(Robust) Edge-Based Semidefinite Programming Relaxation of Sensor Network Localization[1]

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Abstract

Recently Wang, Zheng, Boyd, and Ye [36] proposed a further relaxation of the semidefinite programming (SDP) relaxation of the sensor network localization problem, named edge-based SDP (ESDP). In simulation, the ESDP is solved much faster by interior-point method than SDP relaxation, and the solutions found are comparable or better in approximation accuracy. We study some key properties of the ESDP relaxation, showing that, when distances are exact, zero individual trace is not only sufficient, but also necessary for a sensor to be correctly positioned by an interior solution. We also show via an example that, when distances are inexact, zero individual trace is insufficient for a sensor to be accurately positioned by an interior solution. We then propose a noise-aware robust version of ESDP relaxation for which small individual trace is necessary and sufficient for a sensor to be accurately positioned by a certain analytic center solution, assuming the noise level is sufficiently small. For this analytic center solution, the position error for each sensor is shown to be in the order of the square root of its trace. Lastly, we propose a log-barrier penalty coordinate gradient descent method to find such an analytic center solution. In simulation, this method is much faster than interior-point method for solving ESDP, and the solutions found are comparable in approximation accuracy. Moreover, the method can distribute its computation over the sensors via local communication, making it practical for positioning and tracking in real time.

Key words. Sensor network localization, semidefinite programming relaxation, error bound, log-barrier, coordinate gradient descent.

1 Introduction

A problem that has received considerable attention is that of ad hoc wireless sensor network localization [2, 4, 6, 8–16, 18–21, 25–28, 30–32, 37]. In the basic version of this problem, we have n distinct points in $\mathbb{R}^d$ $(d \geq 1)$. We are given the Cartesian coordinates of the last $n-m$ points (called "anchors") $x_{m+1}, \ldots, x_n$, and an estimate $d_{ij} > 0$ of the Euclidean distance between "neighboring" points i and j for all $(i,j) \in \mathcal{A}$, where $\mathcal{A} \subseteq (\{1,\ldots,m\} \times \{1,\ldots,n\}) \cup (\{1,\ldots,n\} \times \{1,\ldots,m\})$.[4] We wish to estimate the Cartesian coordinates of the first m points (called "sensors"). Typically, $d = 2$ and two points are neighbors if the distance between them is below some threshold (e.g., the radio range). In variants of this problem, the distances may be non-Euclidean [31] or may have measurement errors, and there may be additional constraints on the unknown points [12]. This problem is closely related to distance geometry problems arising in the determination of protein structure [7, 22] and to graph rigidity [1, 13, 32].

The sensor network localization problem is NP-hard in general [29]; also see remark in [22]. This can be proved for $d = 1$ by reduction from the set partition problem, and the proof readily extends for $d > 1$; also see [2, 27] for related studies. Thus, efforts have been directed at solving this problem approximately.

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[4]The set $\mathcal{A}$ is undirected in the sense that $(i,j) = (j,i)$ and $d_{ij} = d_{ji}$ for all $(i,j) \in \mathcal{A}$.

These include heuristics based on Euclidean geometry, shortest path, and local improvement; see [25,27, 28,30,37] and references therein. A different approach involves solving a convex relaxation, and then refining the resulting solution through local improvement. This has been effective in simulation and, under appropriate assumptions, the solution is provably exact/accurate. For example, a second-order cone programming (SOCP) relaxation can be efficiently solved and yields good approximation when the anchors are "spread out" [12,34]. Here we are interested in semidefinite programming (SDP) relaxations since they are better approximations than SOCP relaxations [34, Proposition 3.1], [36, Theorem 4.5], though SDPs are also more difficult to solve than SOCPs.

In the SDP approach of Biswas and Ye [8,9], the original problem is formulated as the following nonconvex minimization problem:

$$\upsilon_{\text{opt}} := \min_{x_1,\dots,x_m} \sum_{(i,j)\in\mathcal{A}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right|, \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm. Letting $X := \begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix}$ and I_d denote the $d \times d$ identity matrix, Biswas and Ye considered the following SDP relaxation of (1):

$$\begin{aligned} \upsilon_{\text{sdp}} := \quad & \min_{Z} \sum_{(i,j)\in\mathcal{A}} \left| \ell_{ij}(Z) - d_{ij}^2 \right| \\ & \text{s.t.} \quad Z = \begin{pmatrix} Y & X^T \\ X & I_d \end{pmatrix}, \quad Z \succeq 0, \end{aligned} \tag{2}$$

where $Y = \left(y_{ij}\right)_{1\le i,j\le m}$, "s.t." is short for "subject to", and

$$\ell_{ij}(Z) := \begin{cases} y_{ii} - 2y_{ij} + y_{jj} & \text{if } (i,j) \in \mathcal{A}^s; \\ y_{ii} - 2x_i^T x_j + \|x_j\|^2 & \text{if } i \le m < j, \end{cases} \tag{3}$$

with $\mathcal{A}^s := \{(i,j) \in \mathcal{A} \mid i,j \le m\}$. It can be seen that (2) reduces to (1) when we add the constraint $\operatorname{rank} Z = d$. Properties of (2) and its solutions are studied in [8,32].[5] In particular, Biswas and Ye [8, Section 4] introduced the notion of individual traces, defined as

$$\operatorname{tr}_i(Z) := y_{ii} - \|x_i\|^2, \quad i = 1,\dots,m.$$

These individual traces are equivalently the diagonals of the Schur complement $Y - X^T X$. In [9, Section 2] and [10, Section 3], they were used to evaluate the accuracy of the estimated positions $x_1,\dots,x_m$, with smaller trace indicating higher accuracy. So and Ye [32, Theorem 2] proved in the case of $\upsilon_{\text{sdp}} = 0$ that the sensors are "uniquely localizable" if and only if, for any interior solution Z (equivalently, Z has maximum rank), all individual traces of Z are zero, i.e., $Y = X^T X$. (Throughout, "interior solution" means a point in the relative interior of the solution set.) Moreover, for any interior solution Z, $\operatorname{tr}_i(Z) = 0$ implies x_i is invariant over the solution set and hence equals the true position of sensor i when $\upsilon_{\text{opt}} = 0$ [34, Proposition 4.1]. Other extensions and refinements of the above SDP approach are described in [3–7,10,20].

While (2) is a good approximation of the original problem (1), it cannot be solved in reasonable time for $m \ge 500$, and domain decomposition methods have been proposed to solve many small SDP subproblems and refine the solutions using local improvement heuristics [9,10,20]. These methods tend to work well if many anchors are uniformly distributed; see [36, Section 5.4]. This contrasts with SOCP relaxation which can be solved in under 6 minutes for $n = 4000$ using a smoothing coordinate gradient descent method [34]. Recently, Wang, Zheng, Boyd, and Ye [36] proposed a further relaxation of the SDP relaxation (2), called edge-based SDP (ESDP) relaxation. The ESDP relaxation is stronger than

[5] Throughout, "solution" of an optimization problem means a global optimal solution.

the SOCP relaxation and, can be solved in under 11 minutes for $n = 4000$ using SeDuMi [33] (not counting problem setup time), and yields solution comparable or better in approximation accuracy to the SDP relaxation; see [36, Section 5], [37, Section 7]. The ESDP relaxation is obtained by relaxing the constraint $Z \succeq 0$ in (2) to require only those principal submatrices of Z associated with $\mathcal{A}$ to be positive semidefinite. Specifically, the ESDP relaxation is

$$\begin{aligned} \upsilon_{\text{esdp}} := \quad & \min_{Z} \quad \sum_{(i,j)\in\mathcal{A}} \left|\ell_{ij}(Z) - d_{ij}^2\right| \\ & \text{s.t.} \quad Z = \begin{pmatrix} Y & X^T \\ X & I_d \end{pmatrix}, \\ & \qquad \begin{pmatrix} y_{ii} & y_{ij} & x_i^T \\ y_{ij} & y_{jj} & x_j^T \\ x_i & x_j & I_d \end{pmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}^s, \\ & \qquad \begin{pmatrix} y_{ii} & x_i^T \\ x_i & I_d \end{pmatrix} \succeq 0 \qquad \forall i \leq m. \end{aligned} \tag{4}$$

Notice that the objective function and the positive semidefinite constraints in (4) do not depend on y_{ij}, $(i,j) \notin \mathcal{A}$. Also, the third constraint in (4) is redundant for those $i \leq m$ such that $(i,j) \in \mathcal{A}$ for some $j \leq m$ (i.e., sensor i has another sensor as neighbor) since it is implied by the second constraint.

In [11, 19], a variant of (1) and (2) is considered whereby $|\cdot|$ is replaced with $|\cdot|^2$, and a primal-dual interior-point method is applied to solve the SDP and its dual in a certain reduced/projected form. Simulation results with up to $m = 20$ sensors are reported. Nie [26] considered the same problem variant and proposed a sparse sum-of-square (SOS) relaxation which is equivalent to a certain sparse SDP. In simulation with $m = 500$ sensors and exact distances, accurate solutions were found in about 1.5 hours using SeDuMi. Recently, Kim, Kojima, and Waki [18] reformulated this problem variant as a constrained quadratic optimization problem, and used a positive definite matrix completion technique to reduce the SOS relaxation of order 1 into an SDP having analogous form as (4), but with each principal submatrix of Z associated with a maximal clique of a chordal extension of a minimal subgraph. In simulation with $m = 4000$ sensors and exact distances, accurate solutions were found in 80–1000 seconds using SeDuMi.

In practice, due to limited transmission power of the sensors, measured distances may be inexact, i.e.,

$$d_{ij}^2 = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2 + \delta_{ij} \quad \forall (i,j) \in \mathcal{A}, \tag{5}$$

where $\delta = (\delta_{ij})_{(i,j)\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|}$ denotes the measurement noise, and x_i^{true} denotes the true position of the ith point (so that $x_i = x_i^{\text{true}}$ for $i > m$); see [14, Eq. (2)], [15, Eq. (3a)–(3f)], [21, Section 2]. Methods for sensor network localization can be highly sensitive to such noises. Our aims are three-fold. First, we study the approximation accuracy of SDP relaxation (2) and ESDP relaxation (4), as measured by the individual traces of interior solutions. We show that, when distances are exact (i.e., $\delta = 0$), zero individual trace is not only sufficient, but also necessary for a sensor to be correctly positioned by an interior solution of the ESDP relaxation; see Theorem 1. On the other hand, we show via an example that, when distances are inexact (i.e., $\delta \neq 0$), zero individual trace is insufficient for a sensor to be accurately positioned by an interior solution of the SDP/ESDP relaxation; see Example 2. This somewhat surprising result shows that SDP and ESDP relaxations are more noise-sensitive than SOCP relaxation (compare with [34, Proposition 7.2]). Second, we propose a noise-aware robust version of ESDP relaxation for which small individual trace is necessary and sufficient for a sensor to be accurately positioned by an analytic center solution, assuming $\|\delta\|$ is sufficiently small. Moreover, we show that the position error for each sensor is in the order of the square root of the individual trace; see Theorems 3 and 4. Third, we propose a log-barrier penalty coordinate gradient descent method to find such an analytic center solution; see

Section 6. In our simulation, this method is much faster than interior-point method for solving ESDP, and the solutions found are comparable in approximation accuracy; see Section 7. Moreover, this method is implementable in a distributed manner, with each sensor updating its position estimate knowing only the current position estimates of its neighbors, the measured distance between them, and a few other quantities. This is an important consideration for real-time implementation [16, page 65], [21, 25, 28]. In contrast, existing SDP-based methods [5, 6, 8–11, 18–20, 36] require some level of centralization. Thus, our method is efficient, potentially implementable in real time, and can handle noise and certify which sensors are accurately positioned. The positions of remaining sensors can be refined using any number of local improvement heuristics, such as those used in [5, 6, 10, 18, 20, 36], though their accuracy cannot be certified.

Throughout, $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, and ^{T} denotes transpose. For a vector $x \in \mathbb{R}^p$, $\|x\|$ and $\|x\|_\infty$ denote the Euclidean norm of x and the ∞-norm of x, respectively. For $A \in \mathbb{R}^{p\times q}$, a_{ij} denotes the (i,j)th entry of A, and $\|A\|_F$ denotes the Fröbenius norm of A. For $A, B \in \mathcal{S}^p$, $A \succeq B$ means $A - B$ is positive semidefinite. For $A \in \mathcal{S}^p$ and $\mathcal{I} \subseteq \{1, \dots, p\}$, $A_{\mathcal{I}} = \left(a_{ij}\right)_{i,j\in\mathcal{I}}$ denotes the principal submatrix of A comprising the rows and columns of A indexed by $\mathcal{I}$. We will abbreviate "$m+1, m+2, \dots, m+d$" as "m^+." Thus, $Z_{\{i,j,m^+\}}$ and $Z_{\{i,m^+\}}$ are, respectively, the $(2+d)\times(2+d)$ and $(1+d)\times(1+d)$ principal submatrices of Z appearing in the second and third constraint of (4). For any finite set $\mathcal{J}$, $|\mathcal{J}|$ denotes the cardinality of $\mathcal{J}$. For any $\mathcal{I} \subseteq \{1, \dots, m\}$, we denote the set of its neighbors and the set of edges to its neighbors by

$$\begin{aligned} \mathcal{N}(\mathcal{I}) &:= \{j \notin \mathcal{I} \mid (i,j) \in \mathcal{A} \text{ for some } i \in \mathcal{I}\}, \\ \mathcal{A}(\mathcal{I}) &:= \{(i,j) \in \mathcal{A} \mid i \in \mathcal{I},\ j \notin \mathcal{I}\}. \end{aligned}$$

2 Trace test for uniquely positioned sensors by SDP and ESDP

Let $\mathcal{S}^\delta_{\text{sdp}}$ denote the solution set of (2). Let $\mathcal{S}^\delta_{\text{esdp}}$ denote the solution set of (4) with y_{ij} set arbitrarily to zero for all $(i,j) \notin \mathcal{A}$ (see the remark following (4)). The latter simplifies certain compactness arguments later, e.g., the proof of Proposition 3. Both $\mathcal{S}^\delta_{\text{sdp}}$ and $\mathcal{S}^\delta_{\text{esdp}}$ are closed convex, and hence their relative interior $\text{ri}(\mathcal{S}^\delta_{\text{sdp}})$ and $\text{ri}(\mathcal{S}^\delta_{\text{esdp}})$ are well defined. As in [32, Proposition 1] and [34, page 162], we make the reasonable assumption that each sensor is connected, directly or indirectly, to some anchor.

Assumption 1. *Each connected component of the graph $\mathcal{G} := (\{1, \dots, n\}, \mathcal{A})$ contains an anchor index.*

Assumption 1 is necessary and sufficient for $\mathcal{S}^\delta_{\text{sdp}}$ and $\mathcal{S}^\delta_{\text{esdp}}$ to be bounded–an important consideration when solving (2) or (4) by an interior-point method. Assumption 1 is reasonable since if a connected component of $\mathcal{G}$ does not contain an anchor index, then the location of the corresponding sensors can be determined only up to a common translation factor. In applications such as 3D protein structure prediction, the unknown points only need to be determined up to common translation and rotation factors, so Assumption 1 can be made without loss of generality; see [9].

In what follows, we define

$$\mathcal{I}^\delta_{\text{esdp}} := \left\{ i \in \{1, \dots, m\} \;\middle|\; x_i \text{ is invariant over all } Z = \begin{pmatrix} Y & X^T \\ X & I_d \end{pmatrix} \in \mathcal{S}^\delta_{\text{esdp}} \right\}.$$

We define $\mathcal{I}^\delta_{\text{sdp}}$ analogously. In the noiseless case ($\delta = 0$), those sensors indexed by $\mathcal{I}^\delta_{\text{esdp}}$ (respectively, $\mathcal{I}^\delta_{\text{sdp}}$) are correctly positioned by any ESDP solution (respectively, SDP solution). Thus it is of interest to identify these index sets. The following result from [34, Proposition 4.1] shows that a subset of $\mathcal{I}^\delta_{\text{esdp}}$ is identified by zero individual traces at an interior solution of SDP (2); see [32, Theorem 2] for related results in the case of $\delta = 0$ and all individual traces being zero at interior solutions.

Proposition 1. *For any* $\delta \in \mathbb{R}^{|\mathcal{A}|}$, $Z \in \mathrm{ri}(\mathcal{S}^{\delta}_{\mathrm{sdp}})$ *and* $i \in \{1, \dots, m\}$, *if* $\mathrm{tr}_i(Z) = 0$, *then* $i \in \mathcal{I}^{\delta}_{\mathrm{sdp}}$.

An analogous result can be proved for the ESDP relaxation; also see [36, Theorem 2].

Proposition 2. *For any* $\delta \in \mathbb{R}^{|\mathcal{A}|}$, $Z \in \mathrm{ri}(\mathcal{S}^{\delta}_{\mathrm{esdp}})$ *and* $i \in \{1, \dots, m\}$, *if* $\mathrm{tr}_i(Z) = 0$, *then* $i \in \mathcal{I}^{\delta}_{\mathrm{esdp}}$.

The proofs of Propositions 1 and 2 are based on the following simple properties of the individual trace. Let $\mathcal{F}_{\mathrm{esdp}}$ denote the feasible set of (4) with y_{ij} set to zero for all $(i,j) \notin \mathcal{A}$. For any $Z \in \mathcal{F}_{\mathrm{esdp}}$, we have from the third constraint in (4) that

$$\mathrm{tr}_i(Z) \geq 0, \quad i = 1, \dots, m.$$

We also note the following key identity for individual traces. For any $Z, Z' \in \mathcal{F}_{\mathrm{esdp}}$ and any $\alpha \in [0,1]$, we have $Z^{\alpha} := \alpha Z + (1-\alpha) Z' \in \mathcal{F}_{\mathrm{esdp}}$ and

$$\mathrm{tr}_i(Z^{\alpha}) = \alpha \mathrm{tr}_i(Z) + (1-\alpha)\mathrm{tr}_i(Z') + \alpha(1-\alpha)\|x_i - x_i'\|^2, \quad i = 1, \dots, m. \tag{6}$$

Thus each individual trace is a concave function on $\mathcal{F}_{\mathrm{esdp}}$. The following result follows from the concavity and nonnegativity of the individual trace on $\mathcal{F}_{\mathrm{esdp}}$.

Lemma 1. *For any* $\delta \in \mathbb{R}^{|\mathcal{A}|}$, *if* $\mathrm{tr}_i(Z) = 0$ *for some* $Z \in \mathrm{ri}(\mathcal{S}^{\delta}_{\mathrm{esdp}})$, *then* $\mathrm{tr}_i(Z) = 0$ *for all* $Z \in \mathcal{S}^{\delta}_{\mathrm{esdp}}$.

3 Trace test for correctly positioned sensors by ESDP: necessity in the noiseless case

In this section we show that the converse of Proposition 2 holds in the noiseless case ($\delta = 0$). In other words, in the noiseless case, the condition $\mathrm{tr}_i(Z) = 0$ is not only sufficient, but also necessary for x_i to equal x_i^{true} for any $Z \in \mathrm{ri}(\mathcal{S}^{0}_{\mathrm{esdp}})$. The proof is divided into two parts. In the first part, we show by induction that if a sensor $i \in \mathcal{I}^0_{\mathrm{esdp}}$ is connected to some anchor through neighboring sensors also in $\mathcal{I}^0_{\mathrm{esdp}}$, then $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathcal{S}^0_{\mathrm{esdp}}$; see Lemma 3. In the second part, we show that if there exists a sensor $i \in \mathcal{I}^0_{\mathrm{esdp}}$ with $\mathrm{tr}_i(Z) > 0$ for some $Z \in \mathcal{S}^0_{\mathrm{esdp}}$, then x_i can be rotated to obtain another ESDP solution, contradicting the definition of $\mathcal{I}^0_{\mathrm{esdp}}$. We begin with the following two lemmas relating the traces of neighboring sensors.

Lemma 2. **(a)** *For any* $Z \in \mathcal{F}_{\mathrm{esdp}}$, *we have*

$$\begin{pmatrix} y_{ii} - \|x_i\|^2 & y_{ij} - x_i^T x_j \\ y_{ij} - x_i^T x_j & y_{jj} - \|x_j\|^2 \end{pmatrix} \succeq 0. \tag{7}$$

(b) *Suppose* $\delta = 0$. *For any* $Z \in \mathcal{S}^0_{\mathrm{esdp}}$ *and* $(i,j) \in \mathcal{A}^s$, *if* $\|x_i - x_j\| = d_{ij}$, *then* $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$.

Proof. (a) Since $Z \in \mathcal{F}_{\mathrm{esdp}}$ so that it satisfies the second constraint in (4), a basic property of Schur complement yields (7).

(b) Since $\delta = 0$ so that $\upsilon_{\mathrm{esdp}} = 0$ and $Z \in \mathcal{S}^0_{\mathrm{esdp}}$, we have $\ell_{ij}(Z) = d_{ij}^2$. Since $(i,j) \in \mathcal{A}^s$, (3) implies

$$y_{ii} - 2y_{ij} + y_{jj} = d_{ij}^2.$$

This together with $\|x_i - x_j\| = d_{ij}$ implies that

$$y_{ii} - \|x_i\|^2 + y_{jj} - \|x_j\|^2 = 2(y_{ij} - x_i^T x_j). \tag{8}$$

As in the proof of [34, Proposition 3.1], by setting $a = y_{ii} - \|x_i\|^2$, $b = y_{jj} - \|x_j\|^2$ and $c = y_{ij} - x_i^T x_j$, we have from (7) that $a, b \geq 0$, $ab - c^2 \geq 0$. Then $(a+b)^2 = (a-b)^2 + 4ab \geq (a-b)^2 + 4c^2 \geq 4c^2$. By (8), we also have $a + b = 2c$, so that $(a+b)^2 = 4c^2$. Hence $a = b$, i.e., $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$. ■

In what follows, we denote

$$X^{\mathrm{true}} := \begin{pmatrix} x_1^{\mathrm{true}} & \cdots & x_m^{\mathrm{true}} \end{pmatrix}, \quad y_{ij}^{\mathrm{true}} := \begin{cases} 0 & \text{if } (i,j) \notin \mathcal{A}; \\ (x_i^{\mathrm{true}})^T x_j^{\mathrm{true}} & \text{else}, \end{cases} \quad Z^{\mathrm{true}} := \begin{pmatrix} Y^{\mathrm{true}} & (X^{\mathrm{true}})^T \\ X^{\mathrm{true}} & I_d \end{pmatrix}. \tag{9}$$

Thus $Z^{\mathrm{true}} \in \mathcal{S}^0_{\mathrm{esdp}} \subseteq \mathcal{F}_{\mathrm{esdp}}$.

Lemma 3. *Suppose $\delta = 0$.*

(a) *For any $(i,j) \in \mathcal{A}$ with $i, j \in \mathcal{I}^0_{\mathrm{esdp}}$, we have $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$ for all $Z \in \mathcal{S}^0_{\mathrm{esdp}}$.*

(b) *For any $(i,j) \in \mathcal{A}$ with $i \in \mathcal{I}^0_{\mathrm{esdp}}$ and $j > m$, we have $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathcal{S}^0_{\mathrm{esdp}}$.*

Proof. (a) Fix any $Z \in \mathcal{S}^0_{\mathrm{esdp}}$. Since $i, j \in \mathcal{I}^0_{\mathrm{esdp}}$ and $Z^{\mathrm{true}} \in \mathcal{S}^0_{\mathrm{esdp}}$, we have $x_i = x_i^{\mathrm{true}}$ and $x_j = x_j^{\mathrm{true}}$. Hence $\|x_i - x_j\| = \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\| = d_{ij}$. By Lemma 2(b), $\mathrm{tr}_i(Z) = \mathrm{tr}_j(Z)$.

(b) Fix any $Z \in \mathcal{S}^0_{\mathrm{esdp}}$. Since $i \in \mathcal{I}^0_{\mathrm{esdp}}$, $j > m$ and $Z^{\mathrm{true}} \in \mathcal{S}^0_{\mathrm{esdp}}$, we have $x_i = x_i^{\mathrm{true}}$ and $x_j = x_j^{\mathrm{true}}$. Also $\upsilon_{\mathrm{esdp}} = 0$, so that $\ell_{ij}(Z) = d_{ij}^2$. Since $j > m$, (3) implies $y_{ii} - 2x_i^T x_j + \|x_j\|^2 = d_{ij}^2$. Hence

$$\mathrm{tr}_i(Z) \;=\; y_{ii} - \|x_i\|^2 \;=\; d_{ij}^2 + 2x_i^T x_j - \|x_j\|^2 - \|x_i\|^2 \;=\; d_{ij}^2 - \|x_i - x_j\|^2 \;=\; d_{ij}^2 - \|x_i^{\mathrm{true}} - x_j^{\mathrm{true}}\|^2 \;=\; 0.$$

■

Lemma 3 shows that if a sensor $i \in \mathcal{I}^0_{\mathrm{esdp}}$ is connected to some anchor by a path in $\mathcal{G}$ whose intermediate nodes are all in $\mathcal{I}^0_{\mathrm{esdp}}$, then $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathcal{S}^0_{\mathrm{esdp}}$. We will show in Theorem 2 that in fact all sensors $i \in \mathcal{I}^0_{\mathrm{esdp}}$ have this property. We also need the following matrix identity about Schur complement.

Lemma 4. *For any $\bar{A}, A \in \mathbb{R}^{d \times k}$, $\bar{B}, B \in \mathbb{R}^{k \times k}$, and $\alpha \in [0,1]$, we have upon letting $X^\alpha = \alpha\bar{A} + (1-\alpha)A$ and $Y^\alpha = \alpha\bar{B} + (1-\alpha)B$ that*

$$Y^\alpha - (X^\alpha)^T X^\alpha \;=\; \alpha(\bar{B} - \bar{A}^T\bar{A}) + (1-\alpha)(B - A^T A) + \alpha(1-\alpha)\left(\bar{A} - A\right)^T\left(\bar{A} - A\right).$$

We are now ready to prove the main result of this section, showing that the converse of Proposition 2 holds in the noiseless case. The proof uses Assumption 1, and Lemmas 1, 3 and 4. In particular, we show that if there exist $\bar{i} \in \mathcal{I}^0_{\mathrm{esdp}}$ and $Z \in \mathcal{S}^0_{\mathrm{esdp}}$ with $\mathrm{tr}_{\bar{i}}(Z) > 0$, then we can rotate $x_{\bar{i}}$ to obtain another element of $\mathcal{S}^0_{\mathrm{esdp}}$, thus contradicting the definition of $\mathcal{I}^0_{\mathrm{esdp}}$.

Theorem 1. *For any $i \in \mathcal{I}^0_{\mathrm{esdp}}$, we have $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathcal{S}^0_{\mathrm{esdp}}$.*

Proof. Fix any $\bar{i} \in \mathcal{I}^0_{\mathrm{esdp}}$ and $\bar{Z} \in \mathcal{S}^0_{\mathrm{esdp}}$. Let $\bar{\mathcal{I}}$ be the set of all $i \in \mathcal{I}^0_{\mathrm{esdp}}$ that are joined to $\bar{i}$ by a path in the subgraph of $\mathcal{G}$ induced by $\mathcal{I}^0_{\mathrm{esdp}}$ (i.e., $i \in \bar{\mathcal{I}}$ if and only if i is joined to $\bar{i}$ by a path in $\mathcal{G}$ consisting only of nodes in $\mathcal{I}^0_{\mathrm{esdp}}$). By Assumption 1, $\mathcal{N}(\bar{\mathcal{I}}) \neq \emptyset$. If there exists an $i \in \bar{\mathcal{I}}$ with $\mathrm{tr}_i(\bar{Z}) = 0$ or a $j \in \mathcal{N}(\bar{\mathcal{I}})$ with $j > m$, then, by Lemma 3, $\mathrm{tr}_i(\bar{Z}) = 0$ for all $i \in \bar{\mathcal{I}}$ and, in particular, $\mathrm{tr}_{\bar{i}}(\bar{Z}) = 0$. Suppose that no such i or j exists, so that $\mathrm{tr}_i(\bar{Z}) > 0$ for all $i \in \bar{\mathcal{I}}$ and, by the definition of $\bar{\mathcal{I}}$,

$$\mathcal{N}(\bar{\mathcal{I}}) \subseteq \{1, \ldots, m\} \setminus \mathcal{I}^0_{\mathrm{esdp}}. \tag{10}$$

We will arrive at a contradiction below.

By (10), there exists a $Z \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$ such that $x_j \neq x_j^{\mathrm{true}}$ for all $j \in \mathcal{N}(\bar{\mathcal{I}})$. Since $Z^{\mathrm{true}} \in \mathcal{S}^0_{\mathrm{esdp}}$ and $\mathcal{S}^0_{\mathrm{esdp}}$ is convex, we have

$$Z^\alpha := \alpha Z^{\mathrm{true}} + (1-\alpha)Z \in \mathcal{S}^0_{\mathrm{esdp}} \quad \forall\, 0 \leq \alpha \leq 1.$$

Fix any $(i,j) \in \mathcal{A}(\bar{\mathcal{I}})$ with $i \in \bar{\mathcal{I}}$. By (10), $j \leq m$ so that $(i,j) \in \mathcal{A}^s$. Applying Lemma 4 with $\bar{A} = \begin{pmatrix} x_i^{\mathrm{true}} & x_j^{\mathrm{true}} \end{pmatrix}$, $A = \begin{pmatrix} x_i & x_j \end{pmatrix}$, $\bar{B} = Z^{\mathrm{true}}_{\{i,j\}}$, $B = Z_{\{i,j\}}$ and using $x_i = x_i^{\mathrm{true}}$ (since $i \in \mathcal{I}^0_{\mathrm{esdp}}$) yield

$$(Y^\alpha - X^{\alpha T} X^\alpha)_{\{i,j\}} \;=\; (1-\alpha)\left(\begin{pmatrix} \mathrm{tr}_i(Z) & y_{ij} - {x_i}^T x_j \\ y_{ij} - {x_i}^T x_j & \mathrm{tr}_j(Z) \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 \\ 0 & \|x_j - x_j^{\mathrm{true}}\|^2 \end{pmatrix} \right).$$

Since $\mathrm{tr}_i(\bar{Z}) > 0$, Lemma 1 implies $\mathrm{tr}_i(Z) > 0$. Since $x_j \neq x_j^{\mathrm{true}}$ and the first matrix on the right-hand side is positive semidefinite (since $Z_{\{i,j,m+\}} \succeq 0$), the right-hand side is nonsingular or, equivalently, $Z^\alpha_{\{i,j,m+\}}$ is nonsingular for all $0 < \alpha \leq 1$ sufficiently small. Choose a $0 < \alpha \leq 1$ such that $Z^\alpha_{\{i,j,m+\}}$ is nonsingular (and hence positive definite) for all $(i,j) \in \mathcal{A}(\bar{\mathcal{I}})$. We now construct a feasible perturbation of Z^α. By translating all n points by a common factor if necessary, we can assume that $x_i^\alpha \neq 0$ for all $i \in \bar{\mathcal{I}}$. For each $\theta > 0$, let $U_\theta \in \mathbb{R}^{d \times d}$ be an orthogonal matrix satisfying $0 < \|U_\theta - I_d\|_F = O(\theta)$ (e.g., U_θ corresponds to a rotation by angle θ). Then, for $\theta > 0$ sufficiently small, we have

$$\begin{pmatrix} y_{ii}^\alpha & y_{ij}^\alpha & (U_\theta x_i^\alpha)^T \\ y_{ij}^\alpha & y_{jj}^\alpha & {x_j^\alpha}^T \\ U_\theta x_i^\alpha & x_j^\alpha & I_d \end{pmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}(\bar{\mathcal{I}}).$$

Fix any such θ. For each $(i,j) \in \mathcal{A}$ with $i,j \in \bar{\mathcal{I}}$, we have from $Z^\alpha \in \mathcal{F}_{\mathrm{esdp}}$ that

$$\begin{pmatrix} y_{ii}^\alpha & y_{ij}^\alpha \\ y_{ij}^\alpha & y_{jj}^\alpha \end{pmatrix} - (U_\theta \begin{pmatrix} x_i^\alpha & x_j^\alpha \end{pmatrix})^T U_\theta \begin{pmatrix} x_i^\alpha & x_j^\alpha \end{pmatrix} = \begin{pmatrix} y_{ii}^\alpha & y_{ij}^\alpha \\ y_{ij}^\alpha & y_{jj}^\alpha \end{pmatrix} - \begin{pmatrix} x_i^\alpha & x_j^\alpha \end{pmatrix}^T \begin{pmatrix} x_i^\alpha & x_j^\alpha \end{pmatrix} \succeq 0,$$

from which it follows that

$$\begin{pmatrix} y_{ii}^\alpha & y_{ij}^\alpha & (U_\theta x_i^\alpha)^T \\ y_{ij}^\alpha & y_{jj}^\alpha & (U_\theta x_j^\alpha)^T \\ U_\theta x_i^\alpha & U_\theta x_j^\alpha & I_d \end{pmatrix} \succeq 0.$$

Thus, replacing x_i^α in Z^α by $U_\theta x_i^\alpha$ for all $i \in \bar{\mathcal{I}}$ yields a $\tilde{Z}^\alpha$ that is feasible for (4). Moreover, $\tilde{Z}^\alpha$ is optimal for (4) (with $\delta = 0$) since, by (3) and $\bar{\mathcal{I}} \cup \mathcal{N}(\bar{\mathcal{I}}) \subseteq \{1, \ldots, m\}$ (see (10)), the objective function of (4) does not depend on x_i for $i \in \bar{\mathcal{I}}$. Thus $\tilde{Z}^\alpha \in \mathcal{S}^0_{\mathrm{esdp}}$ but its $U_\theta x_i^\alpha$ component differs from the x_i^α component of Z^α for all $i \in \bar{\mathcal{I}}$, contradicting the definition of $\bar{\mathcal{I}}$. ∎

Notice that Lemmas 1, 2, 3 readily extend to the SDP relaxation (2). It is an open question whether Theorem 1 extends to the SDP relaxation. By using Proposition 2, Lemmas 1 and 2(a), and Theorem 1, we show below that every sensor $i \in \mathcal{I}^0_{\mathrm{esdp}}$ is connected to some anchor through neighboring sensors also in $\mathcal{I}^0_{\mathrm{esdp}}$. This result is analogous to [34, Proposition 5.1(b)] for an SOCP relaxation of (1), though here we consider only the noiseless case. The proof, like the proof of Theorem 1, involves a feasible perturbation of x_i for all $i \in \bar{\mathcal{I}}$, where $\bar{\mathcal{I}}$ is some subset of $\mathcal{I}^0_{\mathrm{esdp}}$. However, we make use of $\mathrm{tr}_j(Z) > 0$ for all $j \in \mathcal{N}(\bar{\mathcal{I}})$ instead of $\mathrm{tr}_i(Z) > 0$ for all $i \in \bar{\mathcal{I}}$, and the perturbation involves a contraction instead of a rotation.

Theorem 2. *Every $i \in \mathcal{I}^0_{\mathrm{esdp}}$ is joined to some $j > m$ by a path in $\mathcal{G}$ whose intermediate nodes are all in $\mathcal{I}^0_{\mathrm{esdp}}$.*

Proof. Fix any $\bar{i} \in \mathcal{I}^0_{\rm esdp}$. Let $\bar{\mathcal{I}}$ be the set of all $i \in \mathcal{I}^0_{\rm esdp}$ that are joined to $\bar{i}$ by a path in the subgraph of $\mathcal{G}$ induced by $\mathcal{I}^0_{\rm esdp}$. By Assumption 1, $\mathcal{N}(\bar{\mathcal{I}}) \neq \emptyset$. If there exists a $j \in \mathcal{N}(\bar{\mathcal{I}})$ with $j > m$, then the conclusion follows. Suppose that no such j exists. Then, by the definition of $\bar{\mathcal{I}}$, (10) holds.

Fix any $Z \in \mathrm{ri}(\mathcal{S}^0_{\rm esdp})$. By translating all n points by a common factor if necessary, we can assume that $x_i \neq 0$ for all $i \in \bar{\mathcal{I}}$. Also, by (10), Proposition 2 and Lemma 1, we have that $\mathrm{tr}_j(Z) > 0$ for all $j \in \mathcal{N}(\bar{\mathcal{I}})$. For each $0 < \epsilon \leq 1$, define

$$x_i^\epsilon := (1-\epsilon)x_i \quad \forall i \in \bar{\mathcal{I}}.$$

For each $(i,j) \in \mathcal{A}(\bar{\mathcal{I}})$ with $i \in \bar{\mathcal{I}}$, we have from $i \in \mathcal{I}^0_{\rm esdp}$ and Theorem 1 that $y_{ii} = \|x_i\|^2$. Since $Z \in \mathcal{F}_{\rm esdp}$, so that (7) holds, this implies $y_{ij} = {x_i}^T x_j$. Then $y_{ii} - \|x_i^\epsilon\|^2 = \|x_i\|^2 - (1-\epsilon)^2\|x_i\|^2 = (2\epsilon - \epsilon^2)\|x_i\|^2 > 0$ and

$$\begin{aligned} \det \begin{pmatrix} y_{ii} - \|x_i^\epsilon\|^2 & y_{ij} - {x_i^\epsilon}^T x_j \\ y_{ij} - {x_i^\epsilon}^T x_j & y_{jj} - \|x_j\|^2 \end{pmatrix} &= (2\epsilon - \epsilon^2)\|x_i\|^2 \mathrm{tr}_j(Z) - \epsilon^2({x_i}^T x_j)^2 \\ &= \epsilon\left(2\|x_i\|^2\mathrm{tr}_j(Z) - \epsilon\|x_i\|^2\mathrm{tr}_j(Z) - \epsilon({x_i}^T x_j)^2\right), \end{aligned}$$

which is positive for all ϵ sufficiently small. Hence we can choose $0 < \epsilon \leq 1$ so that

$$\begin{pmatrix} y_{ii} & y_{ij} & x_i^\epsilon \\ y_{ij} & y_{jj} & x_j \\ x_i^\epsilon & x_j & I_d \end{pmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A}(\bar{\mathcal{I}}) \text{ with } i \in \bar{\mathcal{I}}.$$

For each $(i,j) \in \mathcal{A}$ with $i,j \in \bar{\mathcal{I}}$, we have from Theorem 1 that $y_{ii} = \|x_i\|^2$, $y_{jj} = \|x_j\|^2$ and hence (7) implies $y_{ij} = {x_i}^T x_j$. Then

$$\begin{pmatrix} y_{ii} & y_{ij} \\ y_{ij} & y_{jj} \end{pmatrix} - \begin{pmatrix} \|x_i^\epsilon\|^2 & {x_i^\epsilon}^T x_j^\epsilon \\ {x_i^\epsilon}^T x_j^\epsilon & \|x_j^\epsilon\|^2 \end{pmatrix} = (2\epsilon - \epsilon^2)\begin{pmatrix} \|x_i\|^2 & {x_i}^T x_j \\ {x_i}^T x_j & \|x_j\|^2 \end{pmatrix},$$

which is positive semidefinite for $0 < \epsilon \leq 1$. Hence

$$\begin{pmatrix} y_{ii} & y_{ij} & x_i^\epsilon \\ y_{ij} & y_{jj} & x_j^\epsilon \\ x_i^\epsilon & x_j^\epsilon & I_d \end{pmatrix} \succeq 0 \quad \forall (i,j) \in \mathcal{A} \text{ with } i,j \in \bar{\mathcal{I}}.$$

Thus, replacing x_i in Z by x_i^ϵ for all $i \in \bar{\mathcal{I}}$ yields a $\tilde{Z}$ that is feasible for (4). Moreover, $\tilde{Z}$ is optimal for (4) (with $\delta = 0$) since, by (3) and (10), the objective function of (4) does not depend on x_i for $i \in \bar{\mathcal{I}}$. Thus $\tilde{Z} \in \mathcal{S}^0_{\rm esdp}$ but its x_i^ϵ component differs from the x_i component of Z for all $i \in \bar{\mathcal{I}}$, contradicting the definition of $\bar{\mathcal{I}}$. ∎

4 Trace test for accurately positioned sensors by SDP and ESDP: failure in the noisy case

We saw from Proposition 2 and Theorem 1 that, in the noiseless case, $\mathrm{tr}_i(Z) = 0$ for any interior ESDP solution Z implies x_i is invariant over all ESDP solutions (and hence $x_i = x_i^{\rm true}$) and conversely. Thus, by computing an interior ESDP solution (using, say, an interior-point method) and checking the individual traces, we can determine exactly which sensors are correctly positioned. Can this be extended to the noisy case? For example, if the noise level is sufficiently low and Z is the analytic center of the ESDP solution set, does $\mathrm{tr}_i(Z) = 0$ imply x_i is near $x_i^{\rm true}$? However, the examples below show that this is false for ESDP and SDP relaxations. Thus, ESDP and SDP relaxations are more sensitive to noises than the SOCP relaxation.

Our first example shows that Theorem 1 is false when $\mathcal{I}^0_{\rm esdp}$ and $\mathcal{S}^0_{\rm esdp}$ are replaced by $\mathcal{I}^\delta_{\rm esdp}$ and $\mathcal{S}^\delta_{\rm esdp}$, regardless of how small $\|\delta\|$ is.

Example 1. *Let $m = 1$, $n = 4$, $x_1^{\rm true} = (0,0)^T$, and x_2, x_3, x_4 be non-collinear points in $\mathbb{R}^2$. Let $\delta_{1i} = \epsilon$ for $i = 2,3,4$ ($\epsilon \geq 0$). Then $d_{1i}^2 = \|x_i\|^2 + \epsilon$ for $i = 2,3,4$. Here $\epsilon = 0$ corresponds to the noiseless case.*

The corresponding ESDP relaxation (4)*, which is equivalent to the SDP relaxation* (2)*, is the following problem:*

$$\begin{array}{ll} \min\limits_{Z} & \sum_{i=2}^4 |y_{11} - 2x_i^T x_1 + \|x_i\|^2 - d_{1i}^2| \\ \text{s.t.} & Z = \begin{pmatrix} y_{11} & x_1^T \\ x_1 & I_2 \end{pmatrix}, \quad y_{11} \geq \|x_1\|^2. \end{array} \tag{11}$$

We claim that

$$\mathcal{S}^\delta_{\rm esdp} = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & I_2 \end{pmatrix} \right\} \quad \forall \epsilon \geq 0.$$

To see this, note that the unique element of the above set is feasible for (11) *with zero objective value. Thus the optimal value of* (11) *is* 0*. Hence, for Z to be a solution of* (11)*, it must satisfy*

$$0 = y_{11} - 2x_i^T x_1 + \|x_i\|^2 - d_{1i}^2 = y_{11} - 2x_i^T x_1 - \epsilon, \quad i = 2,3,4. \tag{12}$$

Since x_2, x_3, x_4 are not collinear, the vectors $(1, -2x_i^T)$, $i = 2,3,4$, are linearly independent, implying that (12) *has a unique solution. Hence $\mathcal{I}^\delta_{\rm esdp} = \{1\}$. However, for $\epsilon > 0$, we have $\delta \neq 0$ and $\mathrm{tr}_1(Z) = \epsilon - 0 > 0$ for all $Z \in \mathcal{S}^\delta_{\rm esdp}$.*

In Example 1, as $\epsilon \to 0$, we have $x_1 = x_1^{\rm true}$ and $\mathrm{tr}_1(Z) \to 0$ for all $Z \in \mathcal{S}^\delta_{\rm esdp} = \mathcal{S}^\delta_{\rm sdp}$. In general, if $\delta \approx 0$, does $\mathrm{tr}_i(Z) = 0$ for some $Z \in \mathrm{ri}(\mathcal{S}^\delta_{\rm esdp})$ (or $Z \in \mathrm{ri}(\mathcal{S}^\delta_{\rm sdp})$) imply $x_i \approx x_i^{\rm true}$? Our second example below shows that this is false even when Z is the unique solution of the SDP/ESDP relaxation and $\|\delta\|$ is arbitrarily small. This contrasts with an SOCP relaxation of (1), for which such a result does hold [34, Proposition 7.2].

Example 2. *Let $m = 2$, $n = 6$, $x_1^{\rm true} = (2,0)^T$, $x_2^{\rm true} = (0,-1)^T$, $x_3 = (2,-1)^T$, $x_4 = (2,1)^T$, $x_5 = (-1,0)^T$, $x_6 = (1,0)^T$. Let $\delta_{12} = \sqrt{4 + (1-\epsilon)^2} - \sqrt{5}$, $\delta_{13} = \epsilon$, $\delta_{14} = -\epsilon$, and $\delta_{25} = \delta_{26} = 0$ ($0 \leq \epsilon < \frac{1}{2}$). Then $d_{12} = \sqrt{4 + (1-\epsilon)^2}$, $d_{13} = 1 + \epsilon$, $d_{14} = 1 - \epsilon$, $d_{25} = d_{26} = \sqrt{2}$. Here $\epsilon = 0$ corresponds to the noiseless case.*

The corresponding ESDP relaxation (4)*, which is equivalent to the SDP relaxation* (2)*, is the following problem:*

$$\begin{array}{ll} \min\limits_{Z} & |y_{11} - 2x_3^T x_1 + 5 - (1+\epsilon)^2| + |y_{11} - 2x_4^T x_1 + 5 - (1-\epsilon)^2| \\ & + |y_{22} - 2x_5^T x_2 - 1| + |y_{22} - 2x_6^T x_2 - 1| + |y_{11} - 2y_{12} + y_{22} - 4 - (1-\epsilon)^2| \\ \text{s.t.} & Z = \begin{pmatrix} y_{11} & y_{12} & x_1^T \\ y_{12} & y_{22} & x_2^T \\ x_1 & x_2 & I_2 \end{pmatrix} \succeq 0. \end{array} \tag{13}$$

We claim that

$$\mathcal{S}^\delta_{\rm esdp} = \left\{ \begin{pmatrix} 4+\epsilon^2 & \epsilon & 2 & \epsilon \\ \epsilon & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ \epsilon & 1 & 0 & 1 \end{pmatrix} \right\} \quad \forall 0 < \epsilon < \frac{1}{2}.$$

To see this, note that the unique element of the above set has zero objective value and it is feasible for (13) *because*

$$\begin{pmatrix} 4+\epsilon^2 & \epsilon \\ \epsilon & 1 \end{pmatrix} - \begin{pmatrix} 2 & \epsilon \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ \epsilon & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the optimal value of (13) *is* 0*. We show below that* (13) *has a unique solution.*

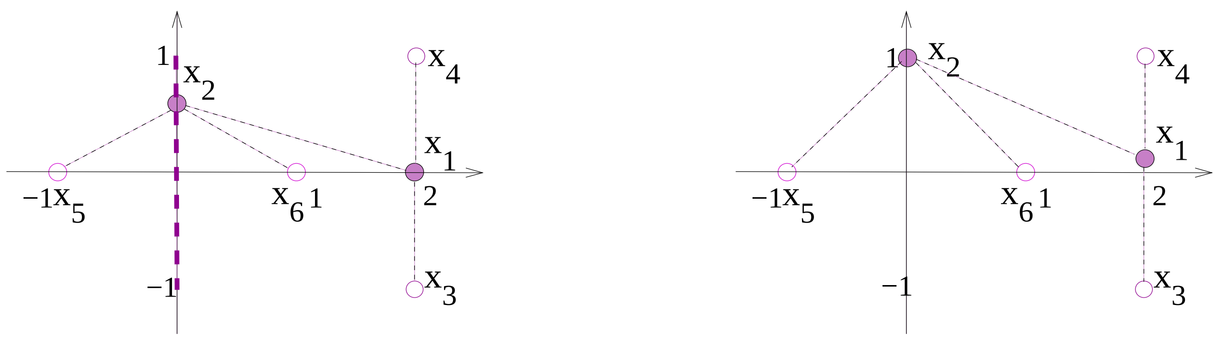

Figure 1: An example showing that, when distance measurements have noise, zero individual trace is uninformative of sensor position accuracy in ESDP and SDP solutions.

Since the optimal value of (13) *is zero, the expressions inside the absolute values in the objective function must be zero when evaluated at any* $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$*. Then we have from* $y_{11} - 2x_3^T x_1 + 5 - (1+\epsilon)^2 = y_{11} - 2x_4^T x_1 + 5 - (1-\epsilon)^2 = 0$ *and the constraint* $y_{11} \geq \|x_1\|^2$ *that* $x_1 = (2, \epsilon)^T$*,* $y_{11} = 4 + \epsilon^2$*, from* $y_{22} - 2x_5^T x_2 - 1 = y_{22} - 2x_6^T x_2 - 1 = 0$ *that* $x_2 = (0, t)^T$ *for some* $t \in \mathbb{R}$ *and* $y_{22} = 1$*, and from* $y_{11} - 2y_{12} + y_{22} - 4 - (1-\epsilon)^2 = 0$ *that* $y_{12} = \dfrac{y_{11} + y_{22} - 4 - (1-\epsilon)^2}{2} = \epsilon$*. Hence each* $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$ *must have the form*

$$Z = \begin{pmatrix} 4+\epsilon^2 & \epsilon & 2 & \epsilon \\ \epsilon & 1 & 0 & t \\ 2 & 0 & 1 & 0 \\ \epsilon & t & 0 & 1 \end{pmatrix}.$$

Since $Z \succeq 0$ *and* $\text{tr}_1(Z) = 0$*, we must have* $y_{12} - x_1^T x_2 = \epsilon - \begin{pmatrix} 2 & \epsilon \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} = 0$*, i.e.,* $t = 1$*. Thus, for* $\epsilon \in (0, \frac{1}{2})$*,* $\mathcal{S}^{\delta}_{\text{esdp}}$ *is a singleton and* $\mathcal{I}^{\delta}_{\text{esdp}} = \{1, 2\}$*. Moreover,* $\text{tr}_1(Z) = \text{tr}_2(Z) = 0$ *for all* $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$*. However, while* $x_1 = (2, \epsilon)^T$ *approaches* $x_1^{\text{true}} = (2, 0)^T$ *as* $\epsilon \to 0$*,* $x_2 = (0, 1)^T$ *does not approach* $x_2^{\text{true}} = (0, -1)^T$ *as* $\epsilon \to 0$*.*

By using the observation that

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1-t^2 \end{pmatrix} \succeq 0 \quad \forall t \in [-1, 1],$$

it is straightforward to verify that

$$\mathcal{S}^{0}_{\text{esdp}} = \left\{ \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & t \\ 2 & 0 & 1 & 0 \\ 0 & t & 0 & 1 \end{pmatrix} \;\middle|\; t \in [-1, 1] \right\}.$$

Hence $\mathcal{I}^{0}_{\text{esdp}} = \{1\}$*, i.e., only* x_1 *is invariant over* $\mathcal{S}^{0}_{\text{esdp}}$*.*

Example 2 shows that individual traces are uninformative of the accuracy of the ESDP solution in the presence of noise. In fact, we know of no easy way to judge which computed sensor positions are accurate in this case. In the next section, we propose a robust version of ESDP that overcomes this difficulty. We close this section with some compactness and semicontinuity properties of $\mathcal{S}^{\delta}_{\text{esdp}}$ with respect to δ. These properties will be used to prove Lemma 5.

Proposition 3.

(a) $\limsup_{\delta\to 0} \mathcal{S}^{\delta}_{\text{esdp}} \subseteq \mathcal{S}^{0}_{\text{esdp}}$.

(b) *For any bounded set* $\Delta \subseteq \mathbb{R}^{|\mathcal{A}|}$*,* $\bigcup_{\delta\in\Delta} \mathcal{S}^{\delta}_{\text{esdp}}$ *is a bounded set.*

(c) *For each* $\epsilon > 0$ *there exists a scalar* $\overline{\delta} > 0$ *such that*

$$\min_{Z_0\in\mathcal{S}^{0}_{\text{esdp}}} \|Z - Z_0\|_F \le \epsilon \quad \forall Z \in \mathcal{S}^{\delta}_{\text{esdp}},\ \forall\ 0 \le \|\delta\|_\infty < \overline{\delta}. \tag{14}$$

Proof. (a) Let $f^{\delta}(Z)$ denote the objective function of (4) with d_{ij}^2 given by (5). Fix any $\bar{Z} \in \limsup_{\delta\to 0} \mathcal{S}^{\delta}_{\text{esdp}}$. Then there exist sequences $\delta_k \in \mathbb{R}^{|\mathcal{A}|}$ and $Z_k \in \mathcal{S}^{\delta_k}_{\text{esdp}}$, $k = 1, 2, \ldots$, such that $\lim_{k\to\infty} \delta_k = 0$, $\lim_{k\to\infty} Z_k = \bar{Z}$. Since $Z_k \in \mathcal{F}_{\text{esdp}}$ and $\mathcal{F}_{\text{esdp}}$ is closed, $\bar{Z} \in \mathcal{F}_{\text{esdp}}$. Fix any $Z \in \mathcal{S}^{0}_{\text{esdp}}$. Since $Z_k \in \mathcal{S}^{\delta_k}_{\text{esdp}}$ and $Z \in \mathcal{F}_{\text{esdp}}$, we have

$$f^{\delta_k}(Z_k) \ \le\ f^{\delta_k}(Z), \quad k = 1, 2, \ldots$$

Taking limit yields

$$0 \ \le\ f^0(\bar{Z}) \ =\ \lim_{k\to\infty} f^{\delta_k}(Z_k) \ \le\ \lim_{k\to\infty} f^{\delta_k}(Z) \ =\ f^0(Z) \ =\ 0.$$

Hence $\bar{Z} \in \mathcal{S}^{0}_{\text{esdp}}$.

(b) Owing to the positive semidefinite constraints in (4) and $y_{ij} = 0$ for $(i,j) \notin \mathcal{A}$ and $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$, it suffices to show that y_{ii} is uniformly bounded over $\delta \in \Delta$ and $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$, for $i = 1, \ldots, m$. Fix any $\varrho > 0$ such that $\Delta \subseteq [-\varrho, \varrho]^{|\mathcal{A}|}$.

Since $Z^{\text{true}} \in \mathcal{F}_{\text{esdp}}$, we have for any $\delta \in \Delta$ and $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$ that

$$\begin{aligned}
\sum_{(i,j)\in\mathcal{A}} \left|\ell_{ij}(Z) - d_{ij}^2\right| \ &\le\ \sum_{(i,j)\in\mathcal{A}} \left|\ell_{ij}(Z^{\text{true}}) - d_{ij}^2\right| \\
&=\ \sum_{(i,j)\in\mathcal{A}} \left|\|x_i^{\text{true}} - x_j^{\text{true}}\|^2 - d_{ij}^2\right| \\
&=\ \sum_{(i,j)\in\mathcal{A}} |\delta_{ij}| \ \le\ |\mathcal{A}|\varrho,
\end{aligned}$$

where the equalities use (3), (9), and (5). Thus (3) yields

$$\left.\begin{array}{lll} y_{ii} - 2y_{ij} + y_{jj} & \le d_{ij}^2 + |\mathcal{A}|\varrho & \forall (i,j) \in \mathcal{A}^s, \\ y_{ii} - 2x_i^T x_j + \|x_j\|^2 & \le d_{ij}^2 + |\mathcal{A}|\varrho & \forall (i,j) \in \mathcal{A} \text{ with } i \le m < j, \end{array}\right\} \quad \forall Z \in \mathcal{S}^{\delta}_{\text{esdp}},\ \forall \delta \in \Delta. \tag{15}$$

Fix any $\delta \in \Delta$ and $Z \in \mathcal{S}^{\delta}_{\text{esdp}}$. We first consider those $i \le m$ such that $(i,j) \in \mathcal{A}$ for some $j > m$ (i.e., i is a neighbor of some anchor). We have from $\text{tr}_i(Z) \ge 0$ and (15) that

$$\|x_i\|^2 - 2\|x_i\|\|x_j\| + \|x_j\|^2 \le y_{ii} - 2x_i^T x_j + \|x_j\|^2 \le d_{ij}^2 + |\mathcal{A}|\varrho \le (d_{ij}^{\text{true}})^2 + \bar{\varrho},$$

where we let $d_{ij}^{\text{true}} := \|x_i^{\text{true}} - x_j^{\text{true}}\|$ and $\bar{\varrho} := (1+|\mathcal{A}|)\varrho$. By writing the left-hand side as $(\|x_i\| - \|x_j\|)^2$, we obtain $\|x_i\| \le \sqrt{(d_{ij}^{\text{true}})^2 + \bar{\varrho}} + \|x_j\|$, which together with the second inequality above yields

$$y_{ii} \le (d_{ij}^{\text{true}})^2 + \bar{\varrho} + 2\|x_j\|\sqrt{(d_{ij}^{\text{true}})^2 + \bar{\varrho}} + \|x_j\|^2.$$

We next consider those $i \le m$ such that $(i,j) \in \mathcal{A}$ for some $j \le m$. We have from $Z_{\{i,j\}} \succeq 0$ and (15) that

$$(\sqrt{y_{ii}} - \sqrt{y_{jj}})^2 = y_{ii} - 2\sqrt{y_{ii}y_{jj}} + y_{jj} \le y_{ii} - 2y_{ij} + y_{jj} \le d_{ij}^2 + |\mathcal{A}|\varrho \le (d_{ij}^{\text{true}})^2 + \bar{\varrho},$$

from which it follows that $0 \le y_{ii} \le \left(\sqrt{(d_{ij}^{\text{true}})^2 + \bar{\varrho}} + \sqrt{y_{jj}}\right)^2$. It then follows from induction and Assumption 1 that each y_{ii} is uniformly bounded, independent of δ.

(c) If the statement were false, then there would exist an $\epsilon > 0$ such that, for each integer $k > 0$ there exist $\delta_k \in \mathbb{R}^{|\mathcal{A}|}$ with $\|\delta_k\|_\infty \le \frac{1}{k}$ and $Z_k \in \mathcal{S}_{\text{esdp}}^{\delta_k}$ satisfying

$$\min_{Z_0 \in \mathcal{S}_{\text{esdp}}^0} \|Z_k - Z_0\|_F > \epsilon. \tag{16}$$

By part (b), $\{Z_k\}$ is bounded. Since $\{\delta_k\} \to 0$, part (a) implies that every cluster point of $\{Z_k\}$ lies in $\mathcal{S}_{\text{esdp}}^0$, so that

$$\min_{Z_0 \in \mathcal{S}_{\text{esdp}}^0} \|Z_k - Z_0\|_F \to 0 \quad \text{as} \quad k \to \infty.$$

This contradicts (16). ■

Proposition 3(c) implies that, when the noise level $\|\delta\|$ is low, the computed position x_i from $Z \in \mathcal{S}_{\text{esdp}}^\delta$ is near its true position x_i^{true} for all $i \in \mathcal{I}_{\text{esdp}}^0$. However, in practice we are unlikely to know $\mathcal{I}_{\text{esdp}}^0$ and, as Example 2 shows, there is no easy way to estimate $\mathcal{I}_{\text{esdp}}^0$ from $Z \in \mathcal{S}_{\text{esdp}}^\delta$, however small $\|\delta\|$ is.

5 A robust ESDP relaxation

We saw from Example 2 that SDP and ESDP relaxations have the defect that individual traces are uninformative of sensor position accuracy in the presence of noise. In this section we propose a noise-aware robust version of the ESDP relaxation that dampens sensitivity to noise by expanding the solution set to include the noiseless solutions. In particular, let $\mathcal{S}_{\text{resdp}}^{\rho,\delta}$ denote the set of Z satisfying

$$Z \in \mathcal{F}_{\text{esdp}} \quad \text{and} \quad |\ell_{ij}(Z) - d_{ij}^2| \le \rho_{ij} \quad \forall (i,j) \in \mathcal{A}. \tag{17}$$

with $\rho = (\rho_{ij})_{(i,j)\in\mathcal{A}} \ge 0$. Notice that each Z satisfying (17) belongs to $\mathcal{S}_{\text{esdp}}^u$, where $u_{ij} = \ell_{ij}(Z) - d_{ij}^2 + \delta_{ij}$ for all $(i,j) \in \mathcal{A}$. Since $|u| \le \rho + |\delta|$, where $|\cdot|$ is taken componentwise, this implies

$$\mathcal{S}_{\text{resdp}}^{\rho,\delta} \subseteq \bigcup_{|u| \le \rho + |\delta|} \mathcal{S}_{\text{esdp}}^u. \tag{18}$$

By Proposition 3(b), the right-hand side is bounded. Hence (18) implies that $\mathcal{S}_{\text{resdp}}^{\rho,\delta}$ is bounded. Moreover, if $\rho \ge |\delta|$ (i.e., $\rho_{ij} \ge |\delta_{ij}|$ for all $(i,j) \in \mathcal{A}$), then

$$\mathcal{S}_{\text{esdp}}^0 \subseteq \mathcal{S}_{\text{resdp}}^{\rho,\delta}. \tag{19}$$

(since $Z \in \mathcal{S}_{\text{esdp}}^0$ implies $\ell_{ij}(Z) = \|x_i^{\text{true}} - x_j^{\text{true}}\|^2$ and hence $|\ell_{ij}(Z) - d_{ij}^2| = |\delta_{ij}|$ for all $(i,j) \in \mathcal{A}$). Then any x_i that is not invariant over $\mathcal{S}_{\text{esdp}}^0$ would also not be invariant over $\mathcal{S}_{\text{resdp}}^{\rho,\delta}$. In applications, each $|\delta_{ij}|$

may be estimated by d_{ij}^2/σ_{ij}, where σ_{ij} is the signal-to-noise ratio for communication between sensors i and j.

The following lemma shows that the robust ESDP generalizes two key properties of ESDP in the noiseless case (i.e., Proposition 2 and Theorem 1) to the noisy case. Its proof uses Theorem 1, Propositions 2 and 3, as well as (18) and (19).

Lemma 5.

(a) $\lim_{|\delta|<\rho\to 0} \mathcal{S}_{\text{resdp}}^{\rho,\delta} = \mathcal{S}_{\text{esdp}}^{0}$.

(b) *For each $i \in \{1,\ldots,m\}$, $i \in \mathcal{I}_{\text{esdp}}^0$ if and only if for every $\eta > 0$, there exists a $\bar{\rho} > 0$ such that*

$$\text{tr}_i(Z) < \eta \quad \forall Z \in \mathcal{S}_{\text{resdp}}^{\rho,\delta},\ \forall |\delta| < \rho < \bar{\rho}e, \tag{20}$$

where $e := (1,...,1)^T \in \mathbb{R}^{|\mathcal{A}|}$.

Proof. Part (a) follows readily from (18), (19), and Proposition 3(a). We prove part (b) below.

Fix any $i \in \{1,\ldots,m\}$. Suppose that for every $\eta > 0$ there exists a $\bar{\rho} > 0$ such that (20) holds. Fix any $Z_0 \in \text{ri}(\mathcal{S}_{\text{esdp}}^0)$. For any $\eta > 0$, by taking $|\delta| < \rho$ sufficiently small, we have from (19) that $Z_0 \in \mathcal{S}_{\text{resdp}}^{\rho,\delta}$ and from (20) that $\text{tr}_i(Z_0) < \eta$. Hence $\text{tr}_i(Z_0) = 0$, so that Proposition 2 yields $i \in \mathcal{I}_{\text{esdp}}^0$.

Conversely, suppose that $i \in \mathcal{I}_{\text{esdp}}^0$. We have from (18) that

$$\mathcal{S}_{\text{resdp}}^{\rho,\delta} \subseteq \bigcup_{|u|<2\rho} \mathcal{S}_{\text{esdp}}^{u} \subseteq \mathcal{F} := \bigcup_{|u|<2e} \mathcal{S}_{\text{esdp}}^{u} \qquad \forall |\delta| < \rho \le e. \tag{21}$$

By Proposition 3(b), $\mathcal{F}$ is bounded. Since $\text{tr}_i(Z)$ is continuous in Z, this implies that $\text{tr}_i(Z)$ is uniformly continuous over $Z \in \mathcal{F}$, i.e., for any $\eta > 0$, there exists an $\epsilon > 0$ such that

$$|\text{tr}_i(Z) - \text{tr}_i(Z')| < \eta \qquad \forall Z, Z' \in \mathcal{F} \text{ with } \|Z - Z'\|_F \le \epsilon. \tag{22}$$

By Proposition 3(c), there exists a $\bar{\delta} > 0$ satisfying (14). Take $\bar{\rho} = \min\{1, \bar{\delta}\}/2$. Then (21), (14), (22), together with $\text{tr}_i(Z_0) = 0$ for all $Z_0 \in \mathcal{S}_{\text{esdp}}^0$ (see Theorem 1), yield (20). ■

Lemma 5(c) says that we can determine whether $i \in \mathcal{I}_{\text{esdp}}^0$ by checking $\text{tr}_i(Z)$ for all $Z \in \mathcal{S}_{\text{resdp}}^{\rho,\delta}$ and all $|\delta| < \rho$ near 0. This is clearly an impractical way to find $\mathcal{I}_{\text{esdp}}^0$. Below we consider a more practical way based on computing, for a single $|\delta| < \rho$ near 0, a $Z \in \mathcal{S}_{\text{resdp}}^{\rho,\delta}$ that is "most interior" and hence least sensitive to noise. In particular, for each $|\delta| < \rho$, let $Z^{\rho,\delta}$ be the unique solution of the following log-barrier problem:

$$\min_{Z \in \mathcal{S}_{\text{resdp}}^{\rho,\delta}} \quad B(Z) := -\sum_{(i,j)\in\mathcal{A}^s} \ln\det(Z_{\{i,j,m+\}}) - \sum_{i=1}^{m} \ln \text{tr}_i(Z). \tag{23}$$

Since $|\delta| < \rho$, there exists a $Z \in \mathcal{S}_{\text{resdp}}^{\rho,\delta}$ satisfying $B(Z) < \infty$ (e.g., take any $Z \in \mathcal{S}_{\text{esdp}}^0$ and increase y_{ii}, $i = 1,\ldots,m$, by a sufficiently small amount). Moreover, the objective function of (23) is a strictly convex function and $\mathcal{S}_{\text{resdp}}^{\rho,\delta}$ is compact. Hence $Z^{\rho,\delta}$, which may be viewed as a variant of the analytic center of $\mathcal{S}_{\text{resdp}}^{\rho,\delta}$, is well defined, unique, and $B(Z^{\rho,\delta}) < \infty$. The following result justifies the term of robust ESDP, showing that $\text{tr}_i(Z^{\rho,\delta}) \approx 0$ and $x_i^{\rho,\delta} \approx x_i^{\text{true}}$ whenever $|\delta| < \rho \approx 0$, for all $i \in \mathcal{I}_{\text{esdp}}^0$. Its proof uses Theorem 1 and Lemma 5(a).

Theorem 3.

(a) *Every cluster point of* $\{Z^{\rho,\delta}\}$*, as* $|\delta| < \rho \to 0$*, belongs to* $\mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$.

(b) *For each* $i \in \mathcal{I}^0_{\mathrm{esdp}}$,

$$\lim_{|\delta|<\rho\to 0} \mathrm{tr}_i(Z^{\rho,\delta}) = 0 \quad \text{and} \quad \lim_{|\delta|<\rho\to 0} x_i^{\rho,\delta} = x_i^{\mathrm{true}}. \tag{24}$$

Proof. (a) Since $B \succeq 0$ and $\bar{B} \succeq 0$ imply that $\mathrm{Null}(B+\bar{B}) = \mathrm{Null}(B) \cap \mathrm{Null}(\bar{B})$, we see that, for each $(i,j) \in \mathcal{A}^s$ and $l \in \{1,...,m\}$, $\mathrm{rank}(Z_{\{i,j,m^+\}})$ and $\mathrm{rank}(Z_{\{l,m^+\}})$ are constant over all $Z \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$, which we denote by r_{ij} and r_l, respectively. Then, $\mathrm{rank}(Z_{\{i,j,m^+\}}) \le r_{ij}$ and $\mathrm{rank}(Z_{\{l,m^+\}}) \le r_l$ for all $(i,j) \in \mathcal{A}^s$, $l \in \{1,...,m\}$, and $Z \in \mathcal{S}^0_{\mathrm{esdp}}$. Moreover,

$$Z \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}}) \iff Z \in \mathcal{S}^0_{\mathrm{esdp}},\ \mathrm{rank}(Z_{\{i,j,m^+\}}) = r_{ij}\ \forall (i,j) \in \mathcal{A}^s,\ \mathrm{rank}(Z_{\{i,m^+\}}) = r_i\ \forall i. \tag{25}$$

For any $Z \in \mathcal{F}_{\mathrm{esdp}}$ and $i \in \{1,...,m\}$, since $\mathrm{tr}_i(Z)$ is the Schur complement of I_d in $Z_{\{i,m^+\}}$, we have $\mathrm{rank}(Z_{\{i,m^+\}}) = d + \mathrm{rank}(\mathrm{tr}_i(Z))$. Then (25), together with Proposition 2 and Theorem 1, implies that

$$r_i = \begin{cases} d & \text{if } i \in \mathcal{I}^0_{\mathrm{esdp}}; \\ d+1 & \text{if } i \notin \mathcal{I}^0_{\mathrm{esdp}}. \end{cases}$$

Hence (25) is equivalent to

$$Z \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}}) \iff Z \in \mathcal{S}^0_{\mathrm{esdp}},\ \mathrm{rank}(Z_{\{i,j,m^+\}}) = r_{ij}\ \forall (i,j) \in \mathcal{A}^s,\ \mathrm{tr}_i(Z) > 0\ \forall i \notin \mathcal{I}^0_{\mathrm{esdp}}. \tag{26}$$

For any $W \in \mathcal{S}^p$ $(p \ge 1)$, let $\lambda_k(W)$ denote the kth eigenvalue of W, arranged in decreasing order. Let

$$\begin{aligned} \lambda_k^{i,j}(Z) &:= \lambda_k\left(Z_{\{i,j,m^+\}}\right), \quad k = 1,\ldots,d+2,\ (i,j) \in \mathcal{A}^s, \\ \mathcal{J}^a &:= \{(i,j,k) \mid (i,j) \in \mathcal{A}^s,\ 1 \le k \le r_{ij}\}, \\ B^a(Z) &:= -\sum_{(i,j,k)\in\mathcal{J}^a} \ln \lambda_k^{i,j}(Z) - \sum_{i\notin\mathcal{I}^0_{\mathrm{esdp}}} \ln \mathrm{tr}_i(Z). \end{aligned}$$

Then, by (26), $\mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}}) = \mathcal{S}^0_{\mathrm{esdp}} \cap \mathrm{dom} B^a$.

Let $\bar{Z}$ be any cluster point of $\{Z^{\rho,\delta}\}$ as $|\delta| < \rho \to 0$. By Lemma 5(a), $\bar{Z} \in \mathcal{S}^0_{\mathrm{esdp}}$. Suppose to the contrary that $\bar{Z} \notin \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$, so that $\bar{Z} \notin \mathrm{dom}B^a$. Consider any sequence $|\delta_t| < \rho_t$, $t = 1,2,\ldots$, such that $\{\rho_t\} \to 0$ and $\{Z^{\rho_t,\delta_t}\} \to \bar{Z}$. Fix any $Z^a \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$. Hence $\frac{\bar{Z}+Z^a}{2} \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$. Since $\bar{Z} \notin \mathrm{dom}B^a$, we have $\{B^a(Z^{\rho_t,\delta_t})\} \to \infty$. Since $\frac{\bar{Z}+Z^a}{2} \in \mathrm{dom}B^a$ so that B^a is continuous there, we also have $\left\{B^a\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right)\right\} \to B^a\left(\frac{\bar{Z}+Z^a}{2}\right) < \infty$. Thus

$$B^a\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) - B^a(Z^{\rho_t,\delta_t}) \to -\infty \quad \text{as} \quad t \to \infty. \tag{27}$$

On the other hand, for $(i,j,k) \notin \mathcal{J}^a$, since $Z^a_{\{i,j,m^+\}} \succeq 0$ and $\lambda_k^{i,j}(\cdot)$ is operator monotone (see [17, Corollary 4.3.3]), we have

$$\lambda_k^{i,j}\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) \ge \lambda_k^{i,j}\left(\frac{1}{2}Z^{\rho_t,\delta_t}\right)$$

for all t. Since $-\ln(\cdot)$ is nonincreasing, this implies that

$$-\ln\lambda_k^{i,j}\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) \le -\ln\lambda_k^{i,j}\left(\frac{1}{2}Z^{\rho_t,\delta_t}\right) = -\ln\lambda_k^{i,j}(Z^{\rho_t,\delta_t}) + \ln 2. \tag{28}$$

For $i = 1, \ldots, m$, we have from (6) that

$$\mathrm{tr}_i\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) = \frac{1}{2}\mathrm{tr}_i(Z^{\rho_t,\delta_t}) + \frac{1}{2}\mathrm{tr}_i(Z^a) + \frac{1}{4}\|x_i^{\rho_t,\delta_t} - x_i^a\|^2 \geq \frac{1}{2}\mathrm{tr}_i(Z^{\rho_t,\delta_t})$$

for all t. Since $-\ln(\cdot)$ is nonincreasing, this implies that

$$-\ln \mathrm{tr}_i\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) \leq -\ln \frac{1}{2}\mathrm{tr}_i\left(Z^{\rho_t,\delta_t}\right) = -\ln \mathrm{tr}_i(Z^{\rho_t,\delta_t}) + \ln 2. \tag{29}$$

Combining (28) and (29) yields that

$$\begin{aligned}
& B\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) \\
= \; & B^a\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) - \sum_{(i,j,k)\notin\mathcal{J}^a} \ln \lambda_k^{i,j}\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) - \sum_{i\in\mathcal{I}^0_{\mathrm{esdp}}} \ln \mathrm{tr}_i\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) \\
\leq \; & B^a\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) - \sum_{(i,j,k)\notin\mathcal{J}^a} \ln \lambda_k^{i,j}(Z^{\rho_t,\delta_t}) - \sum_{i\in\mathcal{I}^0_{\mathrm{esdp}}} \ln \mathrm{tr}_i(Z^{\rho_t,\delta_t}) + (|\mathcal{A}^s|(2+d)+m)\ln 2 \\
= \; & B^a\left(\frac{Z^{\rho_t,\delta_t}+Z^a}{2}\right) - B^a(Z^{\rho_t,\delta_t}) + B(Z^{\rho_t,\delta_t}) + (|\mathcal{A}^s|(2+d)+m)\ln 2.
\end{aligned}$$

By (27), the right-hand side is less than $B(Z^{\rho_t,\delta_t})$ for all t sufficiently large. Since $Z^a \in \mathcal{S}^0_{\mathrm{esdp}} \subseteq \mathcal{S}^{\rho_t,\delta_t}_{\mathrm{resdp}}$ (see (19)) so that $\frac{Z^{\rho_t,\delta_t}+Z^a}{2} \in \mathcal{S}^{\rho_t,\delta_t}_{\mathrm{resdp}}$, this contradicts the definition of Z^{ρ_t,δ_t} as the solution of (23). Thus, $B^a(\bar{Z}) < \infty$ and hence $\bar{Z} \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$.

(b) By (21), $\{Z^{\rho,\delta}\}_{|\delta|<\rho\leq e}$ lies in the bounded set $\mathcal{F}$. By Lemma 5(a), as $|\delta| < \rho \to 0$, all cluster points of $\{Z^{\rho,\delta}\}$ are in $\mathcal{S}^0_{\mathrm{esdp}}$. For each $i \in \mathcal{I}^0_{\mathrm{esdp}}$, since x_i is invariant over $\mathcal{S}^0_{\mathrm{esdp}}$, we have $\bar{x}_i = x_i^{\mathrm{true}}$ for every cluster point $\bar{x}_i$ of $\{x_i^{\rho,\delta}\}$ as $|\delta| < \rho \to 0$. Since $\{x_i^{\rho,\delta}\}$ lies in a bounded set, this implies that

$$\lim_{|\delta|<\rho\to 0} x_i^{\rho,\delta} = x_i^{\mathrm{true}}.$$

Similarly, Theorem 1 implies that $\mathrm{tr}_i(Z) = 0$ for all $Z \in \mathcal{S}^0_{\mathrm{esdp}}$. Since $\{x_i^{\rho,\delta}\}$ and $\{y_{ii}^{\rho,\delta}\}$ lie in a bounded set, this implies that

$$\lim_{|\delta|<\rho\to 0} \mathrm{tr}_i(Z^{\rho,\delta}) = 0.$$

■

It is an open question whether $\{Z^{\rho,\delta}\}$ converges as $|\delta| < \rho \to 0$ and, if yes, what the limit is; see [24] and references therein for related results. The following result shows that $\mathcal{I}^0_{\mathrm{esdp}}$ is identified by those i with $\mathrm{tr}_i(Z^{\rho,\delta}) \approx 0$ for any $|\delta| < \rho \approx 0$. It also shows that the distance from $x_i^{\rho,\delta}$ to its true position x_i^{true} is $O\left(\sqrt{\mathrm{tr}_i(Z^{\rho,\delta})}\right)$. Its proof uses Proposition 2, Lemma 4, and Theorem 3.

Theorem 4.

(a) *There exists* $\bar{\eta} > 0$, $\bar{\rho} > 0$ *such that*

$$\begin{aligned}
\mathrm{tr}_i(Z^{\rho,\delta}) < \bar{\eta} \;\; \text{for some } |\delta| < \rho \leq \bar{\rho}e \quad &\Longrightarrow \quad i \in \mathcal{I}^0_{\mathrm{esdp}}, \\
\mathrm{tr}_i(Z^{\rho,\delta}) \geq 0.1\bar{\eta} \;\; \text{for some } |\delta| < \rho \leq \bar{\rho}e \quad &\Longrightarrow \quad i \notin \mathcal{I}^0_{\mathrm{esdp}},
\end{aligned}$$

where $e := (1, \ldots, 1)^T \in \mathbb{R}^{|\mathcal{A}|}$.

(b) *For $i \in \{1, \ldots, m\}$,*

$$\|x_i^{\rho,\delta} - x_i^{\mathrm{true}}\| \le \sqrt{2|\mathcal{A}^s| + m} \left(\mathrm{tr}_i(Z^{\rho,\delta})\right)^{\frac{1}{2}} \quad \forall |\delta| < \rho. \tag{30}$$

Proof. (a) Let $\mathcal{C}$ denote the set of all cluster points of $Z^{\rho,\delta}$ as $|\delta| < \rho \to 0$. By Theorem 3, $\mathcal{C} \subseteq \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$. Then $\mathcal{C}$, being a closed subset of $\mathcal{S}^0_{\mathrm{esdp}}$, is compact. Define

$$\bar{\eta} := \frac{1}{2} \min_{i \notin \mathcal{I}^0_{\mathrm{esdp}}} \inf_{Z \in \mathcal{C}} \mathrm{tr}_i(Z). \tag{31}$$

Since $\mathrm{tr}_i(Z)$ is continuous in Z and $\mathcal{C}$ is compact, for each $i \in \{1, \ldots, m\} \setminus \mathcal{I}^0_{\mathrm{esdp}}$, there exists $Z^i \in \mathcal{C}$ such that $\inf_{Z \in \mathcal{C}} \mathrm{tr}_i(Z) = \mathrm{tr}_i(Z^i)$ and, by Proposition 2, $\mathrm{tr}_i(Z^i) > 0$. Hence $\bar{\eta} > 0$.

We claim that there exists a $\bar{\rho} > 0$ such that

$$\mathrm{tr}_i(Z^{\rho,\delta}) \ge \bar{\eta} \quad \forall i \notin \mathcal{I}^0_{\mathrm{esdp}}, \ \forall |\delta| < \rho \le \bar{\rho} e.$$

If this claim were false, then there would exist some $i \notin \mathcal{I}^0_{\mathrm{esdp}}$ and sequence $|\delta_t| < \rho_t$, $t = 1, 2, \ldots$, such that $\{\rho_t\} \to 0$ and $\mathrm{tr}_i(Z^{\rho_t,\delta_t}) < \bar{\eta}$ for all t. Then taking the limit would yield $\mathrm{tr}_i(\bar{Z}) \le \bar{\eta}$, where $\bar{Z}$ is any cluster point of $\{Z^{\rho_t,\delta_t}\}$. Since $\bar{Z} \in \mathcal{C}$, (31) would imply $\bar{\eta} \le \frac{1}{2}\mathrm{tr}_i(\bar{Z})$, a contradiction of $\bar{\eta} > 0$.

By Theorem 3(b), for each $i \in \mathcal{I}^0_{\mathrm{esdp}}$, we have that $\lim_{|\delta| < \rho \to 0} \mathrm{tr}_i(Z^{\rho,\delta}) = 0$. Combining this with the preceding claim, we conclude that there exists $\bar{\rho} > 0$ such that

$$\mathrm{tr}_i(Z^{\rho,\delta}) < 0.1\bar{\eta} \quad \forall i \in \mathcal{I}^0_{\mathrm{esdp}} \quad \text{and} \quad \mathrm{tr}_i(Z^{\rho,\delta}) \ge \bar{\eta} \quad \forall i \notin \mathcal{I}^0_{\mathrm{esdp}}, \quad \forall |\delta| < \rho \le \bar{\rho} e.$$

(b) Fix any $|\delta| < \rho$. Let $\alpha \in (0, 1)$. For any $i \in \{1, \ldots, m\}$, we have from (6) and $\mathrm{tr}_i(Z^{\mathrm{true}}) = 0$ (see (9)) that

$$\mathrm{tr}_i(\alpha Z^{\rho,\delta} + (1-\alpha) Z^{\mathrm{true}}) = \alpha \mathrm{tr}_i(Z^{\rho,\delta}) + \alpha(1-\alpha)\|x_i^{\rho,\delta} - x_i^{\mathrm{true}}\|^2. \tag{32}$$

For any $(i, j) \in \mathcal{A}^s$, letting $\bar{A} = \begin{pmatrix} x_i^{\rho,\delta} & x_j^{\rho,\delta} \end{pmatrix}$, $A = \begin{pmatrix} x_i^{\mathrm{true}} & x_j^{\mathrm{true}} \end{pmatrix}$, $\bar{B} = Z^{\rho,\delta}_{\{i,j\}}$, $B = Z^{\mathrm{true}}_{\{i,j\}}$, we have

$$\begin{aligned}
& \det\left(\alpha Z^{\rho,\delta}_{\{i,j,m+\}} + (1-\alpha) Z^{\mathrm{true}}_{\{i,j,m+\}}\right) \\
= \ & \det \begin{pmatrix} \alpha\bar{B} + (1-\alpha)B & (\alpha\bar{A} + (1-\alpha)A)^T \\ \alpha\bar{A} + (1-\alpha)A & I_d \end{pmatrix} \\
= \ & \det\left((\alpha\bar{B} + (1-\alpha)B) - (\alpha\bar{A} + (1-\alpha)A)^T(\alpha\bar{A} + (1-\alpha)A)\right) \\
= \ & \det\left(\alpha(\bar{B} - \bar{A}^T\bar{A}) + (1-\alpha)(B - A^TA) + \alpha(1-\alpha)\left(\bar{A} - A\right)^T\left(\bar{A} - A\right)\right) \\
\ge \ & \det\left(\alpha(\bar{B} - \bar{A}^T\bar{A})\right) \\
= \ & \alpha^2 \det(Z^{\rho,\delta}_{\{i,j,m+\}}),
\end{aligned} \tag{33}$$

where the third equality uses Lemma 4; the fourth equality uses $\det\begin{pmatrix} \bar{B} & \bar{A}^T \\ \bar{A} & I_d \end{pmatrix} = \det(\bar{B} - \bar{A}^T\bar{A})$ and the second equality uses an analogous identity; the inequality uses $B - A^TA \succeq 0$, $(\bar{A} - A)^T(\bar{A} - A) \succeq 0$, and the monotonicity of $\det(\cdot)$ with respect to $\succeq$ over positive semidefinite matrices. Note that the solution $Z^{\rho,\delta}$ of (23) equivalently solves

$$\max_{Z \in \mathcal{S}^{\rho,\delta}_{\mathrm{resdp}}} G(Z) := \prod_{(i,j) \in \mathcal{A}^s} \det Z_{\{i,j,m+\}} \prod_{i=1}^m \mathrm{tr}_i(Z).$$

Since $Z^{\text{true}} \in \mathcal{S}^0_{\text{esdp}}$, (19) implies that $\alpha Z^{\rho,\delta} + (1-\alpha)Z^{\text{true}} \in \mathcal{S}^{\rho,\delta}_{\text{resdp}}$. Hence, for any $\bar{i} \in \{1,\ldots,m\}$, we have

$$\begin{aligned} G(Z^{\rho,\delta}) &\geq G(\alpha Z^{\rho,\delta} + (1-\alpha)Z^{\text{true}}) \\ &\geq \left(\prod_{(i,j)\in\mathcal{A}^s} \alpha^2 \det(Z^{\rho,\delta}_{\{i,j,m+\}})\right)\left(\prod_{i\neq\bar{i}} \alpha \mathrm{tr}_i(Z^{\rho,\delta})\right)\left(\alpha \mathrm{tr}_{\bar{i}}(Z^{\rho,\delta}) + \alpha(1-\alpha)\|x^{\rho,\delta}_{\bar{i}} - x^{\text{true}}_{\bar{i}}\|^2\right) \\ &= \alpha^{2|\mathcal{A}^s|+m} G(Z^{\rho,\delta}) + \alpha^{2|\mathcal{A}^s|+m}(1-\alpha)\frac{G(Z^{\rho,\delta})}{\mathrm{tr}_{\bar{i}}(Z^{\rho,\delta})}\|x^{\rho,\delta}_{\bar{i}} - x^{\text{true}}_{\bar{i}}\|^2, \end{aligned}$$

where the inequality uses (32) and (33). It follows that

$$\|x^{\rho,\delta}_{\bar{i}} - x^{\text{true}}_{\bar{i}}\|^2 \leq \frac{1-\alpha^{2|\mathcal{A}^s|+m}}{\alpha^{2|\mathcal{A}^s|+m}(1-\alpha)} \mathrm{tr}_{\bar{i}}(Z^{\rho,\delta}) \quad \forall \alpha \in (0,1).$$

Letting $\alpha \to 1$ and using $\lim_{\alpha\to 1} \frac{1-\alpha^r}{1-\alpha} = r$ $(r \geq 1)$ yields (30). ■

Remark 5.1. *It can be seen that* (32) *and* (33) *hold for any* $Z \in \mathcal{S}^{\rho,\delta}_{\text{resdp}}$ *in place of* Z^{true}. *Hence, following the proof of Theorem 4(b), we have for each* $i = 1,\ldots,m$ *that,*

$$\sup_{Z,Z' \in \mathcal{S}^{\rho,\delta}_{\text{resdp}}} \|x_i - x'_i\| \leq 2\sqrt{2|\mathcal{A}^s| + m}(\mathrm{tr}_i(Z^{\rho,\delta}))^{\frac{1}{2}}.$$

This suggests that $\mathrm{tr}_i(Z^{\rho,\delta})$ *will likely increase as* ρ *increases since the set* $\mathcal{S}^{\rho,\delta}_{\text{resdp}}$ *will be enlarged.*

6 An LPCGD method for solving the robust ESDP relaxation

The results of Section 5 suggest solving (23), with ρ small but above the noise level, and then checking the individual traces of the solution to determine which sensors are accurately positioned. How can (23) be efficiently solved? An interior-point method can be used, but it cannot easily exploit the problem structure and distribute the computation over sensors—an important consideration for practical implementation. In this section, we propose a method for solving (23) that can distribute the computation over sensors by exploiting the partially separable structure of the problem. This method is a block-coordinate gradient descent method [35], similar to the one used in [34, Section 8] for an SOCP relaxation, applied to an unconstrained reformulation of (23) using quadratic penalization. In our simulation (see Section 7), this method is significantly faster than solving the ESDP relaxation (4) by an interior-point method.

We first reformulate (17) as a smooth convex optimization problem over $\mathcal{F}_{\text{esdp}}$ by introducing a smooth convex penalty function for its second set of constraints. For any scalar $r > 0$, let

$$h_r(t) := \frac{1}{2}\max\{0, t-r\}^2 + \frac{1}{2}\max\{0, -t-r\}^2 = \frac{1}{2}\max\{0, |t|-r\}^2.$$

Then h_r is smooth (i.e., continuously differentiable), convex, nonnegative-valued, and $h_r(t) = 0$ if and only if $|t| \leq r$. For any $\rho = (\rho_{ij})_{(i,j)\in\mathcal{A}} > 0$, define the smooth convex penalty function

$$f_\rho(Z) := \sum_{(i,j)\in\mathcal{A}} h_{\rho_{ij}}(\ell_{ij}(Z) - d^2_{ij}). \tag{34}$$

Then when $\rho \geq |\delta|$, $Z \in \mathcal{S}^{\rho,\delta}_{\text{resdp}}$ if and only if $Z \in \mathcal{F}_{\text{esdp}}$ and $f_\rho(Z) = 0$ (i.e., Z is a minimizer of f_ρ over $\mathcal{F}_{\text{esdp}}$ with zero objective function value). We augment f_ρ by a scalar $\mu > 0$ multiple of the log-barrier function B from (23) to obtain the following log-barrier penalty function:

$$f^\mu_\rho(Z) := f_\rho(Z) + \mu B(Z). \tag{35}$$

Then f_ρ^μ is convex, twice differentiable on dom$B$, partially separable (i.e., a sum of functions, each of few variables), and $f_\rho^\mu(Z) \to \infty$ as Z approaches any boundary point of dom$B$. A standard argument shows that $\arg\min_Z f_\rho^\mu(Z) \to Z^{\rho,\delta}$ as $\mu \to 0$, assuming $\rho > |\delta|$. If $\rho \not> |\delta|$, then it can still be shown that every cluster point of $\arg\min_Z f_\rho^\mu(Z)$ as $\mu \to 0$ is a solution of

$$\min_{Z \in \mathcal{F}_{\text{esdp}}} f_\rho(Z). \tag{36}$$

Since $h_0(t) = \frac{1}{2}t^2$, we see that, in the special case of $\rho = 0$, (36) is equivalent to the variant of (4) whereby $|\cdot|$ is replaced with $|\cdot|^2$. Thus (36) may be viewed as a noise-aware generalization of this variant.

By a slight abuse of notation, we denote by Z_i the subvector of variables $x_i, y_{ii}, \{y_{ij} \mid (i,j) \in \mathcal{A}^s\}$ and by $\nabla_{Z_i} f_\rho^\mu$ the gradient of f_ρ^μ with respect to Z_i, $i = 1, \dots, m$. Notice that B is twice differentiable on dom$B$. We denote its Hessian with respect to Z_i by $\nabla^2_{Z_i} B$. Although the quadratic penalty function h_r is not twice differentiable, ∇h_r is Lipschitz continuous and piecewise-linear. Thus the generalized Hessian $\partial^2 h_r$ is well defined and given by

$$\partial^2 h_r(t) = \begin{cases} 1 & \text{if } |t| > r; \\ [0,1] & \text{if } |t| = r; \\ 0 & \text{else.} \end{cases}$$

For our method, we make the (somewhat arbitrary) selection of 1 if $|t| > r$ and 0 else. This yields, via (34) and the chain rule, a selection of $\partial^2_{Z_i} f_\rho(Z)$, which we denote by $H_{i,\rho}(Z)$. The corresponding selection of $\partial^2_{Z_i} f_\rho^\mu(Z)$ is

$$H_{i,\rho}^\mu(Z) \ := \ H_{i,\rho}(Z) + \mu \nabla^2_{Z_i} B(Z).$$

Since $H_{i,\rho}(Z) \succeq 0$ and $\nabla^2_{Z_i} B(Z) \succ 0$, we have $H_{i,\rho}^\mu(Z) \succ 0$ for $Z \in$ dom$B$. Moreover, $H_{i,\rho}^\mu(Z)$ has an "arrow" sparsity structure:

$$\begin{array}{c} \\ y_{ij_1} \\ \vdots \\ y_{ij_k} \\ x_i \\ y_{ii} \end{array} \begin{array}{c} \begin{array}{ccccc} y_{ij_1} & \cdots & y_{ij_k} & x_i^T & y_{ii} \end{array} \\ \begin{pmatrix} * & & & * & * \\ & \ddots & & \vdots & \vdots \\ & & * & * & * \\ * & \cdots & * & * & * \\ * & \cdots & * & * & * \end{pmatrix} \end{array},$$

where $\mathcal{N}(i) = \{j_1, \dots, j_k\}$, so its Cholesky factorization can be efficiently computed in linear time.

Our method, which we call the log-barrier penalty coordinate gradient descent (LPCGD) method, is based on applying a block-coordinate gradient descent method [35] to minimize f_ρ^μ inexactly, with Z_i as coordinate block and with μ decreased periodically; see [34, Section 8] for a related method for an SOCP relaxation of (1). We describe this method below.

0. Choose initial $\mu > 0$ and $Z \in$ dom$B$ with $Z_{\{m+\}} = I_d$. Choose $\mu^{\text{final}} > 0$ and a continuous function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{\mu \downarrow 0} \psi(\mu) = 0$. Choose stepsize parameters $0 < \beta < 1$ and $0 < \omega < \frac{1}{2}$. Go to step 1.

1. If there exists $i \in \{1, \dots, m\}$ such that $\|\nabla_{Z_i} f_\rho^\mu(Z)\| > \psi(\mu)$, then construct the block-coordinate generalized Newton direction:

$$D_i = -(H_{i,\rho}^\mu(Z))^{-1} \nabla_{Z_i} f_\rho^\mu(Z),$$

and repeat step 1 with

$$Z^{\text{new}} = Z[\alpha],$$

where $Z[\alpha]$ is obtained from Z by replacing Z_i with $Z_i + \alpha D_i$ and α is the largest element of $\{1, \beta, \beta^2, \cdots\}$ satisfying

$$f_\rho^\mu(Z[\alpha]) \leq f_\rho^\mu(Z) + \alpha\omega D_i^T \nabla_{Z_i} f_\rho^\mu(Z).$$

Otherwise, go to step 2.

2. If $\mu \leq \mu^{\text{final}}$, then stop. Otherwise, decrease μ and return to step 1.

The LPCGD method is highly parallelizable since, for any $i, j \in \{1, \ldots, m\}$ that share no neighbor, Z_i and Z_j share no variable and can be updated simultaneously. Moreover, the computation distributes over the sensors since each sensor i needs to communicate only with its neighbors in order to update Z_i. This is an important practical consideration, especially when tracking the position of moving sensors in real time, since the coordination of communication/computation over all sensors is expensive and the graph topology may change; see [16, 21, 25, 28]. Only the changing of μ needs centralized coordination among all sensors, but this needs to be done only infrequently. For tracking, μ can conceivably be held fixed at a small value, especially when sensors are moving slowly relative to the frequency of computation and one-hop communication.

In the noiseless case ($\delta = 0$), the LPCGD method with $\rho = 0$ computes an interior solution of the ESDP relaxation (4) within a desired accuracy.

7 Implementation and simulation results

In this section, we describe an implementation of the LPCGD method of Section 6 and present simulation results for the ρ-ESDP relaxation (23), as solved by the LPCGD method, and compare them with those for the ESDP relaxation (4), as solved by an interior-point method [33], and for the SOCP relaxation, as solved by the SCGD method in [34, Section 8].

7.1 Problem generation

To facilitate comparison with existing work, we follow [8, 9, 34, 36] and generate $x_1^{\text{true}}, \cdots, x_n^{\text{true}}$ independently according to a uniform distribution on the unit square $[-.5, .5]^2$, and set $m = 0.9n$ (i.e., 10% of the points are anchors), $\mathcal{A} = \{(i,j) : \|x_i^{\text{true}} - x_j^{\text{true}}\| < rr\}$, and

$$d_{ij} = \|x_i^{\text{true}} - x_j^{\text{true}}\| \cdot |1 + \epsilon_{ij} \cdot \sigma| \quad \forall (i,j) \in \mathcal{A},$$

where ϵ_{ij} is a random variable, $rr \in (0,1)$ is the radio range, and $\sigma \in [0,1]$ is the noisy factor. As in [8, 9, 34, 36], each ϵ_{ij} is normally distributed with mean 0 and variance 1, and we use the parameter values of $\sigma = 0, .001, .01$ and $rr = .06$ for $n = 1000, 2000$, $rr = .035$ for $n = 4000$, $rr = .02$ for $n = 10000$; see Table 1. While an additive Gaussian noise model is standard, the standard deviation is often assumed to be independent of the distances [15, Eq. (3a)–(3d)], [21, Section 6]. Still, for radio signal, the standard deviation increases with distance and the above noise model seems reasonable [38].

7.2 Implementation of the LPCGD method

We coded in Fortran-77 the LPCGD method of Section 6, with initial $\mu = 10$ and

$$\mu^{\text{final}} = 10^{-14}, \quad \psi(\mu) = \begin{cases} \mu & \text{if } \mu > 10^{-7}; \\ 10^{-7} & \text{if } \mu^{\text{final}} \leq \mu \leq 10^{-7}, \end{cases} \quad \beta = 0.5, \quad \omega = 0.1. \tag{37}$$

P	n	σ	$\lvert\mathcal{A}\rvert$
1	1000	0	5063
2	1000	.001	5288
3	1000	.01	5212
4	2000	0	21122
5	2000	.001	21070
6	2000	.01	20897
7	4000	0	29547
8	4000	.001	29342
9	4000	.01	29892
10	10000	0	61124
11	10000	.001	61038
12	10000	.01	61124

Table 1: Input parameters for the test problems. ($rr = .06$ for $n = 1000, 2000$, $rr = .035$ for $n = 4000$, $rr = .02$ for $n = 10000$.)

We choose i in Step 1 in a cyclic order, compute D_i using a Cholesky factorization of $H^{\mu}_{i,\rho}(Z)$, and decrease μ by a factor of 10 in Step 2. These choices were made with little experimentation and can conceivably be improved. As in [34], initially $x_i = x_i^{\text{true}} + \Delta_i$, with the components of Δ_i randomly generated from the square $[-.2, .2]^2$. We then set $y_{ii} = \|x_i\|^2 + 1$ and $y_{ij} = x_i^T x_j$.

Since the Gaussian distribution has unbounded support, the condition $\rho > |\delta|$ for ρ-ESDP is not guaranteed to hold for a fixed $\rho > 0$. On the other hand, the tail of the Gaussian beyond 2 standard deviations is below 5% and, in particular, Prob$(|\epsilon_{ij}| < 2) = .9545$.. Thus we will estimate $|\delta_{ij}|$ under the assumption that $|\epsilon_{ij}| < 2$. We have

$$\begin{aligned} |\delta_{ij}| &= |d_{ij}^2 - \|x_i^{\text{true}} - x_j^{\text{true}}\|^2| \\ &= d_{ij}^2 \left| 1 - \frac{1}{(1+\epsilon_{ij}\cdot\sigma)^2} \right| \\ &< d_{ij}^2 \max_{|t|\le 2} \left| 1 - \frac{1}{(1+t\cdot\sigma)^2} \right| \\ &= d_{ij}^2 \left(\frac{1}{(1-2\sigma)^2} - 1 \right), \end{aligned}$$

where the last equality is obtained by dividing into two cases $t \in [0, 2]$ and $t \in [-2, 0]$ and comparing the respective maximum found (at $t = 2$ and $t = -2$). Accordingly, we set

$$\rho_{ij} = d_{ij}^2 \left(\frac{1}{(1-2\hat{\sigma})^2} - 1 \right) \quad \forall (i,j) \in \mathcal{A}, \tag{38}$$

where $0 \le \hat{\sigma} < \frac{1}{2}$ is our estimate of σ. If $\hat{\sigma} \ge \sigma > 0$, then $\rho_{ij} > |\delta_{ij}|$ for over 95% of the edges on average.

For each Z found by our LPCGD code, we judge a sensor i to be accurately positioned if

$$\text{tr}_i(Z) \le (a_0 + a_1\hat{\sigma})\ \bar{d_i}^{\,2}, \tag{39}$$

where $\bar{d}_i := \frac{1}{|\mathcal{N}(i)|}\sum_{j\in\mathcal{N}(i)} d_{ij}$ and a_0, a_1 are positive constants. This is patterned after the trace test used for SOCP relaxation [34, Section 9]; see also (40). Here, the distance is squared so that (39) is invariant under scaling of the points and distances. The test (39) is justified by Proposition 2, Theorems 1 and 4, and the remark following (36). Specifically, when $\delta = 0$ and we set $\hat{\sigma} = 0$, we have Z approximately equal to some $Z_0 \in \mathrm{ri}(\mathcal{S}^0_{\mathrm{esdp}})$ and, by Proposition 2 and Theorem 1, $i \in \mathcal{I}^0_{\mathrm{esdp}}$ if and only if $\mathrm{tr}_i(Z_0) = 0$, implying $\mathrm{tr}_i(Z) \approx 0$. When $\delta \neq 0$ is sufficiently small and we set $\hat{\sigma}$ such that $|\delta| < \rho$ and ρ is sufficiently small, we have from Theorem 4(a) that $i \in \mathcal{I}^0_{\mathrm{esdp}}$ if and only if $\mathrm{tr}_i(Z^{\rho,\delta})$ is sufficiently small, implying $\mathrm{tr}_i(Z)$ is sufficiently small. We settled on the constants of $a_0 = 0.01$ and $a_1 = 30$ after some experimentation.

7.3 Simulation results

In Table 2, we compare the ρ-ESDP relaxation (23), as solved by LPCGD method, with an SOCP relaxation, as solved by the SCGD method [34, Section 8] and the ESDP relaxation (4), as solved by a primal-dual interior-point method, namely, SeDuMi (Version 1.05) by Jos Sturm [33]. In the LPCGD method, we assume knowledge of σ and set $\hat{\sigma} = \sigma$. As in [34, Section 9], for each interior SOCP solution $x_1, \ldots, x_m, (y_{ij})_{(i,j)\in\mathcal{A}}$ found, a sensor i is judged to be accurately positioned if there exists a $j \in \mathcal{N}(i)$ satisfying

$$\left| \|x_i - x_j\|^2 - y_{ij} \right| \leq 10^{-7} d_{ij}, \tag{40}$$

(with $x_i = x_i^{\mathrm{true}}$ for $i > m$). For each interior ESDP solution Z found, a sensor i is judged to be accurately positioned if (39) is satisfied. Although Example 2 shows that (39) may wrongly judge a sensor to be accurately positioned when there is noise, in our simulation this test showed good predictive power.

Analogous to [34, Section 9], we denote by m_{ap} the number of sensors that are judged to be accurately positioned. We check the accuracy of these computed positions by computing the maximum error between them and the true positions:

$$err_{\mathrm{ap}} \;=\; \max_{\substack{i \text{ accurately} \\ \text{positioned}}} \|x_i - x_i^{\mathrm{true}}\|.$$

For comparison, we also compute the maximum error and the root-mean-square deviation between computed positions and true positions of all sensors:

$$err = \max_{i=1,\ldots,m} \|x_i - x_i^{\mathrm{true}}\|,$$

$$\mathrm{RMSD} = \left(\frac{1}{m} \sum_{i=1}^{m} \|x_i - x_i^{\mathrm{true}}\|^2 \right)^{\frac{1}{2}}.$$

In Table 2, we report the number of iterations and the cpu time (in seconds) for LPCGD, SCGD, and SeDuMi on the test problems from Table 1. For each solution found, we report m_{ap}, err_{ap}, err, and RMSD. For LPCGD and SCGD, the number of iterations is shown in ten thousands. Like LPCGD, SCGD is coded in Fortran, while SeDuMi is coded in C. SeDuMi is interfaced with a Matlab code, written by Wang et al. [36], that constructs the SDP data in SeDuMi format from the anchor positions and distance measurements. The code further drops some edges $(i, j) \in \mathcal{A}^s$ to keep the number of neighboring sensors below a user-specified threshold, suggested to be between 5 and 10. We set the threshold to 5 for faster solution time; also see [18, Section 5]. The total time **cpu** shown includes the time to run the interface, as well as the SeDuMi run time (which is indicated by $\mathbf{cpu}_S$). The results in Table 2 are obtained using a 2006 version of the Matlab interface, sent to the second author by Yinyu Ye in a private communication, instead of the current public-domain version available from `http://www.stanford.edu/~yyye/`. This is because the 2006 version does not postprocess the ESDP solution using local improvement and includes y_{ii} in its output, thus allowing for a direct comparison of ρ-ESDP solution with ESDP solution and a test of solution accuracy using trace. We distinguish the two versions by the suffixes "I06" and "I08". A comparison of LPCGD with SeDuMi-I08 is given in the next subsection on refining solutions using local improvement. The Fortran codes were compiled by Gnu F-77 compiler (Version 3.2.57). All codes were run on an HP DL360 workstation, under Red Hat Linux 3.5 and installed with Matlab Version 7.2.

	ρ-ESDP (LPCGD)	SOCP (SCGD)	ESDP (SeDuMi-I06)
P	**iter*/cpu/m_{ap}/err_{ap}/err/RMSD**	**iter*/cpu/m_{ap}/err_{ap}/err/RMSD**	**iter/cpu(cpu$_S$)/m_{ap}/err_{ap}/err/RMSD**
1	66/7/662/1.7e-3/.13/1.7e-2	207/13/385/4.1e-4/.14/2.6e-2	23/182(104)/669/2.1e-3/.12/1.7e-2
2	52/6/667/4.1e-3/.18/2.4e-2	500/34/443/2.2e-3/.19/3.3e-2	22/177(93)/736/3.4e-3/.18/2.2e-2
3	43/5/660/2.2e-2/.10/1.7e-2	603/40/438/6.3e-3/.12/2.2e-2	19/119(42)/720/3.1e-2/.11/1.7e-2
4	110/26/1762/3.1e-4/.04/1.7e-3	547/83/1463/8.8e-5/.10/9.6e-3	21/1145(397)/1742/3.9e-4/.03/1.4e-3
5	81/19/1729/1.7e-3/.05/2.6e-3	843/122/1500/2.7e-3/.11/9.9e-3	21/1196(457)/1758/1.8e-3/.05/2.2e-3
6	89/20/1699/1.4e-2/.05/5.5e-3	2307/324/1556/2.2e-2/.07/9.0e-3	21/966(233)/1746/2.4e-2/.05/4.5e-3
7	192/36/3440/2.8e-4/.03/8.1e-4	1003/110/2913/3.9e-4/.04/4.2e-3	21/3296(660)/3250/8.1e-4/.03/1.4e-3
8	158/29/3340/1.1e-3/.11/5.8e-3	1271/136/2859/2.2e-3/.11/8.2e-3	19/3057(496)/3313/2.2e-3/.09/5.1e-3
9	144/27/3396/1.9e-2/.08/5.8e-3	3156/337/3046/7.4e-3/.08/8.4e-3	21/3157(529)/3458/2.2e-2/.08/4.9e-3
10	435/77/7844/2.3e-3/.05/3.0e-3	2916/278/6397/4.9e-4/.05/4.4e-3	20/16411(1297)/6481/2.5e-3/.04/2.6e-3
11	389/69/8117/2.5e-3/.04/2.2e-3	3658/373/6569/1.5e-3/.04/3.8e-3	19/16317(1096)/7960/1.7e-3/.04/2.4e-3
12	354/63/8336/1.0e-2/.05/3.7e-3	5706/584/7176/5.7e-3/.05/4.4e-3	20/16368(1264)/8593/8.7e-3/.04/3.0e-3

Table 2: Comparing ρ-ESDP, SOCP, and ESDP as solved by LPCGD, SCGD, and SeDuMi-I06, respectively. cpu times are in seconds. In the LPCGD and SCGD columns, **iter*** represents iterations in ten thousands. In the SeDuMi-I06 column, cpu and cpu$_S$ denote the total time to solve ESDP and the time to run SeDuMi, respectively.

We see from Table 2 that LPCGD is generally faster than SCGD and much faster than SeDuMi-I06. The accuracy of the solutions found by LPCGD is generally better than solutions found by SCGD (i.e., m_{ap} is larger, err_{ap} is comparable, err and RMSD are lower) and almost comparable to solutions found by SeDuMi-I06, though the later tends to have lower RMSD. This is also illustrated in Figure 2. Notice that the cpu time for LPCGD increases about linearly with n.

		ρ-ESDP (LPCGD)
P	$\hat{\sigma}$	**iter*/cpu/m_{ap}/err_{ap}/err/RMSD**
1	.005	44/4.7/637/1.4e-2/.14/2.0e-2
1	.01	39/4.3/660/1.9e-2/.14/2.2e-2
1	.02	35/3.9/686/2.8e-2/.14/2.3e-2
2	.005	41/4.8/674/1.1e-2/.19/2.7e-2
2	.01	37/4.4/704/2.0e-2/.19/2.8e-2
2	.02	34/4.0/733/3.1e-2/.19/3.0e-2
3	.015	40/4.6/674/2.4e-2/.11/1.8e-2
3	.02	37/4.3/686/2.9e-2/.11/1.9e-2
3	.03	34/3.9/703/3.7e-2/.11/2.1e-2

Table 3: Comparing ρ-ESDP, as solved by LPCGD, for varying $\hat{\sigma}$. cpu times are in seconds. **iter*** represents iterations in ten thousands.

In Table 2, we set $\hat{\sigma} = \sigma$, which may be restrictive since it assumes an accurate knowledge of σ. In Table 3, we report the performance of ρ-ESDP on the first three problems from Table 1 with varying $\hat{\sigma}$. Not too surprisingly, when $\hat{\sigma}$ is larger than σ, err and err_{ap} are larger. Intuitively, as $\hat{\sigma}$ increases, each ρ_{ij} increases according to (38), and $\mathcal{S}^{\rho,\delta}_{\text{resdp}}$ expands. Then, the Z found by LPCGD, a sort of "center" of this set, would tend to be further away from Z^{true}, and err would increase. On the other hand, the number of iterations and the cpu time for LPCGD decreases with increasing $\hat{\sigma}$.

We next compare ρ-ESDP, SOCP, and ESDP in the presence of high noise. We consider an example used in [5, 34] of 60 sensors, 4 anchors (at $(\pm.45, \pm.45)$), $rr = 0.3$, and $\sigma = 0.1, 0.2$. For LPCGD, we

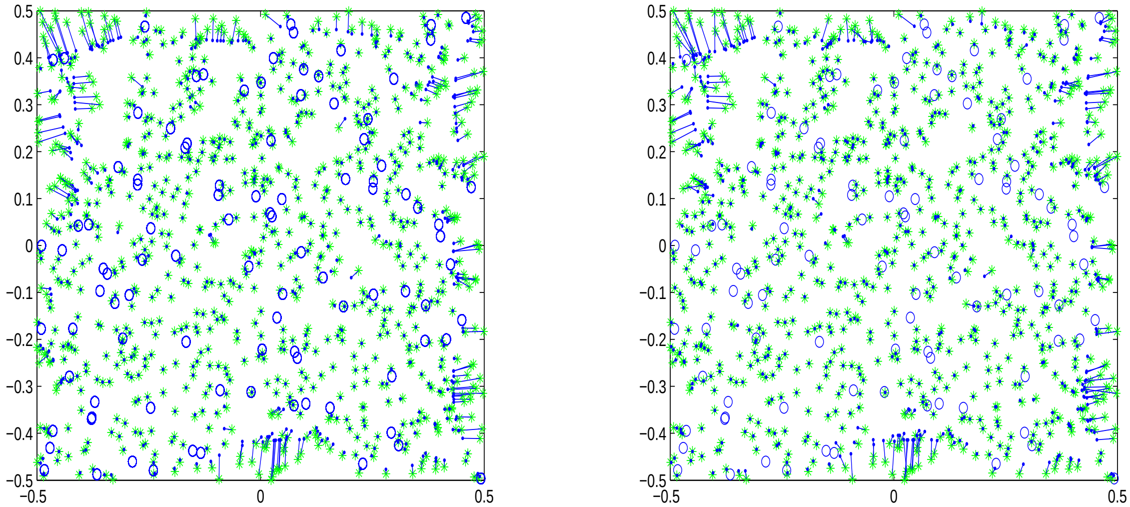

Figure 2: The left figure shows the anchor ("∘") and the solution found by LPCGD for problem 3 in Table 1 ($n = 1000$, $\sigma = .01$). Each sensor position ("·") found is joined to its true position ("$*$") by a line. The right figure shows the same information for the solution of ESDP found by SeDuMi-I06 for the same problem.

choose ρ_{ij} as in (38) with $\hat{\sigma} = \sigma$. The results are reported in Table 4. We see from Table 4 that the solution accuracy is comparable for all three convex relaxations.

	ρ-ESDP (LPCGD)	SOCP (SCGD)	ESDP (SeDuMi-I06)
σ	iter/cpu/$m_{\rm ap}$/$err_{\rm ap}$/ err/RMSD	iter/cpu/$m_{\rm ap}$/$err_{\rm ap}$/ err/RMSD	iter/cpu(cpu$_S$)/$m_{\rm ap}$/$err_{\rm ap}$/ err/RMSD
0.1	24240/0.3/58/2.1e-1/.32/7.9e-2	995811/7.7/51/1.1e-1/.28/7.4e-2	12/2.9(1.9)/60/3.1e-1/.31/8.7e-2
0.2	24910/0.3/51/2.5e-1/.35/1.1e-1	1180512/8.3/47/1.7e-1/.35/1.1e-1	12/1.9(1.4)/60/3.4e-1/.34/1.0e-1

Table 4: Comparing ρ-ESDP, SOCP, and ESDP as solved by LPCGD, SCGD, and SeDuMi-I06, respectively, on small problems with high distance measurement noise.

Lastly, we solved the ESDP relaxation from Example 2 (with noise $\epsilon = 0.01$) using SeDuMi-I06. When the termination tolerance par.eps in SeDuMi is set to 1e-3, it outputs a Z with $\mathrm{tr}_2(Z) = 0.232$ and $x_2 = (0, 0.876)$. When par.eps is decreased to 1e-7, it outputs a Z with $\mathrm{tr}_2(Z) = 0.011$ and $x_2 = (0, 0.994)$. When par.eps is further decreased below 1e-7, SeDuMi encounters numerical difficulty. Thus, in the presence of distance measurement noise, the solution obtained by solving SDP/ESDP to a higher accuracy can be more misleading of the true sensor position (when proximity to the true position is measured by individual trace)!

7.4 Refinements

When the graph $\mathcal{G}$ is dense, Wang et al. [36] proposed removing some of the edges joining sensors so as to keep the number of neighboring sensors below a user-specified bound $\deg_{\rm bd}$, say, 5 or 10. This can significantly speed up the ESDP solution time without significantly compromising solution accuracy. Such preprocessing was also used by Nie [26] and Kim, Kojima, Waki [18] in solving sparse SOS relaxations. We have implemented this preprocessing for LPCGD. In fact, since LPCGD updates each sensor position using only information from its neighbors, removed edges can be added back dynamically. We experimented with two versions: Version I does not add back edges. Version II in Step 2 of LPCGD (when μ is decreased) adds back those edges $(i,j) \in \mathcal{A}^s$ with $|\ell_{ij}(Z) - d_{ij}^2| > 0$, where y_{ij} is chosen to minimize this quantity subject to $Z_{\{i,j,m^+\}} \succeq 0$. We denote these two versions by LPCGD($\deg_{\rm bd}$,I) and LPCGD($\deg_{\rm bd}$,II), respectively. In our tests, we set $\deg_{\rm bd}$ to be either 5 or m.

As in [5, 10, 18, 20], local improvement heuristics can be used to refine the solution found by LPCGD and improve its RMSD. In [5], a steepest descent method is applied to locally minimize the error function

$$\hat{f}(X) := \sum_{(i,j)\in\mathcal{A}} (\|x_i - x_j\| - d_{ij})^2.$$

To maintain the distributed nature of our method, we apply a block-coordinate steepest descent method to locally minimize $\hat{f}$. At each iteration, the method chooses an $i \in \{1, \dots, m\}$ with $\|\nabla_{x_i}\hat{f}(X)\| > 10^{-3}$ and updates x_i by

$$x_i \leftarrow x_i - \alpha\nabla_{x_i}\hat{f}(X),$$

and the stepsize α is chosen by an Armijo rule analogously as in Step 1 of LPCGD. We experimented with two versions: Version A omits updating x_i if x_i is judged to be accurately positioned by the trace test (39). Version B makes no such omission.

	LPCGD(5,I)	**LPCGD(5,II)**	**LPCGD(m,I)**
P	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$
1	2/2.6e-2/2/2.1e-2/2/2.1e-2	7/1.8e-2/7/1.9e-2/7/1.8e-2	7/1.7e-2/7/2.1e-2/7/2.0e-2
2	2/3.4e-2/2/2.5e-2/2/2.6e-2	6/2.4e-2/6/2.8e-2/6/2.8e-2	6/2.4e-2/6/2.8e-2/6/2.8e-2
3	1/3.1e-2/1/1.9e-2/1/1.8e-2	4/1.7e-2/4/1.1e-2/4/9.0e-3	4/1.7e-2/4/1.1e-2/5/9.1e-3
4	4/9.4e-3/4/4.6e-3/4/4.9e-3	24/1.8e-3/24/4.0e-5/24/6.9e-5	25/1.7e-3/25/4.5e-5/25/5.4e-5
5	3/7.3e-3/3/4.4e-3/3/4.3e-3	18/2.6e-3/18/2.4e-3/18/2.4e-3	19/2.6e-3/19/2.4e-3/19/2.4e-3
6	2/1.3e-2/2/5.5e-3/2/3.9e-3	19/5.5e-3/19/2.4e-3/19/1.0e-3	20/5.5e-3/20/2.4e-3/20/1.0e-3
7	10/7.8e-3/10/3.2e-3/10/3.3e-3	35/7.8e-4/35/8.2e-4/35/8.3e-4	36/8.1e-4/36/9.9e-4/36/8.3e-4
8	7/1.1e-2/7/7.0e-3/8/7.3e-3	28/5.8e-3/28/5.9e-3/28/5.9e-3	28/5.8e-3/28/5.9e-3/28/5.9e-3
9	6/1.0e-2/6/4.9e-3/6/5.9e-3	24/5.8e-3/24/5.0e-3/24/4.7e-3	27/5.8e-3/27/5.0e-3/27/4.7e-3
10	24/6.6e-3/24/3.8e-3/25/3.5e-3	71/3.0e-3/71/3.5e-3/71/3.5e-3	77/3.0e-3/77/3.5e-3/77/3.5e-3
11	23/7.5e-3/23/3.1e-3/24/3.1e-3	67/2.2e-3/67/1.7e-3/67/1.8e-3	69/2.2e-3/69/1.8e-3/69/1.8e-3
12	17/6.6e-3/17/4.0e-3/18/3.9e-3	59/3.7e-3/59/3.7e-3/60/3.7e-3	63/3.7e-3/63/3.7e-3/63/3.7e-3

Table 5: Comparing the time and solution RMSD for LPCGD with refinements on the problems from Table 1.

We applied LPCGD with the preceding two refinements to the problems in Tables 1 and 4. For LPCGD, we choose ρ_{ij} as in (38) with $\hat{\sigma} = \sigma$. The cpu times (in seconds) and the solution RMSD are reported in Tables 5 and 6. Here, cpu denotes the time to run LPCGD and RMSD denotes the RMSD of the resulting solution; cpu$_A$ denotes the time to run LPCGD with version A of local improvement, and RMSD$_A$ denotes the RMSD of the resulting solution; cpu$_B$ and RMSD$_B$ have analogous meanings. We see from Tables 5 and 6 that LPCGD(5,I) is signficantly faster than LPCGD(5,II) and LPCGD(m,I), but its solution RMSD is generally higher. Thus, if speed is more important than solution accuracy, then LPCGD(5,I) would be preferrable. Otherwise, either LPCGD(5,II) or LPCGD(m,I) should be used. Not surprisingly, version A of local improvement is faster than version B. In the presence of low noise or no noise, the improvements in RMSD obtained by the two versions are comparable and somewhat marginal; see Table 5. However, in the presence of high noise, version B tends to yield a significantly lower RMSD; see Table 6. These improvements in the RMSD are also illustrated in Figures 3 and 4.

	LPCGD(5,I)	LPCGD(5,II)	LPCGD(m,I)
σ	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$
0.1	.06/1.6e-1/.06/1.4e-1/.09/2.1e-2	.26/7.9e-2/.27/7.5e-2/.27/2.1e-2	.33/7.9e-2/.33/6.5e-2/.33/2.1e-2
0.2	.06/1.5e-1/.07/1.3e-1/.08/8.1e-2	.27/1.1e-1/.27/9.4e-2/.28/9.3e-2	.33/1.1e-1/.33/9.5e-2/.34/9.3e-2

Table 6: Comparing the time and solution RMSD for LPCGD with refinements on the problems from Table 4.

Lastly, we compare LPCGD(5,II) with SeDuMi-I08, which also has a local improvement heuristic for refining the ESDP solution, as well as SeDuMi-I06, which does not have such a heuristic. We generate six problem instances using the same input parameters as problems 1–3 and 7–9 in Table 1. We set degree bound to 5 for SeDuMi-I06 and SeDuMi-I08. The results are reported in Table 7. For SeDuMi, we also report the SDP objective value ("obj") for comparison. We see that SeDuMi-I08 is roughly 1.1-2.2 times faster than SeDuMi-I06 while LPCGD(5,II) is much faster than both. The solution RMSD found by SeDuMi-I08 tends to be lower, however. The SeDuMi-08 run times are higher than those reported in [36, Table 5.3], which may be explained by the older server we use. The objective values and RMSD found by SeDuMi-I08 are consistently higher than those reported in [36, Table 5.3]. We do not yet have an explanation for this.

		ρ-ESDP (LPCGD(5,II))	ESDP (SeDuMi-I06)	ESDP (SeDuMi-I08)
n	σ	cpu/RMSD/cpu$_A$/RMSD$_A$/cpu$_B$/RMSD$_B$	cpu(cpu$_S$)/obj/RMSD	cpu(cpu$_S$)/obj/RMSD
1000	0	8/2.2e-2/8/2.3e-2/8/2.3e-2	169(87)/4.6e-2/2.0e-2	102(38)/4.7e-2/1.5e-2
1000	.001	5/3.6e-2/5/4.1e-2/5/4.1e-2	158(76)/3.9e-2/3.4e-2	100(38)/3.9e-2/3.4e-2
1000	.01	4/2.0e-2/4/1.6e-2/4/1.5e-2	128(48)/6.0e-2/1.7e-2	97(35)/5.5e-2/1.1e-2
4000	0	34/5.7e-3/34/5.5e-3/34/5.5e-3	3118(482)/5.1e-2/6.1e-3	2419(348)/5.1e-2/5.1e-3
4000	.001	30/3.2e-3/30/3.2e-3/30/3.3e-3	3166(589)/5.7e-2/3.3e-3	2566(509)/5.5e-2/3.3e-3
4000	.01	29/6.5e-3/29/6.9e-3/29/6.9e-3	3177(527)/1.3e-1/5.8e-3	2318(285)/1.1e-1/8.0e-3

Table 7: Comparing the time and solution RMSD for LPCGD(5,II) and SeDuMi on randomly generated problems with $n = 1000$, $rr = .06$ and $n = 4000$, $rr = .035$, with varying noisy factor σ.

8 Extensions and open questions

Instead of absolute error in (1), squared error can also be used, as in [5, 10, 11, 19, 26]. Our results can be extended accordingly.

Can our analysis and method be extended to the sparse SOS relaxations studied in [18, 26]? Can they be extended to incorporate upper and lower bounds on the distances [10, 19], and angle of arrival (AoA) information [4, 23, 25]? It has been shown in [4] and [3, Chapter 5] that the SDP relaxation (2) can be extended to incorporate AoA information, but the resulting SDP appears more difficult to solve; see [4, Section 5] and [3, Section 5.3.2].

Can Theorem 1 be extended to the SDP relaxation (2)? Does $\{Z^{\rho,\delta}\}$ converge as $|\delta| < \rho \to 0$ and, if yes, what is the limit? Despite Example 2, can the zero trace test for SDP/ESDP solutions, as used in [9, Section 2] and [10, Section 3] (also see the ESDP column in Table 2), be justified theoretically when there is noise?

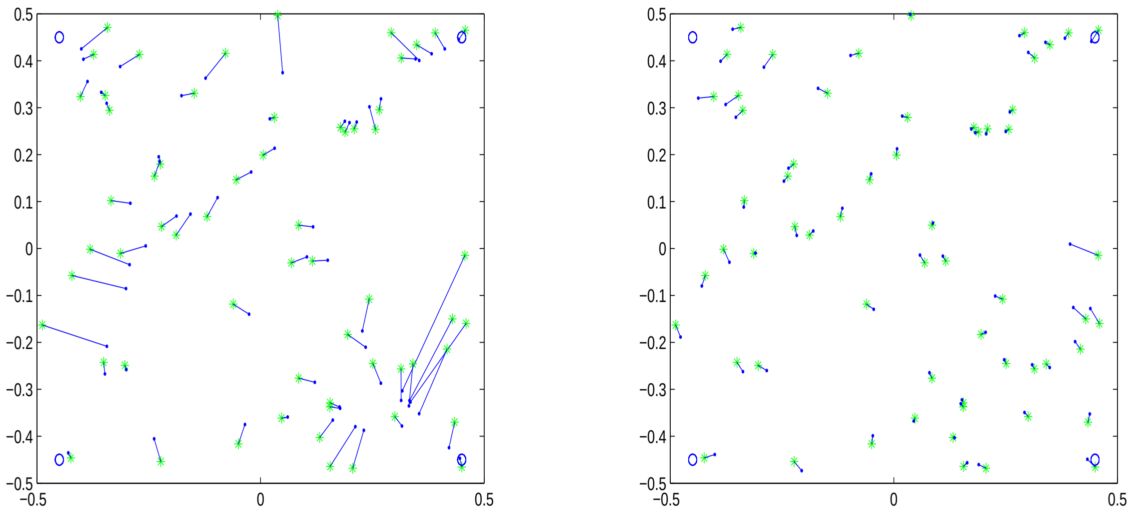

Figure 3: The left figure shows the anchor ("∘") and the solution found by LPCGD for problem 1 in Table 6 ($m = 60$, $\sigma = 0.1$). Each sensor position ("·") found is joined to its true position ("$*$") by a line. The right figure shows the same information for the solution found by LPCGD(m,I) using Version B of local improvement.

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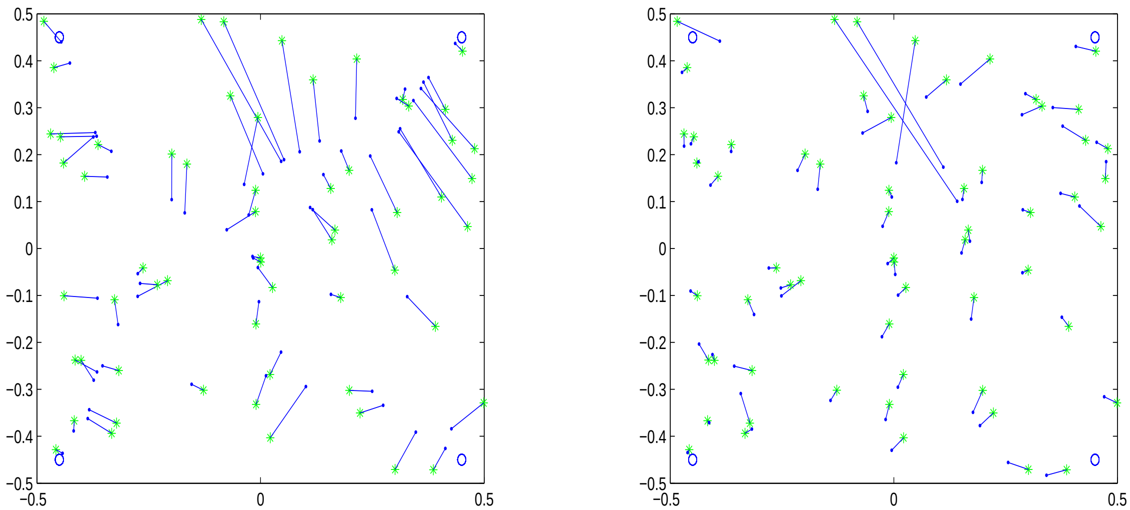

Figure 4: The left figure shows the anchor ("∘") and the solution found by LPCGD for problem 2 in Table 6 ($m = 60$, $\sigma = 0.2$). Each sensor position ("·") found is joined to its true position ("$*$") by a line. The right figure shows the same information for the solution found by LPCGD(m,I) using Version B of local improvement.

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