Second-Order Cone Programming Relaxation of Sensor Network Localization[1]

(in memory of Jos Sturm)

August 14, 2005 (revised July 28, 2006)

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Abstract

The sensor network localization problem has been much studied. Recently Biswas and Ye proposed a semidefinite programming (SDP) relaxation of this problem which has various nice properties and for which a number of solution methods have been proposed. Here, we study a second-order cone programming (SOCP) relaxation of this problem, motivated by its simpler structure and its potential to be solved faster than SDP. We show that the SOCP relaxation, though weaker than the SDP relaxation, has nice properties that make it useful as a problem preprocessor. In particular, sensors that are uniquely positioned among interior solutions of the SOCP relaxation are accurate up to the square root of the distance error. Thus, these sensors, which are easily identified, are accurately positioned. In our numerical simulation, the interior solution found can accurately position up to 80-90% of the sensors. We also propose a smoothing coordinate gradient descent method for finding an interior solution that is faster than an interior-point method.

Key words. Sensor network localization, semidefinite program, second-order cone program, approximation algorithm, error bound.

1 Introduction

A problem that has received considerable attention is that of ad hoc wireless sensor network localization [3, 10, 11, 16, 17, 22, 28, 30, 31]. The basic version of this problem can be described as follows:

> There are n distinct points in $\Re^d$ ($d \geq 1$). We know the Cartesian coordinates of the last $n-m$ points ('anchors') $x_{m+1}, ..., x_n$ and the Euclidean distance $d_{ij} > 0$ between 'neighboring' points i and j for $(i,j) \in \mathcal{A}$, where $\mathcal{A} \subseteq (\{1,...,n\} \times \{1,...,m\}) \cup (\{1,...,m\} \times \{1,...,n\})$.[3] We wish to estimate

[1]This work is supported by the National Science Foundation Grant DMS-0511283.

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[3]The set $\mathcal{A}$ is undirected in the sense that $(i,j) = (j,i)$ and $d_{ij} = d_{ji}$ for all $(i,j) \in \mathcal{A}$.

the Cartesian coordinates of the first m points ("sensors").

Typically, $d = 2$ and two points are neighbors if the distance between them is below some threshold (the radio range). In variants of this problem, the distances may be non-Euclidean [30] or may have measurement errors, and there may be additional constraints on the unknown points [16]. This problem is closely related to distance geometry problems arising in the determination of protein structure [8, 24] and to graph rigidity [1, 17, 31].

It is known that the sensor network localization problem is NP-hard in general [29]; also see remark in [24]. A proof for $d = 1$ is by reduction from the set partition problem, which is readily generalized to $d > 1$. Additional studies are given in [3, 28]. Thus, efforts have been directed at solving this problem approximately. A method based on second-order cone programming (SOCP) relaxation was proposed in [16]. In the case where the anchors lie on the "perimeter", a distributed relaxation method was proposed in [28]. The performance of these methods were tested through simulations.

Recently, Biswas and Ye proposed an approach to sensor network localization based on semidefinite programming (SDP) relaxation [10, 11]. In this approach, the problem is formulated as the following nonconvex minimization problem:

$$\upsilon_{\text{opt}} \stackrel{\text{def}}{=} \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm. Introduce

$$X \stackrel{\text{def}}{=} [x_1 \ \cdots \ x_m], \quad A \stackrel{\text{def}}{=} [x_{m+1} \ \cdots \ x_n].$$

Then, for each $(i,j) \in \mathcal{A}$,

$$\begin{aligned} \|x_i - x_j\|^2 &= (x_i - x_j)^T (x_i - x_j) \\ &= (e_i - e_j)^T \begin{bmatrix} X^T \\ A^T \end{bmatrix} [\,X \quad A\,] (e_i - e_j) \\ &= b_{ij}^T \begin{bmatrix} X^T \\ I_d \end{bmatrix} [\,X \quad I_d\,] b_{ij} \\ &= \left\langle b_{ij} b_{ij}^T, \begin{bmatrix} X^T X & X^T \\ X & I_d \end{bmatrix} \right\rangle_F \end{aligned}$$

where e_i is the ith coordinate vector in $\Re^n$ and $b_{ij} \stackrel{\text{def}}{=} \begin{bmatrix} I_m & 0 \\ 0 & A \end{bmatrix} (e_i - e_j)$. Throughout, I_k is the $k \times k$ identity matrix and $\langle A, B\rangle_F \stackrel{\text{def}}{=} \text{trace}[AB]$ for any symmetric real matrices A, B of same dimension. It is not difficult to see that

$$\begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix} \succeq 0 \text{ has rank } d \quad \Longleftrightarrow \quad Y = X^T X.^4$$

Thus (1) may be reformulated as

$$\begin{aligned} \upsilon_{\text{opt}} = \quad & \min_Z \sum_{(i,j)\in\mathcal{A}} \left| \left\langle b_{ij} b_{ij}^T, Z \right\rangle_F - d_{ij}^2 \right| \\ & \text{s.t. } [Z_{ij}]_{i,j\geq n-d} = I_d, \ Z \succeq 0, \ \text{rank} Z = d. \end{aligned} \tag{2}$$

Relaxing the rank-d constraint yields the convex problem:

$$\begin{aligned} \upsilon_{\text{sdp}} \stackrel{\text{def}}{=} \quad & \min_Z \sum_{(i,j)\in\mathcal{A}} \left| \left\langle b_{ij} b_{ij}^T, Z \right\rangle_F - d_{ij}^2 \right| \\ & \text{s.t. } \ [Z_{ij}]_{i,j\geq n-d} = I_d, \ Z \succeq 0, \end{aligned} \tag{3}$$

which is an SDP. In particular, by introducing slack variables, this can be written in the standard conic form:

$$\begin{aligned} \min \quad & \sum_{(i,j)\in\mathcal{A}} u_{ij} + v_{ij} \\ \text{s.t.} \quad & \left\langle b_{ij} b_{ij}^T, Z \right\rangle_F - u_{ij} + v_{ij} = d_{ij}^2 \quad \forall (i,j) \in \mathcal{A}, \\ & [Z_{ij}]_{i,j\geq n-d} = I_d, \\ & u_{ij} \geq 0, \ v_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A}, \ Z \succeq 0, \end{aligned} \tag{4}$$

which has $(m + d)(m + d + 1)/2 + 2|\mathcal{A}|$ variables and $|\mathcal{A}| + d(d + 1)/2$ equality constraints. Here $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. In sensor network localization, $|\mathcal{A}| = \Omega(m)$ and $d = 2$, so that (4) has $\Omega(m^2)$ variables and $\Omega(m)$ equality constraints. Properties of the SDP relaxation and its solutions are studied in [10, 31].[5] As is noted in [11], the SDP relaxation can be solved by existing SDP solvers for $m \leq 100$ but not for much larger m. Thus, a distributed (domain decomposition) method is proposed to solve larger SDP relaxation. In [22], to further improve the speed and accuracy, the distributed SDP method is terminated early and then a gradient search method is used to locally refine the approximate solution. Simulation results show that this method can more quickly and accurately position most sensors, even in the presence of distance errors.

The challenge in solving the SDP relaxation motivates us to consider SOCP relaxation, first studied by Doherty et al. [16], since SOCP can be solved to much

[4]In general, $\begin{bmatrix} u \\ v \end{bmatrix} \in \text{Null} \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix}$ if and only if $u \in \text{Null}(Y - X^T X)$, $v = -Xu$.

[5]Throughout, "solution" of an optimization problem means a global optimal solution.

larger size than SDP [2, 27]. In fact, there has been little study of SOCP relaxation, compared to SDP relaxation, for nonconvex optimization. Besides [16], which presented models and simulation results with SOCP relaxations of sensor network localization (assuming no distance error), Kojima et al. [20, 21] studied SOCP relaxations of certain special classes of SDP and quadratic optimization problems, but their results do not apply to sensor network localization. Here, we present a study, both theoretical and numerical, of the SOCP relaxation of the sensor network localization problem (1), allowing for distance errors. In particular, we show that an interior solution of the SOCP relaxation can be used to accurately position a high percentage of the sensors.[6] To motivate the SOCP relaxation, we reformulate (1) as

$$\begin{aligned} \upsilon_{\text{opt}} = \quad & \min_{x_1,\ldots,x_m,y_{ij}} \sum_{(i,j)\in\mathcal{A}} |y_{ij} - d_{ij}^2| \\ & \text{s.t.} \quad y_{ij} = \|x_i - x_j\|^2 \quad \forall\ (i,j) \in \mathcal{A}. \end{aligned} \tag{5}$$

Relaxing the equality constraints to "$\geq$" inequality constraints yields the convex problem:

$$\begin{aligned} \upsilon_{\text{socp}} \stackrel{\text{def}}{=} \quad & \min_{x_1,\ldots,x_m,y_{ij}} \sum_{(i,j)\in\mathcal{A}} |y_{ij} - d_{ij}^2| \\ & \text{s.t.} \quad y_{ij} \geq \|x_i - x_j\|^2 \quad \forall (i,j) \in \mathcal{A}, \end{aligned} \tag{6}$$

which is an SOCP. In particular, by noting that $y_{ij} \geq d_{ij}^2$ in any solution of (6) and introducing slack variables, this can be written in the standard conic form:

$$\begin{aligned} \min \quad & \sum_{(i,j)\in\mathcal{A}} u_{ij} \\ \text{s.t.} \quad & x_i - x_j - w_{ij} = 0 \quad \forall (i,j) \in \mathcal{A}, \\ & y_{ij} - u_{ij} = d_{ij}^2 \quad \forall (i,j) \in \mathcal{A}, \\ & \alpha_{ij} = \tfrac{1}{2} \quad \forall (i,j) \in \mathcal{A}, \\ & u_{ij} \geq 0, \ (\alpha_{ij}, y_{ij}, w_{ij}) \in \text{Rcone}^{d+2} \quad \forall (i,j) \in \mathcal{A}, \end{aligned} \tag{7}$$

where $\text{Rcone}^{d+2} \stackrel{\text{def}}{=} \{(\alpha, y, w) \in \Re \times \Re \times \Re^d : 2\alpha y \geq \|w\|^2\}$ [32]. This is an SOCP since

$$y \geq \|w\|^2 \iff (y + \frac{1}{4})^2 \geq (y - \frac{1}{4})^2 + \|w\|^2 \iff y + \frac{1}{4} \geq \|(y - \frac{1}{4}, w)\|$$

(see [4, page 88] or [25, page 221]). The SOCP (7) has $(d+3)|\mathcal{A}| + md$ variables and $(d+2)|\mathcal{A}|$ equality constraints. In sensor network localization, $|\mathcal{A}| = \Omega(m)$ and

[6] Throughout, "interior solution" means an element in the relative interior of the optimal solution set.

$d = 2$, so that (7) has $\Omega(m)$ variables and $\Omega(m)$ equality constraints. Thus, SOCP relaxation has smaller size than SDP relaxation.

How good approximation is the SOCP relaxation? Can it be efficiently solved? We will show that the SOCP relaxation is always weaker than the SDP relaxation and that any interior solution of the SOCP relaxation (which can be found by, say, an interior-point method) will accurately position (up to square root distance error) those sensors that are uniquely positioned; see Propositions 1, 5, and 6. Moreover, the aforementioned sensors (which lie in the convex hull of the anchors) can be easily identified; see Propositions 3 and 4. In our simulations, described in Section 9, up to 80-90% of the sensors are accurately positioned using this technique. Thus, the SOCP relaxation can act as a useful pre-processor by accurately positioning most of the sensors, thus greatly reducing the problem size. The remaining sensors can be positioned by other means, such as SDP relaxation. In Section 8, we propose a smoothing coordinate gradient descent method that computes an interior solution of the SOCP relaxation faster than an interior-point method. In Sections 10 and 11, we present a mixed SDP-SOCP relaxation of (1), which can flexibly mediate between strength of relaxation and problem size, and discuss alternative problem formulations. In particular, other objective functions can be used in (1), for which SOCP relaxation may be more "natural" than SDP relaxation. However, changing the objective function of (1) changes its solution. Here we consider (1) so to better compare with existing SDP relaxation approach (Propositions 1, 2) and to introduce the mixed SDP-SOCP relaxation. In addition, the SOCP relaxation is a useful problem pre-processor even if it is weaker than SDP relaxation.

Throughout, $\Re^n$ denotes the space of n-dimensional real column vectors (sometimes written horizontally for convenience), $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, and ^{T} denotes transpose. For $A \in \Re^{m\times n}$, A_{ij} denotes the (i,j)th entry of A. For $A, B \in \mathcal{S}^n$, $A \succeq B$ means $A - B$ is positive semidefinite. "conv" means the convex hull.

2 An Ilustrative Example

To understand properties of SDP and SOCP relaxations, it is instructive to look at an example. Consider the following example of Ye, with $d = 2$, $n = 3$, $m = 1$, and

$$x_2 = (-1, 0), \quad x_3 = (1, 0), \quad d_{12} = d_{13} = 2.$$

The optimization problem (1) is:

$$\min_{x_1=(\alpha,\beta)\in\Re^2} \quad |(1-\alpha)^2 + \beta^2 - 4| + |(-1-\alpha)^2 + \beta^2 - 4|.$$

It has two solutions at $x_1 = (0, \sqrt{3})$, $x_1 = (0, -\sqrt{3})$.

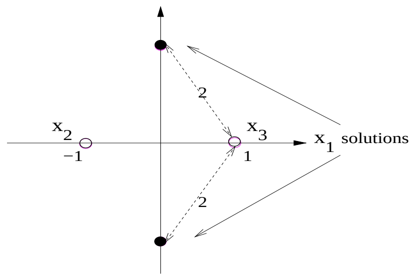

Figure 1: The localization problem has two solutions at $(0, \pm\sqrt{3})$.

The SDP relaxation (3) is:

$$\begin{array}{ll} \min\limits_{\substack{x_1=(\alpha,\beta)\in\Re^2 \\ y\in\Re}} & |y - 2\alpha - 3| + |y + 2\alpha - 3| \\ \text{s.t.} & \begin{bmatrix} y & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} \succeq 0. \end{array}$$

Its solutions have the form $y = 3$ and x_1 is any point on the line segment joining $(0, -\sqrt{3})$ and $(0, \sqrt{3})$. If we solve the corresponding SDP (4) by an interior-point method, then it will find the solution that maximizes the barrier (see [25, page 235], [4, page 384])

$$\det \begin{bmatrix} 3 & 0 & \beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix} = 3 - \beta^2.$$

The maximum is attained at $\beta = 0$. The corresponding x_1-solution $(0, 0)$ is the analytic center of the SDP solution set.

The SOCP relaxation (6) is:

$$\begin{array}{ll} \min\limits_{\substack{x_1=(\alpha,\beta)\in\Re^2 \\ y,z\in\Re}} & |y - 4| + |z - 4| \\ \text{s.t.} & y \geq (\alpha - 1)^2 + \beta^2, \\ & z \geq (\alpha + 1)^2 + \beta^2. \end{array}$$

Its solutions have the form $y = z = 4$ and x_1 is any point in the intersection of the two disks of radius 2 and centered at $(-1, 0)$ and $(1, 0)$. If we solve the corresponding

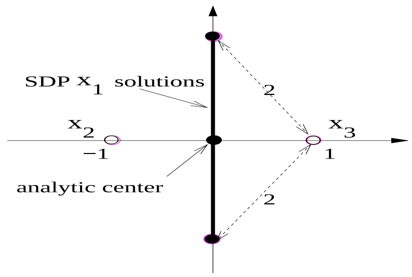

Figure 2: The SDP relaxation has the entire line segment as its x_1-solution set.

SOCP (7) by an interior-point method, then it will find the solution that maximizes the barrier (see [25, page 223], [4, page 384], and also Section 6)

$$\log(4 - (\alpha - 1)^2 - \beta^2) + \log(4 - (\alpha + 1)^2 - \beta^2).$$

This maximization is attained at $\alpha = \beta = 0$. The corresponding x_1-solution $(0, 0)$ is the analytic center of the SOCP solution set. In general, finding the analytic center may be more efficient and accurate than the bounding approach suggested in [16], which entails solving an SOCP $2d$ times with different linear objective functions.

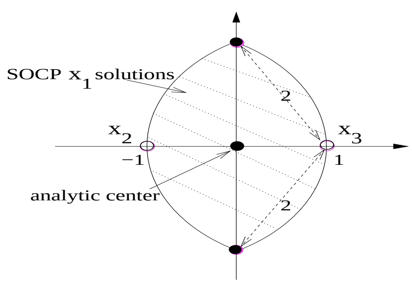

Figure 3: The SOCP relaxation has the intersection of two disks as its x_1-solution set.

From the above example, we make the following observations:

- The SDP x_1-solution set is contained in the SOCP x_1-solution set.
- The analytic center of the SOCP x_1-solution set lies in the convex hull of its neighbors x_2 and x_3.

We will now study in more generality these observed properties of the SDP and SOCP relaxations.

3 Properties of SDP and SOCP Relaxations

We show below that the SDP $(x_1, ..., x_m)$-solution set is contained in the SOCP $(x_1, ..., x_m)$-solution set, so that the SOCP relaxation is weaker than the SDP relaxation.

Proposition 1 *If* $Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix}$ *is feasible for the SDP relaxation* (3)*, then*

$$\begin{array}{rcl} x_i & = & i\text{th column of } X, \quad i = 1, ..., m, \\ y_{ij} & = & \begin{cases} Y_{ii} - 2Y_{ij} + Y_{jj}, & \text{if } (i,j) \in \mathcal{A},\ i < j \leq m; \\ \|x_i\|^2 - 2x_i^T x_j + Y_{jj}, & \text{if } (i,j) \in \mathcal{A},\ j \leq m < i, \end{cases} \end{array}$$

is feasible for the SOCP relaxation (6) *with the same objective function value.*

Proof. Since Z is feasible for (3), we have $Z \succeq 0$, so that $Y - X^T X \succeq 0$. Then any 2×2 principal submatrix of $Y - X^T X$ is positive semidefinite, so that, for any $(i, j) \in \mathcal{A}$ with $i < j \leq m$,

$$\begin{bmatrix} Y_{ii} - \|x_i\|^2 & Y_{ij} - x_i^T x_j \\ Y_{ij} - x_i^T x_j & Y_{jj} - \|x_j\|^2 \end{bmatrix} \succeq 0,$$

implying that

$$Y_{ii} \geq \|x_i\|^2, \quad Y_{jj} \geq \|x_j\|^2, \quad (Y_{ii} - \|x_i\|^2)(Y_{jj} - \|x_j\|^2) \geq (Y_{ij} - x_i^T x_j)^2. \qquad (8)$$

For any $a \geq 0, b \geq 0, ab \geq c^2$, we have $(a + b)^2 = 4ab + (a - b)^2 \geq 4c^2$ and hence $a + b \geq 2|c|$. Thus (8) implies

$$Y_{ii} - \|x_i\|^2 + Y_{jj} - \|x_j\|^2 \geq 2|Y_{ij} - x_i^T x_j| \geq 2(Y_{ij} - x_i^T x_j).$$

Hence

$$y_{ij} = Y_{ii} - 2Y_{ij} + Y_{jj} \geq \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = \|x_i - x_j\|^2.$$

Similarly, any diagonal entry of $Y - X^T X$ is nonnegative, so that, for any $(i, j) \in \mathcal{A}$ with $j \leq m < i$, $Y_{jj} - \|x_j\|^2 \geq 0$ and hence

$$y_{ij} = \|x_i\|^2 - 2x_i^T x_j + Y_{jj} \geq \|x_i\|^2 - 2x_i^T x_j + \|x_j\|^2 = \|x_i - x_j\|^2.$$

Thus $x_1, ..., x_m$, $(y_{ij})_{(i,j) \in \mathcal{A}}$ is feasible for (6).

Lastly, we have from the definition of b_{ij} and y_{ij} that

$$\begin{aligned}
& \sum_{(i,j)\in\mathcal{A}} | \left\langle b_{ij}b_{ij}^T, Z\right\rangle_F - d_{ij}^2| \\
= & \sum_{\substack{i<j\le m\\(i,j)\in\mathcal{A}}} |Y_{ii} - 2Y_{ij} + Y_{jj} - d_{ij}^2| + \sum_{\substack{j\le m<i\\(i,j)\in\mathcal{A}}} |\|x_i\|^2 - 2x_i^T x_j + Y_{jj} - d_{ij}^2| \\
= & \sum_{\substack{i<j\le m\\(i,j)\in\mathcal{A}}} |y_{ij} - d_{ij}^2| + \sum_{\substack{j\le m<i\\(i,j)\in\mathcal{A}}} |y_{ij} - d_{ij}^2|.
\end{aligned}$$

Thus, Z and $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ have the same objective function value for (3) and (6), respectively. ∎

Proposition 1 shows that (i) $\upsilon_{\text{SDP}} \ge \upsilon_{\text{SOCP}}$ and (ii) if $\upsilon_{\text{SDP}} = \upsilon_{\text{SOCP}}$, then the set of SDP solutions is contained in the set of SOCP solutions when projected on to the $x_1, ..., x_m$-space.

It is well known that the solution set of (3) is closed and convex, and the same is true of (6). An interior solution can be found by, say, applying an interior-point method to (4) and (7). We will see that such an interior solution has desirable properties for identifying sensors that are accurately positioned.

When solving SDP or SOCP by an interior-point method, the solution set must be bounded. It is readily seen that the solution set of (3) is bounded if and only if the solution set of (4) is bounded. Similarly, the solution set of (6) is bounded if and only if the solution set of (7) is bounded. In [31, Proposition 1], it is shown in the case of $\upsilon_{\text{opt}} = 0$ (i.e., no distance error) that the solution set of (3) is bounded if the following assumption holds.

Assumption 1 *Each connected component of the graph* $\mathcal{G} \stackrel{\text{def}}{=} (\{1, ..., n\}, \mathcal{A})$ *contains an anchor index.*

It is not difficult to see that this remains true when $\upsilon_{\text{opt}} > 0$ and that the converse also holds. Similarly, it is readily shown that the set of solutions of (6) is bounded if and only if Assumption 1 holds. This is summarized in the following lemma.

Lemma 1 **(a)** *The solution set of* (3) *is bounded if and only if Assumption 1 holds.*

(b) *The solution set of* (6) *is bounded if and only if Assumption 1 holds.*

Assumption 1 is reasonable since if a connected component of $\mathcal{G}$ does not contain an anchor index, then the corresponding sensors cannot be accurately positioned. In the absence of anchor (i.e., $m = n$), as arises in protein structure prediction, the solution set is unbounded and, in particular, each solution can be rotated and translated to yield another solution. In [8], an optimization formulation is proposed to remove the translation factor and ensure bounded solution set (assuming no distance error) and an extension of the distributed SDP method in [11] is proposed, in which points in overlapping "sub-configurations" are further rotated and translated to match closely on the overlap.

4 Interior Solution of the SDP Relaxation

Let $Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix}$ be any solution of the SDP relaxation (3). Biswas and Ye introduced the notion of individual traces of Z, defined by

$$\mathrm{tr}_i[Z] \stackrel{\mathrm{def}}{=} Y_{ii} - \|x_i\|^2, \quad i = 1, ..., m,$$

where x_i is the ith column of X. Since $Z \succeq 0$ so that $Y - X^T X \succeq 0$, we have

$$\mathrm{tr}_i[Z] \geq 0, \quad i = 1, ..., m.$$

These individual traces were given a probabilistic interpretation in [10, Section 4] as the variance of random points $\tilde{x}_i$ with $\mathrm{E}[\tilde{x}_i] = x_i$ and $\mathrm{E}[\tilde{x}_i^T \tilde{x}_j] = Y_{ij}$. In [11, Section 2], they were used to evaluate the accuracy of the estimated positions x_i, $i = 1, ..., m$, with smaller trace indicating higher accuracy. So and Ye [31, Theorem 2] proved in the case of $\upsilon_{\mathrm{sdp}} = 0$ that the sensors are "uniquely localizable" if and only if, for any interior solution Z (equivalently, Z is a solution of maximum rank), all individual traces of Z are zero, i.e., $Y = X^T X$.

The following proposition provides some justification for using individual traces to evaluate accuracy of computed sensor positions. It shows that, for any interior solution of (3), if the ith trace is zero, then the ith sensor is uniquely positioned by the SDP relaxation (and hence is correctly positioned when $\upsilon_{\mathrm{sdp}} = 0$). This result gives a local generalization of the "if" direction in [31, Theorem 2] that is analogous to [31, Theorem 4].

Proposition 2 *Let* $Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix}$ *be an interior solution of* (3)*. For each* $i \in \{1, ..., m\}$, *if* $\mathrm{tr}_i[Z] = 0$, *then* x_i *is invariant over all solutions of* (3), *where* x_i *is the* i*th column of* X*. Moreover,* $Y_{JJ} - X_J^T X_J = 0$, *where* $J \stackrel{\mathrm{def}}{=} \{i \leq m : \mathrm{tr}_i[Z] = 0\}$, Y_{JJ} *is the principal submatrix of* Y *indexed by* J, *and* X_J *is submatrix of* X *comprising the columns indexed by* J.

Proof. Consider any solution Z' of (3). Since Z is an interior solution, then

$$Z^1 \stackrel{\text{def}}{=} Z + \epsilon(Z' - Z), \quad Z^2 \stackrel{\text{def}}{=} Z - \epsilon(Z' - Z)$$

are both solutions of (3) for any sufficiently small $\epsilon > 0$. Write them in the forms:

$$Z^1 = \begin{bmatrix} Y^1 & (X^1)^T \\ X^1 & I_d \end{bmatrix} \quad \text{and} \quad Z^2 = \begin{bmatrix} Y^2 & (X^2)^T \\ X^2 & I_d \end{bmatrix}.$$

Since $Z = (Z^1 + Z^2)/2$, this implies that, for any $i \in \{1, ..., m\}$,

$$\begin{aligned} \text{tr}_i[Z] &= \text{tr}_i[(Z^1 + Z^2)/2] \\ &= (Y^1_{ii} + Y^2_{ii})/2 - \|(x^1_i + x^2_i)/2\|^2 \\ &= (Y^1_{ii} + Y^2_{ii})/2 - \left(\|x^1_i\|^2 + \|x^2_i\|^2 - \|x^1_i - x^2_i\|^2/2\right)/2 \\ &= (\text{tr}_i[Z^1] + \text{tr}_i[Z^2])/2 + \|x^1_i - x^2_i\|^2/4, \end{aligned}$$

where x^1_i, x^2_i are the ith column of X^1, X^2, respectively. Since $\text{tr}_i[Z^1] \geq 0$ and $\text{tr}_i[Z^2] \geq 0$, if $\text{tr}_i[Z] = 0$, then $x^1_i = x^2_i$ and hence $x'_i = x_i$.

Since $Y - X^T X \succeq 0$, we have $Y_{JJ} - X^T_J X_J \succeq 0$. Since $0 = \text{tr}_i[Z] = [Y - X^T X]_{ii}$ for all $i \in J$ so that $Y_{JJ} - X^T_J X_J$ has zero diagonals, this implies $Y_{JJ} - X^T_J X_J = 0$. ■

Proposition 2 shows that any interior solution identifies some subset of sensors that are uniquely positioned by the SDP relaxation. It is an open question whether the converse of Proposition 2 holds, i.e., if Z is an interior solution of (3) and $\text{tr}_i[Z] > 0$, then x_i is not invariant over all solutions of (3). We will prove in Section 5 an analogous result for the SOCP relaxation (6).

When an interior-point method is used to solve the SDP relaxation (4), it will find not only an interior solution, but an interior solution that maximizes the nonzero traces in some sense. Using such a solution should make the zero-trace test more robust under computation errors. A rigorous study of this topic requires knowledge of the asymptotic behavior of the central path for SDP, which is not fully understood– see [26] and references therein. On the other hand, the simpler structure of the SOCP relaxation (7) makes possible such a study, as we will do in Section 5.

5 Interior Solution of the SOCP Relaxation

Since the SOCP is a convex minimization problem, there exists a maximal subset of constraints that are tight/active at every solution. In particular, there exists a unique $\mathcal{B} \subseteq \mathcal{A}$ such that

$$\|x_i - x_j\|^2 = y_{ij} \ \forall \text{ solutions } x_1, ..., x_m,\ (y_{ij})_{(i,j)\in\mathcal{A}} \text{ of } (6) \quad \Longleftrightarrow \quad (i,j) \in \mathcal{B}. \quad (9)$$

Any solution that satisfies strictly the remaining constraints of (6) lies in the relative interior of the solution set, i.e., it is an interior solution.

In what follows, we denote the set of neighbors of $i \in \{1, ..., m\}$ relative to any $\mathcal{B} \subseteq \mathcal{A}$ by

$$N_{\mathcal{B}}(i) \stackrel{\text{def}}{=} \{j \in \{1, ..., n\} \ : \ (i,j) \in \mathcal{B}\}.$$

Also,

$$M_{\mathcal{B}} \stackrel{\text{def}}{=} \{i \in \{1, ..., m\} \ : \ N_{\mathcal{B}}(i) \neq \emptyset\}.$$

The next result is key for identifying those sensors that are uniquely positioned by the SOCP relaxation.

Proposition 3 *Let $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ be any interior solution of* (6)*. Let $\mathcal{B}$ be given by* (9)*. The following results hold.*

(a) For each $i \in M_{\mathcal{B}}$,

$$x_i \in \text{conv}\, \{x_j\}_{j\in N_{\mathcal{B}}(i)}\,. \quad (10)$$

(b) Each connected component of the graph $G_{\mathcal{B}} \stackrel{\text{def}}{=} (M_{\mathcal{B}} \cup \{m+1, ..., n\}, \mathcal{B})$ contains an anchor index $i \in \{m+1, ..., n\}$.

(c) For each $i \in \{1, ..., m\}$, x_i is invariant over all solutions of (6) *if and only if $i \in M_{\mathcal{B}}$.*

Proof. (a) We argue by contradiction. Suppose that (10) fails to hold for some $i \in M_{\mathcal{B}}$. Let p_i denote the nearest-point projection of x_i onto $\text{conv}\, \{x_j\}_{j\in N_{\mathcal{B}}(i)}$. Then, $p_i \neq x_i$ and, for each $j \in N_{\mathcal{B}}(i)$, we have $(x_i - p_i)^T(p_i - x_j) \geq 0$, implying

$$\begin{aligned} \|x_i - x_j\|^2 &= \|x_i - p_i + p_i - x_j\|^2 \\ &= \|x_i - p_i\|^2 + \|p_i - x_j\|^2 + 2(x_i - p_i)^T(p_i - x_j) \\ &> \|p_i - x_j\|^2. \end{aligned}$$

For $\epsilon \in (0, 1)$, let

$$x_i^\epsilon = (1-\epsilon)x_i + \epsilon p_i.$$

Since $\|x_i - x_j\|^2 = y_{ij}$ for all $j \in N_{\mathcal{B}}(i)$ and $\|x_i - x_j\|^2 < y_{ij}$ for all $j \in N_{\mathcal{A}\setminus\mathcal{B}}(i)$, the convexity and continuity of $\|\cdot\|^2$ yield that

$$\|x_i^\epsilon - x_j\|^2 < y_{ij} \quad \forall j \in N_{\mathcal{A}}(i),$$

for all ϵ sufficiently small. Thus, replacing x_i by x_i^ϵ yields another solution of (6), and it satisfies strictly the constraints corresponding to $j \in N_{\mathcal{B}}(i)$. This contradicts the assumption that $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ is an interior solution.

(b) Choose any $\bar{i} \in M_{\mathcal{B}}$ and initialize $\bar{M} \leftarrow \{\bar{i}\}$. Then, whenever there is an $i \in \bar{M} \cap M_{\mathcal{B}}$ with $N_{\mathcal{B}}(i) \not\subseteq \bar{M}$, we add $N_{\mathcal{B}}(i)$ to $\bar{M}$, i.e., $\bar{M} \leftarrow \bar{M} \cup N_{\mathcal{B}}(i)$, until no such i exists. Since, for each $i \in M_{\mathcal{B}}$, each $j \in N_{\mathcal{B}}(i)$ either indexes an anchor or else belongs to $M_{\mathcal{B}}$ (since $N_{\mathcal{B}}(j) \neq \emptyset$), we see that $\bar{M} \subseteq M_{\mathcal{B}} \cup \{m+1, ..., n\}$. Moreover, for each $i \in \bar{M} \cap M_{\mathcal{B}}$, we have $N_{\mathcal{B}}(i) \subseteq \bar{M}$ and, by (a), (10) holds, so that

$$x_i \in \text{conv}\,\{x_j\}_{j\in N_{\mathcal{B}}(i)} \subseteq \text{conv}\,\{x_j\}_{j\in\bar{M}},$$

implying that x_i is not an extreme point of $\{x_j\}_{j\in\bar{M}}$. Thus, all extreme points of $\text{conv}\,\{x_j\}_{j\in\bar{M}}$ are anchors. Let

$$\bar{\mathcal{A}} = \{(i,j) \ : \ i \in \bar{M} \cap M_{\mathcal{B}}, \ j \in N_{\mathcal{B}}(i)\}.$$

Then $(\bar{M}, \bar{\mathcal{A}})$ is a connected subgraph of $G_{\mathcal{B}}$, and it contains an anchor index; see Figure 4 for an illustrative example. Thus the connected component of $G_{\mathcal{B}}$ that contains this subgraph contains an anchor index. Since the choice of $\bar{i} \in M_{\mathcal{B}}$ was arbitrary, this shows that every connected component of $G_{\mathcal{B}}$ contains an anchor index.

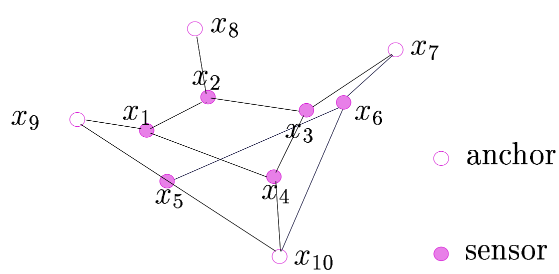

Figure 4: In this example, $\mathcal{B}$ is shown as lines and $M_{\mathcal{B}} = \{1, 2, ..., 6\}$. For $\bar{i} \in \{1, 2, 3, 4\}$, we have $\bar{M} = \{1, 2, 3, 4, 7, 8, 9, 10\}$, $\bar{\mathcal{A}} = \{(1, 2), (1, 4), (1, 9), (2, 3), (2, 8), (3, 7), (3, 4), (4, 10)\}$. For $\bar{i} \in \{5, 6\}$, we have $\bar{M} = \{5, 6, 7, 9, 10\}$, $\bar{\mathcal{A}} = \{(5, 6), (5, 9), (5, 10), (6, 7), (6, 10)\}$.

(c) If $x'_1, ..., x'_m$, $(y'_{ij})_{(i,j)\in\mathcal{A}}$ is any solution of (6), then for each $(i,j) \in \mathcal{B}$,

$$\|x'_i - x'_j\|^2 = y'_{ij}$$

(with $x'_i = x_i$ for $i > m$). Combining this with $\|x_i - x_j\|^2 = y_{ij}$ yields

$$\frac{y_{ij} + y'_{ij}}{2} = \left\| \frac{x_i - x_j}{2} + \frac{x'_i - x'_j}{2} \right\|^2 + \left\| \frac{x_i - x_j}{2} - \frac{x'_i - x'_j}{2} \right\|^2. \tag{11}$$

Since the solution set of (6) is convex, so that $\frac{1}{2}(x_1 + x'_1), ..., \frac{1}{2}(x_m + x'_m)$, $(\frac{1}{2}(y_{ij} + y'_{ij}))_{(i,j)\in\mathcal{A}}$ also forms a solution, $(i,j) \in \mathcal{B}$ implies the right-most term in (11) must be zero. This in turn implies that

$$x'_i - x'_j = x_i - x_j.$$

Thus, for each $(i,j) \in \mathcal{B}$ there exists $\Delta_{ij} \in \Re^d$ such that

$$x'_i - x'_j = \Delta_{ij} \quad \forall \text{ solutions } x'_1, ..., x'_m, (y'_{ij})_{(i,j)\in\mathcal{A}} \text{ of (6) (with } x'_i = x_i \text{ for } i > m). \tag{12}$$

Let $(\bar{M}, \bar{\mathcal{B}})$ be any connected component of $G_{\mathcal{B}}$. By (b), there exists $i \in \bar{M} \cap \{m+1, ..., n\}$, i.e., x_i is an anchor. For each $j \in N_{\bar{\mathcal{B}}}(i)$, we have $(i,j) \in \mathcal{B}$ so that (12) implies

$$x_i - x'_j = \Delta_{ij} = x_i - x_j,$$

for all solutions $x'_1, ..., x'_m$, $(y'_{ij})_{(i,j)\in\mathcal{A}}$ of (6). Hence $x'_j = x_j$. Since $j \in \bar{M}$, we can repeat the above argument with j in place of i and so on. This yields $x'_j = x_j$ for all $j \in \bar{M}$. Since the choice of the connected component was arbitrary, this shows that $x'_j = x_j$ for all $j \in M_{\mathcal{B}}$.

If $i \le m$ and $i \notin M_{\mathcal{B}}$, then $N_{\mathcal{B}}(i) = \emptyset$. This implies

$$\|x_i - x_j\|^2 < y_{ij} \quad \forall j \in N_{\mathcal{A}}(i).$$

Then, we can perturb x_i and obtain another solution $x'_1, ..., x'_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ of (6) with $x'_i \ne x_i$. ∎

As a corollary of Proposition 3(c), we have that the solution of (6) is unique if and only if each connected component of the graph $(\{1, ..., n\}, \mathcal{B})$ contains an anchor index (i.e., $M_{\mathcal{B}} = \{1, ..., m\}$).

Proposition 3 shows that those points x_i with $i \in M_{\mathcal{B}}$ have the following three properties: (i) they satisfy (10), (ii) $\|x_i - x_j\|^2 = y_{ij}$ for all $j \in N_{\mathcal{B}}(i)$, and (iii) their positions are uniquely determined by the anchors $x_{m+1}, ..., x_n$ and $(y_{ij})_{(i,j)\in\mathcal{B}}$. Might the first two properties (i), (ii) imply property (iii)? This question is related to graph rigidity and uniqueness of graph realizability. However, the following example in Figure 5, suggested by Bob Connelly [15], shows that this is not true. The outer three points are anchors, the edges of $\mathcal{B}$ are as shown, and the inner three points (sensors) form a triangle that can be twisted slightly clockwise or counterclockwise to be in two different positions, both of which have properties (i) and (ii).

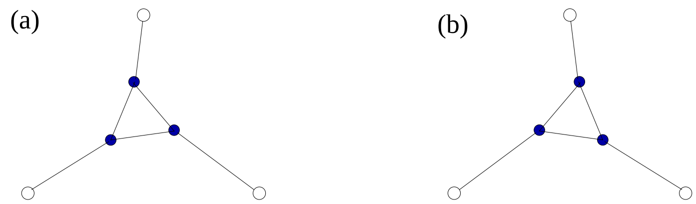

Figure 5: An example in $\Re^2$ of non-unique sensor positions satisfying (10) and perserving distances.

6 Analytic Center Solution of the SOCP Relaxation

As is mentioned in Section 2, when we solve (7) using an interior-point method, the method will generally find not only an interior solution, but an analytic center of the solution set. We study this in more depth below. We first need the following lemma to relate the solutions of (6) and (7).

Lemma 2 $(y_{ij})_{(i,j)\in\mathcal{A}}$ *is invariant over all solutions of* (6).

Proof. Let $\mathcal{B}$ be given by (9). Suppose we have two solutions of (6): $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ and $x'_1, ..., x'_m$, $(y'_{ij})_{(i,j)\in\mathcal{A}}$. Then, for each $(i,j) \in \mathcal{B}$, $y_{ij} = \|x_i - x_j\|^2$ and $y'_{ij} = \|x'_i - x'_j\|^2$ (with $x'_i = x_i$ for $i > m$), so that

$$\frac{y_{ij} + y'_{ij}}{2} = \left\| \frac{x_i - x_j}{2} + \frac{x'_i - x'_j}{2} \right\|^2 + \left\| \frac{x_i - x_j}{2} - \frac{x'_i - x'_j}{2} \right\|^2.$$

Since the solution set is convex so that $\frac{1}{2}(x_1 + x'_1), ..., \frac{1}{2}(x_m + x'_m)$, $(\frac{1}{2}(y_{ij} + y'_{ij}))_{(i,j)\in\mathcal{A}}$ also forms a solution of (6), the right-most term must be zero, i.e., $x_i - x_j = x'_i - x'_j$. Thus $y_{ij} = y'_{ij}$.

For each $(i,j) \in \mathcal{A}\backslash\mathcal{B}$, we have $y_{ij} > \|x_i - x_j\|^2$ for all interior solutions $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ of (6), implying $y_{ij} = d_{ij}^2$. (If $y_{ij} \neq d_{ij}^2$, then y_{ij} can be perturbed to decrease $|y_{ij} - d_{ij}^2|$ and hence decrease the objective function value.) Taking closure yields that $y_{ij} = d_{ij}^2$ for all solutions $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ of (6) so that y_{ij} is unique. ∎

By using Lemma 2, we see that $(u_{ij})_{(i,j)\in\mathcal{A}}$ is invariant over all solutions of (7). Then, under Assumption 1, the limiting point of the central path for (7) would be an interior solution of (7) that maximizes (see [25, page 223], [4, page 384])

$$\sum_{(i,j)\in\mathcal{A}\setminus\mathcal{B}} \log\left((y_{ij}+\frac{1}{4})^2 - \|(y_{ij}-\frac{1}{4}, w_{ij})\|^2\right) = \sum_{(i,j)\in\mathcal{A}\setminus\mathcal{B}} \log\left(y_{ij} - \|x_i - x_j\|^2\right).$$

Accordingly, we define an *analytic center solution* of (6) to be an interior solution of (6) that maximizes

$$\sum_{(i,j)\in\mathcal{A}\setminus\mathcal{B}} \log\left(y_{ij} - \|x_i - x_j\|^2\right) \tag{13}$$

over all interior solutions. Thus, an analytic center solution in some sense maximizes the slacks $y_{ij} - \|x_i - x_j\|^2$ for all inactive constraints $(i,j) \in \mathcal{A}\setminus\mathcal{B}$. Its existence is guaranteed by Assumption 1. It is unique because of Proposition 3(c) and that, by Lemma 2 and the strict concavity of $\log(y_{ij} - \|\cdot\|^2)$, $x_i - x_j$ is unique for all $(i,j) \in \mathcal{A} \setminus \mathcal{B}$. This is the interior solution that a log-barrier interior-point method will likely find. If a barrier method based on a different barrier function is used to solve (7), then the interior solution found need not be the analytic center.

The next proposition verifies one of our observations from the example in Section 2. This is further illustrated in Figure 7.

Proposition 4 *If* $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ *is the analytic center solution of* (6)*, then*

$$x_i \in \text{conv}\left\{x_j\right\}_{j\in N_{\mathcal{A}}(i)}, \quad i = 1, ..., m.$$

Proof. We argue by contradiction. Suppose there exists $i \in \{1, ..., m\}$ such that

$$x_i \notin \text{conv}\left\{x_j\right\}_{j\in N_{\mathcal{A}}(i)}.$$

Let p_i denote the nearest-point projection of x_i onto this convex hull. Then, as in the proof of Proposition 3(a), we have

$$\|p_i - x_j\|^2 \;<\; \|x_i - x_j\|^2 \quad \forall j \in N_{\mathcal{A}}(i).$$

Thus, replacing x_i by p_i would yield another interior solution of (6). Moreover, if $(i,j) \in \mathcal{A} \setminus \mathcal{B}$, then $y_{ij} - \|p_i - x_j\|^2 > y_{ij} - \|x_i - x_j\|^2 > 0$, so that

$$\log\left(y_{ij} - \|p_i - x_j\|^2\right) \;>\; \log\left(y_{ij} - \|x_i - x_j\|^2\right).$$

Summing these inequalities yields

$$\sum_{j\in N_{\mathcal{A}\setminus\mathcal{B}}(i)} \log\left(y_{ij} - \|p_i - x_j\|^2\right) \;>\; \sum_{j\in N_{\mathcal{A}\setminus\mathcal{B}}(i)} \log\left(y_{ij} - \|x_i - x_j\|^2\right).$$

This contradicts our assumption that $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ is the analytic center solution of (6). ■

It is an open question whether a result analogous to Lemma 2 holds for the SDP relaxation (3), namely, is $\left(\left\langle b_{ij}b_{ij}^T, Z\right\rangle_F\right)_{(i,j)\in\mathcal{A}}$ invariant over all solutions of (3)? There does not appear to be a result analogous to Proposition 4 for SDP relaxation. In particular, (3) need not have any solution satisfying the convex hull condition of Proposition 4; see an example in Figure 1(a) of [31].

7 Error Analysis for the SOCP Relaxation

In practice, the distance d_{ij} has measurement error, i.e.,

$$d_{ij}^2 = \bar{y}_{ij} + \delta_{ij} \quad \forall (i,j) \in \mathcal{A},$$

where $\delta_{ij} \in \Re$ and $\bar{y}_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2$ for some $x_1^{\text{true}}, ..., x_m^{\text{true}}$ representing the true positions of the sensors, and with $x_i^{\text{true}} = x_i$ for $i > m$. What is the corresponding error in the solution of (6)? We study this question in this section.

In what follows, we denote for simplicity $x = (x_1, ..., x_m) \in \Re^d \times \cdots \times \Re^d$ and

$$q_{ij}(x) \stackrel{\text{def}}{=} \|x_i - x_j\|^2 - \bar{y}_{ij} \quad \forall (i,j) \in \mathcal{A}. \tag{14}$$

Also,

$$\Xi \stackrel{\text{def}}{=} \{x : q_{ij}(x) \le 0 \;\; \forall\ (i,j) \in \mathcal{A}\}. \tag{15}$$

Then Ξ contains the true solution $x^{\text{true}} = (x_1^{\text{true}}, ..., x_m^{\text{true}})$. By the convexity of q_{ij}, Ξ is a convex set and there exists $\bar{\mathcal{B}} \subseteq \mathcal{A}$ such that

$$q_{ij}(x) = 0\ \forall x \in \Xi \quad \Longleftrightarrow \quad (i,j) \in \bar{\mathcal{B}}.$$

Since x^{true}, $(\bar{y}_{ij})_{(i,j)\in\mathcal{A}}$ is feasible for (6), any solution $x = (x_1, ..., x_m)$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ of (6) satisfies

$$\sum_{(i,j)\in\mathcal{A}} |y_{ij} - d_{ij}^2| \le \sum_{(i,j)\in\mathcal{A}} |\bar{y}_{ij} - d_{ij}^2| = \sum_{(i,j)\in\mathcal{A}} |\delta_{ij}|. \tag{16}$$

Since $\|x_i - x_j\|^2 \le y_{ij}$ so that $q_{ij}(x) \le y_{ij} - \bar{y}_{ij}$, this yields

$$\begin{aligned} \sum_{(i,j)\in\mathcal{A}} q_{ij}(x)_+ &\le \sum_{(i,j)\in\mathcal{A}} (y_{ij} - \bar{y}_{ij})_+ \\ &\le \sum_{(i,j)\in\mathcal{A}} |y_{ij} - \bar{y}_{ij}| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{(i,j)\in\mathcal{A}} (|y_{ij} - d_{ij}^2| + |d_{ij}^2 - \bar{y}_{ij}|) \\
&\leq 2 \sum_{(i,j)\in\mathcal{A}} |\delta_{ij}|, \qquad (17)
\end{aligned}
$$

where $\alpha_+ \stackrel{\text{def}}{=} \max\{0, \alpha\}$.

Using Proposition 3, we show below that if the distance error is small so the right-hand side of (17) is small, then $(x_i)_{i\in M_{\mathcal{B}}}$ in a solution of (6) has small error (in fact, proportional to the square root of distance error), where $\mathcal{B}$ is given by (9); see Propositions 5 and 6. Moreover, we can find $\mathcal{B}$ from an interior solution of (6); also see Section 9. Although there exist sensivity analysis results for convex quadratic inequalities of the form (15), the results either make the restrictive assumption that Ξ has nonempty interior [23] or prove a much weaker result that the solution error is proportional to the $2^{|\mathcal{A}|+1}$th root of distance error [33]; see discussions following Corollary 1. Existing sensitivity analysis results for general nonlinear programs make technical assumptions that either do not hold or are difficult to verify for (15); see, e.g., [12, Sections 5.2, 5.3].

For any $\mathcal{B} \subseteq \mathcal{A}$,

$$\Xi_{\mathcal{B}} \stackrel{\text{def}}{=} \{x \in \Xi \ : \ q_{ij}(x) = 0 \ \forall (i,j) \in \mathcal{B}\}.$$

For any nonempty closed subset Ξ' of Ξ, let

$$\text{dist}((x_1, ..., x_m), \Xi') \stackrel{\text{def}}{=} \min_{(\bar{x}_1,...,\bar{x}_m)\in\Xi'} \max_{i=1,...,m} \|x_i - \bar{x}_i\|.$$

Proposition 5 *(a) For each $\epsilon > 0$, there exists a scalar $\delta > 0$ such that*

$$\Xi_{\mathcal{B}} \neq \emptyset \qquad \text{and} \qquad \text{dist}\,(x, \Xi_{\mathcal{B}}) \leq \epsilon$$

whenever $\mathcal{B}$ satisfies (9), $x = (x_1, ..., x_m)$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ is a solution of (6), and $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \leq \delta$.

(b) There exists an $\bar{\epsilon} > 0$ such that, for each $0 < \epsilon < \bar{\epsilon}$, there exists a scalar $\delta > 0$ such that

$$\mathcal{B} \subseteq \bar{\mathcal{B}} \qquad \text{and} \qquad \|x_i - x_i^{\text{true}}\| \leq \epsilon \quad \forall i \in M_{\mathcal{B}},$$

whenever $\mathcal{B}$ satisfies (9), $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ is a solution of (6), and $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \leq \delta$.

Proof. (a) Fix any $\epsilon > 0$. If the desired δ does not exist, there would exist $(\delta_{ij}^t)_{(i,j)\in\mathcal{A}}$, $t = 1, 2, ...$, with $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}^t| \to 0$, and a $\mathcal{B} \subseteq \mathcal{A}$ satisfying (9) with

$d_{ij}^2 = \bar{y}_{ij} + \delta_{ij}^t$ for $(i,j) \in \mathcal{A}$ in (6), $t = 1, 2, ...$ In addition, for each $t = 1, 2, ...,$ there would exist a solution $x^t = (x_1^t, ..., x_m^t)$, $y^t = (y_{ij}^t)_{(i,j)\in\mathcal{A}}$, of (6) with $d_{ij}^2 = \bar{y}_{ij} + \delta_{ij}^t$, and yet the set $\Xi_{\mathcal{B}}$ is either empty or $\text{dist}(x^t, \Xi_{\mathcal{B}}) > \epsilon$ for all t.

We see from (16) that $\{y_{ij}^t\} \to \bar{y}_{ij}$ for all $(i,j) \in \mathcal{A}$. Also, we can assume without loss of generality that $\{x^t\}$ is bounded.[7] By passing to a subsequence if necessary, we assume that $\{x^t\}$ converges to some $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$. By (17),

$$\sum_{(i,j)\in\mathcal{A}} q_{ij}(x^t)_+ \le 2 \sum_{(i,j)\in\mathcal{A}} |\delta_{ij}^t|, \quad t = 1, 2, ...$$

This yields in the limit $\sum_{(i,j)\in\mathcal{A}} q_{ij}(\bar{x})_+ \le 0$, implying $\bar{x} \in \Xi$. Since $\mathcal{B}$ satisfies (9) with $d_{ij}^2 = \bar{y}_{ij} + \delta_{ij}^t$ for $(i,j) \in \mathcal{A}$ in (6), we also have

$$\|x_i^t - x_j^t\|^2 = y_{ij}^t \quad \forall (i,j) \in \mathcal{B},\ t = 1, 2, ... \quad (\text{with } x_i^t = x_i\ \forall i > m).$$

This yields in the limit

$$\|\bar{x}_i - \bar{x}_j\|^2 = \bar{y}_{ij} \quad \forall (i,j) \in \mathcal{B} \quad (\text{with } \bar{x}_i = x_i\ \forall i > m),$$

implying $\bar{x} \in \Xi_{\mathcal{B}}$. Moreover $\max_{i=1,...,m} \|x_i^t - \bar{x}_i\| \to 0$. This contradicts $\Xi_{\mathcal{B}} = \emptyset$ or $\text{dist}\,(x^t, \Xi_{\mathcal{B}}) > \epsilon$ for all t.

(b) Since each q_{ij} is convex, Ξ has an interior solution, i.e., $x' = (x_1', ..., x_m') \in \Xi$ satisfying

$$\|x_i' - x_j'\|^2 < \bar{y}_{ij} \quad \forall (i,j) \in \mathcal{A} \setminus \bar{\mathcal{B}}, \tag{18}$$

where we let $x_i' = x_i$ for $i > m$.

By (a), for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any interior solution $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ of (6) with $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \le \delta$, there exists $\bar{x} = (\bar{x}_1, ..., \bar{x}_m) \in \Xi_{\mathcal{B}}$ with $\max_{i=1,...,m} \|x_i - \bar{x}_i\| \le \epsilon$, where $\mathcal{B}$ satisfies (9). Let

$$\begin{aligned} d_i &= \bar{x}_i - x_i, \quad i = 1, ..., n, \\ d_i' &= x_i' - \bar{x}_i, \quad i = 1, ..., n, \end{aligned}$$

with $\bar{x}_i = x_i$ for $i > m$. Since $\bar{x}$ and x' are both in Ξ, we have $\|\bar{x}_i - \bar{x}_j\|^2 = \|x_i' - x_j'\|^2$ for all $(i,j) \in \bar{\mathcal{B}}$, which yields (also see the proof of Lemma 2)

$$d_i' - d_j' = 0 \quad \forall (i,j) \in \bar{\mathcal{B}}. \tag{19}$$

[7]Consider any connected component $\mathcal{C}$ of the graph $\mathcal{G} = (\{1, ..., n\}, \mathcal{A})$. If $\mathcal{C}$ contains an anchor index, then $\{x_i^t\}$ is bounded for all $i \le m$ in $\mathcal{C}$. If $\mathcal{C}$ does not contain an anchor index, then it can be seen that $\{\|x_i^t - x_j^t\|\}$ is bounded for all i and j in $\mathcal{C}$, so we can translate x_i^t for all i in $\mathcal{C}$ by the same displacement (thus preserving the distances between them) so that one of them is at the origin.

For each $(i,j) \in \mathcal{B}$, we have

$$\|x_i' - x_j'\|^2 = \bar{y}_{ij} - s_{ij}$$

for some $s_{ij} \geq 0$, or, equivalently,

$$\|d_i' - d_j' + \bar{x}_i - \bar{x}_j\|^2 = \|\bar{x}_i - \bar{x}_j\|^2 - s_{ij}.$$

Expanding the quadratics yields

$$2(\bar{x}_i - \bar{x}_j)^T(d_i' - d_j') = -\|d_i' - d_j'\|^2 - s_{ij}$$

or, equivalently

$$\begin{aligned} 2(x_i - x_j)^T(d_i' - d_j') &= -\|d_i' - d_j'\|^2 - s_{ij} - 2(d_i - d_j)^T(d_i' - d_j') \\ &\leq -\|d_i' - d_j'\|^2 - s_{ij} + 2\|d_i - d_j\|\|d_i' - d_j'\|. \end{aligned}$$

Since $s_{ij} \geq 0$, this implies that

$$(x_i - x_j)^T(d_i' - d_j') < 0 \quad \text{whenever} \quad \|d_i - d_j\| < \|d_i' - d_j'\|/2. \tag{20}$$

Let us choose

$$\epsilon < \frac{1}{4} \min_{(i,j)\in\mathcal{A}\setminus\bar{\mathcal{B}}} \{\sqrt{\bar{y}_{ij}} - \|x_i' - x_j'\|\}, \tag{21}$$

where the right-hand side is positive by (18). For each $(i,j) \in \mathcal{B}$ with $d_i' - d_j' \neq 0$, we have from (19) that $(i,j) \notin \bar{\mathcal{B}}$. Then, by using $\|d_i\| \leq \epsilon$ and (21), we have

$$\|d_i - d_j\| \leq 2\epsilon < \frac{\sqrt{\bar{y}_{ij}} - \|x_i' - x_j'\|}{2} \leq \frac{\|d_i' - d_j'\|}{2},$$

where the last inequality follows from

$$\|d_i' - d_j'\| = \|x_i' - x_j' - (\bar{x}_i - \bar{x}_j)\| \geq \|\bar{x}_i - \bar{x}_j\| - \|x_i' - x_j'\| = \sqrt{\bar{y}_{ij}} - \|x_i' - x_j'\|.$$

Then, by (20), $(x_i - x_j)^T(d_i' - d_j') < 0$. Thus, for each $(i,j) \in \mathcal{B}$ and for all $\alpha > 0$ sufficiently small, we have

$$\|(x_i + \alpha d_i') - (x_j + \alpha d_j')\|^2 \begin{cases} < \|x_i - x_j\|^2 = y_{ij} & \text{if } d_i' - d_j' \neq 0; \\ = \|x_i - x_j\|^2 = y_{ij} & \text{if } d_i' - d_j' = 0. \end{cases}$$

Also, for each $(i,j) \in \mathcal{A} \setminus \mathcal{B}$, since $\|x_i - x_j\|^2 < y_{ij}$, we have

$$\|(x_i + \alpha d_i') - (x_j + \alpha d_j')\|^2 < y_{ij}$$

for all $\alpha > 0$ sufficiently small. Thus, if $d'_i - d'_j \neq 0$ for some $(i,j) \in \mathcal{B}$, this would contradict the definition of $\mathcal{B}$. Hence $d'_i - d'_j = 0$ for all $(i,j) \in \mathcal{B}$. This in turn implies

$$\|x'_i - x'_j\|^2 = \|\bar{x}_i - \bar{x}_j\|^2 = \bar{y}_{ij} \quad \forall (i,j) \in \mathcal{B}.$$

Hence (18) yields that $\mathcal{B} \subseteq \bar{\mathcal{B}}$.

Since $\bar{x} \in \Xi$, by applying Proposition 3(c) we have

$$\bar{x}_i = x_i^{\text{true}} \quad \forall i \in M_{\bar{\mathcal{B}}}.$$

Since $\mathcal{B} \subseteq \bar{\mathcal{B}}$, $M_{\mathcal{B}} \subseteq M_{\bar{\mathcal{B}}}$. Thus

$$\|x_i - x_i^{\text{true}}\| = \|x_i - \bar{x}_i\| \le \epsilon \quad \forall i \in M_{\mathcal{B}}.$$

■

From the proof of Proposition 5(b) we see that we can take $\bar{\epsilon}$ to be the right-hand side of (21), maximized over all $(x'_1, ..., x'_m) \in \Xi$. Proposition 5(b) says that if the distance error is not too large, then the error in the position of those sensors indexed by $M_{\mathcal{B}}$ is also not too large. However, it does not say how fast the position error grows with the distance error. We show below that the position error grows at most like the square root of the distance error.

We say that $\mathcal{B} \subseteq \bar{\mathcal{B}}$ is *active with respect to* $\mathcal{M} \subseteq \{1, ..., m\}$ if

$$q_{ij}(x) \le 0 \;\; \forall (i,j) \in \mathcal{B}, \quad x_i = x_i^{\text{true}} \;\; \forall i \notin \mathcal{M} \quad \Longrightarrow \quad q_{ij}(x) = 0 \;\; \forall (i,j) \in \mathcal{B}.$$

We say that $\mathcal{B}$ is *minimally active with respect to* $\mathcal{M}$ if there is no proper subset of $\mathcal{B}$ that is active with respect to $\mathcal{M}$.

Proposition 6 *There exist a constant* $K > 0$ *such that*

$$\max_{i \in M_{\bar{\mathcal{B}}}} \|x_i - x_i^{\text{true}}\| \le K \max_{(i,j) \in \bar{\mathcal{B}}} q_{ij}(x)_+^{1/2} \quad \forall x = (x_1, ..., x_m).$$

Proof. If $\bar{\mathcal{B}} = \emptyset$, then our proof is complete. Otherwise, by its definition, $\bar{\mathcal{B}}$ is active with respect to $\{1, ..., m\}$. Then, there exists nonempty $\mathcal{B}_1 \subseteq \bar{\mathcal{B}}$ that is minimally active with respect to $\{1, ..., m\}$. By using Gordan's theorem as in the proof of [33, Theorem 3.1], there exist $\lambda_{ij} > 0$, $(i,j) \in \mathcal{B}_1$, satisfying

$$\sum_{(i,j) \in \mathcal{B}_1} \nabla q_{ij}(x^{\text{true}}) \lambda_{ij} = 0.^8 \tag{22}$$

[8] Why? Since $\mathcal{B}_1$ is active with respect to $\{1, ..., m\}$ and $q_{ij}(x^{\text{true}}) = 0$ for all $(i,j) \in \mathcal{B}_1$, the

Fix any $x = (x_1, ..., x_m)$. For each $(i,j) \in \mathcal{B}_1$, we have from $q_{ij}(x^{\text{true}}) = 0$ that

$$q_{ij}(x) = \nabla q_{ij}(x^{\text{true}})^T(x - x^{\text{true}}) + \|x_i - x_j - (x_i^{\text{true}} - x_j^{\text{true}})\|^2.$$

Multiplying both sides by λ_{ij} and summing over all $(i,j) \in \mathcal{B}_1$ and using (22) yields

$$\sum_{(i,j)\in\mathcal{B}_1} q_{ij}(x)\lambda_{ij} = \sum_{(i,j)\in\mathcal{B}_1} \|x_i - x_j - (x_i^{\text{true}} - x_j^{\text{true}})\|^2\lambda_{ij}.$$

Thus

$$\|x_i - x_j - (x_i^{\text{true}} - x_j^{\text{true}})\|^2\lambda_{ij} \le \sum_{(i,j)\in\mathcal{B}_1} \lambda_{ij} \cdot \max_{(i,j)\in\mathcal{B}_1} q_{ij}(x)_+ \quad \forall (i,j) \in \mathcal{B}_1.$$

This in turn implies

$$\|x_i - x_j - (x_i^{\text{true}} - x_j^{\text{true}})\| \le C_1 \max_{(i,j)\in\mathcal{B}_1} q_{ij}(x)_+^{1/2} \quad \forall (i,j) \in \mathcal{B}_1, \tag{23}$$

where

$$C_1 \stackrel{\text{def}}{=} \left(\frac{\sum_{(i,j)\in\mathcal{B}_1} \lambda_{ij}}{\min_{(i,j)\in\mathcal{B}_1} \lambda_{ij}}\right)^{1/2}.$$

We can then apply Proposition 3(b) with d_{ij}, $\mathcal{A}$, $\mathcal{B}$, $\{1, ..., m\}$ replaced by, respectively, $\sqrt{y_{ij}}$, $\mathcal{B}_1$, $\mathcal{B}_1$, $\mathcal{M}_1 \stackrel{\text{def}}{=} \{i \in \{1, ..., m\} : N_{\mathcal{B}_1}(i) \neq \emptyset\} = M_{\mathcal{B}_1}$. This yields that each connected component of the graph $\mathcal{G}_1 \stackrel{\text{def}}{=} (\mathcal{M}_1 \cup \{m+1, ..., n\}, \mathcal{B}_1)$ contains an anchor index $j \in \{m+1, ..., n\}$. (In fact, this graph is connected since $\mathcal{B}_1$ is minimally active with respect to $\mathcal{M}_1$.) Then, for each $i \in N_{\mathcal{B}_1}(j)$, we have from (23) and $x_j = x_j^{\text{true}}$ that

$$\|x_i - x_i^{\text{true}}\| \le C_1 \max_{(i,j)\in\mathcal{B}_1} q_{ij}(x)_+^{1/2}.$$

Continuing this argument with each neighbor of i in $\mathcal{G}_1$, and so on, we obtain that

$$\|x_i - x_i^{\text{true}}\| \le C_1 D_1 \max_{(i,j)\in\mathcal{B}_1} q_{ij}(x)_+^{1/2} \quad \forall i \in \mathcal{M}_1, \tag{24}$$

---

linear system $\nabla q_{ij}(x^{\text{true}})^T d < 0$, $(i,j) \in \mathcal{B}_1$, is infeasible. By Gordan's theorem [13, page 23], there exist $\lambda_{ij} \ge 0$ for $(i,j) \in \mathcal{B}_1$, not all zero, satisfying (22). Let $\hat{\mathcal{B}}_1 \stackrel{\text{def}}{=} \{(i,j) \in \mathcal{B}_1 : \lambda_{ij} > 0\}$. By Gordan's theorem again, the linear system $\nabla q_{ij}(x^{\text{true}})^T d < 0$, $(i,j) \in \hat{\mathcal{B}}_1$, is infeasible. If $\hat{\mathcal{B}}_1 \neq \mathcal{B}_1$, then the quadratic system $q_{ij}(x) < 0$, $(i,j) \in \hat{\mathcal{B}}_1$, would be feasible. (Otherwise there would exist a nonempty $\tilde{\mathcal{B}}_1 \subseteq \hat{\mathcal{B}}_1$ such that $q_{ij}(x) = 0$, $(i,j) \in \tilde{\mathcal{B}}_1$, whenever $q_{ij}(x) \le 0$, $(i,j) \in \hat{\mathcal{B}}_1$. Choose $\tilde{\mathcal{B}}_1$ to be maximal. Then $\tilde{\mathcal{B}}_1$ would be active with respect to $\{1, ..., m\}$, contradicting $\mathcal{B}_1$ being minimally active with respect to $\{1, ..., m\}$.) Then the linear system $\nabla q_{ij}(x^{\text{true}})^T d < 0$, $(i,j) \in \hat{\mathcal{B}}_1$, would be feasible, a contradiction. Thus $\hat{\mathcal{B}}_1 = \mathcal{B}_1$.

where $D_1 \stackrel{\text{def}}{=} \max_{i\in\mathcal{M}_1} \min_{j\notin\mathcal{M}_1} (\text{minimum \# edges in a path between } i \text{ and } j \text{ in } \mathcal{G}_1)$.

If $\bar{\mathcal{B}} = \mathcal{B}_1$, then our proof is complete. Otherwise, $\bar{\mathcal{B}} \setminus \mathcal{B}_1$ is active with respect to $\{1, ..., m\} \setminus \mathcal{M}_1$. Then, there exists nonempty $\mathcal{B}_2 \subseteq \bar{\mathcal{B}} \setminus \mathcal{B}_1$ that is minimally active with respect to $\{1, ..., m\} \setminus \mathcal{M}_1$. Repeating the above argument, we obtain that

$$\|x_i - x_j - (x_i^{\text{true}} - x_j^{\text{true}})\| \leq C_2 \max_{(i,j)\in\mathcal{B}_2} q_{ij}(x)_+^{1/2} \quad \forall (i,j) \in \mathcal{B}_2, \tag{25}$$

with C_2 defined analogously as C_1 and with $x_i = x_i^{\text{true}}$ for $i \in \mathcal{M}_1$. We can then apply Proposition 3(b) with d_{ij}, $\mathcal{A}$, $\mathcal{B}$, $\{1, ..., m\}$ replaced by, respectively, $\sqrt{\bar{y}_{ij}}$, $\mathcal{B}_2$, $\mathcal{B}_2$, $\mathcal{M}_2 \stackrel{\text{def}}{=} \{i \in \{1, ..., m\} \setminus \mathcal{M}_1 : N_{\mathcal{B}_2}(i) \neq \emptyset\}$. This yields that each connected component of the graph $\mathcal{G}_2 \stackrel{\text{def}}{=} (\mathcal{M}_1 \cup \mathcal{M}_2 \cup \{m+1, ..., n\}, \mathcal{B}_2)$ contains a node $j \in \mathcal{M}_1 \cup \{m+1, ..., n\}$. Then, for each $i \in N_{\mathcal{B}_2}(j)$, we have from (25) and (24) that

$$\|x_i - x_i^{\text{true}}\| \leq C_1 D_1 \max_{(i,j)\in\mathcal{B}_1} q_{ij}(x)_+^{1/2} + C_2 \max_{(i,j)\in\mathcal{B}_2} q_{ij}(x)_+^{1/2}.$$

Continuing this argument with each neighbor of i in $\mathcal{G}_2$, and so on, we obtain that

$$\|x_i - x_i^{\text{true}}\| \leq C_1 D_1 \max_{(i,j)\in\mathcal{B}_1} q_{ij}(x)_+^{1/2} + C_2 D_2 \max_{(i,j)\in\mathcal{B}_2} q_{ij}(x)_+^{1/2} \quad \forall i \in \mathcal{M}_2,$$

with D_2 defined analogously as D_1.

Continuing the above argument inductively completes the proof. ■

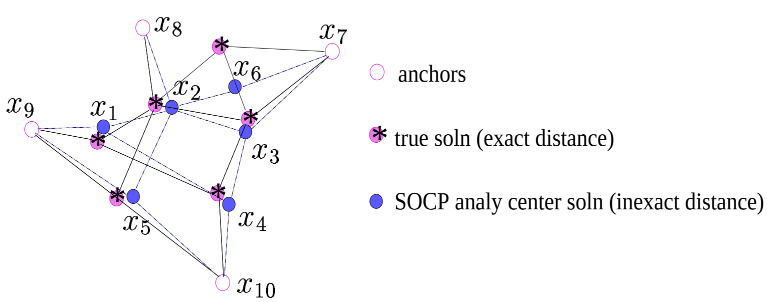

Figure 6: In this example, $\mathcal{M}_1 = \{1, 2, 3, 4\}$, $\mathcal{B}_1 = \{(1, 2), (1, 4), (1, 9), (2, 3), (2, 8), (3, 7), (3, 4), (4, 10)\}$, $\mathcal{M}_2 = \{5\}$, $\mathcal{B}_2 = \{(5, 2), (5, 9), (5, 10)\}$, and $\bar{\mathcal{B}} = \mathcal{B}_1 \cup \mathcal{B}_2$. Removing a point indexed by $\mathcal{M}_1$ affects points indexed by $\mathcal{M}_2$ but not conversely.

The proof of Proposition 6 shows that the points indexed by $\mathcal{M}_1$, which are the sensors 'nearest' to the anchors, are the least sensitive to distance measurement errors. An important (and intuitively reasonable) result shown by Proposition 6 is that the errors affect the sensor positions additively as they percolate to $\mathcal{M}_2$, and so on; see Figure 6 for an illustrative example.

Corollary 1 *There exist a constant $L > 0$ such that*

$$\mathrm{dist}(x, \Xi) \leq L \max_{(i,j)\in\mathcal{A}\setminus\bar{\mathcal{B}}} q_{ij}(x)_+ + LK \max_{(i,j)\in\bar{\mathcal{B}}} q_{ij}(x)_+^{1/2} \quad \forall x = (x_1, ..., x_m),$$

where K is defined as in Proposition 6.

Proof. Consider the system of convex quadratic inequalities and linear equations in $x = (x_1, ..., x_m)$:

$$q_{ij}(x) \leq 0 \ \ \forall (i,j) \in \mathcal{A} \setminus \bar{\mathcal{B}}, \qquad x_i = x_i^{\text{true}} \ \ \forall i \in M_{\bar{\mathcal{B}}}.$$

By applying Proposition 3(c) with d_{ij} replaced by $\sqrt{\bar{y}_{ij}}$, we see that Ξ equals the solution set of this system. Moreover, each interior solution of Ξ satisfies the quadratic inequalities strictly. Thus, applying a result of Luo and Luo [23], there exists $L > 0$ such that

$$\mathrm{dist}(x, \Xi) \leq L \max_{(i,j)\in\mathcal{A}\setminus\bar{\mathcal{B}}} q_{ij}(x)_+ + L \max_{i\in M_{\bar{\mathcal{B}}}} \|x_i - x_i^{\text{true}}\| \quad \forall x = (x_1, ..., x_m).$$

Using Proposition 6 to bound the second term on the right-hand side completes the proof. ■

The error bound in Corollary 1 sharpens the Hölderian error bound of Wang and Pang [33] for general convex quadratic inequalities. In particular, a direct application of the result in [33] yields the existence of $\tau > 0$ and integer $\ell \leq |\mathcal{A}| + 1$ such that

$$\mathrm{dist}(x, \Xi) \leq \kappa \max_{(i,j)\in\mathcal{A}} \left(q_{ij}(x)_+ + q_{ij}(x)_+^{1/2^\ell} \right) \quad \forall x = (x_1, ..., x_m).$$

An example in [33] shows that, for general convex quadratic functions q_{ij}, $\ell = |\mathcal{A}|$ is possible. Corollary 1 in effect shows that we can take $\ell = 1$ in the special case where each q_{ij} has the form (14). It is an open question whether the active set index $\bar{\mathcal{B}}$ can be identified using the Lipschitzian error bound. The difficulty lies in that $\bar{y}_{ij}$ is unknown, so that $q_{ij}(x)$ cannot be directly evaluated.

Lastly, we show that the x-component of the analytic center solution of (6) converges to the analytic center solution of Ξ as the distance error goes to zero.

Proposition 7 *Under Assumption 1, let* $x^c = (x_1^c, ..., x_m^c)$, $(y_{ij}^c)_{(i,j)\in\mathcal{A}}$ *be the analytic center solution of* (6). *As* $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \to 0$, x^c *converges to the analytic center* $\bar{x}^c = (\bar{x}_1^c, ..., \bar{x}_m^c)$ *of* Ξ.

Proof. For $i = 1, ..., m$, let

$$\tilde{x}_i \stackrel{\text{def}}{=} \begin{cases} x_i^c & \text{if } i \in M_{\bar{\mathcal{B}}}; \\ \bar{x}_i^c & \text{if } i \notin M_{\bar{\mathcal{B}}}. \end{cases}$$

By $\bar{x}^c \in \Xi$ and Proposition 6, we have $\bar{x}_i^c = x_i^{\text{true}}$ for all $i \in M_{\bar{\mathcal{B}}}$. Let

$$\rho \stackrel{\text{def}}{=} - \max_{(i,j)\in\mathcal{A}\setminus\bar{\mathcal{B}}} q_{ij}(\bar{x}^c) > 0. \tag{26}$$

Suppose $\sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \le \delta$ for some $\delta > 0$. By (16), (17), and Proposition 6, we have

$$\max_{(i,j)\in\mathcal{A}} |y_{ij}^c - \bar{y}_{ij}| \le \delta, \qquad \max_{i\in M_{\bar{\mathcal{B}}}} \|x_i^c - \bar{x}_i^c\| \le K\sqrt{2\delta}. \tag{27}$$

For each $(i,j) \in \mathcal{A}$, consider the following three cases: (i) If $i \in M_{\bar{\mathcal{B}}}$ and $j \in M_{\bar{\mathcal{B}}}$, then

$$\|\tilde{x}_i - \tilde{x}_j\|^2 = \|x_i^c - x_j^c\|^2 \le y_{ij}^c.$$

(ii) If $i \notin M_{\bar{\mathcal{B}}}$ and $j \notin M_{\bar{\mathcal{B}}}$, then $(i,j) \notin \bar{\mathcal{B}}$ and hence (26), (27) yield

$$\|\tilde{x}_i - \tilde{x}_j\|^2 = \|\bar{x}_i^c - \bar{x}_j^c\|^2 \le \bar{y}_{ij} - \rho \le y_{ij}^c + \delta - \rho.$$

(iii) If $i \in M_{\bar{\mathcal{B}}}$ and $j \notin M_{\bar{\mathcal{B}}}$, then $(i,j) \notin \bar{\mathcal{B}}$ and hence (26), (27) yield

$$\begin{aligned} \|\tilde{x}_i - \tilde{x}_j\|^2 &= \|x_i^c - \bar{x}_j^c\|^2 \\ &\le (\|x_i^c - \bar{x}_i^c\| + \|\bar{x}_i^c - \bar{x}_j^c\|)^2 \\ &\le (K\sqrt{2\delta} + \|\bar{x}_i^c - \bar{x}_j^c\|)^2 \\ &\le 2K^2\delta + 2K\sqrt{2\delta}\sqrt{\bar{y}_{ij}} + \bar{y}_{ij} - \rho \\ &\le 2K^2\delta + 2K\sqrt{2\delta}\sqrt{\bar{y}_{ij}} + y_{ij}^c + \delta - \rho. \end{aligned}$$

Notice that $(i,j) = (j,i)$, so the case of $i \notin M_{\bar{\mathcal{B}}}$ and $j \in M_{\bar{\mathcal{B}}}$ is covered by case (iii). Since $x^c = (x_1^c, ..., x_m^c)$, $(y_{ij}^c)_{(i,j)\in\mathcal{A}}$ is a solution of (6) and $\rho > 0$, the above analysis shows that, for δ sufficiently small, $\tilde{x}_1, ..., \tilde{x}_m$, $(y_{ij}^c)_{(i,j)\in\mathcal{A}}$ is an interior solution of (6). Since $x^c = (x_1^c, ..., x_m^c)$, $(y_{ij}^c)_{(i,j)\in\mathcal{A}}$ is the analytic center solution of (6), this implies

$$\sum_{(i,j)\in\mathcal{A}\setminus\mathcal{B}} \log\left(y_{ij}^c - \|x_i^c - x_j^c\|^2\right) \ge \sum_{(i,j)\in\mathcal{A}\setminus\mathcal{B}} \log\left(y_{ij}^c - \|\tilde{x}_i - \tilde{x}_j\|^2\right),$$

where $\mathcal{B}$ satisfies (9) (with $x_i^c = \tilde{x}_i = x_i$ for $i > m$). By Proposition 5(b), we have $\mathcal{B} \subseteq \bar{\mathcal{B}}$ for δ sufficiently small. For $(i,j) \in \bar{\mathcal{B}}$, since $i \in M_{\bar{\mathcal{B}}}$ and $j \in M_{\bar{\mathcal{B}}}$, we have $\|\tilde{x}_i - \tilde{x}_j\|^2 = \|x_i^c - x_j^c\|^2$. Thus we further have

$$\sum_{(i,j)\in\mathcal{A}\backslash\bar{\mathcal{B}}} \log\left(y_{ij}^c - \|x_i^c - x_j^c\|^2\right) \geq \sum_{(i,j)\in\mathcal{A}\backslash\bar{\mathcal{B}}} \log\left(y_{ij}^c - \|\tilde{x}_i - \tilde{x}_j\|^2\right).$$

Taking $\delta \to 0$, we have from (27) that $y_{ij}^c \to \bar{y}_{ij}$ for all $(i,j) \in \mathcal{A}\backslash\bar{\mathcal{B}}$ and $\tilde{x}_i = x_i^c \to \bar{x}_i^c$ for all $i \in M_{\bar{\mathcal{B}}}$. Also, $\tilde{x}_i = \bar{x}_i^c$ for all $i \notin M_{\bar{\mathcal{B}}}$. Thus, we obtain in the limit that any cluster point $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ of x^c (which exists since x^c is uniformly bounded by Assumption 1) belongs to Ξ (using (17) and Corollary 1) and satisfies

$$\sum_{(i,j)\in\mathcal{A}\backslash\bar{\mathcal{B}}} \log\left(\bar{y}_{ij} - \|\bar{x}_i - \bar{x}_j\|^2\right) \geq \sum_{(i,j)\in\mathcal{A}\backslash\bar{\mathcal{B}}} \log\left(\bar{y}_{ij} - \|\bar{x}_i^c - \bar{x}_j^c\|^2\right)$$

(with $\bar{x}_i = \bar{x}_i^c = x_i$ for $i > m$). This shows that $\bar{x}$ is an analytic center of Ξ, so that in fact $\bar{x} = \bar{x}^c$. ∎

It is an open question whether the results of this section extend to the SDP relaxation (3).

8 Methods for Solving the SOCP Relaxation

We saw in previous sections that the SOCP relaxation (6), though weaker than the SDP relaxation (3), has the advantage of a smaller problem size and its interior solutions are useful for identifying sensors that are accurately positioned. What method would best solve (6) and, in particular, find an interior solution? Primal-dual interior-point method can find an analytic center solution of SOCP with good accuracy. However, as we will see in Section 9, applying an interior-point method directly to (7) can be slow, due to the large size of the SOCP. We tried adapting the distributed SDP method of Biswas and Ye [11] to the SOCP relaxation. However, possibly due to the weaker SOCP relaxation, the resulting distributed SOCP method was not satisfactory. Further studies are needed. Below we describe a third method, based on smoothing and (block) coordinate gradient descent, which can find an interior solution faster, as we will see in Section 9. This method has the nice feature that its computations easily distribute over many processors in parallel.

First, we observe that, for any $d \in \Re$,

$$\min_{y\geq z} |y - d^2| = [z - d^2]_+ \quad \forall z \in \Re,$$

where $[t]_+ = \max\{0, t\}$. Thus, we can rewrite the SOCP relaxation (6) as the unconstrained optimization problem:

$$\upsilon_{\text{socp}} = \min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} [\|x_i - x_j\|^2 - d_{ij}^2]_+. \tag{28}$$

The objective function is convex, but nonsmooth due to the term $\max\{0,\cdot\}$. It is well known in the context of complementarity problems that a smoothing approach can be effective in handling this type of nonsmoothness; see [14, 18] and references therein. In particular, for any function $h : \Re \to \Re$ that is smooth, convex, and satisfying $\lim_{t\to-\infty} h(t) = \lim_{t\to\infty} h(t) - t = 0$, we have that

$$\lim_{\mu\downarrow 0} \mu h(t/\mu) \;=\; [t]_+.$$

Thus, for $\mu > 0$ and small, we have $\mu h(t/\mu) \approx [t]_+$. In our numerical tests, we use a popular choice of h due to Chen, Harker, Kanzow and Smale:

$$h(t) = ((t^2+4)^{1/2} + t)/2.$$

Thus, the nonsmooth problem (28) is approximated by the smooth problem, parameterized by $\mu > 0$:

$$\min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \mu h\left(\frac{\|x_i - x_j\|^2 - d_{ij}^2}{\mu}\right). \tag{29}$$

For each $\mu > 0$, the objective function is smooth, convex, and as $\mu \to 0$, any cluster point of the solution of (29) is a solution of (28).

Since we wish to find an interior solution, following the interior-point approach, we add a log-barrier term and consider

$$\min_{y\geq z} |y - d^2| - \mu\log(y - z) = [z + \mu - d^2]_+ - \mu\log\left(\mu + [d^2 - z - \mu]_+\right) \quad \forall z \in \Re.$$

This is a convex function of z. Upon smoothing $[\cdot]_+$ by $\mu h(\cdot/\mu)$, we obtain the corresponding smooth barrier problem:

$$\min_{x=(x_1,\ldots,x_m)} f_\mu(x) \;\stackrel{\text{def}}{=}\; \sum_{(i,j)\in\mathcal{A}} \mu h\left(\frac{t_{ij}}{\mu}\right) - \mu\log\left(1 + h\left(\frac{-t_{ij}}{\mu}\right)\right)_{t_{ij}=\|x_i-x_j\|^2+\mu-d_{ij}^2}. \tag{30}$$

Here, for simplicity, we used the same parameter μ for the log-barrier and the smoothing function. Notice that the objective function f_μ is partially separable, being a sum of functions each of which depends only on the difference of neighboring points. This suggests that a block-coordinate descent approach may be efficient for solving (30), whereby at each iteration the objective function f_μ is minimized with respect to x_i, for some $i \in \{1, ..., m\}$, while the other points are held fixed at their current value. Since exact minimization is expensive, the minimization is done only inexactly. In particular, we minimize a quadratic approximation of f_μ with respect to x_i to generate the descent direction d_i and then minimize f_μ inexactly along d_i using an Armijo stepsize rule [6]. We decrease μ whenever $\|\nabla f_\mu(x)\|$ is small relative to μ. The method, which we refer to as the smoothing coordinate gradient descent (SCGD) method, is described more precisely below.

0. Initialize $\mu > 0$ and $x = (x_1, ..., x_m)$. Choose $\mu^{\text{final}} > 0$ and a continous function $\psi : (0, \infty) \to (0, \infty)$ satisfying $\lim_{\mu \downarrow 0} \psi(\mu) = 0$. Choose stepsize parameters $0 < \beta < 1$, $0 < \sigma < \frac{1}{2}$. Go to Step 1.

1. If there exists an $i \in \{1, ..., m\}$ satisfying $\|\nabla_{x_i} f_\mu(x)\| > \psi(\mu)$, then set

$$d_i = -[H_i]^{-1} \nabla_{x_i} f_\mu(x),$$

update

$$x_i^{\text{new}} = x_i + \alpha d_i,$$

and repeat Step 1, where $H_i \in \Re^{d \times d}$ is a user-chosen symmetric positive definite matrix, and α is the largest element of $\{1, \beta, (\beta)^2, ...\}$ satisfying

$$f_\mu(x_1, ..., x_i + \alpha d_i, ..., x_m) \leq f_\mu(x) - \alpha \sigma d_i^T \nabla_{x_i} f_\mu(x).$$

Otherwise, go to Step 2.

2. If $\mu \leq \mu^{\text{final}}$, then stop. Otherwise decrease μ, and return to Step 1.

The SCGD method is highly parallelizable since updating x_i only requires knowledge of neighboring points $\{x_j\}_{j \in N_{\mathcal{A}}(i)}$, so non-neighbors can update their positions simultaneously. Thus the computation can be distributed over the sensors, with each sensor communicating with its neighbors only.

In our current implementation of the SCGD method, we choose

$$H_i = \nabla^2_{x_i x_i} f_\mu(x),$$

which can be verified to be positive definite. Both $\nabla_{x_i} f_\mu(x)$ and H_i can be efficiently evaluated using network data structure for $\mathcal{G}$.

9 Numerical Simulation Results

In this section, we present simulation results based on the SOCP relaxations (6) and (7). Following Biswas and Ye [10, 11], we generate the true positions of the points $x_1^{\text{true}}, ..., x_n^{\text{true}}$ independently according to a uniform distribution on the unit square $[-.5, .5]^2$, and set $m = 0.9n$ (i.e., 10% of the points are anchors), $\mathcal{A} = \{(i,j) : \|x_i^{\text{true}} - x_j^{\text{true}}\| < radiorange\}$, and

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot |1 + \epsilon_{ij} \cdot noisyfactor| \quad \forall (i,j) \in \mathcal{A},$$

where ϵ_{ij} is a random variable representing measurement noise, and $radiorange \in (0,1)$, $noisyfactor \in [0,1]$. Similar to [10, 11], each ϵ_{ij} is normally distributed, and we use the parameter values of $noisyfactor = 0, .001, .01$ and $radiorange = .06$ for $n = 1000, 2000$, $radiorange = .035$ for $n = 4000$.[9]

P	n	$noisyfactor$	$\lvert\mathcal{A}\rvert$	**SOCP dim**
1	1000	0	5318	21472×28590
2	1000	.001	5068	20472×27340
3	1000	.01	5276	21304×28380
4	2000	0	21010	84440×109050
5	2000	.001	20859	83836×108295
6	2000	.01	20859	83836×108295
7	4000	0	29322	118088×154610
8	4000	.001	29322	118088×154610
9	4000	.01	29322	118088×154610

Table 1: Input parameters for the test problems and the corresponding SOCP (7) dimensions. ($radiorange = .06$ for $n = 1000, 2000$, $radiorange = .035$ for $n = 4000$.)

We wrote two codes to compute an interior solution of the SOCP relaxation (6). The first code is written in Matlab and calls SeDuMi (Version 1.05) by Jos Sturm [32], a C implementation of a predictor-corrector primal-dual interior-point method for solving SDP/SOCP, to find an interior solution of (7).[10] The second code is written in Fortran-77 and implements the SCGD method described in Section 8,

[9] Other noise models can also be used. We use the model from [10, 11] to facilitate comparison with previous work.

[10] We also tried a new version 1.1 of SeDuMi, maintained by the Advanced Optimization Laboratory at McMaster University, but it gave wrong answers on our SOCP problems.

whereby we initialize $\mu = 10^{-5}$, and $x_i = x_i^{\text{true}} + \Delta_i$, with the components of Δ_i randomly generated from the square $[-.2, .2]^2$. We choose

$$\mu^{\text{final}} = 10^{-9}, \quad \psi(\mu) = \max\{10\mu, 10^{-7}\}, \quad \beta = 0.5, \quad \sigma = 0.1.$$

We choose i in Step 1 in a cyclic order, and we decrease μ by a factor of 10 in Step 2. These choices were made with little experimentation. Conceivably the performance can be improved with more judicious choices (e.g., replacing the cyclic order by a queue, as in the Bellman-Ford method for shortest path [5, Section 2.4]).

For the interior solution $x_1, ..., x_m$, $(y_{ij})_{(i,j)\in\mathcal{A}}$ found, the position of the ith sensor is judged to be uniquely positioned (using Propositions 5 and 6) if there exists a $j \in N_{\mathcal{A}}(i)$ satisfying

$$\left| \|x_i - x_j\|^2 - y_{ij} \right| \leq 10^{-7} d_{ij}$$

(with $x_i = x_i^{\text{true}}$ for $i > m$). In what follows, m_{up} is the number of sensors that are judged to be uniquely positioned by this test. To check the accuracy of these sensors, we compute the maximum error between their computed positions and their true positions:

$$err_{\text{up}} = \max_{i \text{ is uniquely positioned}} \|x_i - x_i^{\text{true}}\|.$$

For comparison, we also compute the maximum error between computed positions and true positions of all sensors:

$$err = \max_{i=1,...,m} \|x_i - x_i^{\text{true}}\|.$$

Table 2 reports the iteration count, cpu time, the final SOCP objective value, m_{up}, err_{up}, err for the two codes. We see from Table 2 that SCGD is consistently faster than SeDuMi, though it uses more iterations. SCGD is more sensitive to $noisyfactor$ than SeDuMi. We do not have a good explanation for this yet. On the other hand, the cpu times for SCGD are still high on problems with higher distance errors. These times can conceivably be further reduced by fine tuning the algorithm parameters and/or distributing the computations over multiple processors. This is a topic for future research. One idea would be to adapt the approach in [22] by terminating the SOCP method early and then applying a local descent method to the original problem (1) to refine the solution. Or we can find new methods to solve the SOCP (6), as is discussed in Section 12.

We also see from Table 2 that err_{up} is much smaller than err and decreases with $noisyfactor$, which corroborates Propositions 5 and 6. For the larger problems 4–9, m_{up} is large (80-90% of m), showing that a large number of sensors are accurately positioned (with error err_{up}) by the interior solutions found. Of course, m_{up} depends

	SeDuMi	**SCGD**
P	**iter/cpu/obj**/m_{up}/err_{up}/err	**iter/cpu/obj**/m_{up}/err_{up}/err
1	22/3.6/7.8e-6/402/7.2e-4/.11	1803189/.2/2.1e-06/357/3.8e-5/.11
2	22/3.2/9.1e-4/473/1.8e-3/.17	3523150/.4/9.1e-4/442/1.5e-3/.17
3	22/3.9/1.0e-2/554/1.5e-2/.17	14381707/1.6/1.0e-2/518/1.1e-2/.17
4	25/176.7/6.0e-6/1534/4.3e-4/.058	3482697/0.8/1.5e-5/1541/3.3e-4/.077
5	25/208.6/1.1e-2/1464/3.6e-3/.088	7894112/1.8/1.1e-2/1466/3.6e-3/.090
6	17/161.8/1.30/1710/5.1e-2/.093	12113931/2.9/1.30/1707/5.1e-2/.094
7	27/202.5/4.5e-5/2851/4.0e-4/.099	9345127/1.6/2.1e-5/2844/3.2e-4/.099
8	25/193.8/4.7e-3/2938/3.2e-3/.099	29304035/5.1/4.7e-3/2894/3.0e-3/.099
9	25/196.3/4.9e-2/3073/1.0e-2/.099	34650852/6.1/4.9e-2/3020/9.1e-3/.099

Table 2: Times to solve SOCP relaxation and accuracy of sensors judged to be uniquely positioned. cpu times are in minutes on an HP DL360 workstation, running Matlab (Version 7.0) and Gnu F-77 compiler (Version 3.2.57) under Red Hat Linux 3.5.

on *radiorange* also. If *radiorange* is small, so the graph $\mathcal{G}$ has low connectivity, then m_{up} would be small.

The true sensor positions and the computed positions for problems 1 and 3 are shown in Figure 7. Notice the close match of sensors whose true positions lie in the convex hull of "nearby" anchors. The positions are least accurate on the boundary, as we expect. The computed position of each sensor lies in the convex hull of its neighbors, corroborating Proposition 4.

At the suggestion of a referee, we also compare the SOCP solutions with solutions of the SDP (3) when n, m are small, *noisyfactor* is large, and ϵ_{ij}s have different distributions. In particular, we apply SeDuMi to compute a solution $Z = \begin{bmatrix} Y & X^T \\ X & I_d \end{bmatrix}$ of the SDP (4), which is likely to be the analytic center solution, and extract the sensor positions $X = [x_1 \ \cdots \ x_m]$. To compare with existing work, we follow a recent study by Biswas et al. [9] of SDP solution under noisy distance measurements and choose $n = 64$, $m = 60$, $radiorange = 0.3$, with 4 anchors at $(\underline{+}.45, \underline{+}.45)$. We also choose $noisfyfactor \in \{0.1, 0.2\}$ and choose ϵ_{ij}s to be either (i) normally distributed or (ii) uniformly distributed on $[-\sqrt{3}, \sqrt{3}]$ or (iii) distributed as an additive-Gaussian where, with probability $\frac{1}{2}$, is normally distributed with mean 1 (otherwise with mean -1). Thus ϵ_{ij} has mean 0 and variance 1. Table 3 reports the final objective value for SDP and SOCP, as well as

$$err_{\text{rms}} = \sum_{i=1}^{m} \|x_i - x_i^{\text{true}}\|^2$$

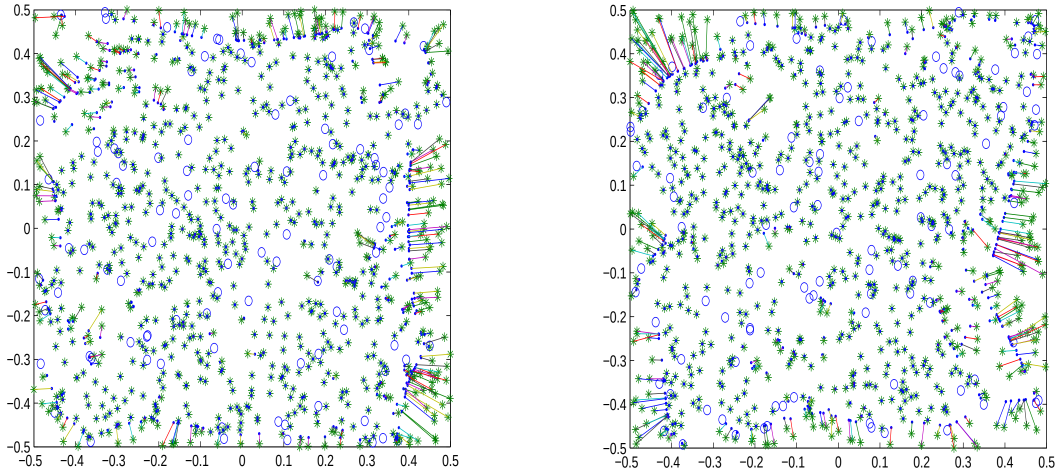

Figure 7: The left figure shows the anchors ("∘") and the analytic center solution found by SCGD for problem 1 ($n = 1000$). Each sensor position found ("·") is joined to its true position ("$*$") by a line. The right figure shows the same information for problem 3.

and $m_{\rm up}$, $err_{\rm up}$ for SOCP.

We see from Table 3 that objective value is higher and $err_{\rm rms}$ is lower for SDP solution than for SOCP solution, corroborating Proposition 1. The $err_{\rm rms}$ is higher for both SDP and SOCP solutions under additive Gaussian noise. We do not yet have a good explanation for this. Figures 8-10 display the SDP solutions and SOCP solutions for the case of $noisyfactor = .2$. These results suggest that, for small randomly generated problems where the points are irregularly spaced, SDP (3) is much preferrable over SOCP (6). This situation could change with alternative problem formulations (see Section 11), so further studies would be needed. In general, SOCP relaxation and mixed SDP-SOCP relaxation (see next section) seem most useful for larger problems where SDP relaxation is expensive to solve. Also, the SCGD method for solving (6) can be implemented in a highly distributed manner, with each sensor communicating with its neighbors only; see discussions at the end of Section 8. This may help to reduce communication and synchronization delays among sensors in practice.

		SOCP	SDP
Noise Pdf	$noisyfactor$	**obj**$/err_{\mathrm{rms}}/m_{\mathrm{up}}/err_{\mathrm{up}}$	**obj**$/err_{\mathrm{rms}}$
Normal	.1	.28/.24/52/.10	1.78/.06
	.2	.43/.48/48/.17	2.82/.23
Uniform	.1	.09/.41/30/.07	.88/.13
	.2	.24/.29/41/.11	2.26/.16
Additive-	.1	.34/.63/42/.17	2.30/.41
Gaussian	.2	.82/.71/52/.22	3.86/.50

Table 3: Comparing analytic center solutions of SDP and SOCP for smaller problems and more noisy distance measurements.

10 A Mixed SDP-SOCP Relaxation

Instead of an SDP or an SOCP relaxation, we can more generally consider a mixed SDP-SOCP relaxation of (1). Let $\mathcal{N}$ be any subset of $\{1, ..., m\}$. By renumbering the points if necessary, we assume that

$$\mathcal{N} = \{\hat{m} + 1, ..., m\},$$

with $0 \le \hat{m} \le m$. Let

$$\hat{\mathcal{A}} \stackrel{\text{def}}{=} \{(i,j) \in \mathcal{A} \;:\; i \in \mathcal{N} \text{ or } j \in \mathcal{N}\}.$$

Then the mixed SDP-SOCP relaxation associated with $\mathcal{N}$ is

$$\begin{array}{cl} \displaystyle\min_{x_1,\ldots,x_m,y_{ij},Z} & \displaystyle\sum_{(i,j)\in\mathcal{A}\setminus\hat{\mathcal{A}}} \left|\left\langle \hat{b}_{ij}\hat{b}_{ij}^T, Z\right\rangle_F - d_{ij}^2\right| + \sum_{(i,j)\in\hat{\mathcal{A}}} |y_{ij} - d_{ij}^2| \\ \text{s.t.} & [Z_{ij}]_{i,j\ge n-d} = I_d, \quad Z \succeq 0, \\ & [Z_{ij}]_{i\ge n-d, j\le \hat{m}} = [x_1 \ \cdots \ x_{\hat{m}}], \\ & y_{ij} \ge \|x_i - x_j\|^2 \quad \forall (i,j) \in \hat{\mathcal{A}}, \end{array}$$

where $\hat{b}_{ij} \stackrel{\text{def}}{=} \begin{bmatrix} I_{\hat{m}} & 0 & 0 \\ 0 & 0 & A \end{bmatrix} (e_i - e_j)$. Notice that $Z \in \mathcal{S}^{d+\hat{m}}$. This relaxation reduces to the SDP relaxation (3) if $\hat{\mathcal{A}} = \emptyset$ and reduces to the SOCP relaxation (6) if $\hat{\mathcal{A}} = \mathcal{A}$.

Such a mixed SDP-SOCP relaxation mediates between approximation accuracy and solution efficiency. In particular, Propositions 3 and 4 suggest putting into $\hat{\mathcal{A}}$ those pairs $(i,j) \in \mathcal{A}$ of sensors that are estimated to lie in the convex hull of their neighbors. Can the results in Sections 4–7 be extended to the mixed SDP-SOCP relaxation? Can we design efficient methods to find interior solutions? These are topics for future research.

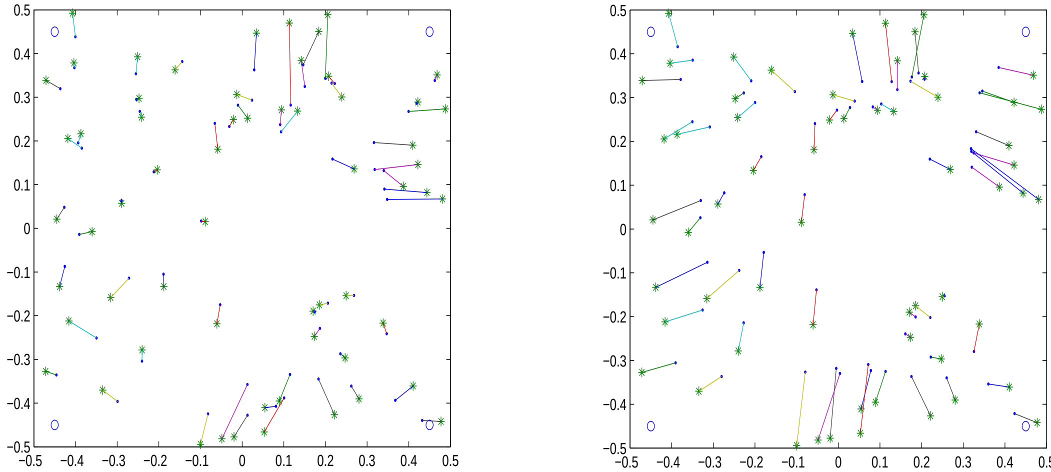

Figure 8: The left figure shows the anchors ("∘") and the analytic center solution of SDP found by SeDuMi for normally distributed noise and $noisyfactor = .2$ (row 2 of Table 3). Each sensor position found ("·") is joined to its true position ("∗") by a line. The right figure shows the same information for the analytic center solution of SOCP found by SCGD.

11 Variants of the Basic Problem

If 'sum' is replaced by 'max', then (1) becomes

$$\min_{x_1,\ldots,x_m} \quad \max_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|, \tag{31}$$

and the SDP relaxation (3) and SOCP relaxation (6) change accordingly. In general, if the objective function is a convex piecewise linear/quadratic function of $\|x_i - x_j\|^2$, $(i,j) \in \mathcal{A}$, then both an SDP relaxation and an SOCP relaxation can be analogously formulated. If the distances are not squared, then (1) becomes

$$\min_{x_1,\ldots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\| - d_{ij} \right|. \tag{32}$$

If the distances d_{ij} are exact (i.e., $\upsilon_{\text{opt}} = 0$), then (32) is equivalent to (1). In general, (32) puts a smaller penalty on large deviation from d_{ij} and has different solutions than (1). We leave the choice of the objective function to the modeler.

For (32), an SOCP relaxation, which seems more natural than an SDP relaxation,

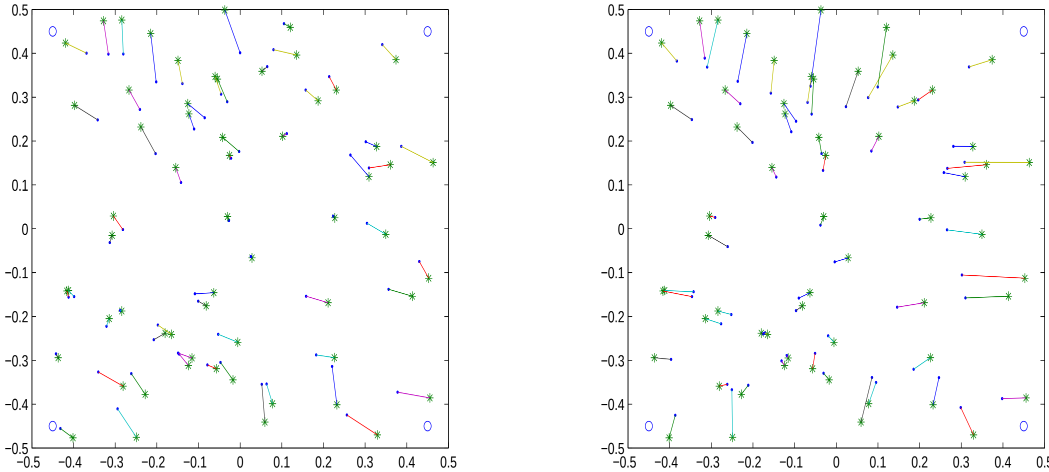

Figure 9: This figure is analogous to Figure 8, but for uniformly distributed noise and $noisyfactor = .2$ (row 4 of Table 3).

is

$$\begin{array}{ll} \min_{x_1,\ldots,x_m,y_{ij}} & \sum_{(i,j)\in\mathcal{A}} |y_{ij} - d_{ij}| \\ \text{s.t.} & y_{ij} \geq \|x_i - x_j\| \quad \forall\ (i,j) \in \mathcal{A}. \end{array} \tag{33}$$

By noting that $y_{ij} \geq d_{ij}$ in any solution of (33), we can write this in the standard conic form:

$$\begin{array}{ll} \min & \sum_{(i,j)\in\mathcal{A}} u_{ij} \\ \text{s.t.} & x_i - x_j - w_{ij} = 0 \quad \forall (i,j) \in \mathcal{A}, \\ & y_{ij} - u_{ij} = d_{ij} \quad \forall (i,j) \in \mathcal{A}, \\ & u_{ij} \geq 0,\ (y_{ij}, w_{ij}) \in \text{Qcone}^{d+1} \quad \forall (i,j) \in \mathcal{A}, \end{array} \tag{34}$$

where $\text{Qcone}^{d+1} \stackrel{\text{def}}{=} \{(y,w) \in \Re \times \Re^d : y \geq \|w\|\}$ [32]. This SOCP has a smaller size than (7). In general, if the objective function is a convex piecewise linear/quadratic function of $\|x_i - x_j\|$, $(i,j) \in \mathcal{A}$, then an SOCP relaxation can be analogously formulated. Other variants of (1) involve replacing the Euclidean (ℓ_2) distance by, say, rectilinear (ℓ_1) distance or ℓ_∞ distance.

When $v_{\text{opt}} = 0$, (33) is equivalent to (6) and, moreover, they have the same analytic center solution.

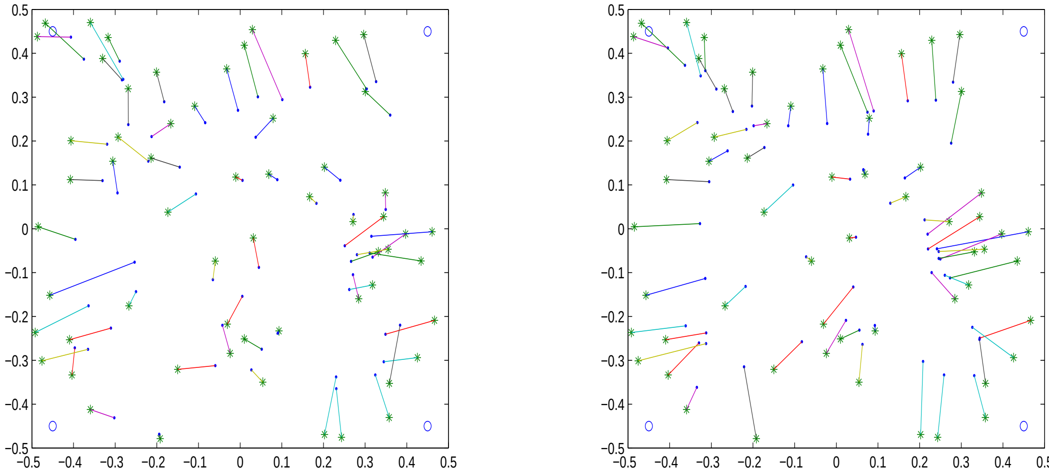

Figure 10: This figure is analogous to Figure 8, but for additive Gaussian noise and $noisyfactor = .2$ (row 6 of Table 3).

12 Future Directions

There are many directions for future research. For example, can our results for (1) be extended to other variants such as (31) and (32)? How do these variants compare under different distance noise distributions? What about additional constraints as discussed in [16] or replacing the 2-norm by a p-norm ($1 \leq p \leq \infty$)? Can our analysis of the SOCP relaxation (6) be extended to the mixed SDP-SOCP relaxation of Section 10? Can finite termination of the SCGD method be proved? Finally, the SOCP relaxation (28) may be interpreted as the Lagrangian dual of a d-commodity convex network flow problem. For $d = 1$, this can be solved very efficiently using an ϵ-relaxation method [5, 7, 19]. Can this method be extended to $d \geq 2$, thus speeding up the solution time of the SOCP relaxation?

Acknowledgement. The author thanks Yinyu Ye for motivating the topic of this paper. He also thanks two anonymous referees for their helpful comments and suggestions.

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