# Traffic Equilibrium: Link Flows, Path Flows and Weakly/Strongly Acyclic Solutions

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**Abstract**

There are two popular formulations of the traffic equilibrium problem with fixed demand: one in terms of path flows and another in terms of multicommodity link flows. These two formulations in general may have very different solutions. In this paper, we give a systematic study of conditions under which the two formulations have the same solutions or, at least, share a solution. These conditions involve the absence of negative cost cycles and are derived using the notion of weakly/strongly acyclic solutions for the link formulation. In cases such as nonnegative continuous pseudomonotone travel costs, we show that we can construct, from a solution to one formulation, a solution to both formulations by, at most, solving a min-cost multicommodity flow problem.

# 1 Introduction

This paper is concerned with the problem of determining an equilibrium distribution of traffic flow in a network where demands for travel are fixed and travelers travel along paths/routes which minimize individual travel cost (e.g., travel time) with no cooperation among travelers. Such an equilibrium distribution is commonly referred to as a user-equilibrium distribution and the problem is generally formulated as a variational inequality (see [10, 16] and references therein). Two formulations have appeared in the literature. In one, the variational inequality is expressed in terms of flows on paths of the network; in the other, the variational inequality is expressed in terms of origin/destination-specific flows on the links (or arcs) of the network. Each formulation has certain advantages, though in general the two formulations may have very different solutions. It is thus of interest to study conditions under which the formulations have the same solution or, at least, share a solution. This question has been well studied when arc travel costs are separable functions of the arc flows, i.e., the travel cost of each arc depends only on the flow on that arc. Here, we study the question in the general case where the travel cost of an arc can also depend on the flows on other arcs.

Specifically, in the traffic equilibrium problem we are given a directed network with nodes indexed $i = 1, 2, \ldots, m$, and links (or arcs) indexed $k = 1, 2, \ldots, n$. There is a fixed demand for travel $d_j > 0$ (measured in number of travelers per time unit) from an origin node $s_j$ to a destination node $t_j$, for $j = 1, 2, \ldots, q$. We refer to travelers associated with each origin/destination pair $j$ as a *commodity*. Associated with each arc $k$ is a travel cost function $F_k : \Re^n_+ \mapsto \Re$. Given the vector $\mathbf{x}$, with $x_k$ specifying the total flow on arc $k$, $F_k(\mathbf{x})$ is the cost per unit flow of using arc $k$. We denote by $\mathbf{F} : \Re^n_+ \mapsto \Re^n$ the vector-valued function with entries $F_k$, $k = 1, 2, \ldots, n$. The separable case corresponds to when $F_k(\mathbf{x})$ depends only on $x_k$, the component of $\mathbf{x}$ corresponding to arc $k$.

We consider only the case in which the cost of a route can be determined by adding the cost of the individual links belonging to the route. In certain situations, more complex models may be necessary. In particular, the link-flow formulation given below may not be relevant, and the path-flow formulation takes a somewhat different form.

In the link-flow formulation, we denote by $x_k^j$ the flow of commodity $j$ on arc $k$ (measured in number of travelers per time unit) and by $\mathbf{x}^j$ the flow vector with entries $x_k^j$, $k = 1, 2, \ldots, n$. Then, the vector of total flows on the arcs is given by the sum $\mathbf{x} = \sum_j \mathbf{x}^j$. Also, each $\mathbf{x}^j$ is required to be nonnegative and satisfy flow conservation for commodity $j$. This leads to the following feasible set for $\mathbf{x}$:

$$L = \{\mathbf{x} \in \Re^n | \exists\ \mathbf{x}^j \geq \mathbf{0}, j = 1, \ldots, q, \text{ satisfying } \mathbf{A}\mathbf{x}^j = \mathbf{b}^j, \mathbf{x} = \sum_j \mathbf{x}^j\}.$$

Here $\mathbf{A}$ is the $m \times n$ node-arc incidence matrix with entry $A_{i,k} = 1$ if arc $k$ is directed out of node $i$, $A_{i,k} = -1$ if arc $k$ is directed into node $i$, and $A_{i,k} = 0$ otherwise. Also, $\mathbf{b}^j \in \Re^m$ is the vector of demands for commodity $j$ with entry $b_i^j = d_j$ if $i = s_j$, $b_i^j = -d_j$ if $i = t_j$, and $b_i^j = 0$ otherwise. The associated link-flow variational inequality (LVI) [16, Remark 3.4] is:

$$\begin{array}{rl} \text{find} & \mathbf{x} \in L \\ \text{satisfying} & \mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0 \quad \forall\ \mathbf{y} \in L. \end{array}$$

In the path-flow formulation, we consider all simple directed paths from $s_j$ to $t_j$, for $j = 1, ..., q$. Let these paths be indexed from 1 up to $p$. We denote by $\alpha_h$ the flow assigned to path $h$ and by $\alpha$ the path-flow vector with entries $\alpha_h$, $h = 1, 2, \ldots, p$. We let $\mathbf{P}$ be the $n \times p$ arc-path incidence matrix with entry $P_{k,h} = 1$ if arc $k$ lies on path $h$, and $P_{k,h} = 0$ otherwise. Then, the vector of total flows on the arcs is given by $\mathbf{x} = \mathbf{P}\alpha$. Also, each $\alpha_h$ is required to be nonnegative and, when summed over all simple directed paths $h$ from $s_j$ to $t_j$, equals the demand $d_j$. This leads to the following feasible set for $\mathbf{x}$:

$$P = \{\mathbf{x} \in \Re^n | \exists\ \alpha \in \Re^p \text{ satisfying } \alpha \geq \mathbf{0}, \mathbf{R}\alpha = \mathbf{d}, \mathbf{x} = \mathbf{P}\alpha\}.$$

Here $\mathbf{R}$ is the $q \times p$ commodity-path incidence matrix with entry $R_{j,h} = 1$ if path $h$ joins $s_j$ to $t_j$, and $R_{j,h} = 0$ otherwise. Also, $\mathbf{d} = (d_1, d_2, \ldots, d_q)$. The associated path-flow variational inequality (PVI), as considered by Smith[18] and Dafermos[8], is:

$$\begin{array}{ll} \text{find} & \mathbf{x} \in P \\ \text{satisfying} & \mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0 \quad \forall\, \mathbf{y} \in P. \end{array}$$

We note that Aashtiani and Magnanti [1] considered an alternative path-flow formulation as follows:

$$\begin{array}{ll} \text{find} & \alpha \in \Lambda \\ \text{satisfying} & \mathbf{P}^T\mathbf{F}(\mathbf{P}\alpha) \cdot (\beta - \alpha) \geq 0 \quad \forall\, \beta \in \Lambda, \end{array}$$

where $\Lambda = \{\alpha \in \Re^p | \alpha \geq \mathbf{0}, \mathbf{R}\alpha = \mathbf{d}\}$. Since $\mathbf{P}^T\mathbf{F}(\mathbf{P}\alpha) \cdot (\beta - \alpha) = \mathbf{F}(\mathbf{P}\alpha) \cdot (\mathbf{P}\beta - \mathbf{P}\alpha)$, we see that this formulation is equivalent to PVI in the sense that if $\alpha$ solves this formulation, then $\mathbf{x} = \mathbf{P}\alpha$ solves PVI and, conversely, if $\mathbf{x}$ solves PVI, then $\mathbf{x} = \mathbf{P}\alpha$ for some $\alpha$ solving this formulation.

All three formulation have been used in the design of algorithms to compute equilibrium flows. For example, Bertsekas and Gafni [7] used the formulation of Aashtiani and Magnanti in adapting a projection method to the traffic equilibrium problem. Dafermos and Nagurney [9] presented an algorithm, based on the path-flow formulation, to find a user-equilibrium distribution of flows. Bergendorff, Hearn, and Ramana [5] use both the path-flow and the link-flow formulation in their analysis of road tolls.

Ford and Fulkerson [11, Chapter 1] considered feasibility conditions for network flows in terms of both link flows and path flows. They pointed out difficulties with a path-flow formulation of exhaustive enumeration of paths, which may require exponential time, and of the obscuring of the inherent network structure when we use an arc-path incidence matrix rather than a node-arc incidence matrix. Ahuja, et al., [2] point out that for multicommodity flows with constant cost per unit flow, and capacity constraints, the number of paths used (paths in the basis), is a small fraction of the number of variables, and thus column generation methods based on a path-flow formulation may be quite efficient. However, when costs are nonlinear, the number of paths used may be very large. Bar-Gera [3] notes that for the Chicago regional network, with 12,982 nodes, 39,018 links, and 2,297,945 origin/destination pairs with positive demands, he obtains a flow distribution which has, for one origin/destination pair, 136,498 used routes. The continuing pro- and con- arguments for path flows versus link flows suggests that there is merit in a systematic study of the relationship between the two formulations.

In general, the two formulations LVI and PVI may have very different solutions. We will study conditions on the travel cost function $\mathbf{F}$ under which LVI and PVI have the same solution or, at least, share a solution. This question has been partially studied in the case where $\mathbf{F}$ is the gradient of a differentiable function $f : \Re^n \mapsto \Re$, i.e.,

$$F_k(\mathbf{x}) = \frac{\partial f}{\partial x_k}(\mathbf{x}) \quad \forall k.$$

This case includes the separable case and the symmetric case where $F$ is continuously differentiable and the Jacobian matrix $\nabla \mathbf{F}(\mathbf{x})$ is symmetric for all $\mathbf{x} \in \Re^n$. In this case, LVI reduces to the stationarity condition for the following min-cost multicommodity flow problem:

$$\begin{array}{rl} \min & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in L. \end{array} \tag{1}$$

In the separable case, for which $f(\mathbf{x}) = \sum_{k=1}^{n} \int_0^{x_k} F_k(\mathbf{y}) dy_k$, this problem was considered by Beckmann et al. [4]. Analogously, PVI reduces to the stationarity condition for the optimization problem:

$$\begin{array}{rl} \min & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in P. \end{array} \tag{2}$$

Patriksson [16, Section 2.3.2] shows that when $f$ is separable, strictly increasing, and $\mathbf{F}$ is continuous, then PVI has a solution which also solves LVI. He also establishes that if $f$ is strictly convex (equivalently, $\mathbf{F}$ is strictly monotone), though possibly non-separable, then a solution to PVI is unique. We can make the general statement that if $f$ is convex and the sum of the costs $F_k$ around a directed cycle is always positive, then the two optimization problems have the same solutions. This result is a consequence of the fact that decreasing a commodity flow around a directed cycle with positive cost yields a lower $f$ value [6, Section 8.6]. Then if all directed cycles have positive cost, an optimal solution to (1) cannot have a commodity with positive flow on all arcs of a directed cycle.

In this paper, we consider the general case where $\mathbf{F}$ need not be the gradient of some function. Thus, the travel cost for an arc may depend on the flows on one or more other arcs in the network and the Jacobian $\nabla \mathbf{F}(\mathbf{x})$ may not exist or may be asymmetric for some $\mathbf{x}$. In this general case, there is no longer a connection with optimization problems of the form of (1) and (2) and a different analysis is needed. Our analysis will be based on a key fact that a solution $\mathbf{x}$ of LVI can be decomposed into the sum of a "strongly acyclic" multicommodity flow and a circulation. We then can show that if the circulation is nonzero, there must exist a directed cycle with nonpositive cost. This decomposition enables us to construct, in the case where arc costs are continuous and nonnegative along directed cycles, a solution to PVI from a solution to LVI and vice versa. Such an equivalence of solutions serves both as an indicator of the robustness of the formulations and as a vehicle for adapting theory/algorithms developed for one formulation to the other. For example, it enables us to connect results from traffic equilibrium, often stated in terms of path flows, with results from network optimization, often stated in terms of arc flows. Accordingly, we may better be able to adapt the many algorithms developed for network optimization to traffic equilibrium.

This paper is organized as follows. Section 2 develops some lemmas necessary for proving the main results. Section 3 presents the main results concerning the relationship between the two formulations, and Section 4 presents counterexamples to certain natural conjectures. Section 5 discusses application of the results to the case of commodity-dependent cost functions. Section 6 summarizes the results and discusses further extensions. In our notation, all vectors and matrices are in bold, $\Re_+$ denotes the nonnegative reals, $^T$ denotes transpose, and $\cdot$ denotes Euclidean inner product.

# 2 Preliminary Results

To begin, we note that $L$ and $P$, which are the feasible sets for LVI and PVI respectively, have the relationship that if one is nonempty then so is the other. (In terms of the network, they are nonempty if for each commodity $j$ there is a directed path from $s_j$ to $t_j$.)

The claim that $P \neq \emptyset \implies L \neq \emptyset$ is a consequence of $P \subseteq L$. To justify the latter statement, let us collect into a matrix $\mathbf{P}^j$ the columns of $\mathbf{P}$ corresponding to paths from $s_j$ to $t_j$ for each commodity $j$, and similarly collect into a vector $\alpha^j$ the entries of $\alpha$. Then, $\mathbf{x}$ together with $\mathbf{x}^j = \mathbf{P}^j \alpha^j$, $j = 1, 2, \ldots, q$, satisfies the constraints defining $L$. The claim that $L \neq \emptyset \implies P \neq \emptyset$ is a consequence of the existence of a strongly acyclic flow in $L$ whenever $L$ is nonempty (see the Decomposition Theorem and Lemma 2.2).

We have observed that $P \subseteq L$. If there exist flows in $L$ which are not in $P$, they are formed by sending flow around a directed cycle. Because of the importance of flows around directed cycles in our analysis, we make the following definitions. Recall that a network is *acyclic* if it has no directed cycles. For any nonnegative $\mathbf{x} \in \Re^n$, we say that $\mathbf{x}$ *uses* arc $k$ if $x_k > 0$. We say that $\mathbf{x}$ *uses* a cycle if it uses every arc in the cycle. For $\mathbf{x} \in L$, we say that $\mathbf{x}$ is *weakly acyclic* (respectively, *strongly acyclic*) if for some (respectively, every) $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^q)$ satisfying the conditions defining $L$, no $\mathbf{x}^j$ uses all the arcs of a directed cycle. Clearly if $\mathbf{x}$ is strongly acyclic, then $\mathbf{x}$ is weakly acyclic. An important fact, shown in Lemma 2.2, is that if $\mathbf{x}$ is weakly acyclic, then $\mathbf{x} \in P$. Thus,

$$\{\mathbf{x} \in L | \mathbf{x} \text{ strongly acyclic}\} \subseteq \{\mathbf{x} \in L | \mathbf{x} \text{ weakly acyclic}\} \subseteq P \subseteq L. \qquad (3)$$

We say that $\mathbf{F}$ is *cycle-wise nonnegative* on $L$ if, for every $\mathbf{x} \in L$, we have $\sum_{k \in C} F_k(\mathbf{x}) \geq 0$ for every directed cycle $C$ of the network (or, equivalently, $\mathbf{F}(\mathbf{x}) \cdot \hat{\mathbf{x}} \geq 0$ for every nonnegative and nonzero circulation $\hat{\mathbf{x}}$ of the network). We say that $\mathbf{F}$ is *cycle-wise positive* on $L$ if "$\geq 0$" can be replaced by "$> 0$" in the above definition. Clearly $\mathbf{F}$ is cycle-wise nonnegative on $L$ if either the network is acyclic or $\mathbf{F}$ is nonnegative on $L$.

Gallager [12] introduced the concept of a strongly acyclic flow in the context of loop-free routing in a data network. He used the term loop-free where we speak of strongly acyclic. He also introduced the decomposition of a multicommodity flow discussed below.

For any nonnegative $\mathbf{x} \in \Re^n$, which may be viewed as a vector of arc flows on the network with node-arc incidence matrix $\mathbf{A}$, there exists a decomposition $\mathbf{x} = \mathbf{z} + \mathbf{w}$ such that $\mathbf{z} \geq \mathbf{0}$, $\mathbf{Az} = \mathbf{0}$, $\mathbf{w} \geq \mathbf{0}$, with the additional property that the arcs $k$ on which $w_k > 0$ form an acyclic subnetwork of the original network [17, Sec. 4B]. Any vector $\mathbf{z}$ satisfying $\mathbf{Az} = \mathbf{0}$ is by definition a *circulation*. In order to deal with the multicommodity flows in our situation, we will use a slight variation of this decomposition. For a given total flow $\mathbf{x} \in L$, we consider the following problem.

**DECOMP($\mathbf{x}$)**: Find a solution $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ to the following optimization problem.

$$\begin{array}{lll} \min & \mathbf{1} \cdot \bar{\mathbf{x}} & \\ \text{subject to} & \bar{\mathbf{x}} - \sum_j \bar{\mathbf{x}}^j = 0 & \\ & \bar{\mathbf{x}} + \hat{\mathbf{x}} = \mathbf{x} & \\ & \hat{\mathbf{x}} \geq \mathbf{0} & \\ & \mathbf{A}\bar{\mathbf{x}}^j = \mathbf{b}^j & \forall\, j \\ & \bar{\mathbf{x}}^j \geq \mathbf{0} & \forall\, j \end{array}$$

This is a capacitated min-cost multicommodity flow problem with per-unit cost of 1 on every arc. An equivalent formulation appears in Gallager's paper [12]. We restate a theorem in his paper in our notation and terminology.

**Decomposition Theorem** (Gallager, 1977). For any $\mathbf{x} \in L$, a solution $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ to DECOMP($\mathbf{x}$) exists and $\bar{\mathbf{x}}$ is strongly acyclic. $\mathbf{x}$ is strongly acyclic if and only if $\mathbf{x} = \bar{\mathbf{x}}$.

It is straightforward to show also that $\hat{\mathbf{x}}$ is a circulation.

We define a vector pair $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ to be a *strongly acyclic decomposition of* $\mathbf{x} \in L$ if $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ satisfies $\mathbf{x} = \bar{\mathbf{x}} + \hat{\mathbf{x}}$, $\bar{\mathbf{x}} \in L$, $\bar{\mathbf{x}}$ is a nonnegative strongly acyclic flow and $\hat{\mathbf{x}}$ is a nonnegative circulation. Thus, Gallager's Decomposition Theorem states that every $\mathbf{x} \in L$ has a strongly acyclic decomposition, obtained by solving the linear program DECOMP($\mathbf{x}$). This program is solvable in time polynomial in the binary representation of $\mathbf{x}$ and $\mathbf{d}$. In general the strongly acyclic decomposition is not unique. We note that this decomposition is different from the one defined by Rockafellar [17], since there may be a circulation in $\bar{\mathbf{x}}$, but if such a circulation exists, it is made up of flows of more than one commodity.

**Lemma 2.1** *Let $L_S = \{\mathbf{x} \in L | \mathbf{x}$ is strongly acyclic$\}$. Then $L_S$ is compact.*

*Proof:* Clearly $L_S$ is bounded, since no arc can carry more than $\sum_j d_j$ units of flow.

To show that $L_S$ is closed, consider any sequence $\mathbf{x}^\ell \in L_S$, $\ell = 1, 2, \ldots$, converging to a vector $\mathbf{x}^*$. Since $\mathbf{x}^\ell \in L$ and $L$ is closed, $\mathbf{x}^* \in L$. Now, in the definition of DECOMP($\mathbf{x}$) with $\mathbf{x} \in L$, $\hat{\mathbf{x}}$ serves as a set of slack variables defined by $\hat{\mathbf{x}} = \mathbf{x} - \bar{\mathbf{x}}$. Therefore we can refer to a (feasible) solution to DECOMP($\mathbf{x}$) in terms of $\bar{\mathbf{x}}$ only. Since $\mathbf{x}^\ell \in L_S$, Gallager's Decomposition Theorem implies $\mathbf{x}^\ell$ is a solution to DECOMP($\mathbf{x}^\ell$). Let $\bar{\mathbf{x}}^*$ be any solution to DECOMP($\mathbf{x}^*$). Since DECOMP($\mathbf{x}^\ell$) and DECOMP($\mathbf{x}^*$) have linear constraints that differ only in the right-hand side by $\mathbf{x}^\ell - \mathbf{x}^*$, a result of Hoffman [14] implies there exists a feasible solution $\bar{\mathbf{x}}^\ell$ to DECOMP($\mathbf{x}^\ell$) such that

$\|\bar{\mathbf{x}}^\ell - \bar{\mathbf{x}}^*\| \leq c\|\mathbf{x}^\ell - \mathbf{x}^*\|$, where $\| \ \|$ denotes the Euclidean norm and $c$ is some positive constant independent of $\ell$. Since $\mathbf{x}^\ell \to \mathbf{x}^*$, this yields $\bar{\mathbf{x}}^\ell \to \bar{\mathbf{x}}^*$. Also, since $\mathbf{x}^\ell$ and $\bar{\mathbf{x}}^\ell$ are, respectively, a solution and a feasible solution of DECOMP($\mathbf{x}^\ell$), then $\mathbf{1} \cdot \mathbf{x}^\ell \leq \mathbf{1} \cdot \bar{\mathbf{x}}^\ell$. In the limit, this yields $\mathbf{1} \cdot \mathbf{x}^* \leq \mathbf{1} \cdot \bar{\mathbf{x}}^*$. Since $\mathbf{x}^* \in L$, it is a feasible solution to DECOMP($\mathbf{x}^*$). Then $\mathbf{x}^*$ solves DECOMP($\mathbf{x}^*$). Applying Gallager's Decomposition Theorem gives $\mathbf{x}^* \in L_S$. ∎

The next lemma shows that, from any weakly acyclic $\mathbf{x} \in L$, we can find an $\alpha \in \Lambda$ such that $\mathbf{x} = \mathbf{P}\alpha$.

**Lemma 2.2** *If $\mathbf{x} \in L$ is weakly acyclic, then $\mathbf{x} \in P$.*

*Proof:* Since $\mathbf{x}$ is weakly acyclic, there exist $\mathbf{x}^j$, $j = 1, \ldots, q$, satisfying the constraints of $L$ and such that the subnetwork of arcs used by $\mathbf{x}^j$ is acyclic for all $j$. The vector $\mathbf{x}^j$ can be decomposed (not uniquely) as a sum of nonnegative flows on simple paths from $s_j$ to $t_j$ [17, Sec. 4B]. Then we can use this decomposition to write $\mathbf{x}^j = \mathbf{P}^j\alpha^j$, with $\alpha^j \geq \mathbf{0}$ and $\mathbf{R}^j\alpha^j = d_j$ where $\mathbf{R}^j$ is the row of $\mathbf{R}$ corresponding to commodity $j$. Then $\mathbf{x} = \sum_j \mathbf{x}^j = \mathbf{P}\alpha$. ∎

**Lemma 2.3** *For any solution $\mathbf{x}$ to LVI and any strongly acyclic decomposition $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ of $\mathbf{x}$, we have $\mathbf{F}(\mathbf{x}) \cdot \hat{\mathbf{x}} \leq 0$.*

*Proof:* Since $\mathbf{x}$ solves LVI, and $\bar{\mathbf{x}} \in L$, then $\mathbf{F}(\mathbf{x}) \cdot (\bar{\mathbf{x}} - \mathbf{x}) \geq 0$, or, substituting $\hat{\mathbf{x}}$ for $\mathbf{x} - \bar{\mathbf{x}}$, we have $\mathbf{F}(\mathbf{x}) \cdot \hat{\mathbf{x}} \leq 0$. ∎

We note that a variant of this lemma establishes that the cost of any used directed cycle must be nonpositive. If $j$ is the commodity using the cycle, and $v$ is the minimum flow of commodity $j$ on any arc of the cycle, we can define $\tilde{\mathbf{x}}$ by $\tilde{x}_k = v$ for any arc $k$ in the cycle and $\tilde{x}_k = 0$ for any other arc. We can then substitute $\tilde{\mathbf{x}}$ for $\hat{\mathbf{x}}$. Since $\mathbf{x} - \tilde{\mathbf{x}}$ is feasible, we have the result that the cost of the cycle must be nonpositive.

In the case that $\mathbf{F}$ is asymmetric, Lemma 2.3 and its variant are not quite obvious, since it is conceivable that commodity $j$ might find it beneficial to traverse the arcs of a directed cycle in order to deter commodity $j'$ from using certain other arcs belonging to routes used by commodity $j$.

We prove Theorems 3.1–3.5 below using the preceding lemmas and Gallager's Decomposition Theorem.

# 3 Relationship Between the Two Formulations

In this section we state and prove the main results relating the two formulations, LVI and PVI.

**Theorem 3.1** *If* $\mathbf{F}$ *is cycle-wise nonnegative on* $L$*, then every solution to PVI solves LVI.*

*Proof:* Suppose $\mathbf{F}$ is cycle-wise nonnegative on $L$. Let $\mathbf{x}$ solve PVI. Then, for any $\mathbf{y} \in L$, upon letting $(\bar{\mathbf{y}}, \hat{\mathbf{y}})$ be a strongly acyclic decomposition of $\mathbf{y}$, we have

$$\begin{aligned}
\mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) &= \mathbf{F}(\mathbf{x}) \cdot (\hat{\mathbf{y}} + \bar{\mathbf{y}} - \mathbf{x}) \\
&= \mathbf{F}(\mathbf{x}) \cdot \hat{\mathbf{y}} + \mathbf{F}(\mathbf{x}) \cdot (\bar{\mathbf{y}} - \mathbf{x}) \\
&\geq \mathbf{F}(\mathbf{x}) \cdot (\bar{\mathbf{y}} - \mathbf{x}),
\end{aligned}$$

since $\mathbf{F}(\mathbf{x}) \cdot \hat{\mathbf{y}}$ is nonnegative. Now $\bar{\mathbf{y}}$ is strongly acyclic, so, by Lemma 2.2, $\bar{\mathbf{y}} \in P$, and

$$\mathbf{F}(\mathbf{x}) \cdot (\bar{\mathbf{y}} - \mathbf{x}) \geq 0,$$

since $\mathbf{x}$ solves PVI. We conclude that $\mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$ for all $\mathbf{y} \in L$. Since $\mathbf{x} \in L$, $\mathbf{x}$ solves LVI. ∎

**Theorem 3.2** *Every weakly acyclic solution to LVI solves PVI.*

*Proof:* Let $\mathbf{x}$ be a weakly acyclic solution to LVI. By Lemma 2.2, $\mathbf{x} \in P$. Then, for any $\mathbf{y} \in P \subseteq L$, we have $\mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$. Hence $\mathbf{x}$ solves PVI. ∎

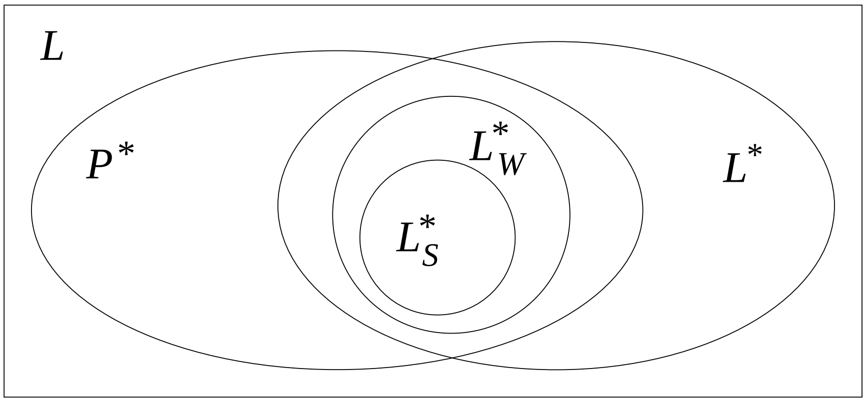


Figure 1: Venn Diagram Relating Solution Sets

To clarify the relationship between the two formulations, let us define the following solution sets:

$$\begin{aligned} P^* &= \{\mathbf{x} \in P | \mathbf{x} \text{ solves PVI}\}, \\ L^* &= \{\mathbf{x} \in L | \mathbf{x} \text{ solves LVI}\}, \\ L_W^* &= \{\mathbf{x} \in L^* | \mathbf{x} \text{ is weakly acyclic}\}, \\ L_S^* &= \{\mathbf{x} \in L^* | \mathbf{x} \text{ is strongly acyclic}\}. \end{aligned}$$

As a consequence of Theorem 3.2, we have the Venn diagram shown in Figure 1. If $\mathbf{F}$ is cycle-wise nonnegative on $L$, Theorem 3.1 implies the following nested containment of the solution sets, analogous to (3) for feasible sets:

$$L_S^* \subseteq L_W^* \subseteq P^* \subseteq L^*. \tag{4}$$

We will see in Section 4 that the above containments can be strict.

The following result, together with Theorems 3.1 and 3.2, shows that the two formulations have the same solutions if $\mathbf{F}$ is cycle-wise positive on $L$. Although this result is considered known (at least in the separable case) [16, Remark 3.4], a formal proof has not appeared in the literature.

**Theorem 3.3** *If* $\mathbf{F}$ *is cycle-wise positive on* $L$*, then every solution to LVI is strongly acyclic.*

*Proof:* Suppose $\mathbf{F}$ is cycle-wise positive on $L$ and $\mathbf{x}$ is a solution to LVI. Lemma 2.3 implies that $\mathbf{F}(\mathbf{x}) \cdot \hat{\mathbf{x}} \leq 0$ for any strongly acyclic decomposition $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ of $\mathbf{x}$. However, since $\mathbf{F}(\mathbf{x})$ is cycle-wise positive and $\hat{\mathbf{x}}$ is nonnegative, we must then have $\hat{\mathbf{x}} = \mathbf{0}$, so, by Gallager's Decomposition Theorem (see Section 2), $\mathbf{x}$ is strongly acyclic. ∎

Thus, if the network is acyclic, then the two formulations have the same solutions. In what follows, we will consider the more likely situation where the network contains directed cycles, e.g., two-way streets. We will focus on the existence/construction of strongly acyclic solutions to LVI, which, by Theorem 3.2, solve both LVI and PVI.

We now consider the existence of strongly acyclic solutions. If $\mathbf{F}$ is continuous on $L$, it is continuous on the compact set $P$. If $P$ is nonempty, a standard existence result for variational inequalities [13, Thm. 3.1] implies PVI has a solution $\mathbf{x}$. If $\mathbf{F}$ is also cycle-wise nonnegative, then Theorem 3.1 implies $\mathbf{x}$ is a solution to LVI. The following result states that $\mathbf{x}$ can in fact be specified to be strongly acyclic.

**Theorem 3.4** *If* $\mathbf{F}$ *is cycle-wise nonnegative and continuous on* $L$*, and* $L$ *is nonempty, then there exists a strongly acyclic solution to LVI.*

*Proof:* Suppose $\mathbf{F}$ is cycle-wise nonnegative and continuous on $L$. Consider the perturbed problems PVI($u$) and LVI($u$), obtained from PVI and LVI by replacing $\mathbf{F}$ with $\mathbf{F} + u\mathbf{1}$, where $u > 0$. Take any sequence of $u^\ell > 0$, $\ell = 1, 2, \ldots$, tending to 0. Since the cost function $\mathbf{F} + u^\ell \mathbf{1}$ is continuous on $P$ and $P$ is compact, convex, and nonempty, the existence result [13, Thm. 3.1] shows that PVI($u^\ell$) has a solution, which we denote by $\mathbf{x}^\ell$. Since $\mathbf{F} + u^\ell \mathbf{1}$ is cycle-wise nonnegative on $L$, Theorem 3.1 yields that $\mathbf{x}^\ell$ is a solution to LVI($u^\ell$). Since $\mathbf{x}^\ell \in P$ and $P$ is compact, by passing to a subsequence if necessary, we can assume that $\mathbf{x}^\ell$ converges to some $\mathbf{x}^* \in P$ as $\ell \to \infty$. We will show that $\mathbf{x}^*$ is a solution to PVI and a strongly acyclic solution to LVI.

For any $\mathbf{y} \in P$, we have

$$(\mathbf{F}(\mathbf{x}^\ell) + u^\ell \mathbf{1}) \cdot (\mathbf{y} - \mathbf{x}^\ell) \geq 0$$

for all $\ell$. Since $\mathbf{F}$ is continuous on $L$, we obtain in the limit as $\ell \to \infty$ that

$$\mathbf{F}(\mathbf{x}^*) \cdot (\mathbf{y} - \mathbf{x}^*) \geq 0.$$

Hence $\mathbf{x}^*$ solves PVI. Since $\mathbf{F}$ is cycle-wise nonnegative on $L$, Theorem 3.1 yields that $\mathbf{x}^*$ solves LVI.

Since $\mathbf{F} + u\mathbf{1}$ is cycle-wise positive on $L$, Theorem 3.3 implies $\mathbf{x}^\ell \in L_S$ for all $\ell$, where $L_S$ is as defined in Lemma 2.1. But $L_S$ is compact by Lemma 2.1, so $\mathbf{x}^* \in L_S$. ∎

We say that $\mathbf{F}$ is *pseudomonotone* on $L$ if $\mathbf{F}(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in L$ with $\mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$. We say that $\mathbf{F}$ is *strictly $f$-monotone* on $L$ if $(\mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) > 0$ for all $\mathbf{x}, \mathbf{y} \in L$ with $\mathbf{F}(\mathbf{y}) \neq \mathbf{F}(\mathbf{x})$.

The following result gives conditions under which a strongly acyclic solution to LVI can be constructed efficiently from any solution to LVI. Thus, in this case PVI can be solved as efficiently as LVI.

**Theorem 3.5** *If $\mathbf{F}$ is cycle-wise nonnegative on $L$ and is either continuous and pseudomonotone on $L$ or strictly $f$-monotone on $L$, then from any solution $\mathbf{x}$ to LVI we can construct a strongly acyclic solution to LVI by solving a linear program whose data size is polynomial in the binary representation of $\mathbf{x}$ and $\mathbf{d}$.*

*Proof:* Suppose $\mathbf{F}$ is cycle-wise nonnegative, continuous, and pseudomonotone on $L$. Since $\mathbf{F}$ is pseudomonotone and continuous on $L$, and $L$ is closed and convex, it is known that an $\mathbf{x} \in L$ solves LVI if and only if $\mathbf{F}(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$ for all $\mathbf{y} \in L$ [13, Prop. 3.1]. Thus, for any solution $\mathbf{x}$ of LVI, upon letting $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ be a strongly acyclic decomposition of $\mathbf{x}$, we have

$$\mathbf{F}(\mathbf{y}) \cdot (\mathbf{y} - \bar{\mathbf{x}} - \hat{\mathbf{x}}) \;=\; \mathbf{F}(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x}) \;\geq\; 0$$

for all $\mathbf{y} \in L$, which together with nonnegativity of $\mathbf{F}(\mathbf{y}) \cdot \hat{\mathbf{x}}$ yields

$$\mathbf{F}(\mathbf{y}) \cdot (\mathbf{y} - \bar{\mathbf{x}}) \ \geq \ \mathbf{F}(\mathbf{y}) \cdot \hat{\mathbf{x}} \ \geq \ 0$$

for all $\mathbf{y} \in L$. Thus $\bar{\mathbf{x}}$, which is strongly acyclic and in $L$, solves LVI.

Suppose instead that $\mathbf{F}$ is cycle-wise nonnegative and strictly $f$-monotone on $L$. For any solution $\mathbf{x}$ to LVI, upon letting $(\bar{\mathbf{x}}, \hat{\mathbf{x}})$ be any strongly acyclic decomposition of $\mathbf{x}$, we have from the strict $f$-monotonicity of $\mathbf{F}$ that either $(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\bar{\mathbf{x}})) \cdot \hat{\mathbf{x}} > 0$ or $\mathbf{F}(\mathbf{x}) = \mathbf{F}(\bar{\mathbf{x}})$. But Lemma 2.3 and the nonnegativity of $\mathbf{F}(\bar{\mathbf{x}}) \cdot \hat{\mathbf{x}}$ eliminate the first possibility, so the second possibility holds. Then, for any $\mathbf{y} \in L$, we have

$$\begin{aligned} \mathbf{F}(\bar{\mathbf{x}}) \cdot (\mathbf{y} - \bar{\mathbf{x}}) &= \mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} - \bar{\mathbf{x}}) \\ &= \mathbf{F}(\mathbf{x}) \cdot (\mathbf{y} + \hat{\mathbf{x}} - \mathbf{x}) \\ &\geq 0, \end{aligned}$$

since $\mathbf{y} + \hat{\mathbf{x}} \in L$ and $\mathbf{x}$ solves LVI. Thus $\bar{\mathbf{x}}$, which is strongly acyclic and in $L$, solves LVI.

We can find $\bar{\mathbf{x}}$ by solving the linear program DECOMP($\mathbf{x}$), whose data size is polynomial in the binary representation of $\mathbf{x}$ and $\mathbf{d}$. ∎

A stronger type of monotonicity than the ones given above is the following. $\mathbf{F}$ is *strictly monotone* on $L$ if $(\mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) > 0$ for all $\mathbf{x}, \mathbf{y} \in L$ with $\mathbf{y} \neq \mathbf{x}$. If $\mathbf{F}$ is cycle-wise nonnegative and strictly monotone, then any solution to LVI is unique [13, Prop. 3.2] and, by Theorem 3.5, must be strongly acyclic.

# 4 Counterexamples to Some Natural Conjectures

In previous sections we have shown that the two formulations LVI and PVI have the same solutions $\mathbf{x}$ when $\mathbf{F}$ is cycle-wise positive or $\mathbf{F}$ is cycle-wise

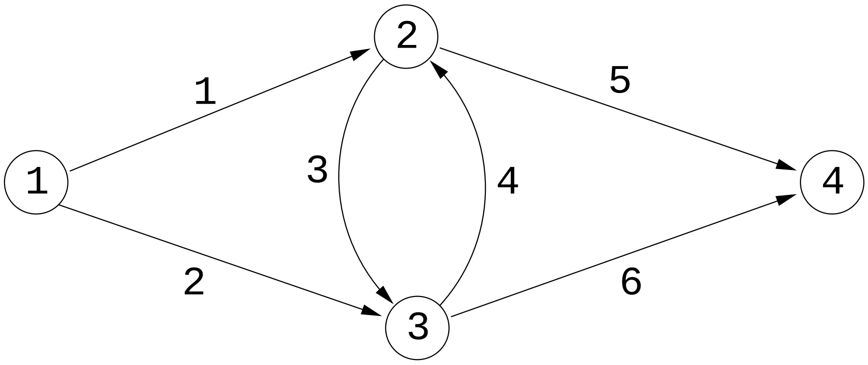


Figure 2: Example Network

nonnegative and strictly monotone. Furthermore, under suitable conditions, a solution to one of these formulations can be constructed efficiently from a solution to the other. In this section, we present counterexamples to certain natural conjectures. These conjectures arise in asking whether all regions shown in Figure 1 can be nonempty and in asking whether the hypotheses of the theorems of Section 3 are as weak as possible.

The examples use the network shown in Figure 2, with $m = 4$ nodes and $n = 6$ arcs. Each arc and node is labeled by its index. Node 1, and sometimes 2, will serve as origin nodes and 4 is the only destination node. In all examples except the last we set

$$F_1(\mathbf{x}) = x_1, \qquad F_2(\mathbf{x}) = x_2, \qquad F_5(\mathbf{x}) = x_5, \qquad F_6(\mathbf{x}) = x_6$$

and vary $F_3$ and $F_4$. All examples except one use separable functions, but the theory does not require this. The notation for solution sets is as given in Section 3.

## 4.1 F is not Cycle-Wise Nonnegative on $L$

This example shows that when the cost around a directed cycle can be negative, the solution to LVI need not be a solution to PVI nor vice versa.

Consider a one-commodity problem with origin $s_1 = 1$, destination $t_1 = 4$, and demand $d_1 = 2$. Let

$$F_3(\mathbf{x}) = x_3 - 2, \qquad F_4(\mathbf{x}) = x_4 - 2.$$

Then **F** is continuous and strictly monotone on $\Re^6$. For this example,

$$\begin{aligned} P^* &= \{(1,1,1,1,1,1)\}, \\ L^* &= \{(1,1,2,2,1,1)\}. \end{aligned}$$

## 4.2 F is Nonnegative, Continuous on $L$

This example shows that when **F** is nonnegative on $L$, so that the solution sets satisfy the containment relation given by (4), the containments can all be strict even if **F** is continuous.

Consider the two-commodity problem with origins $s_1 = 1, s_2 = 2$, destinations $t_1 = t_2 = 4$, and demands $d_1 = 4$, $d_2 = 2$. Let

$$F_3(\mathbf{x}) = 0, \qquad F_4(\mathbf{x}) = 0.$$

Then **F** is linear and strictly $f$-monotone, but not strictly monotone. For this example, LVI has a convex but unbounded solution set, and it can be checked that

$$\begin{aligned} (2,2,1,0,3,3) &\in L_S^*, \\ (2,2,2,1,3,3) &\in L_W^* - L_S^*, \\ (2,2,3,2,3,3) &\in P^* - L_W^*, \\ (2,2,4,3,3,3) &\in L^* - P^*. \end{aligned}$$

## 4.3 F is Nonnegative, Continuous, not Pseudomonotone on $L$

This is an example for which Theorem 3.4 applies but not Theorem 3.5.

Consider a one-commodity problem with origin $s_1 = 1$, destination $t_1 = 4$, and demand $d_1 = 2$. Let

$$F_3(\mathbf{x}) = |x_3 - 2|, \qquad F_4(\mathbf{x}) = |x_4 - 2|.$$

Then $\mathbf{F}$ is continuous and nonnegative on $\Re^6$, but is not pseudomonotone on $L$. For this example,

$$\begin{aligned} L_S^* &= \{(1,1,0,0,1,1)\}, \\ L^* &= \{(1,1,0,0,1,1),(1,1,2,2,1,1)\}. \end{aligned}$$

## 4.4 F is Nonnegative, Strictly Monotone on $L$, not Continuous

This is an example for which Theorem 3.5 applies but not Theorem 3.4.

Consider a one-commodity problem with origin $s_1 = 1$, destination $t_1 = 4$, and demand $d_1 = 2$. Let

$$F_3(\mathbf{x}) = x_3 + \lfloor x_3 \rfloor, \qquad F_4(\mathbf{x}) = x_4 + \lfloor x_4 \rfloor.$$

Then $\mathbf{F}$ is nonnegative and strictly monotone on $L$, but not continuous. For this example,

$$L_S^* = L^* = \{(1,1,0,0,1,1)\}.$$

## 4.5 F is Nonnegative, Monotone on $L$, not Continuous, not Strictly Monotone

This example shows that when $\mathbf{F}$ is nonnegative and monotone on $L$, it is possible for LVI to have a solution while PVI has none.

Consider a one-commodity problem with origin $s_1 = 1$, destination $t_1 = 4$, and demand $d_1 = 2$. Let

$$\begin{aligned}
F_1(\mathbf{x}) &= \begin{cases} 0 & \text{if } x_1 < 1 \\ 1 & \text{if } x_1 = 1 \text{ and } x_3 > 1, \\ 2 & \text{otherwise} \end{cases} \\
F_2(\mathbf{x}) &= \begin{cases} 2 & \text{if } x_2 > 1, \\ 1 & \text{if } x_2 = 1 \text{ and } x_3 > 1 \\ 0 & \text{otherwise} \end{cases} \\
F_3(\mathbf{x}) &= F_4(\mathbf{x}) = F_5(\mathbf{x}) = F_6(\mathbf{x}) = 0.
\end{aligned}$$

Then $\mathbf{F}$ is nonnegative, but not continuous. It is not strictly monotone, but is monotone on $L$, that is, $(\mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in L$. This can be seen by using the fact that $\mathbf{x} \in L$ implies $x_1 + x_2 = 2$ with $x_1 \geq 0, x_2 \geq 0$. For this example, it can be checked that

$$\begin{aligned}
L^* &= \{(1, 1, x + w, x, 1 - w, 1 + w) | -1 \leq w \leq 1, x > 1 - w\}, \\
P^* &= \emptyset.
\end{aligned}$$

# 5 Application to Commodity-Dependent Costs

The results of previous sections are for the case where the travel cost on each arc depends on the sum of the commodity flows. A more detailed model can also be developed to reflect differences in travel costs across commodity types and the dependence of travel costs on individual commodity flows. In such a model, distinct commodities $j$ are defined for each origin/destination pair and commodity type. Then the cost for commodity $j$ to traverse arc $k$ when the commodity flows in the network are $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^q)$ is $F_k^j(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^q)$, where the function $F_k^j : \Re_+^{nq} \mapsto \Re$ is given. Using an observation of Aashtiani and Magnanti [1], we will transform this into a single-commodity problem defined on a larger network, to which our results may then be applied.

More precisely, Aashtiani and Magnanti suggested constructing a super-network comprising $q$ copies of the original network, one for each commodity. In the super-network, flows for commodity $j$ from its origin to its destination occur only in the $j$th copy of the original network, so each arc in the super-network is used by at most one commodity. Also, the cost of traversing arc $k$ in the $j$th copy when the flows in the $q$ copies are $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^q)$ is $F_k^j(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^q)$. Then, a multi-commodity traffic equilibrium problem defined on the original network is equivalent to a single-commodity traffic equilibrium problem defined on this super-network. Interactions between commodities are captured by the dependence of each arc travel cost on all arc flows in the super-network. All our results can be applied to this single-commodity problem defined on the super-network, for which all weakly acyclic flows are strongly acyclic. However, the resulting theory will say nothing about whether solutions are weakly or strongly acyclic in the original network.

# 6 Summary

In this paper, we have studied the relationship between the link-flow and the path-flow variational inequality formulations (LVI and PVI) for the static traffic equilibrium problem with inelastic demand. Variational inequality formulations, or the equivalent, must be used when the travel time function $\mathbf{F} : \Re^n \rightarrow \Re^n$ mapping arc flows into arc costs is not a symmetric function of the arc flows in the network. Our study has focussed on the case where the travel costs are cycle-wise nonnegative-valued. Otherwise, as our first example from Section 4 shows, there is considerably less that can be said about the relationship between the two formulations.

In Theorem 3.1, we have shown that if the cost function $\mathbf{F}$ is nonnegative cycle-wise, then any solution to PVI gives rise to a solution to LVI. The absence of negative-cost directed cycles is also important in examining whether a solution to LVI gives rise to a solution to PVI. We formulate the analysis in terms of weakly and strongly acyclic flows, where a vector of total flows is weakly acyclic if there exists a compatible assignment of acyclic commodity flows, and is strongly acyclic if all compatible assignments of commodity flows are acyclic for each commodity. When $\mathbf{F}$ is nonnegative cycle-wise, we have established that (i) any weakly acyclic solution to LVI solves PVI (Theorem 3.2), (ii) if $\mathbf{F}$ is continuous there exists a strongly acyclic solution to LVI (Theorem 3.4), and (iii) if $\mathbf{F}$ is continuous and pseudomonotone, or if it is strictly $f$-monotone, we can construct a strongly acyclic solution from any solution to LVI by solving a min-cost multicommodity flow problem DECOMP (Theorem 3.5). In the process, we gave a formal proof of the fact that when $\mathbf{F}$ is cycle-wise positive, every solution to LVI is strongly acyclic.

Our analysis suggests that nonnegativity in cycle costs is a key criterion for solution equivalence of LVI and PVI. While this is of theoretical interest, in real models of traffic networks, link costs are typically positive or, if artifical links are added, cycle costs are positive. Thus, for real traffic networks, it might suffice to consider the simpler case of positive cycle costs. Nonetheless, when solutions are computed numerically, a small cost on a link might be treated by the numerical method effectively as zero cost. Our analysis involving zero-cost cycles gives some assurance that, in such a case, a computed solution of one formulation can still yield a solution to the other formulation.

Our work also leaves open some topics for further study. Specifically, the last example from Section 4 shows that when $\mathbf{F}$ is nonnegative and monotone on $L$ (but not continuous nor strictly $f$-monotone), PVI may not have a solution even if LVI does. However, the $\mathbf{F}$ we constructed is not monotone on the entire nonnegative orthant, and it would be interesting to see if one can construct an example having this stronger property. It would also be interesting to extend our results to the important case of elastic demand. Lastly, it may be possible to exploit the relationship between the link-flow and path-flow formulations and thus develop more efficient solution methods for the traffic equilibrium problem. For example, the method in [15] is based on column-generation of cheap paths for a path-flow formulation of the problem. Our results provide a means to interpret this method, under suitable conditions, in terms of a link-flow formulation of the problem. Such an interpretation may provide insights and suggest improvements to the method.

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