| = O(\|v^k\|)$. Let

$$\bar{J}_- = \left\{ j \in \{1, ..., n\} \;\middle|\; \frac{x_j^k}{\|v^k\|} \to -\infty \right\}, \quad \bar{J}_+ = \left\{ j \in \{1, ..., n\} \;\middle|\; \frac{x_j^k}{\|v^k\|} \to \infty \right\}.$$

[9] Specifically, if $x_j^k \geq z_{ij}^k + \delta$, then z_{ij}^k equals the right endpoint of L_{ij} and we can set $\rho_{ij} = \dfrac{f_{ij}(z_{ij}^k + \delta) - f_{ij}(z_{ij}^k)}{\delta} - A_{ij}$. If $x_j^k \leq z_{ij}^k - \delta$, then we can similarly set $\rho_{ij} = \dfrac{f_{ij}(z_{ij}^k - \delta) - f_{ij}(z_{ij}^k)}{\delta} + A_{ij}$.

(Either $\bar{J}_-$ or $\bar{J}_+$ may be empty.) Then, by $\delta \le \delta^k = O(\|v^k\|)$, we have $x_j^k \to \infty\ \forall j \in \bar{J}_+$ and $x_j^k \to -\infty\ \forall j \in \bar{J}_-$, so that $\bar{J}_- \subseteq J_-$, $\bar{J}_+ \subseteq J_+$. Thus, for each $i \in I$, we must have

$$\begin{array}{lcll} \text{(left endpoint of } L_{ij}) & = -\infty & \forall j \in \bar{J}_-, \\ \text{(right endpoint of } L_{ij}) & = \ \infty & \forall j \in \bar{J}_+. \end{array} \tag{32}$$

(Otherwise we would have $|x_j^k|/\delta_{ij}^k \to 1$ and hence $|x_j^k| = O(\delta^k) = O(\|v^k\|)$.) Thus, for all $k \gg 0$, we have $x_j^k \in L_{ij}$ for all $i \in I$ and $j \in \bar{J}_- \cup \bar{J}_+$, implying

$$f_i(x^k) = f_i(x^k) - f_i(\bar{x}) = \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}(x_j^k - \bar{x}_j) + \sum_{j \notin \bar{J}_- \cup \bar{J}_+} (f_{ij}(x_j^k) - f_{ij}(\bar{x}_j)) \quad \forall i \in I. \tag{33}$$

For each $i \in I$ and $j \notin \bar{J}_- \cup \bar{J}_+$, we have that either $f_{ij}(x_j^k) \to f_{ij}(\bar{x}_j)$ if $j \in J_=$ or $f_{ij}(x_j^k) \ge A_{ij}(x_j^k - \bar{x}_j) + f_{ij}(\bar{x}_j)$ for all $k \gg 0$ if $j \notin J_=$ (see (29)). In either case, we obtain from $|x_j^k| = O(\|v^k\|)$ and $\delta \le \delta^k = O(\|v^k\|)$ that there exists $\tau > 0$ for which

$$f_{ij}(x_j^k) \ge f_{ij}(\bar{x}_j) - \tau\|v^k\| \quad \forall i \in I,\ j \notin \bar{J}_- \cup \bar{J}_+,\ \forall k \gg 0.$$

This combined with (33) and $f_i(x^k)^+ \to 0$ for all $i \in I$ yields

$$\sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}(x_j^k - \bar{x}_j) \le n\tau\|v^k\| + o(1) \quad \forall i \in I,\ \forall k \gg 0.$$

By passing to a subsequence, we can assume for each $i \in I$ that

$$\text{either} \quad \left| \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}(x_j^k - \bar{x}_j) \right| = O(\|v^k\|) \quad \text{or} \quad \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}(x_j^k - \bar{x}_j)/\|v^k\| \to -\infty.$$

Let

$$\tilde{I} = \left\{ i \in I \;\middle|\; \left| \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}(x_j^k - \bar{x}_j) \right| = O(\|v^k\|) \right\}.$$

For each $k \gg 0$, consider the linear system

$$\sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}x_j = \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}x_j^k \quad \forall i \in \tilde{I}, \qquad \begin{array}{ll} x_j \le \bar{x}_j & \forall j \in \bar{J}_-, \\ x_j \ge \bar{x}_j & \forall j \in \bar{J}_+, \\ x_j = x_j^k & \forall j \notin \bar{J}_- \cup \bar{J}_+. \end{array} \tag{34}$$

This linear system is consistent since x^k is a solution. Since the right-hand side is $O(\|v^k\|)$, by Hoffman's result [12], it has a solution $\tilde{x}^k$ with $\|\tilde{x}^k\| = O(\|v^k\|)$. Then $d^k = x^k - \tilde{x}^k$ satisfies

$$\sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}d_j^k \begin{cases} = 0 & \forall i \in \tilde{I}, \\ < 0 & \forall i \in I \setminus \tilde{I}, \end{cases} \qquad \begin{array}{ll} d_j^k < 0 & \forall j \in \bar{J}_-, \\ d_j^k > 0 & \forall j \in \bar{J}_+, \\ d_j^k = 0 & \forall j \notin \bar{J}_- \cup \bar{J}_+. \end{array} \tag{35}$$

for all $k \gg 0$. This together with (9), (32), and $\bar{x} \in F$ implies that, for each $i \in I$,

$$f_i(\bar{x} + \alpha d^k) \;=\; f_i(\bar{x}) + \alpha \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij} d_j^k \;\le\; f_i(\bar{x}) \;=\; 0 \quad \forall \alpha > 0,$$

with the inequality being strict if $i \in I \setminus \tilde{I}$. For each $i \in \{1, ..., m\} \setminus I$, we have from $f_i(\bar{x}) < 0$, (35), and $(-\infty, \bar{x}_j] \subseteq \mathrm{dom} f_{ij}$ for $j \in \bar{J}_-$ (since $x_j^k \to -\infty$), $[\bar{x}_j, \infty) \subseteq \mathrm{dom} f_{ij}$ for $j \in \bar{J}_+$ (since $x_j^k \to \infty$) that $f_i(\bar{x} + \alpha d^k) \le 0$ for all $\alpha > 0$ sufficiently small. Thus, $\bar{x} + \alpha d^k \in F$ for all $\alpha > 0$ sufficiently small, with $f_i(\bar{x} + \alpha d^k) < 0$ for $i \in I \setminus \tilde{I}$. Then, $d^k \in S$ and, by the definition of I, we must have

$$\tilde{I} = I. \tag{36}$$

Since $\tilde{x}^k$ satisfies (34) for all $k \gg 0$, we have from (32) and (36) that

$$f_i(\tilde{x}^k) \;=\; \sum_{j \in \bar{J}_- \cup \bar{J}_+} A_{ij}(\tilde{x}_j^k - x_j^k) + f_i(x^k) \;=\; f_i(x^k) \;\; \forall i \in I.$$

Thus $f_i(\tilde{x}^k)^+ \to 0$ for all $i \in I$. Also, we have $\tilde{x}^k = x^k - d^k$ and $d^k \in S$. Let

$$t^k = \frac{1}{\max\{1, \|\tilde{x}^k - \bar{x}\|\}}, \qquad \bar{x}^k = \bar{x} + t^k(\tilde{x}^k - \bar{x}).$$

Notice that $\|\tilde{x}^k - \bar{x}\|$ is uniformly bounded away from zero due to $\delta \le \delta^k = O(\|v^k\|)$ and $u^k - d^k - \bar{x} \in S$ and $v^k \in S^\perp$, so that (31) yields

$$\|\tilde{x}^k - \bar{x}\|^2 = \|u^k - d^k - \bar{x}\|^2 + \|v^k\|^2 \;\; \forall k \gg 0. \tag{37}$$

Then $t^k \in (0, 1]$. Also, it can be seen that $\|\bar{x}^k - \bar{x}\| = \min\{1, \|\tilde{x}^k - \bar{x}\|\}$ for all $k \gg 0$. Thus $\{\bar{x}^k\}$ is bounded and, since $\|\tilde{x}^k - \bar{x}\|$ is uniformly bounded away from zero, the same is true for $\|\bar{x}^k - \bar{x}\|$. Since

$$f_i(\bar{x}^k) \le (1 - t^k) f_i(\bar{x}) + t^k f_i(\tilde{x}^k) \le t^k f_i(\tilde{x}^k)^+ \to 0 \quad \forall i \in I,$$

then, by the lsc property of f_i, any cluster point $\bar{x}^\infty$ of $\{\bar{x}^k\}$ satisfies $f_i(\bar{x}^\infty) \le 0$ for all $i \in I$. Moreover, $\bar{x}^\infty - \bar{x} \ne 0$. Since

$$\bar{x}^k - \bar{x} = t^k(u^k - d^k - \bar{x}) + t^k v^k, \quad u^k - d^k - \bar{x} \in S, \quad v^k \in S^\perp \;\; \forall k \gg 0,$$

and, by (37), $\|u^k - d^k - \bar{x}\| \le \|\tilde{x}^k - \bar{x}\| \le \|\tilde{x}^k\| + \|\bar{x}\| = O(\|v^k\|)$, we also have $\|\bar{x}^k - \bar{x}\| = O(\|t^k v^k\|)$. Hence $t^k v^k \not\to 0$, implying that $\bar{x}^\infty - \bar{x} \notin S$.

Let $x(\alpha) = (1 - \alpha)\bar{x} + \alpha \bar{x}^\infty$ for $\alpha \in [0, 1]$. We have from $f_i(\bar{x}^\infty) \le 0$ for all $i \in I$ that

$$f_i(x(\alpha)) \le (1 - \alpha) f_i(\bar{x}) + \alpha f_i(\bar{x}^\infty) \le 0 \quad \forall \alpha \in [0, 1], \;\; \forall i \in I.$$

Also, we claim that

$$\bar{x}^{\infty} \in \mathrm{dom} f_i, \quad i = 0, 1, ..., m. \tag{38}$$

To see this, notice that

$$\tilde{x}_j^k \leq \bar{x}_j \ \forall k \gg 0, \ \forall j \in \bar{J}_-, \quad \tilde{x}_j^k = x_j^k \to -\infty \ \forall j \in J_- \setminus \bar{J}_-.$$

Thus, for all $j \in J_-$ and $k \gg 0$, we have $\tilde{x}_j^k \leq \bar{x}_j$ and hence $\bar{x}_j^k \leq \bar{x}_j$. This implies $\bar{x}_j^{\infty} \leq \bar{x}_j$. Since $x_j^k \to -\infty$, we also have $(-\infty, \bar{x}_j] \subseteq \mathrm{dom} f_{ij}$ for all i. Thus

$$\bar{x}_j^{\infty} \in \mathrm{dom} f_{ij} \quad \forall i = 0, 1, ..., m.$$

By an analogous argument, this is also true for all $j \in J_+$. For each $j \in J_b$, we have $j \notin (\bar{J}_- \cup \bar{J}_+)$ (since $\bar{J}_- \subseteq J_-$, $\bar{J}_+ \subseteq J_+$), implying $\tilde{x}_j^k = x_j^k \to x_j^{\infty}$. By Lemma 1, $x_j^{\infty} \in \mathrm{dom} f_{0j}$. Thus, $\bar{x}_j^{\infty} \in \mathrm{conv}\{\bar{x}_j, x_j^{\infty}\} \subseteq \mathrm{dom} f_{0j} \subseteq \mathrm{dom} f_{ij}$ for all i. This proves (38). By (38) and $f_i(\bar{x}) < 0$ for all $i \notin I$, we have

$$f_i(x(\alpha)) \leq (1-\alpha) f_i(\bar{x}) + \alpha f_i(\bar{x}^{\infty}) \leq 0 \quad \forall \alpha \in (0,1) \text{ sufficiently small}, \quad \forall i \notin I.$$

Thus $x(\alpha) \in F$ for all $\alpha \in (0,1)$ sufficiently small. Since $\bar{x}^{\infty} - \bar{x} \notin S$, then $x(\alpha) - \bar{x} \notin S$. This contradicts $S = \mathrm{aff}(F - \bar{x})$. ∎

5 Weakly Analytic Functions

The following result refines Thm. 4.2 in [16] (also see [2, Thm. 5.4.2] and a generalization to parametric convex inequalities in [14, Sec. 3]) by allowing $f_0, f_1, ..., f_m$ to be extended-real-valued in A3. For example, the composition of a real-valued convex weakly analytic function with a convex strictly increasing function is convex weakly analytic, but not necessarily real-valued. In what follows, I is defined as in (6) and we define

$$C = \bigcap_{i \notin I} \mathrm{dom} f_i, \qquad F_\upsilon = \{x \in F \mid f_0(x) \leq \upsilon\} \ \ \forall \upsilon \in \Re. \tag{39}$$

Notice that $i \notin I$ means $i \in \{0, 1, ..., m\} \setminus I$.

Theorem 2 *Assume f_i is weakly analytic for all $i \in I$ and*

$$F_\upsilon \cap \mathrm{ri} C \neq \emptyset \quad \forall \ \upsilon > \upsilon_{\mathrm{P}}. \tag{40}$$

Then p is lsc at 0.

Proof. Let $v = \lim_{u\to 0} \inf p(u)$. Then, there exists a sequence $x^k \in \Re^n$, $k = 1, 2, ...,$ such that $f_0(x^k) \to v$ and $f_i(x^k)^+ \to 0$, $i = 1, ..., m$. We show below that $v_{\mathrm{P}} \leq v$ which shows p is lsc at 0.

Since $F \neq \emptyset$, by convexity of f_i, there exists a $\tilde{x} \in F$ such that $f_i(\tilde{x}) < 0$ for all $i \notin I \cup \{0\}$. Fix any $\epsilon \in (0, 1)$. By (40), there exists a $\hat{x} \in F$ satisfying

$$f_0(\hat{x}) \leq v_{\mathrm{P}} + \epsilon, \qquad \hat{x} \in \mathrm{ri}C.$$

Let

$$\bar{x} = \epsilon\ \tilde{x} + (1 - \epsilon)\ \hat{x}.$$

Then $\bar{x} \in F$, $f_i(\bar{x}) < 0$ for all $i \notin I \cup \{0\}$,

$$f_0(\bar{x}) \leq \epsilon f_0(\tilde{x}) + (1 - \epsilon) f_0(\hat{x}) \leq \epsilon f_0(\tilde{x}) + (1 - \epsilon)(v_{\mathrm{P}} + \epsilon), \tag{41}$$

and $\bar{x} \in \mathrm{ri}C$. The latter implies there exists $\rho > 0$ such that

$$\bar{x} + (\rho B \cap L) \ \subseteq\ \mathrm{ri}C, \tag{42}$$

where $L = \mathrm{aff}(C - \bar{x})$, $B = \{x \mid \|x\| = 1\}$, and $\|\cdot\|$ denotes the Euclidean norm.

Let $S = \mathrm{aff}(F - \bar{x})$. Since $F \subseteq C$, $S \subseteq L$. For any $x \in F$ with $x \neq \bar{x}$, we have

$$f_i(x) = f_i(\bar{x}) = 0 \quad \forall i \in I.$$

Since f_i is weakly analytic for all $i \in I$, this implies f_i is zero-valued on the line through x and $\bar{x}$. Thus

$$f_i(x) = 0 \quad \forall x \in S + \bar{x},\ \forall i \in I. \tag{43}$$

Thus, by Thm. 8.6 of [18], any $d \in S$ is a recession direction of f_i for all $i \in I$. Since $-d \in S$, then f_i is in fact constant along d. For each k, since $x^k \in C$ so that $x^k - \bar{x} \in L$, we can decompose x^k uniquely as

$$x^k = \bar{x} + d^k + v^k,$$

where $d^k \in S$ and $v^k \in S^\perp \cap L$. Let

$$z^k = \bar{x} + d^k \quad \forall k.$$

By (43), $f_i(z^k) = 0$ for all $i \in I$ and all k.

We claim that $v^k \to 0$. Suppose the contrary. By passing to a subsequence if necessary, we can assume that there exists $\delta > 0$ such that $\|v^k\| \geq \delta$ for all k. Let

$$t^k = \frac{\rho}{\max\{1, \|v^k\|\}}, \qquad \bar{x}^k = \bar{x} + t^k v^k.$$

It can be seen that $t^k \in (0,1]$ and $\|t^k v^k\| = \rho \min\{1, \|v^k\|\}$ for all k. Thus $\{\bar{x}^k\}$ is bounded and $\rho \geq \|t^k v^k\| \geq \rho \min\{1, \delta\}$. Since $d^k \in S$ so that f_i is constant along d^k for each $i \in I$, we have

$$\begin{aligned} f_i(\bar{x}^k) &= f_i(\bar{x} + d^k + t^k v^k) \\ &= f_i((1-t^k)z^k + t^k x^k) \\ &\leq (1-t^k) f_i(z^k) + t^k f_i(x^k) \\ &= t^k f_i(x^k) \quad \forall i \in I, \ \forall k, \end{aligned}$$

where the last equality uses $f_i(z^k) = 0$ for all $i \in I$ and all k. Since $f_i(x^k)^+ \to 0$, this shows that $f_i(\bar{x}^k)^+ \to 0$. Since $\{\bar{x}^k\}$ is bounded and f_i is lsc, this shows that any cluster point $\bar{x}^\infty$ of $\{\bar{x}^k\}$ satisfies

$$f_i(\bar{x}^\infty) \leq 0 \quad \forall i \in I. \tag{44}$$

Moreover, $\bar{x}^\infty = \bar{x} + w^\infty$ for some $w^\infty \in S^\perp \cap L$ with $\rho \geq \|w^\infty\| \geq \rho \min\{1, \delta\}$. Thus $\bar{x}^\infty - \bar{x} \notin S$. Since $w^\infty \in \rho B \cap L$, (42) implies $\bar{x}^\infty \in C$. Let $x(\alpha) = (1-\alpha)\bar{x} + \alpha \bar{x}^\infty$ for $\alpha \in [0,1]$. We have from (44) that

$$f_i(x(\alpha)) \leq (1-\alpha) f_i(\bar{x}) + \alpha f_i(\bar{x}^\infty) \leq 0 \quad \forall \alpha \in [0,1], \ \forall i \in I.$$

Since $\bar{x}^\infty \in C$ and $f_i(\bar{x}) < 0$ for all $i \notin I \cup \{0\}$, we also have

$$f_i(x(\alpha)) \leq (1-\alpha) f_i(\bar{x}) + \alpha f_i(\bar{x}^\infty) \leq 0 \quad \forall \alpha \in (0,1) \text{ sufficiently small}, \ \forall i \notin I \cup \{0\},$$

as well as $x(\alpha) \in \mathrm{dom} f_0$. Thus $x(\alpha) \in F$ for all $\alpha \in (0,1)$ sufficiently small. Since $\bar{x}^\infty - \bar{x} \notin S$, then $x(\alpha) - \bar{x} \notin S$. This contradicts $S = \mathrm{aff}(F - \bar{x})$.

Let

$$w^k = \bar{x} - v^k.$$

Then $w^k \to \bar{x} \in \mathrm{ri} C$. Since $v^k \in L$, (42) implies $w^k \in \mathrm{ri} C$ for all k sufficiently large, so the continuity of f_i on $\mathrm{ri} C$ [18, Thm. 10.1] yields

$$\begin{aligned} f_i(w^k) &\to f_i(\bar{x}) < 0 \quad \forall i \notin I \cup \{0\}, \\ f_0(w^k) &\to f_0(\bar{x}). \end{aligned}$$

Let

$$y^k = \frac{x^k + w^k}{2} = \frac{z^k + \bar{x}}{2}.$$

Then

$$f_i(y^k) = f_i\left(\frac{z^k + \bar{x}}{2}\right) \leq \frac{f_i(z^k) + f_i(\bar{x})}{2} = 0 \quad \forall i \in I,\ \forall k,$$
$$f_i(y^k) = f_i\left(\frac{x^k + w^k}{2}\right) \leq \frac{f_i(x^k)^+ + f_i(w^k)}{2} \to \frac{f_i(\bar{x})}{2} < 0 \quad \forall i \notin I \cup \{0\},$$
$$f_0(y^k) = f_0\left(\frac{x^k + w^k}{2}\right) \leq \frac{f_0(x^k) + f_0(w^k)}{2} \to \frac{\upsilon + f_0(\bar{x})}{2}.$$

The first two relations imply $y^k \in F$ for all k sufficiently large, in which case $f_0(y^k) \geq \upsilon_{\mathrm{P}}$. This and the third relation yield in the limit that

$$\upsilon_{\mathrm{P}} \leq \frac{\upsilon + f_0(\bar{x})}{2} \leq \frac{\upsilon + \epsilon f_0(\tilde{x}) + (1-\epsilon)(\upsilon_{\mathrm{P}} + \epsilon)}{2},$$

where the last inequality uses (41). Rearranging terms yields

$$\upsilon_{\mathrm{P}} \leq \upsilon + \epsilon(f_0(\tilde{x}) - \upsilon_{\mathrm{P}} + 1 - \epsilon).$$

Since this holds for any $\epsilon > 0$ while $\tilde{x}$ is independent of ϵ, taking $\epsilon \downarrow 0$ yields $\upsilon_{\mathrm{P}} \leq \upsilon$.
■

Note 6: Weakly analytic function was introduced by Kummer [16] as a generalization of faithfully convex function (i.e., is affine on a line whenever it is affine on some open segment of the line) introduced by Rockafellar [19, 20]. An algorithm for solving convex programs with faithfully convex constraints was presented by Wolkowicz [24]. Relaxing the assumption of $f_0, f_1, ..., f_m$ being real-valued allows Thm. 2 to be applied to a broader class of problems. A part of our proof uses ideas from the proof of Lemma 2. The last paragraph of our proof is a refinement of the proof of Lemma 5.4.3 and Theorem 5.4.2 in [2].

Note 7: Dimitri Bertsekas suggested that we can more simply work with

$$\min \quad \tilde{f}_0(x) \quad \text{s.t.} \quad f_i(x) \leq 0, \quad i \in I, \tag{45}$$

where $\tilde{f}_0(x) = f_0(x) + \sum_{i \notin I \cup \{0\}} \mu_i^* f_i(x)$, and $(\mu_i^*)_{i \in I}$ is a Kuhn-Tucker vector associated with

$$\min \quad \hat{f}_0(x) \quad \text{s.t.} \quad f_i(x) \leq 0, \quad i \notin I \cup \{0\}, \tag{46}$$

and $\hat{f}_0(x) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0 \text{ for all } i \in I, \\ \infty & \text{else.} \end{cases}$ Such $(\mu_i^*)_{i \in I}$ exists whenever υ_{P} is finite [18, Cor. 28.2.1]. Accordingly, we work with $C = \mathrm{dom}\tilde{f}_0$, which is effectively the same as C given by (39). Also, $f_0, f_1, ..., f_m$ being separable implies $\tilde{f}_0$ is separable (since it is the sum of separable functions).

Note 8: A reviewer remarks that, for any $J \subseteq \{1, ..., m\}$, (P) can be written equivalently as

$$\begin{array}{rcll} \upsilon_{\mathrm{P}} & = & \min & \hat{f}_0(x) \\ & & \text{s.t.} & f_i(x) \leq 0, \quad i \notin J \cup \{0\}, \end{array}$$

where $\hat{f}_0(x) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0 \text{ for all } i \in J, \\ \infty & \text{else.} \end{cases}$ The corresponding dual problem is

$$\hat{\upsilon}_{\mathrm{D}} = \max_{\hat{\mu} \geq 0} \hat{q}(\hat{\mu}),$$

where $\hat{\mu} = (\mu_i)_{i \notin J \cup \{0\}}$ and $\hat{q}(\hat{\mu}) = \inf_x \{\hat{f}_0(x) + \sum_{i \notin J \cup \{0\}} \mu_i f_i(x)\}$. It is readily seen that $\upsilon_{\mathrm{D}} \leq \hat{\upsilon}_{\mathrm{D}} \leq \upsilon_{\mathrm{P}}$. Thus if $f_0, f_1, ..., f_m$ satisfy assumption A3 so that $\upsilon_{\mathrm{D}} = \upsilon_{\mathrm{P}}$, then we also have $\hat{\upsilon}_{\mathrm{D}} = \upsilon_{\mathrm{P}}$. In other words, assumption A3 can be relaxed to allow f_0 to take the value ∞ outside the feasible set of certain real-valued convex constraints. The domain of such f_0 is necessarily closed, but this can always be assumed for (P) by adding to f_0 the indicator function for $\{x \mid f_0(x) \leq f_0(\bar{x})\}$ with $\bar{x} \in F$, which does not affect υ_{P} or υ_{D}. However, for such f_0, the directional derivative $f_0'(x; y - x)$ must be finite for any $x, y \in \mathrm{dom} f_0$ [18, Thm. 23.4], so it excludes functions such as $f_0(x) = x \ln x$ (with $f_0(0) = 0$ and $\mathrm{dom} f_0 = [0, \infty)$). Moreover, $f_1, ..., f_m$ still need to be real-valued, so it excludes the example in Note 2, which satisfies the assumption of Thm. 2 with $I = \emptyset$, $C = (0, \infty) \times \Re$, $\upsilon_{\mathrm{P}} = 0$, while $f_1(x) = -\ln(x_1) + x_2$ is not real-valued. Thus, the assumption of Thm. 2 is more broadly applicable than A3.

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