ERROR BOUNDS FOR REGULARIZED COMPLEMENTARITY PROBLEMS[*]

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Abstract

An approach to solving a complementarity problem entails regularizing/perturbing the problem by adding to the given mapping another mapping multiplied by a small positive parameter. We study properties of the limit point of the solution to the regularized problem. We also derive local error bounds on the distance from the solution to its limit point, expressed in terms of the regularization parameter.

Keywords: Regularization, complementarity problem, optimization, error bound

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1 Introduction

Consider the complementarity problem (CP) of finding an $x \in \Re^n$ satisfying

$$x \geq 0, \quad F(x) \geq 0, \quad F(x)^T x = 0, \tag{1}$$

where $F : \Re^n_+ \mapsto \Re^n$ is a given continuous mapping. This is a well-known problem in optimization, with many applications [12, 24]. In various regularization/continuation/smoothing approaches to solving this problem, one adds to the mapping F another mapping $G : \Re^n_{++} \mapsto \Re^n$, multiplied by a small positive scalar ϵ, and computes (possibly inexactly) an $x^\epsilon \in \Re^n$ satisfying

$$x^\epsilon \geq 0, \quad F(x^\epsilon) + \epsilon G(x^\epsilon) \geq 0, \quad (F(x^\epsilon) + \epsilon G(x^\epsilon))^T x^\epsilon = 0. \tag{2}$$

Then, one may decrease ϵ and update x^ϵ accordingly. Our interests are in properties of any limit point of x^ϵ (along some sequence of $\epsilon \to 0$) and the distance from x^ϵ to this limit point. There are also the related issues of existence/uniqueness/boundedness of x^ϵ as $\epsilon \to 0$, which we will not focus on.

The regularized CP (2) is closely linked to a regularized smooth optimization problem of the form

minimize $f_0(u) + \epsilon g_0(u)$ subject to $u \geq 0$, $f_i(u) + \epsilon g_i(u) \leq 0$, $i = 1, ..., m$, (3)

where $f_0, f_1, ..., f_m$ are continuously differentiable functions defined on some open set containing $\Re_+^l$ and $g_0, g_1, ..., g_m$ are continuously differentiable functions defined on $\Re_{++}^l$ ($m \geq 0, l \geq 1$). In particular, it is well known that the associated Karush-Kuhn-Tucker condition is exactly (2) with $x = \begin{bmatrix} u \\ v \end{bmatrix}$ and

$$F(x) = \begin{bmatrix} \nabla f_0(u) + \sum_{i=1}^m v_i \nabla f_i(u) \\ -f_i(u)_{i=1}^m \end{bmatrix}, \quad G(x) = \begin{bmatrix} \nabla g_0(u) + \sum_{i=1}^m v_i \nabla g_i(u) \\ -g_i(u)_{i=1}^m \end{bmatrix}.$$

Moreover, if $f_0, f_1, ..., f_m$ are convex (respectively, quadratic) on their respective domains, then this F is monotone (respectively, affine) and continuous on $\Re_+^{l+m}$ [42, Example 8], and similarly for G. A well studied case in this optimization setting is when $g_i \equiv 0$ for $i = 1, ..., m$, i.e., constraint functions are unregularized.

For the regularized CP (2), one popular choice of G is the identity mapping

$$G(x) = x, \tag{4}$$

corresponding to the well-known Tikhonov regularization technique. This choice has been much studied [6, 12, 14, 15, 26, 49, 50, 51, 53], including in the general setting of finding a zero of a maximal monotone operator [2, page 62], [8, Chapter II], [32]. The analogous choice of

$$g_0(u) = \|u\|^2/2$$

for (3) has been considered by Karlin [27], Mangasarian [37, 38, 39] and others [36, 44] in the context of linear programs (LP) and by Tikhonov and various others in the general optimization setting (see [1, 13, 32] and references therein). It was shown in [8, Proposition 2.6(iii)] (also see [12, Theorem 5.6.2(b)],[32, Proposition 6.1], [49], [51, Theorem 2]) that, if F is monotone, then each limit point of x^ϵ (as $\epsilon \to 0$) is the least 2-norm solution of CP. Analogous results were obtained by Mangasarian [38] in the context of LP (also see [19, 40] for extensions to other choices of g_0 in this context) and by Levitin and Polyak and others in the general optimization setting (see [1],[13, pages 30, 37],[32, Proposition 6.1] and references therein). If F is only a P_0-function, Sznader and Gowda [51, Theorem 3] showed that any limit point is weak-Pareto-minimal in the sense that no other solution is componentwise strictly less (so any non-positive solution is weak-Pareto-minimal). A second popular choice of G is the inverse function

$$G(x) = [-1/x_j]_{j=1}^n, \tag{5}$$

corresponding to log-barrier methods and interior-point methods. This choice has been considered Kojima et al. [28, 29, 30, 31] and Güler [23] and, in the

general setting of finding a zero of a maximal monotone operator, by McLinden [42]. The analogous choice of

$$g_0(u) = -\sum_{j=1}^{l} \ln(u_j)$$

for (3) has been much studied in the context of LP (see [20, 41, 43, 54] and references therein). It was shown by McLinden [42, Corollary 2] that if F is monotone and CP has a strictly complementary solution, then any limit point of x^ϵ is a least weighted $-\ln(u)$ solution of CP. Analogus results were obtained by McLinden [41, Theorem 9] and Megiddo [43] in the context of LP. A third choice is the logarithm function

$$G(x) = [\ln(x_j) + 1]_{j=1}^{n}. \tag{6}$$

The analogous choice of

$$g_0(u) = \sum_{j=1}^{l} u_j \ln(u_j) \tag{7}$$

for (3) was considered in the context of LP by Fang et al. [16, 17, 18, 45] and, from a dual exponential penalty view, by Cominetti et al. [10, 11]. It was shown in [11] that any limit point of the solution of the regularized LP is the least $u \ln(u)$-entropy solution of the LP. This result was generalized recently by Auslender et al. [4] to convex programs, with g_0 being a certain kind of separable strictly convex essentially smooth function. A similar result was shown in [52] for the LP case, without the convexity and smoothness assumption. Related results in the general optimization setting are given in [1, 13] and references therein. These results do not assume g_0 to be separable or even continuous, but they do need g_0 to be lower semicontinuous and real-valued at the limit point to be meaningful.

As the preceding discussion shows, there have been many studies of the properties of a limit point $\bar{x}$ of x^ϵ, with particular focus on the cases of G given by (4) or (5) or (6). However, there have been relatively few studies of the distance from x^ϵ to $\bar{x}$, estimated in terms of ϵ. In the context of LP with g_0 given by (7), this distance is known to be in the order of $e^{-\rho/\epsilon}$ for some constant $\rho > 0$ [11, Theorem 5.8]. The same reference also gives distance estimates for the dual LP. If g_0 is more generally a separable strictly convex essentially smooth function, this distance can be estimated in terms of $(\nabla g_0)^{-1}$ and ϵ [52]. If the LP has a multicommodity network flow structure and g_0 is a weighted inverse barrier function whose weights are affine functions of a nonnegative variable, this distance is known to be in the order of $\sqrt{\epsilon}$ [7].

In this paper, we study the above questions in the context of CP and its regularization (2). In particular, we show that if F is pseudo-monotone on $\Re_+^n$ and G is continuous at $\bar{x}$, then $\bar{x}$ solves the variational inequality problem with

mapping G over the solution set. Moreover, if F is analytic on an open set containing $\Re_+^n$, then the generalized distance $(G(x^\epsilon) - G(\bar{x}))^T (x^\epsilon - \bar{x})$ is in the order of ϵ^γ for some $\gamma > 0$, with $\gamma = 1$ if F is affine (see (16)). Alternatively, if

$$G(x) = [G_j(x_j)]_{j=1}^n, \tag{8}$$

where each G_j is strictly increasing and continuous on $\Re_{++}$ but may tend to $-\infty$ at 0 (e.g., G given by (5) or (6)), we show that in each coordinate subspace over which F is pseudo-monotone, $\bar{x}$ solves the variational inequality problem with mapping G over the solution set (see Proposition 3(a)). Moreover, under the assumption that either (i) F is pseudo-monotone on $\Re_+^n$ and $\lim_{t \to 0} t G_j(t) = 0$ for $j \notin J$ (e.g., G given by (6)) or (ii) F is affine with certain principal submatrices of its Jacobian positive semidefinite and spanning the corresponding rows or (iii) F is affine with certain principal submatrix of its Jacobian positive semidefinite and $\limsup_{t \to 0} t G_j(t) < 0$ for $j \notin J$ (e.g., G given by (5)), we estimate $(|x_j^\epsilon - \bar{x}_j|)_{j \in J}$ in terms of $(x_j^\epsilon)_{j \notin J}$ and, in the case where F is affine, we estimate the latter in terms of ϵ, where J is the set of indices j with $G_j(\bar{x}_j) > -\infty$ (see Proposition 3(c1)–(c4)). Thus, our results may be applied to analyze regularization of a convex quadratic program of the form (3). Our study is motivated by a related work in the context of LP [52], although our results and our proofs are quite different from those in [52] due to the different problem structure and regularization.

In our notation, $\Re^n$ denotes the space of n-dimensional real column vectors, $\Re_+^n$ and $\Re_{++}^n$ denote the nonnegative orthant and the positive orthant in $\Re^n$, respectively, and ^{T} denotes transpose. For any $x \in \Re^n$, we denote by x_i the ith component of x, and, for any $I \subseteq N := \{1, ..., n\}$, by x_I the vector obtained by removing from x those x_i with $i \notin I$, and by $(x_I, 0)$ the vector in $\Re^n$ whose ith component is x_i if $i \in I$ and is zero otherwise. [Here and throughout, $:=$ means "define".] We denote by $|I|$ the cardinality of I, and denote $I^c := N \backslash I$, $\|x\| := \sqrt{x^T x}$, $\|x\|_\infty := \max_{i \in N} |x_i|$. For any $M \in \Re^{n \times n}$ and any $I, J \subset N$, we denote by M_I the submatrix of M obtained by removing all rows of M with indices outside of I and by M_{IJ} the submatrix of M_I obtained by removing all columns of M_I with indices outside of J. For any $F : \Re_+^n \mapsto \Re^n$ and any nonempty closed convex set $\Sigma \subset \Re_+^n$, we denote

$$\mathrm{VI}(\Sigma, F) := \{ x \in \Sigma : F(x)^T (y - x) \geq 0 \ \forall y \in \Sigma \}.$$

[Thus x satisfies (1) if and only if $x \in \mathrm{VI}(\Re_+^n, F)$ and x^ϵ satsifies (2) if and only if $x^\epsilon \in \mathrm{VI}(\Re_+^n, F + \epsilon G)$.] We denote by F_i the ith component of F and, for any $I \subseteq N$, by F_I the mapping obtained by removing from F those F_i with $i \notin I$. We say F is pseudo-monotone on $\Re_+^n$ [5, page 121] if

$$x, y \in \Re_+^n \text{ and } F(y)^T (x - y) \geq 0 \quad \Longrightarrow \quad F(x)^T (x - y) \geq 0. \tag{9}$$

For any $x \in \Re^n$ and any nonempty closed set $\Sigma \subset \Re^n$, we denote $\mathrm{dist}(x, \Sigma) := \min_{y \in \Sigma} \|y - x\|$.

2 Error Bounds on Distance to Limiting Solution

First, we have the following bound on the distance from x^ϵ to the solution set of (1) in terms of the regularization $G^\epsilon(x^\epsilon)$. This is a simple consequence of an error bound result for analytic systems [33].

Proposition 1 *Consider an open set $\Omega \subset \Re^n$ containing $\Re^n_+$ and an analytic $F : \Omega \mapsto \Re^n$ with $\Sigma := \mathrm{VI}(\Re^n_+, F)$ nonempty. Then, for every bounded $\Xi \subset \Re^n$, there exist $\tau > 0$ and $\gamma > 0$ such that*

$$\mathrm{dist}(x^\epsilon, \Sigma) \leq \tau \left(\| \max\{0, G^\epsilon(x^\epsilon)\} \| + |G^\epsilon(x^\epsilon)^T x^\epsilon| \right)^\gamma \tag{10}$$

for all $G^\epsilon : \Re^n_+ \mapsto \Re^n$ ($\epsilon > 0$) and all $x^\epsilon \in \Xi \cap \mathrm{VI}(\Re^n_+, F + G^\epsilon)$.

Proof. We have that an $x \in \Sigma$ satisfies

$$-x \leq 0, \quad -F(x) \leq 0, \quad F(x)^T x = 0$$

and that an $x^\epsilon \in \mathrm{VI}(\Re^n_+, F + G^\epsilon)$ satisfies

$$-x^\epsilon \leq 0, \quad -F(x^\epsilon) \leq G^\epsilon(x^\epsilon), \quad F(x^\epsilon)^T x^\epsilon = -G^\epsilon(x^\epsilon)^T x^\epsilon.$$

So if x^ϵ is also in the bounded set Ξ, then since F is analytic on an open set containing $\Re^n_+$, an error bound result of Lojasiewicz, as extended by Luo and Pang for analytic systems [33, Theorem 2.2], yields (10) with $\tau > 0$ and $\gamma > 0$ some constants. ∎

Note 1. Proposition 1 does not say anything about the existence or uniqueness or boundedness of $x^\epsilon \in \mathrm{VI}(\Re^n_+, F + G^\epsilon)$. In the case where F is monotone and affine and $G^\epsilon(x) = M^\epsilon x + q^\epsilon$, Robinson [47, Theorem 2] showed that Σ being nonempty and bounded is both necessary and sufficient for the existence of x^ϵ satisfying $\mathrm{dist}(x^\epsilon, \Sigma) \to 0$ as $\|M^\epsilon\| + \|q^\epsilon\| \to 0$. If F is a continuously differentiable P_0-function, a result of Facchinei [14, Theorem 4.4] implies that Σ being nonempty and bounded is sufficient for the existence of x^ϵ satisfying $\mathrm{dist}(x^\epsilon, \Sigma) \to 0$ as $\epsilon \to 0$, where G^ϵ is continuous and satisfies $\lim_{\epsilon \to 0} \sup_{x : \mathrm{dist}(x, \Sigma) \leq \sigma} \|G^\epsilon(x)\| = 0$ for some $\sigma > 0$. If F is a continuously differentiable P_0-function and $G^\epsilon(x) = \epsilon x$, Facchinei and Kanzow [15, Theorem 3.5] showed existence and uniqueness of x^ϵ for all $\epsilon > 0$ and, if in addition Σ is nonempty and bounded, then x^ϵ is bounded and $\mathrm{dist}(x^\epsilon, \Sigma) \to 0$ as $\epsilon \to 0$. Ravindran and Gowda [46] extended the preceding two results to CP with bound constraints, and they weakened the differentiability assumption on F to continuity. In the case where F is a polynomial P_0-function and $G^\epsilon(x) = \epsilon x$, Sznajder and Gowda [51, Theorem 5] showed that x^ϵ either converges or diverges in norm.

Proposition 1 does not give an estimate of the Hölder constant γ. [In the case where F is affine and monotone, a bound with $\gamma = 1/2$ can be shown [33, Theorem 5.4].] By adding a mild assumption on G^ϵ, we derive below a second distance bound in terms of $G^\epsilon(x^\epsilon)$, with $\gamma = 1$ when F is affine. Moreover, in the case where F is pseudo-monotone on $\Re^n_+$ and $G^\epsilon(x^\epsilon)/\epsilon$ converges, we derive a bound on the distance from x^ϵ to its limit point in terms of ϵ.

Proposition 2 *Consider a continuous $F : \Re^n_+ \mapsto \Re^n$, a sequence of positive scalars $\Upsilon = \{\epsilon^1, \epsilon^2, ...\}$ tending to zero and, for each $\epsilon \in \Upsilon$, a $G^\epsilon : \Re^n_+ \mapsto \Re^n$ and an $x^\epsilon \in \Sigma^\epsilon := \mathrm{VI}(\Re^n_+, F + G^\epsilon)$ such that x^ϵ converges to some $\bar{x}$ and $G^\epsilon(x^\epsilon)/\epsilon$ converges to some g as $\epsilon \in \Upsilon \to 0$. Then $\bar{x} \in \Sigma := \mathrm{VI}(\Re^n_+, F)$, and the following hold.*

(a). *If F is pseudo-monotone on $\Re^n_+$, then $\bar{x} \in \mathrm{VI}(\Sigma, g)$.*

(b). *If F is analytic on an open set containing $\Re^n_+$, then there exist $\tau > 0$ and $\gamma > 0$ such that*

$$\mathrm{dist}(x^\epsilon, \Sigma) \le \tau \|G^\epsilon(x^\epsilon)\|^\gamma \tag{11}$$

for all $\epsilon \in \Upsilon$, with $\gamma = 1$ whenever F is affine.

(c). *If F is pseudo-monotone on $\Re^n_+$ and is analytic on an open set containing $\Re^n_+$, then there exist $\tau > 0$ and $\gamma > 0$ such that*

$$(G^\epsilon(x^\epsilon)/\epsilon - g)^T (x^\epsilon - x^*) \le \tau \|g\| \|G^\epsilon(x^\epsilon)\|^\gamma \tag{12}$$

for all $x^* \in \mathrm{VI}(\Sigma, g)$ and all $\epsilon \in \Upsilon$, with $\gamma = 1$ whenever F is affine.

Proof. Since $x^\epsilon \in \Sigma^\epsilon$ so that $x^\epsilon \ge 0$, $F(x^\epsilon) + G^\epsilon(x^\epsilon) \ge 0$, $(F(x^\epsilon) + G^\epsilon(x^\epsilon))^T x^\epsilon = 0$ for all $\epsilon \in \Upsilon$, we have in the limit (also using $G^\epsilon(x^\epsilon) \to 0$) that $\bar{x} \ge 0$, $F(\bar{x}) \ge 0$, $F(\bar{x})^T \bar{x} = 0$. Thus $\bar{x} \in \Sigma$.

(a). Assume F is pseudo-monotone on $\Re^n_+$. Then Σ is closed convex [5, page 121]. Moreover, for any $y \in \Sigma$, (9) and the fact that $F(y)^T(x^\epsilon - y) \ge 0$ imply

$$0 \le F(x^\epsilon)^T(x^\epsilon - y) \le G^\epsilon(x^\epsilon)^T(y - x^\epsilon),$$

where the second inequality uses $x^\epsilon \in \Sigma^\epsilon$ and $y \in \Re^n_+$. Dividing both sides by ϵ yields in the limit that $0 \le g^T(y - \bar{x})$.

(b). For each $\epsilon > 0$, since $x^\epsilon \in \Sigma^\epsilon$, we have

$$F(x^\epsilon)_I = -G^\epsilon(x^\epsilon)_I, \quad F(x^\epsilon)_{I^c} \ge -G^\epsilon(x^\epsilon)_{I^c}, \quad x^\epsilon_I \ge 0, \quad x^\epsilon_{I^c} = 0, \tag{13}$$

for some $I \subset N$. Let $\Upsilon_I := \{\epsilon \in \Upsilon : (13) \text{ holds}\}$. Consider any $I \subset N$ such that $|\Upsilon_I| = \infty$. Since $G^\epsilon(x^\epsilon) \to 0$ as $\epsilon \in \Upsilon_I \to 0$, then any cluster point x of x^ϵ satisfies

$$F(x)_I = 0, \quad F(x)_{I^c} \ge 0, \quad x_I \ge 0, \quad x_{I^c} = 0. \tag{14}$$

Assume F is analytic on an open set containing $\Re_+^n$. Then an error bound result of Lojasiewicz, as extended by Luo and Pang to analytic systems [33, Theorem 2.2], implies the nonlinear system (14) has a solution y^ϵ satisfying

$$\|x^\epsilon - y^\epsilon\| \leq \tau_I \|G^\epsilon(x^\epsilon)\|^{\gamma_I}, \tag{15}$$

where $\tau_I > 0$ and $\gamma_I > 0$ are constants depending on F and I and and $\sup_{\epsilon \in \Upsilon_I} \|x^\epsilon\|$ only. Thus, $y^\epsilon \in \Sigma$ and, moreover, in the case where F is affine, a lemma of Hoffman [25] implies that $\gamma_I = 1$. For any I with $|\Upsilon_I| < \infty$, let y^ϵ be any fixed element of Σ for all $\epsilon \in \Upsilon_I$ and then, for any $\gamma_I \geq 1$, (15) would hold for a suitable τ_I (since its left-hand side is bounded and its right-hand side is bounded away from zero). Taking $\gamma := \min_I \gamma_I$ and $\tau := \max_I \left\{ \sup_{\epsilon \in \Upsilon_I} \tau_I \|G^\epsilon(x^\epsilon)\|^{\gamma_I - \gamma} \right\}$ yields (11) for all $\epsilon \in \Upsilon = \cup_I \Upsilon_I$, with $\gamma = 1$ whenever F is affine.

(c) Assume F is pseudo-monotone on $\Re_+^n$ and is analytic on an open set containing $\Re_+^n$. Fix any $x^* \in \mathrm{VI}(\Sigma, g)$ and any $\epsilon \in \Upsilon$. Letting $y^\epsilon \in \Sigma$ satisfy $\|x^\epsilon - y^\epsilon\| = \mathrm{dist}(x^\epsilon, \Sigma)$, we have together with (11) in part (b) that

$$
\begin{aligned}
0 \leq g^T(y^\epsilon - x^*) &= g^T(y^\epsilon - x^\epsilon) + g^T(x^\epsilon - x^*) \\
&\leq \tau \|g\| \|G^\epsilon(x^\epsilon)\|^\gamma + g^T(x^\epsilon - x^*)
\end{aligned}
$$

for some constants $\tau > 0$ and $\gamma > 0$. Also, since $x^* \in \Sigma$, we have from (9) and $x^\epsilon \in \Sigma^\epsilon$ that

$$0 \leq F(x^\epsilon)^T(x^\epsilon - x^*), \qquad 0 \leq (F(x^\epsilon) + G^\epsilon(x^\epsilon))^T(x^* - x^\epsilon).$$

Adding the above two inequalities to the previous inequality multiplied by ϵ, we obtain

$$0 \leq \epsilon\tau \|g\| \|G^\epsilon(x^\epsilon)\|^\gamma + (\epsilon g - G^\epsilon(x^\epsilon))^T(x^\epsilon - x^*).$$

Rearranging terms yields (12). ∎

Note 2. Notice that Proposition 2 is stated in the setting of ϵ along a sequence, rather than ϵ in a continuum as in Proposition 1. Although for practical purposes such as analyzing the convergence of an iterative method, the former setting is sufficient, it is nevertheless possible to extend Proposition 2 to the latter setting, provided $\|G^\epsilon(x^\epsilon)\|$ is bounded away from zero whenever ϵ in the continuum is bounded away from zero. Also, Propositions 1 and 2 may possibly be extended to F being piecewise-analytic and, more generally, "subanalytic" [34].

Note 3. In the case where F is an analytic P_0-function and the solution set Σ is nonempty and bounded, [14, Theorem 4.4] implies that x^ϵ is defined and bounded as $\epsilon \to 0$ and so Proposition 2(b) yields that, for any sequence of ϵ along which x^ϵ converges, (11) holds for all ϵ in this sequence, where τ, γ depend on the limit point ($\gamma = 1$ if F is affine). A similar result was shown

earlier by Robinson [47, Theorem 2] in the case of F being monotone and affine. If in addition F is polynomial and $G^\epsilon(x) = \epsilon x$, then x^ϵ converges [51]. Bounds of the type (11) were also derived by Fischer [21, Section 3.2] under similar, though not identical, assumptions on F and G^ϵ. Fischer derived his bounds by applying a stability result of Klatte for parametric optimization. In the case where the set-valued mapping $F \Rightarrow VI(\Re^n_+, F)$ has the *Aubin property* relative to $\{F + G : G : \Re^n_+ \mapsto \Re^n, \sup_{x \in \Re^n_+, \|x - \bar{x}\| \leq 1} \|G(x)\| < 1\}$ at F for $\bar{x}$, a bound similar to (11) with $\gamma = 1$ holds [48, 9F]. However, verifying the Aubin property may be difficult. In the optimization setting, an analogous Lipschitzian property can be shown, under very mild assumptions, for the set of ϵ-approximate solutions [3, Theorem 4.3].

Note 4. In the case where F is pseudo-monotone on $\Re^n_+$ and $G^\epsilon(x) = \epsilon G(x)$ with G continuous at $\bar{x}$, Proposition 2(a),(c) imply $g = G(\bar{x})$ and $\bar{x} \in VI(\Sigma, G)$. This extends previous results [6, Theorem 2.3], [12, Theorem 5.6.2(b)], [49] for the case of F being monotone or pseudo-monotone and $G(x) = x$ (also see [2, page 63], [8, Proposition 2.6(iii)], [32, Proposition 6.1] for analogous results in the context of maximal monotone operators in an infinite-dimensional space). If F is also analytic on an open set containing $\Re^n_+$, taking $x^* = \bar{x}$ in (12) yields

$$\left(G(x^\epsilon) - G(\bar{x})\right)^T (x^\epsilon - \bar{x}) \leq \left(\tau \|G(\bar{x})\| \|G(x^\epsilon)\|^\gamma\right) \epsilon^\gamma. \qquad (16)$$

Thus, if in addition G is strictly monotone at $\bar{x}$ in the sense that there exist $\sigma > 0, \varrho > 0, \delta > 0$ such that

$$\left(G(x) - G(\bar{x})\right)^T (x - \bar{x}) \geq \sigma \|x - \bar{x}\|^\varrho \qquad \forall x \geq 0 \text{ with } \|x - \bar{x}\| \leq \delta, \qquad (17)$$

then (16) would yield the error bound that $\|x^\epsilon - \bar{x}\|$ is in the order of $(\epsilon)^{\gamma/\varrho}$ whenever $\|x^\epsilon - \bar{x}\| \leq \delta$. Notice that G essentially needs to be continuous at $\bar{x}$ in order to satisfy the assumption that $G^\epsilon(x^\epsilon)/\epsilon$ converges as $x^\epsilon \to \bar{x}$.

In deriving the error bound in Proposition 2(c), we have required $G^\epsilon(x^\epsilon) \to 0$ as $\epsilon \to 0$. This rules out the important case of $G^\epsilon(x) = \epsilon G(x)$, where G is given by (5) or (6) or, more generally, (8) with possibly $\lim_{t \downarrow 0} G_j(t) = -\infty$. In Proposition 3 below, we consider this case and we study properties of any limit point $\bar{x}$ of x^ϵ (see part (a)) and derive error bounds on the distance from x^ϵ to $\bar{x}$ (see parts (c1)–(c4)). In particular, parts (c1), (c2), (c4) of this proposition estimate, under various assumptions on F and $G_1, ..., G_n$, the distance $\|(x - \bar{x})_J\|$ in terms of $\|x_{J^c}\|$, and parts (c3) and (c4) estimate, in the case where F is affine, the latter in terms of ϵ, with J being the set of indices j with $G_j(\bar{x}_j) > -\infty$. While these error bounds may be complex, Example 1 below suggests that this complexity is needed to account for the different (relative) growth rates of $G_1, ..., G_n$ near zero and the linkage among the components of x^ϵ as imposed by the complementarity condition (2).

Proposition 3 *Consider a continuous $F : \Re_+^n \mapsto \Re^n$ and a continuous $G : \Re_{++}^n \mapsto \Re^n$ given by (8), where $\lim_{t\downarrow 0} G_j(t) = G_j(0) \in [-\infty, \infty)$ for all $j \in N$. Consider a sequence of positive scalars $\Upsilon = \{\epsilon^1, \epsilon^2, ...\}$ tending to zero and, for each $\epsilon \in \Upsilon$, an $x^\epsilon \in \Sigma^\epsilon := \mathrm{VI}(\Re_+^n, F + \epsilon G)$ converging to some $\bar{x}$ as $\epsilon \in \Upsilon \to 0$. Then $\bar{x} \in \Sigma := \mathrm{VI}(\Re_+^n, F)$ and the following hold with $J := \{j \in N : G_j(\bar{x}_j) > -\infty\}$, $\bar{\Sigma}$ being the path-connected component of Σ containing $\bar{x}$, $\bar{J} := \{j \in N : G_j(x_j) > -\infty$ for some $x \in \bar{\Sigma}\}$ and, for each $I \subset J$, $\Upsilon_I := \{\epsilon \in \Upsilon : F(x^\epsilon)_I + \epsilon G(x^\epsilon)_I = 0, x_{\bar{J}\setminus I}^\epsilon = 0\}$.*

(a). *For each $J \subset H \subset \bar{J}$ such that $x_H \mapsto F(x_H, 0)$ is pseudo-monotone on $\Re_+^{|H|}$, and $\Sigma_H := \{x_H : (x_H, 0) \in \bar{\Sigma}\}$ is convex and has an element $y_H = (y_j)_{j\in H}$ with $G_j(y_j) > -\infty$ for all $j \in H$, we have $\bar{x}_H \in \mathrm{VI}(\Sigma_H, p_H)$ (respectively, $H = J$ and $\bar{x}_H \in \mathrm{VI}(\Sigma_H, \zeta p_H + G_H)$) if $\|q_H^\epsilon\|/\epsilon \to \infty$ (respectively, $\|q_H^\epsilon\|/\epsilon \to$ some $\zeta \in \Re_+$) as $\epsilon \to 0$ along some subsequence of Υ, where p_H is any cluster point of $q_H^\epsilon/\|q_H^\epsilon\|$ along this subsequence and $q^\epsilon := F(x^\epsilon) - F(x_H^\epsilon, 0)$.*

(b). *If F is analytic on an open set containing $\Re_+^n$, then there exist $\tau > 0$ and $\gamma > 0$ such that*

$$\mathrm{dist}(x^\epsilon, \Sigma) \le \tau(\epsilon\|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|)^\gamma \tag{18}$$

for all $\epsilon \in \Upsilon$, with $\gamma = 1$ whenever F is affine.

(c1). *If F is pseudo-monotone on $\Re_+^n$ and $\lim_{t\to 0} t G_j(t) = 0$ for all $j \in \bar{J}^c$, then $\bar{J} = J$ and $\bar{x} \in \mathrm{VI}(\Sigma, G)$. If in addition F is analytic on an open set containing $\Re_+^n$, then there exist $\tau > 0$ and $\gamma > 0$ such that*

$$(G(x^\epsilon) - G(\bar{x}))_J^T(x^\epsilon - \bar{x})_J \le \tau(\epsilon\|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|)^\gamma - G(x^\epsilon)_{J^c}^T x_{J^c}^\epsilon \tag{19}$$

for all $\epsilon \in \Upsilon$, with $\gamma = 1$ whenever F is affine.

(c2). *If $F(x) = Mx + q$ for some $M \in \Re^{n\times n}, q \in \Re^n$, with $M_{\bar{J}\bar{J}^c} \in M_{\bar{J}\bar{J}}(\Re^{|\bar{J}|\times|\bar{J}^c|})$ and $M_{\bar{J}\bar{J}} \in M_{\bar{J}\bar{J}}^T(\Re^{|\bar{J}|\times|\bar{J}|})$, and if $\bar{\Sigma}$ is convex, then $\bar{J} = J$ and $\bar{x} \in \mathrm{VI}(\bar{\Sigma}, G)$. Moreover, for each $I \subset J$ with $|\Upsilon_I| = \infty$ and M_{II} positive semidefinite and $M_{IJ^c} = M_{II} N_{IJ^c}$ for some $N_{IJ^c} \in \Re^{|I|\times|J^c|}$, there exists $\tau_I > 0$ such that*

$$(G(x^\epsilon) - G(\bar{x}))_J^T(x^\epsilon - \bar{x})_J \le \tau_I(\epsilon\|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|) - G(x^\epsilon)_I^T N_{IJ^c} x_{J^c}^\epsilon \tag{20}$$

for all $\epsilon \in \Upsilon_I$.

(c3). *If $F(x) = Mx + q$ for some $M \subset \Re^{n\times n}$, $q \in \Re^n$, and if $\bar{\Sigma}$ is convex and G_j is strictly increasing for all $j \in J^c$, then there exist $\rho > 0$ and $\tau > 0$ such that*

$$x_j^\epsilon \le G_j^{-1}(-\rho/\epsilon) \;\; \forall j \in K, \quad \|x_{L\setminus\bar{J}}^\epsilon\| \le \tau\left(\|x_K^\epsilon\| + \sum_{j\in L} h_j(\epsilon) + \epsilon\right) \tag{21}$$

for all $\epsilon \in \Upsilon$ sufficiently small, where $K \subset J^c$ (K may depend on ϵ), $L := J^c \backslash K$, and $h_j(\epsilon)$ is the unique $\eta > 0$ satisfying $G_j^{-1}(-\eta/\epsilon) = \eta$ for $j \in J^c$. If $M = -M^T$, then (21) remains true without the h_j terms.

(c4). *If $F(x) = Mx + q$ for some $M \subset \Re^{n \times n}$, $q \in \Re^n$, with $M_{\bar{J}\bar{J}}$ positive semidefinite, and if G_j is strictly increasing with $\limsup_{t \to 0} tG_j(t) < 0$ for all $j \in J^c$, then Υ is the union of a countable collection of subsequences $\tilde{\Upsilon}$ for each of which there exist $\tau > 0, \rho > 0$, properly nested $\bar{J} = H_1 \supset \cdots \supset H_{r-1} \supset H_r = J$, and $\lambda_l^\epsilon > 0$ depending on $M_{N(H_l)^c} x_{(H_l)^c}^\epsilon$ and ϵ only ($l = 1, .., r$, $\epsilon \in \tilde{\Upsilon}$) such that $\lambda_l^\epsilon \to 0$ as $\epsilon \in \tilde{\Upsilon} \to 0$ and, for each $l = 1, .., r - 1$,*

$$either \quad x_j^\epsilon \le G_j^{-1}\left(-\rho/\lambda_l^\epsilon\right) \quad or \quad x_j^\epsilon \le \tau \lambda_l^\epsilon \quad \forall j \in H_l \backslash H_{l+1} \qquad (22)$$

for all $\epsilon \in \tilde{\Upsilon}$ sufficiently small. And if in addition G_J is locally Lipschitzian at $\bar{x}_J$ with constant $\kappa > 0$ and satisfies (17) with $\sigma > 0, \delta > 0, \varrho = 1$, then there exist $\tau' > 0$ (independent of κ, σ) such that

$$\|(x^\epsilon - \bar{x})_J\| \le \tau'(\kappa/\sigma + 1 + 1/\sigma)\lambda_r^\epsilon \qquad (23)$$

for all $\epsilon \in \tilde{\Upsilon}$ sufficiently small.

Proof. For each $j \in J$, we have $G_j(x_j^\epsilon) \to G_j(\bar{x}_j)$ as $\epsilon \in \Upsilon \to 0$ so the fact x^ϵ satisfies (2) for all $\epsilon \in \Upsilon$ yields in the limit that $\bar{x}_j \ge 0$, $F(\bar{x})_j \ge 0$, $F(\bar{x})_j \bar{x}_j = 0$. For each $j \in J^c$, we have $\bar{x}_j = 0$ and $F(x^\epsilon)_j = -\epsilon G_j(x_j^\epsilon) \ge 0$ for all $\epsilon \in \Upsilon$ sufficiently small. The latter yields in the limit $F(\bar{x})_j \ge 0$. Thus $\bar{x}$ satisfies (1) and hence $\bar{x} \in \Sigma$.

(a). Consider any $J \subset H \subset \bar{J}$ such that $x_H \mapsto F(x_H, 0)$ is pseudo-monotone on $\Re_+^{|H|}$, and Σ_H is convex and contains an element $y_H = (y_j)_{j \in H}$ with $G_j(y_j) > -\infty$ for all $j \in H$. Then, we have from $(y_H, 0) \in \bar{\Sigma}$ and $x^\epsilon \in \Sigma^\epsilon$ that

$$0 \le F(y_H, 0)_H^T (x_H^\epsilon - y_H), \quad 0 \le (F(x_H^\epsilon, 0)_H + q_H^\epsilon + \epsilon G(x^\epsilon)_H)^T (y_H - x_H^\epsilon),$$

with $q^\epsilon := F(x^\epsilon) - F(x_H^\epsilon, 0)$. Since $x_H^\epsilon \in \Re_+^{|H|}$, the first inequality and the pseudo-monotonicity of $x_H \mapsto F(x_H, 0)$ on $\Re_+^{|H|}$ imply $0 \le F(x_H^\epsilon, 0)_H^T (x_H^\epsilon - y_H)$, which when added to the second inequality yields

$$0 \le (q_H^\epsilon + \epsilon G(x^\epsilon)_H)^T (y_H - x_H^\epsilon). \qquad (24)$$

Consider any subsequence of Υ along which either (i) $\|q_H^\epsilon\|/\epsilon \to \infty$ or (ii) $\|q_H^\epsilon\|/\epsilon \to \zeta \in \Re_+$, and let p_H be any cluster point of $q_H^\epsilon/\|q_H^\epsilon\|$ along this subsequence. In case (i), dividing both sides of (24) by $\|q_H^\epsilon\|$ and using $y_j > 0 = \bar{x}_j$ and $G_j(x_j^\epsilon) \to -\infty$ for all $j \in J^c$ yield in the limit that

$$0 \le p_H^T (y_H - \bar{x}_H).$$

Since Σ_H is convex, this holds for all $y_H \in \Sigma_H$, so $\bar{x}_H \in \mathrm{VI}(\Sigma_H, p_H)$. In case (ii), dividing both sides of (24) by ϵ and arguing as in case (i) yield in the limit that $H = J$ and

$$0 \leq (\zeta p_H + G(\bar{x})_H)^T (y_H - \bar{x}_H).$$

Since Σ_H is convex, this holds for all $y_H \in \Sigma_H$, so $\bar{x}_H \in \mathrm{VI}(\Sigma_H, \zeta p_H + G_H)$.

(b). For each $j \in J^c$, we have $x_j^\epsilon > 0$ for all $\epsilon \in \Upsilon$ and $G_j(x_j^\epsilon) < 0$ for all $\epsilon \in \Upsilon$ below some $\bar{\epsilon}$. Consider any $I \subset J$ such that $|\Upsilon_I| = \infty$. For each $\epsilon \in \Upsilon_I$ below $\bar{\epsilon}$, since $x^\epsilon \in \Sigma^\epsilon$, we have

$$F(x^\epsilon)_I = -\epsilon G(x^\epsilon)_I, \quad F(x^\epsilon)_{J \setminus I} \geq -\epsilon G(x^\epsilon)_{J \setminus I}, \quad F(x^\epsilon)_{J^c} \geq 0,$$
$$x_I^\epsilon \geq 0, \quad x_{J \setminus I}^\epsilon = 0, \quad x_{J^c}^\epsilon = x_{J^c}. \tag{25}$$

Since $F(x^\epsilon) \to F(\bar{x})$ and $G(x^\epsilon)_J \to G(\bar{x})_J$ as $\epsilon \in \Upsilon_I \to 0$, (25) yields in the limit that $\bar{x}$ satisfies

$$F(x)_I = 0, \quad F(x)_{I^c} \geq 0, \quad x_I \geq 0, \quad x_{I^c} = 0. \tag{26}$$

Assume F is analytic on an open set containing $\Re_+^n$. Then an error bound result of Lojasiewicz, as extended by Luo and Pang to analytic systems [33, Theorem 2.2], implies the nonlinear system (26) has a solution y^ϵ satisfying

$$\|y^\epsilon - x^\epsilon\| \leq \tau_I (\epsilon \|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|)^{\gamma_I}, \tag{27}$$

where $\tau_I > 0$ and $\gamma_I > 0$ are constants depending on F and I and $\sup_{\epsilon \in \Upsilon_I} \|x^\epsilon\|$ only. Thus, $y^\epsilon \in \Sigma$ and, in the case where F is affine, a lemma of Hoffman [25] implies $\gamma_I = 1$. If Υ_I is finite but nonempty, let y^ϵ be any fixed element of Σ for all $\epsilon \in \Upsilon_I$ and then (27) would hold for any $\gamma_I \geq 1$ and a sufficiently large τ_I. Taking $\gamma := \min_I \gamma_I$ and

$$\tau := \max_I \left\{ \sup_{\epsilon \in \Upsilon_I} \tau_I (\epsilon \|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|)^{\gamma_I - \gamma} \right\}$$

yields (18) for all $\epsilon \in \Upsilon = \cup_I \Upsilon_I$.

(c1). Assume F is pseudo-monotone on $\Re_+^n$ and $\lim_{t \to 0} t G_j(t) = 0$ for all $j \in \bar{J}^c$. Then Σ is closed convex [5, page 121] so $\Sigma = \bar{\Sigma}$ and there exists $y \in \Sigma$ such that $G_j(y_j) > -\infty$ for all $j \in \bar{J}$. For each $\epsilon \in \Upsilon$, since $y \in \Sigma$, (9) and $x^\epsilon \in \Sigma^\epsilon$ imply

$$0 \leq F(x^\epsilon)^T (x^\epsilon - y) \leq \epsilon G(x^\epsilon)^T (y - x^\epsilon). \tag{28}$$

Also, by (a), $\bar{x} \in \Sigma$, so $J \subset \bar{J}$. For each $j \in J$, we have $G_j(x_j^\epsilon)(y_j - x_j^\epsilon)$ converges as $\epsilon \in \Upsilon \to 0$. For each $j \in \bar{J} \setminus J$, we have $y_j > 0$ and $G_j(x_j^\epsilon) \to -\infty$, so $G_j(x_j^\epsilon)(y_j - x_j^\epsilon) \to -\infty$ as $\epsilon \in \Upsilon \to 0$. For each $j \in \bar{J}^c$, we have $x_j^\epsilon \to y_j = 0$ so our assumption on G_j yields $G_j(x_j^\epsilon)(y_j - x_j^\epsilon) \to 0$ as $\epsilon \in \Upsilon \to 0$. Hence, (28) implies $\bar{J} \setminus J = \emptyset$, i.e., $J = \bar{J}$. Now, for any $y \in \Sigma$, (9) and $x^\epsilon \in \Sigma^\epsilon$ imply (28)

holds. Dividing both sides of (28) by ϵ and using $y_j = 0$ and $G_j(x_j^\epsilon)x_j^\epsilon \to 0$ for all $j \in \bar{J}^c = J^c$ yields in the limit that

$$0 \le G(\bar{x})_J^T (y - \bar{x})_J = G(\bar{x})^T (y - \bar{x})$$

(here $\infty \cdot 0 = 0$), so $\bar{x} \in \text{VI}(\Sigma, G)$.

Assume in addition F is analytic on an open set containing $\Re_+^n$, so that, by part (a), there exist $\tau > 0$ and $\gamma > 0$ such that (18) holds for all $\epsilon \in \Upsilon$. Let $y^\epsilon \in \Sigma$ satisfy $\|x^\epsilon - y^\epsilon\| = \text{dist}(x^\epsilon, \Sigma)$. Now, for each $\epsilon \in \Upsilon$, since $\bar{x} \in \Sigma$, (9) and $x^\epsilon \in \Sigma^\epsilon$ imply

$$0 \le F(x^\epsilon)^T (x^\epsilon - \bar{x}), \qquad 0 \le (F(x^\epsilon) + \epsilon G(x^\epsilon))^T (\bar{x} - x^\epsilon).$$

Adding these two inequalities and dividing by ϵ gives $0 \le G(x^\epsilon)^T (\bar{x} - x^\epsilon)$. Also, $\bar{x} \in \text{VI}(\Sigma, G)$ and $y^\epsilon \in \Sigma$ imply $0 \le G(\bar{x})_J^T (y^\epsilon - \bar{x})_J$ (since $\bar{J} = J$). Adding these two inequalities and using $\bar{x}_{J^c} = 0$ gives

$$0 \le G(\bar{x})_J^T (y^\epsilon - x^\epsilon)_J + (G(\bar{x}) - G(x^\epsilon))_J^T (x^\epsilon - \bar{x})_J - G(x^\epsilon)_{J^c}^T x_{J^c}^\epsilon.$$

Combining this with (18) and renaming $\tau \|G(\bar{x})_J\|$ as τ yields (19).

(c2). Assume $F(x) = Mx + q$ for some $M \in \Re^{n \times n}, q \in \Re^n$, and assume $\bar{\Sigma}$ is convex. Consider any $I \subset J$ such that $|\Upsilon_I| = \infty$ and $M_{\bar{J}\bar{J}^c} = M_{\bar{J}\bar{J}} N_{\bar{J}\bar{J}^c}$ and $M_{\bar{J}\bar{J}} = M_{\bar{J}\bar{J}}^T N_{\bar{J}\bar{J}}$ for some $N_{\bar{J}\bar{J}^c} \in \Re^{|\bar{J}| \times |\bar{J}^c|}$ and some $N_{\bar{J}\bar{J}} \in \Re^{|\bar{J}| \times |\bar{J}|}$. First, we have $M_{\bar{J}} y + q_{\bar{J}} = 0$ for all $y \in \bar{\Sigma}$. [If $M_i y + q_i > 0$ for some $i \in N$ and some $y \in \bar{\Sigma}$, then the convexity of $\bar{\Sigma}$ would imply $x_i = 0$ for all $x \in \bar{\Sigma}$, so $i \in \bar{J}^c$.] Fix any $y \in \bar{\Sigma}$. Since $\bar{x} \in \bar{\Sigma}$, then $d = y - \bar{x}$ satisfies $d_{\bar{J}^c} = 0$ and $M_{\bar{J}\bar{J}} d_{\bar{J}} = 0$. Moreover, for each $i \in \bar{J}$ with $\bar{x}_i = 0$ we have $d_i \ge 0$. Thus, there exists $\alpha > 0$ such that $z^\epsilon := x^\epsilon + \alpha d \ge 0$ for all $\epsilon \in \Upsilon$ sufficiently small. Then, $x^\epsilon \in \Sigma^\epsilon$ and $\bar{x} \in \bar{\Sigma}$ imply

$$
\begin{aligned}
0 \ &\le \ (Mx^\epsilon + q + \epsilon G(x^\epsilon))^T (z^\epsilon - x^\epsilon) \\
&= \ \alpha (M_{\bar{J}} x^\epsilon + q_{\bar{J}} + \epsilon G(x^\epsilon)_{\bar{J}})^T d_{\bar{J}} \\
&= \ \alpha (M_{\bar{J}}(x^\epsilon - \bar{x}) + \epsilon G(x^\epsilon)_{\bar{J}})^T d_{\bar{J}} \\
&= \ \alpha \left(M_{\bar{J}\bar{J}}((x^\epsilon - \bar{x})_{\bar{J}} + N_{\bar{J}\bar{J}^c}(x^\epsilon - \bar{x})_{\bar{J}^c}) + \epsilon G(x^\epsilon)_{\bar{J}} \right)^T d_{\bar{J}} \\
&= \ \epsilon \alpha G(x^\epsilon)_{\bar{J}}^T d_{\bar{J}},
\end{aligned}
$$

where the last equality uses $M_{\bar{J}\bar{J}}^T d_{\bar{J}} = N_{\bar{J}\bar{J}}^T M_{\bar{J}\bar{J}} d_{\bar{J}} = 0$. This shows that $G(x^\epsilon)_{\bar{J}}^T d_{\bar{J}} \ge 0$. Since $\bar{x} \in \bar{\Sigma}$ so that $\bar{J} \supset J$, if $\bar{J} \neq J$, then the convexity of $\bar{\Sigma}$ would imply the existence of $y \in \bar{\Sigma}$ with $y_{\bar{J} \setminus J} > 0$. Using this y in the above argument would yield $d_{\bar{J} \setminus J} > 0$ and hence $G(x^\epsilon)_{\bar{J}}^T d_{\bar{J}} \to -\infty$, a contradiction. Thus $\bar{J} = J$. Then, G_j is continuous at $\bar{x}_j$ for all $j \in \bar{J}$ and the above inequality yields in the limit as $\epsilon \to 0$ that

$$0 \le G(\bar{x})_{\bar{J}}^T d_{\bar{J}} = G(\bar{x})^T (y - \bar{x})$$

(here $\infty \cdot 0 = 0$), so $\bar{x} \in \text{VI}(\bar{\Sigma}, G)$.

Consider any $I \subset J$ with $|\Upsilon_I| = \infty$, M_{II} positive semidefinite and $M_{IJ^c} = M_{II} N_{IJ^c}$ for some $N_{IJ^c} \in \Re^{|I| \times |J^c|}$. Then $\bar{x}$ satisfies (26). For each $\epsilon \in \Upsilon_I$, Hoffman's lemma implies (26) has a solution y^ϵ satisfying (27), with $\tau_I > 0$ and $\gamma_I = 1$. Then $y^\epsilon \in \bar{\Sigma}$ (since the line segment joining $\bar{x}$ and y^ϵ lies in Σ) so the fact $\bar{x} \in \text{VI}(\bar{\Sigma}, G)$ implies

$$
\begin{aligned}
0 & \leq & G(\bar{x})^T (y^\epsilon - \bar{x}) \\
& = & G(\bar{x})_I^T (y^\epsilon - x^\epsilon)_I + G(\bar{x})_I^T (x^\epsilon - \bar{x})_I \\
& \leq & \tau_I \|G(\bar{x})_I\| (\epsilon \|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|) + G(\bar{x})_I^T (x^\epsilon - \bar{x})_I .
\end{aligned}
\tag{29}
$$

Also, we have from (25) and $\bar{x}$ satisfying (26) that

$$
0 = (M\bar{x} + q)_I, \qquad 0 = (Mx^\epsilon + q + \epsilon G(x^\epsilon))_I ,
$$

which when subtracted and using $x_{J \setminus I}^\epsilon = 0$ and $\bar{x}_{I^c} = 0$ yields $0 = M_{II}((x^\epsilon - \bar{x})_I + N_{IJ^c} x_{J^c}^\epsilon) + \epsilon G(x^\epsilon)_I$. This and the positive semidefinite property of M_{II} yield

$$
\begin{aligned}
0 & \leq & ((x^\epsilon - \bar{x})_I + N_{IJ^c} x_{J^c}^\epsilon)^T M_{II} ((x^\epsilon - \bar{x})_I + N_{IJ^c} x_{J^c}^\epsilon) \\
& = & -\epsilon ((x^\epsilon - \bar{x})_I + N_{IJ^c} x_{J^c}^\epsilon)^T G(x^\epsilon)_I .
\end{aligned}
$$

Dividing the above inequality by ϵ and adding it to the inequality (29), we obtain

$$
0 \leq \tau_I \|G(\bar{x})_I\| (\epsilon \|G(x^\epsilon)_J\| + \|x_{J^c}^\epsilon\|) + (G(\bar{x}) - G(x^\epsilon))_I^T (x^\epsilon - \bar{x})_I - G(x^\epsilon)_I^T N_{IJ^c} x_{J^c}^\epsilon .
$$

Using $(x^\epsilon - \bar{x})_{J \setminus I} = 0$ and renaming $\tau_I \|G(\bar{x})_I\|$ as τ_I yield (20).

(c3). Assume $F(x) = Mx + q$ for some $M \subset \Re^{n \times n}$, $q \in \Re^n$, and assume $\bar{\Sigma}$ is convex and each G_j is strictly increasing. Fix any $I \subset J$ such that $|\Upsilon_I| = \infty$. For each $\epsilon \in \Upsilon_I$, Hoffman's lemma implies (26) has a solution y^ϵ satisfying (27) with $\tau_I > 0$ and $\gamma_I = 1$. Let Ψ denote the set of $x \in \Re^n$ satisfying (26) and $\|x\| \leq \sup_{\epsilon \in \Upsilon_I} \|y^\epsilon\|$. We claim that there exists a scalar $\rho > 0$ such that, for every $y \in \Psi$ there exists a $K \subset J^c$ such that

$$
M_j y + q_j > 2\rho \; \forall j \in K \quad \text{and} \quad M_L \psi + q_L = 0 \text{ for some } \psi \in \Psi ,
\tag{30}
$$

where $L := J^c \setminus K$ (cf. proof of [52, Proposition 2]). If not, then for every sequence of scalars $\rho^k > 0$, $k = 1, 2, \ldots$, tending to zero, there would exist a $\nu^k \in \Psi$ such that, for every $K \subset J^c$ we have

$$
M_j \nu^k + q_j \leq 2\rho^k \text{ for some } j \in K \quad \text{or} \quad M_L \psi + q_L \neq 0 \; \forall \psi \in \Psi ,
$$

where $L := J^c \setminus K$. Since Ψ is bounded and closed, then ν^k, $k = 1, 2, \ldots$, has a cluster point $\nu \in \Psi$ such that, for every $K \subset J^c$ we have

$$
M_j \nu + q_j = 0 \text{ for some } j \in K \quad \text{or} \quad M_L \psi + q_L \neq 0 \; \forall \psi \in \Psi ,
$$

where $L := J^c \backslash K$. However, this cannot be true since the above relations fail to hold for $K = \{j \in J^c : M_j \nu + q_j > 0\}$ and $\psi = \nu$.

For each $\epsilon \in \Upsilon_I$, we have $y^\epsilon \in \Psi$ and hence there exists a $K \subset J^c$ such that (30) holds with $y = y^\epsilon$ and $L := J^c \backslash K$. Since the number of such subset K is finite, by passing into a subsequence if necessary, we can assume it is the same K for all $\epsilon \in \Upsilon_I$. Since (27) and $x^\epsilon \to \bar{x}$ imply $x^\epsilon - y^\epsilon \to 0$ as $\epsilon \in \Upsilon_I \to 0$, we have $M_j x^\epsilon + q_j \geq \rho$ for all $j \in K$ and sufficiently small $\epsilon \in \Upsilon_I$, in which case $M_j x^\epsilon + q_j + \epsilon G_j(x_j^\epsilon) = 0$ and the strictly increasing property of G_j would imply

$$x_j^\epsilon = G_j^{-1}\left(-(M_j x^\epsilon + q_j)/\epsilon\right) \leq G_j^{-1}\left(-\rho/\epsilon\right) \quad \forall j \in K. \tag{31}$$

For each $\epsilon \in \Upsilon_I$, let $L_1 := \{j \in L : M_j x^\epsilon + q_j \geq h_j(\epsilon)\}$ and let $L_2 := L \backslash L_1$. Since there is only a finite number of different L_1 and L_2, by passing to a subsequence if necessary, we can assume that L_1 and L_2 are the same for all $\epsilon \in \Upsilon_I$. Then, we have as argued above that

$$x_j^\epsilon \leq G_j^{-1}\left(-h_j(\epsilon)/\epsilon\right) = h_j(\epsilon) \quad \forall j \in L_1. \tag{32}$$

We claim there exists constant $\tau_1 > 0$ such that

$$\|x_{L_2 \backslash \bar{J}}^\epsilon\| \leq \tau_1 \left(\|x_{K \cup L_1}^\epsilon\| + \sum_{j \in L_2} h_j(\epsilon) + \epsilon\right) \tag{33}$$

for all $\epsilon \in \Upsilon_I$. If not, then there would exist a subsequence of $\epsilon \in \Upsilon_I$ along which $(\|x_{K \cup L_1}^\epsilon\| + \sum_{j \in L_2} h_j(\epsilon) + \epsilon)/\|x_{L_2 \backslash \bar{J}}^\epsilon\| \to 0$. By (30), there exists $\psi \in \Psi$ satisfying $M_L \psi + q_L = 0$. Then ψ would satisfy (26), which together with (25) implies

$$
\begin{array}{llll}
M_I(x^\epsilon - \psi) & = -\epsilon G(x^\epsilon)_I, & (x^\epsilon - \psi)_{I'} & \geq 0, \\
M_{J'}(x^\epsilon - \psi) & \geq -\epsilon G(x^\epsilon)_{J'}, & (x^\epsilon - \psi)_{J \backslash I} & = 0, \\
M_{K' \cup L_1}(x^\epsilon - \psi) & \geq 0, & (x^\epsilon - \psi)_{K \cup L_1} & = x_{K \cup L_1}^\epsilon, \\
M_{L_2}(x^\epsilon - \psi) & = M_{L_2} x^\epsilon + q_{L_2}, & (x^\epsilon - \psi)_{L_2} & \geq 0,
\end{array}
$$

where $I' := \{i \in I : \psi_i = 0\}$, $J' := \{i \in J \backslash I : M_i \psi + q_i = 0\}$, and $K' := \{i \in K : M_i \psi + q_i = 0\}$. Dividing both sides by $\|x_{L_2 \backslash \bar{J}}^\epsilon\|$ and using $0 \leq M_j x^\epsilon + q_j < h_j(\epsilon)$ for $j \in L_2$ would yield in the limit that

$$
\begin{array}{llll}
M_I u & = 0, & u_{I'} & \geq 0, \\
M_{J'} u & \geq 0, & u_{J \backslash I} & = 0, \\
M_{K' \cup L_1} u & \geq 0, & u_{K \cup L_1} & = 0, \\
M_{L_2} u & = 0, & u_{L_2} & \geq 0,
\end{array}
$$

for some $u \in \Re^n$ with $u_{L_2 \backslash \bar{J}} \neq 0$. Then, since $\psi \in \Psi$ and $M_L \psi + q_L = 0$, the vector $\psi + \alpha u$ would be in Σ for all $\alpha > 0$ sufficiently small. Since ψ and $\bar{x}$ satisfy (26) so that $\psi \in \bar{\Sigma}$, this vector is also in $\bar{\Sigma}$. Since $u_{L_2 \backslash \bar{J}} \neq 0$, this would contradict the fact that $x_{\bar{J}^c} = 0$ for all $x \in \bar{\Sigma}$.

Assume $M = -M^T$. We claim that there exists constant $\tau_1 > 0$ such that

$$\|x^\epsilon_{L\setminus\bar{J}}\| \leq \tau_1(\|x^\epsilon_K\| + \epsilon) \tag{34}$$

for all $\epsilon \in \Upsilon_I$. If not, then there would exist a subsequence of $\epsilon \in \Upsilon_I$ along which $(\|x^\epsilon_K\| + \epsilon)/\|x^\epsilon_{L\setminus\bar{J}}\| \to 0$. By (30), there exists $\psi \in \Psi$ satisfying $M_L\psi + q_L = 0$. Then ψ would satisfy (26), which together with (25) implies

$$
\begin{array}{llll}
M_I(x^\epsilon - \psi) & = -\epsilon G(x^\epsilon)_I, & (x^\epsilon - \psi)_{I'} & \geq 0, \\
M_{J'}(x^\epsilon - \psi) & \geq -\epsilon G(x^\epsilon)_{J'}, & (x^\epsilon - \psi)_{J\setminus I} & = 0, \\
M_{K'}(x^\epsilon - \psi) & \geq 0, & (x^\epsilon - \psi)_K & = x^\epsilon_K, \\
M_L(x^\epsilon - \psi) & \geq 0 & (x^\epsilon - \psi)_L & \geq 0.
\end{array}
$$

where $I' := \{i \in I : \psi_i = 0\}$, $J' := \{i \in J\setminus I : M_i\psi + q_i = 0\}$, and $K' = \{i \in K : M_i\psi + q_i = 0\}$. Dividing both sides by $\|x^\epsilon_{L\setminus\bar{J}}\|$ would yield in the limit that

$$
\begin{array}{llll}
M_I u & = 0, & u_{I'} & \geq 0, \\
M_{J'} u & \geq 0, & u_{J\setminus I} & = 0, \\
M_{K'} u & \geq 0, & u_K & = 0, \\
M_L u & \geq 0, & u_L & \geq 0,
\end{array}
$$

for some $u \in \Re^n$ with $u_{L\setminus\bar{J}} \neq 0$. Since $M = -M^T$ so that $u^T M u = 0$, the above implies $u_j(Mu)_j = 0$ for all $j \in L$. Then, since $\psi \in \Psi$ and $M_L\psi + q_L = 0$, the vector $\psi + \alpha u$ would be in Σ for all $\alpha > 0$ sufficiently small and hence, as argued earlier, would be in $\bar{\Sigma}$. Since $u_{L\setminus\bar{J}} \neq 0$, this would contradict the fact that $x_{\bar{J}^c} = 0$ for all $x \in \bar{\Sigma}$.

(c4). Assume $F(x) = Mx + q$ for some $M \subset \Re^{n\times n}$, $q \in \Re^n$, with $M_{\bar{J}\bar{J}}$ positive semidefinite. Also assume G_j is strictly increasing with $\limsup_{t\to 0} tG_j(t) < 0$ for all $j \in J^c$. Fix any $I \subset J$ such that $|\Upsilon_I| = \infty$. Let $H_1 := \bar{J}$. Initialize $\tilde{\Upsilon}$ to comprise all sufficiently small $\epsilon \in \Upsilon_I$ so that $G_j(x^\epsilon_j) < 0$ for all $j \in J^c$.

Given H_l for some $l \geq 1$, we construct below (by passing to a subsequence of $\tilde{\Upsilon}$ if necessary) a proper subset H_{l+1} of H_l having the desired properties (22), until $H_l = J$. For notational simplicity, we will write H_l and H_{l+1} as H and H^{new} respectively, dropping the subscript l. First, by passing to a subsequence if necessary, we assume there exist $q_h \in \Re^n$ and $\alpha^\epsilon_h > 0$ ($h = 1, ..., n$, $\epsilon \in \tilde{\Upsilon}$) satisfying

$$M_{NH^c} x^\epsilon_{H^c} = \sum_{h=1}^n \alpha^\epsilon_h q_h \ \forall \epsilon \in \tilde{\Upsilon}, \quad \alpha^\epsilon_1 \to 0, \quad \left(\frac{\alpha^\epsilon_{h+1}}{\alpha^\epsilon_h}\right)_{h=1,\ldots,n-1} \to 0 \text{ as } \epsilon \in \tilde{\Upsilon} \to 0. \tag{35}$$

[To see that such a decomposition exists, let $q^\epsilon := M_{NH^c} x^\epsilon_{H^c}$. If $q^\epsilon = 0$ for all $\epsilon \in \tilde{\Upsilon}$ small, then choose $q_h = 0$ and $\alpha^\epsilon_h = (\epsilon)^h$ for $h = 1, ..., n$. Otherwise take any subsequence of $\tilde{\Upsilon}$ along which $q^\epsilon \neq 0$ and $q^\epsilon/\|q^\epsilon\|$ converges. Let q_1 be its limit and let $\tilde{q}^\epsilon$ be the orthogonal projection of q^ϵ onto the subspace

orthogonal to q_1. Then $\tilde{q}^\epsilon = q^\epsilon - \alpha_1^\epsilon q_1$ for some α_1^ϵ satisfying $\|q^\epsilon\|/\alpha_1^\epsilon \to 1$ along the subsequence. Apply the above construction inductively to $\tilde{q}^\epsilon$ (restricted to the above subspace) yields (35).] For each $\epsilon \in \tilde{\Upsilon}$, we have from $\epsilon \in \Upsilon_I$ and $G_j(x_j^\epsilon) < 0$ for all $j \in J^c$ and $J \subset H$ that x^ϵ satisfies (cf. (25))

$$
\begin{aligned}
M_{LH} x_H^\epsilon + M_{LH^c} x_{H^c}^\epsilon + q_L &= -\epsilon G(x^\epsilon)_L, & x_I^\epsilon &\geq 0, \\
M_{J\setminus I H} x_H^\epsilon + M_{J\setminus I H^c} x_{H^c}^\epsilon + q_{J\setminus I} &\geq -\epsilon G(x^\epsilon)_{J\setminus I}, & x_{J\setminus I}^\epsilon &= 0, \\
M_{H^c H} x_H^\epsilon + M_{H^c H^c} x_{H^c}^\epsilon + q_{H^c} &\geq 0, & x_{H\setminus J}^\epsilon &> 0,
\end{aligned}
$$

where for convenience we let $L := I \cup (H\setminus J)$. Letting $\alpha_0^\epsilon := 1$, $q_0 := q$, $I_0 := J\setminus I$, $K_0 := I$ and $L_0 := H^c$ and $y_0^\epsilon := x^\epsilon$, we see from the above relations and (35) that the following holds with $k = 0$:

$$
\begin{aligned}
M_{LH}(y_k^\epsilon)_H + \sum_{h=k}^n \alpha_h^\epsilon (q_h)_L = -\epsilon G(x^\epsilon)_L, & \quad (y_k^\epsilon)_{K_k} \geq 0, \\
M_{I_k H}(y_k^\epsilon)_H + \sum_{h=k}^n \alpha_h^\epsilon (q_h)_{I_k} \geq -\epsilon G(x^\epsilon)_{I_k}, & \quad (y_k^\epsilon)_{J\setminus I} = 0, \\
M_{L_k H}(y_k^\epsilon)_H + \sum_{h=k}^n \alpha_h^\epsilon (q_h)_{L_k} \geq 0, & \quad (y_k^\epsilon)_{H\setminus J} > 0,
\end{aligned}
\tag{36}
$$

$$
y_k^\epsilon = x^\epsilon - \sum_{h=0}^{k-1} \alpha_h^\epsilon u_h, \qquad I_k \subset J,
\tag{37}
$$

for all $\epsilon \in \tilde{\Upsilon}$. Now, suppose that (36)–(37) hold for some $k \geq 0$ for all $\epsilon \in \tilde{\Upsilon}$. By further passing to a subsequence if necessary, we can assume one of the following two cases occurs.

<u>Case 1.</u> There exist $j \in H\setminus J$ and $\rho > 0$ such that $\epsilon G_j(x_j^\epsilon)/\alpha_k^\epsilon \leq -\rho$ for all $\epsilon \in \tilde{\Upsilon}$.

In this case, let $H^{new} := H\setminus\{j\}$ and we have that H^{new} is a proper subset of H and contains J. Moreover, the strictly increasing property of G_j implies

$$
x_j^\epsilon \leq G_j^{-1}(-\rho \alpha_k^\epsilon/\epsilon).
\tag{38}
$$

<u>Case 2.</u> For all $j \in H\setminus J$, $\epsilon G_j(x_j^\epsilon)/\alpha_k^\epsilon \to 0$ as $\epsilon \in \tilde{\Upsilon} \to 0$.

In this case, by further passing to a subsequence, we can assume either $\alpha_k^\epsilon/\epsilon \to \infty$ or $\alpha_k^\epsilon/\epsilon$ converges, as $\epsilon \in \tilde{\Upsilon} \to 0$.

Suppose $\alpha_k^\epsilon/\epsilon \to \infty$ as $\epsilon \in \tilde{\Upsilon} \to 0$. Since (36) holds for all $\epsilon \in \tilde{\Upsilon}$, dividing all sides by α_k^ϵ and using $L = I \cup (H\setminus J)$ and the fact we are in Case 2 yield in the limit

$$
\begin{aligned}
M_{LH}(u_k)_H + (q_k)_L &= 0, & (u_k)_{K_k} &\geq 0, \\
M_{I_k H}(u_k)_H + (q_k)_{I_k} &\geq 0, & (u_k)_{J\setminus I} &= 0, \\
M_{L_k H}(u_k)_H + (q_k)_{L_k} &\geq 0, & (u_k)_{H\setminus J} &\geq 0,
\end{aligned}
\tag{39}
$$

for some $u_k \in \Re^n$. [Notice that $u_0 = \bar{x}$, so $(u_0)_{J^c} = 0$.] By further passing to a subsequence if necessary, we can assume one of the following two subcases occurs.

<u>Subcase 2a.</u> There exist $j \in H \backslash J$ and $\tau > 0$ such that $(y_k^\epsilon - \alpha_k^\epsilon u_k)_j / \alpha_1^\epsilon \leq \tau$ for all $\epsilon \in \tilde{\Upsilon}$.

In this subcase, let $H^{new} := H \backslash \{j\}$ and we have that H^{new} is a proper subset of H and contains J. Moreover, $(u_0)_j = \bar{x}_j = 0$, so (37) yields

$$\tau \alpha_1^\epsilon \geq (y_k^\epsilon - \alpha_k^\epsilon u_k)_j = (x^\epsilon - \sum_{h=1}^{k} \alpha_h^\epsilon u_h)_j. \tag{40}$$

<u>Subcase 2b.</u> For all $j \in H \backslash J$, $(y_k^\epsilon - \alpha_k^\epsilon u_k)_j / \alpha_1^\epsilon \to \infty$ as $\epsilon \in \tilde{\Upsilon} \to 0$.

In this subcase, let $I_{k+1} := \{i \in I_k : M_{iH}(u_k)_H + (q_k)_i = 0\}$ and $K_{k+1} := \{i \in K_k : (u_k)_i = 0\}$ and $L_{k+1} := \{i \in L_k : M_{iH}(u_k)_H + (q_k)_i = 0\}$. Then (36) and (39) yield

$$\begin{aligned}
M_{LH}(y_k^\epsilon - \alpha_k^\epsilon u_k)_H + \sum_{h=k+1}^{n} \alpha_h^\epsilon (q_h)_L &= -\epsilon G(x^\epsilon)_L, \quad (y_k^\epsilon - \alpha_k^\epsilon u_k)_{K_{k+1}} \geq 0, \\
M_{I_{k+1}H}(y_k^\epsilon - \alpha_k^\epsilon u_k)_H + \sum_{h=k+1}^{n} \alpha_h^\epsilon (q_h)_{I_{k+1}} &\geq -\epsilon G(x^\epsilon)_{I_{k+1}}, \quad (y_k^\epsilon - \alpha_k^\epsilon u_k)_{J \backslash I} = 0, \\
M_{L_{k+1}H}(y_k^\epsilon - \alpha_k^\epsilon u_k)_H + \sum_{h=k+1}^{n} \alpha_h^\epsilon (q_h)_{L_{k+1}} &\geq 0, \quad (y_k^\epsilon - \alpha_k^\epsilon u_k)_{H \backslash J} > 0,
\end{aligned} \tag{41}$$

for all $\epsilon \in \tilde{\Upsilon}$ sufficiently small. Letting $y_{k+1}^\epsilon := y_k^\epsilon - \alpha_k^\epsilon u_k$ and we see that (36)–(37) hold with k replaced by $k+1$. Below we show that $k < n$ so that we can repeat the above construction with k replaced by $k+1$. Suppose not, so that $k = n$. Then, dividing all sides of (41) by $\min_{j \in H \backslash J}(y_{n+1}^\epsilon)_j$ and using the fact that we are in Subcase 2b (so that $\epsilon G_j(x_j^\epsilon)/\alpha_1^\epsilon \to 0$ for all $j \in L$ and $\alpha_1^\epsilon/(y_{n+1}^\epsilon)_j \to 0$ for all $j \in H \backslash J$ as $\epsilon \in \tilde{\Upsilon} \to 0$) yield in the limit that

$$M_{LH}(u_{n+1})_H = 0, \quad (u_{n+1})_{J \backslash I} = 0, \quad (u_{n+1})_j \geq 1 \ \forall j \in H \backslash J,$$

for some $u_{n+1} \in \Re^n$. Then, using this and (41) and $H = L \cup (J \backslash I)$, we see that $y^\epsilon := y_{n+1}^\epsilon - 2\|(y_{n+1}^\epsilon)_{H \backslash J}\|_\infty u_{n+1}$ satisfies

$$M_{LL}y_L^\epsilon = -\epsilon G(x^\epsilon)_L.$$

Also, $y_{n+1}^\epsilon = x^\epsilon - \bar{x} - \sum_{h=1}^{n} \alpha_h^\epsilon u_h \to 0$ so that $y^\epsilon \to 0$. Multiplying the above equation on the left by $(y_L^\epsilon)^T$ and using the positive semidefinite property of M_{LL} (since $M_{\bar{J}\bar{J}}$ is positive semidefinite and $L \subset \bar{J}$) yields $0 \leq -\epsilon (y_L^\epsilon)^T G(x^\epsilon)_L$. Dividing both sides by ϵ and using $L = I \cup (H \backslash J)$ gives

$$\sum_{j \in H \backslash J} y_j^\epsilon G_j(x_j^\epsilon) \leq -\sum_{j \in I} y_j^\epsilon G_j(x_j^\epsilon). \tag{42}$$

For each $j \in H \backslash J$, we have $y_j^\epsilon = (y_{n+1}^\epsilon)_j - 2\|(y_{n+1}^\epsilon)_{H \backslash J}\|_\infty (u_{n+1})_j \leq -(y_{n+1}^\epsilon)_j = -x_j^\epsilon - \sum_{h=1}^{n} \alpha_h^\epsilon (u_h)_j$ so that (also using $G_j(x_j^\epsilon) < 0$)

$$y_j^\epsilon G_j(x_j^\epsilon) \geq -\left(1 + \sum_{h=1}^{n} (\alpha_h^\epsilon/x_j^\epsilon)(u_h)_j\right) x_j^\epsilon G_j(x_j^\epsilon).$$

Since we are in Subcase 2b with $k = n$, then for each $j \in H \backslash J$ we have $(y_{n+1}^\epsilon)_j / \alpha_1^\epsilon \to \infty$ as $\epsilon \in \tilde{\Upsilon} \to 0$, so that $(y_{n+1}^\epsilon)_j = x_j^\epsilon + \sum_{h=1}^n \alpha_h^\epsilon (u_h)_j$ yields $x_j^\epsilon / \alpha_1^\epsilon \to \infty$. This together with the above inequality and $\limsup_{t \to 0} t G_j(t) < 0$ implies the left-hand side of (42) is positive and bounded away from zero. On the other hand, we have $G(x^\epsilon)_I \to G(\bar{x})_I$ and $y^\epsilon \to 0$ as $\epsilon \in \tilde{\Upsilon} \to 0$, so the right-hand side of (42) tends to zero, a contradiction.

Suppose instead $\alpha_k^\epsilon / \epsilon$ converges to some $c \in \Re_+$ as $\epsilon \in \tilde{\Upsilon} \to 0$. Then, $k \geq 1$ and, since we are in Case 2 (and $G_j(x_j^\epsilon) \to -\infty$ for $j \in H \backslash J$), it must be that $H \backslash J = \emptyset$, i.e., $H = J$. The first equation in (36) can then be written using $(y_k^\epsilon)_{J \backslash I} = 0$ as

$$M_{II}(y_k^\epsilon)_I + \sum_{h=k}^n \alpha_h^\epsilon (q_h)_I = -\epsilon G(x^\epsilon)_I.$$

Dividing this by ϵ yields in the limit

$$M_{II}(u_k)_I + c(q_k)_I = -G(\bar{x})_I,$$

for some $u_k \in \Re^n$. Combining the above two equations yields

$$-M_{II}\Delta_I + (\alpha_k^\epsilon - c\epsilon)(q_k)_I + \sum_{h=k+1}^n \alpha_h^\epsilon (q_h)_I = \epsilon(G(\bar{x})_I - G(x^\epsilon)_I),$$

where $\Delta := \epsilon u_k - y_k^\epsilon = \bar{x} - x^\epsilon + \sum_{h=1}^{k-1} \alpha_h^\epsilon u_h + \epsilon u_k$. Multiplying the left-hand side by Δ_I^T and using the positive semidefinite property of M_{II} (since M_{JJ} is positive semidefinite and $I \subset \bar{J}$) yields

$$\Delta_I^T \left((\alpha_k^\epsilon - c\epsilon)(q_k)_I + \sum_{h=k+1}^n \alpha_h^\epsilon (q_h)_I \right) \geq \epsilon \Delta_I^T (G(\bar{x})_I - G(x^\epsilon)_I).$$

Thus, dividing both sides by ϵ and expanding Δ_I yields

$$
\begin{aligned}
&(\bar{x} - x^\epsilon)_I^T (G(\bar{x})_I - G(x^\epsilon)_I) \\
\leq\quad &-\left(\sum_{h=1}^{k-1} \alpha_h^\epsilon (u_h)_I + \epsilon(u_k)_I \right)(G(\bar{x})_I - G(x^\epsilon)_I) \\
+\quad &\left((\bar{x} - x^\epsilon)_I + \sum_{h=1}^{k-1} \alpha_h^\epsilon (u_h)_I + \epsilon(u_k)_I \right)^T \left((\frac{\alpha_k^\epsilon}{\epsilon} - c)(q_k)_I + \sum_{h=k+1}^n \frac{\alpha_h^\epsilon}{\epsilon}(q_h)_I \right) \\
\leq\quad &\left\| \sum_{h=1}^{k-1} \alpha_h^\epsilon (u_h)_I + \epsilon(u_k)_I \right\| \|G(\bar{x})_I - G(x^\epsilon)_I\| \\
+\quad &\left(\|(\bar{x} - x^\epsilon)_I\| + \left\| \sum_{h=1}^{k-1} \alpha_h^\epsilon (u_h)_I + \epsilon(u_k)_I \right\| \right) \left\| (\frac{\alpha_k^\epsilon}{\epsilon} - c)(q_k)_I + \sum_{h=k+1}^n \frac{\alpha_h^\epsilon}{\epsilon}(q_h)_I \right\|.
\end{aligned}
$$

Suppose in addition G_J is locally Lipschitzian at $\bar{x}_J$ with constant $\kappa > 0$ and satisfies (17) with $\sigma > 0, \delta > 0, \varrho = 1$. Then for all $\epsilon \in \tilde{\Upsilon}$ sufficiently small so that $\|(\bar{x} - x^\epsilon)_I\| \leq \delta$ and $\|G(\bar{x})_I - G(x^\epsilon)_I\| \leq \kappa \|(\bar{x} - x^\epsilon)_I\|$, the preceding inequality yields

$$\sigma \xi^2 \leq (r\kappa + s)\xi + rs,$$

where $r := \|\sum_{h=1}^{k-1} \alpha_h^\epsilon (u_h)_I + \epsilon(u_k)_I\|$, $s := \left\|(\frac{\alpha_k^\epsilon}{\epsilon} - c)(q_k)_I + \sum_{h=k+1}^n \frac{\alpha_h^\epsilon}{\epsilon}(q_h)_I\right\|$, and $\xi := \|(\bar{x} - x^\epsilon)_I\|$. Solving this using the quadratic formula yields

$$\xi \leq \frac{(r\kappa + s) + \sqrt{(r\kappa + s)^2 + 4\sigma rs}}{2\sigma} \leq \frac{r\kappa + s}{\sigma} + \sqrt{r}\sqrt{\frac{s}{\sigma}} \leq \frac{r\kappa + s}{\sigma} + \frac{r + s/\sigma}{2},$$

where the last two inequalities use the identities $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ and $ab \leq (a^2 + b^2)/2$. This and $x^\epsilon_{J\setminus I} = 0$ and, by taking $\tau' > 0$ sufficiently large so that $r \leq \tau'(\alpha_1^\epsilon + \epsilon)$ for all $\epsilon \in \tilde{\Upsilon}$, yields

$$\|(x^\epsilon - \bar{x})_J\| \leq \tau'\left(\frac{\kappa}{\sigma} + \frac{1}{2}\right)(\alpha_1^\epsilon + \epsilon) + \frac{3}{2\sigma}\left|\frac{\alpha_k^\epsilon}{\epsilon} - c\right|\|(q_k)_J\| + \sum_{h=k+1}^n \frac{\alpha_h^\epsilon}{\epsilon}\|(q_h)_J\|. \quad (43)$$

Thus, letting $\lambda^\epsilon := \epsilon/\alpha_k^\epsilon$ in case (38) and letting $\lambda^\epsilon := \alpha_1^\epsilon + \sum_{h=1}^k \alpha_h^\epsilon(u_h)_j$ in case (40) yield (22). Similarly, (43) yields (23) for suitable choice of λ^ϵ.

By repeating the above argument with $\Upsilon_I \setminus \tilde{\Upsilon}$ in place of Υ_I, we can extract another subsequence of Υ_I having the same properties as $\tilde{\Upsilon}$ and so on. We do this for all $I \subset J$ with $|\Upsilon_I| = \infty$, thus yielding a countable collection of subsequences whose union is Υ. ∎

Note 5. A few words about the assumptions in Proposition 3 are in order. First, the assumptions on $F(x_H, 0)$ and Σ_H in part (a) are satisfied by $H = \bar{J}$ if F is pseudo-monotone on $\Re^n_+$ (since $\Sigma = \bar{\Sigma}$ is convex in this case). Second, the assumption of $\lim_{t\to 0} tG_j(t) = 0$ in part (c1) is equivalent to $\lim_{\epsilon\to 0} G_j^{-1}(-1/\epsilon)/\epsilon = 0$ if G_j is strictly increasing. This is because $G_j^{-1}(-1/\epsilon)/\epsilon \geq c > 0$ implies, by G_j being strictly increasing, that $-c \geq c\epsilon G_j(c\epsilon)$. Conversely, $tG_j(t) \leq -c$ for some $c > 0$ implies, by G_j^{-1} being strictly increasing, that $c \leq G_j^{-1}(-c/t)/(t/c)$. Third, the assumptions on M in part (c2) are satisfied by any $I \subset \bar{J} \subset N$ if M is symmetric positive semidefinite (see, e.g., [35, Lemma 5]) or if M is symmetric nondegenerate (i.e., M_{II} is nonsingular for all $I \subset N$). It is also satisfied by any $I \subset \bar{J} \subset N$ if $M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, which is neither symmetric nor nondegenerate. Fourth, for the h_j defined in part (c3), direct calculation finds that, for G given by (5), $h_j(\epsilon) = \sqrt{\epsilon}$ and, for G given by (6), $h_j(\epsilon)$ is the uniqe η satisfying $-\eta/\ln(\eta) = \epsilon$,

so that $h_j(\epsilon) = o(\epsilon^\beta)$ for any fixed $\beta \in (0,1)$. To see that the bound (21) is reasonable, notice that for $n = 1$ and $F(x) = 2$ and $G(x) = -1/x$, we have $2 + \epsilon G(x^\epsilon) = 0$ so that $x^\epsilon = G^{-1}(-2/\epsilon) = \epsilon/2$. Similarly, for $n = 1$ and $F(x) = x$ and $G(x) = -1/x$, we have $x^\epsilon + \epsilon G(x^\epsilon) = 0$ so that $x^\epsilon = \sqrt{\epsilon}$. Notice that the skew symmetry assumption $M = -M^T$ is satisfied when an LP is formulated as a CP. The dependence of K on ϵ cannot be removed, as is shown by an example in [52] in the context of LP. Fifth, the nesting of index sets in part (c4) reflects a nested dependence of the convergence rate of some components of x^ϵ (indexed by H_l) on the remaining components. Intuitively, if x_i^ϵ converges more slowly than x_j^ϵ, then the term $M_{ij}x_j^\epsilon$ can influence what the limit $\bar{x}_i$ will be and the rate at which x_i^ϵ converges to this limit.

Note 6. If $\bar{x}$ in Proposition 3 satisfies strict complementarity, i.e., $\bar{x} + F(\bar{x}) > 0$, then parts (c3) and (c4) of this proposition simplify considerably. In particular, we have $F(\bar{x})_j > 0$ as well as $F(x^\epsilon)_j + \epsilon G_j(x_j^\epsilon) = 0$ for all $j \in J^c$, so that $F(x^\epsilon)_j \to F(\bar{x})_j$ and the strictly increasing property of G_j yield $x_j^\epsilon \leq G_j^{-1}(-F(x^\epsilon)_j/\epsilon) \leq G_j^{-1}(-\rho/\epsilon)$ for all $\epsilon \in \Upsilon$ sufficiently small, where $\rho := \min_{j \in J} F(\bar{x})_j/2$.

Note 7. If F is affine and pseudo-monotone on $\Re_+^n$ and G_j is strictly increasing with $\lim_{t \to 0} tG_j(t) = 0$ for all $j \in J^c$, then $\Sigma = \bar{\Sigma}$ and Proposition 3(c1),(c3) yield the error bound (19) and (21) for all $\epsilon \in \Upsilon$ sufficiently small, with $\bar{J} = J$, $\gamma = 1$ and $\tau > 0, \rho > 0$ some constants, and with $K \subset J^c$ depending on ϵ and $L := J^c \backslash K$. Similarly, if F is affine and monotone on $\Re_+^n$ and G_j is strictly increasing with $\limsup_{t \to 0} tG_j(t) < 0$ for all $j \in J^c$ and G_j is Lipschitz continuous and strongly monotone near $\bar{x}_j$ for all $j \in J$, then $\Sigma = \bar{\Sigma}$ and Proposition 3(c1),(c4) yield the error bound (21), (22), (23) for all sufficiently small ϵ along some subsequence $\tilde{\Upsilon}$, etc. Moreover, there exists a $c > 0$ such that $tG_j(t) \leq -c$ for all $j \in J^c$ and all $t > 0$ sufficiently small, implying $t \leq G_j(-c/t)$. Thus, the second case in (22) implies the first case.

Note 8. Proposition 3 does not say anything about existence or uniqueness or boundedness of $x^\epsilon \in \mathrm{VI}(\Re_+^n, F + \epsilon G)$. In the case where G is given by (5), it was shown by Kojima et al. [29, Theorem 4.4] that F being a continuous P_0-function and satisfying strict feasibility (i.e., $x > 0, F(x) > 0$ has a solution) and a boundedness condition implies the existence and uniqueness of x^ϵ for all ϵ (also see [30, Theorem 4.4] for the case of affine F and see [28] for extensions to other types of F). Analogous results were shown earlier by McLinden in the context of convex programs [41] and, more generally, when F is a maximal monotone operator [42]. These results were further improved and extended by Kojima et al. [31] and Güler [23]. Recently, Chen et al. [9, Corollary 3.14] showed that F being a continuously differentiable P_0-function and Σ being nonempty and bounded is sufficient for the existence and uniqueness of x^ϵ for all ϵ sufficiently small. Subsequently, Gowda and Tawhid [22, Theorems 8 and 9] weakened the differentiability assumption on F to continuity and considered more general regularizations on F.

We illustrate Proposition 3 with the following example with $n = 3$ variables.

Example 1. Consider

$$F(x) = Mx + q, \; M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \; q = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \; G(x) = \begin{bmatrix} G_1(x_1) \\ G_2(x_2) \\ G_3(x_3) \end{bmatrix} = \begin{bmatrix} x_1 - .5 \\ x_2 - .5 \\ -1/x_3^\beta \end{bmatrix},$$

with $\beta > 0$. Notice that M is positive semidefinite and it can be checked that $\Sigma = \{(t, 1 - t, 0) : 0 \le t \le 1\}$.

(i). Suppose $\beta < 1$ so that $\lim_{t \to 0} t G_3(t) = 0$. Then either by direct calculation or by using Proposition 3(c1), we find that $x^\epsilon \to \bar{x} = (.5, .5, 0) \in \text{VI}(\Sigma, G)$ as $\epsilon \to 0$. Thus $J = \bar{J} = \{1, 2\}$ and, for ϵ sufficiently small, we have $x^\epsilon > 0$ and hence $F(x^\epsilon) + \epsilon G(x^\epsilon) = 0$. Then, direct calculation yields

$$x_3^\epsilon = G_3^{-1}(-(x_2^\epsilon + x_3^\epsilon)/\epsilon) \approx G_3^{-1}(-.5/\epsilon) = (2\epsilon)^{1/\beta}$$

and $(x_1^\epsilon - .5, x_2^\epsilon - .5) = -x_3^\epsilon(1 + 2\epsilon, \epsilon - 1)/(2\epsilon + \epsilon^2) = O(\epsilon^{1/\beta - 1})$. This illustrates parts (c1) and (c3) of Proposition 3.

(ii). Suppose $\beta > 1$ so that $\lim_{t \to 0} t G_3(t) = -\infty$. Then direct calculation finds that, for all ϵ sufficiently small, we have $x_1^\epsilon = 0$ and $x_2^\epsilon = (1 + .5\epsilon - x_3^\epsilon)/(1 + \epsilon)$ with x_3^ϵ satisfying $x_3^\epsilon + (1 + \epsilon)G_3(x_3^\epsilon) = -(1 + .5\epsilon)/\epsilon$. Thus $x^\epsilon \to \bar{x} = (0, 1, 0)$ and $J = \bar{J} = \{1, 2\}$. Moreover, $\bar{x}_J \in \text{VI}(\Sigma_J, p_J)$ with $\Sigma_J = \{(t, 1 - t) : 0 \le t \le 1\}$ and $p_J = \begin{bmatrix} 2 \\ 1 \end{bmatrix}/\sqrt{5}$. Lastly, we have

$$x_3^\epsilon = G_3^{-1}\left(-(1 + .5\epsilon + \epsilon x_3^\epsilon)/\epsilon(1 + \epsilon)\right) = \left(\epsilon(1 + \epsilon)/(1 + .5\epsilon + \epsilon x_3^\epsilon)\right)^{1/\beta} \approx \epsilon^{1/\beta}$$

and hence $x_2^\epsilon - 1 \approx -x_3^\epsilon \approx -\epsilon^{1/\beta}$. This illustrates parts (a) and (c3) of Proposition 3.

(iii). Suppose $\beta > 1$ and G_1, G_2 are changed to $G_1(t) = G_2(t) = -1/t$. First, we claim that, for each $\epsilon > 0$, x^ϵ exists and is unique. To see this, let $I := \{1, 2\}$ and note that $M_{II} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is positive semidefinite and

$$M_{I3} x_3 + q_I = \begin{bmatrix} 2x_3 - 1 \\ x_3 - 1 \end{bmatrix} \in \{b_I \in \Re^2 : b_I = y_I - M_{II} x_I \text{ for some } x_I, y_I \in \Re^2_{++}\}$$

for all $x_3 \ge 0$, so a result of Kojima et al. [31, Corollary 1.2, Theorem 3.3] implies that, for each $x_3 \ge 0$, the equation $M_{II} x_I + M_{I3} x_3 + q_I + \epsilon G(x)_I = 0$ has a unique solution $x_I(x_3) > 0$ which is continuous in x_3 and is bounded as $x_3 \to 0$. Then the equation

$$M_{3I} x_I(x_3) + M_{33} x_3 + q_3 + \epsilon G_3(x_3) = x_2(x_3) + x_3 - \epsilon/(x_3)^\beta = 0$$

has a solution $x_3^\epsilon > 0$ since the left-hand side is continuous in $x_3 > 0$ and tends to $-\infty$ as $x_3 \to 0$ and tends to ∞ as $x_3 \to \infty$. Then $x^\epsilon := [x_I(x_3^\epsilon) \; x_3^\epsilon]^T > 0$

satisfies

$$F(x^\epsilon) + \epsilon G(x^\epsilon) = \begin{bmatrix} x_1^\epsilon + x_2^\epsilon + 2x_3^\epsilon - 1 + \epsilon G_1(x_1^\epsilon) \\ x_1^\epsilon + x_2^\epsilon + x_3^\epsilon - 1 + \epsilon G_2(x_2^\epsilon) \\ x_2^\epsilon + x_3^\epsilon + \epsilon G_3(x_3^\epsilon) \end{bmatrix} = 0. \qquad (44)$$

Uniqueness of x^ϵ follows from $F + \epsilon G$ being strictly monotone on $\Re_{++}^n$. Now, (44) and $x^\epsilon > 0$ imply $x_1^\epsilon < 1 + \epsilon/x_1^\epsilon$, $x_2^\epsilon < 1 + \epsilon/x_2^\epsilon$, $x_3^\epsilon < \epsilon/(x_3^\epsilon)^\beta$ for all $\epsilon > 0$, so x^ϵ is bounded as $\epsilon \to 0$. Then, x^ϵ has a cluster point $\bar{x}$ which, by Proposition 3 is in Σ. Since M is positive semidefinite so that $\Sigma = \bar{\Sigma}$ is convex and $\bar{J} = \{1, 2\}$, Proposition 3(a) with $H = \bar{J}$ implies either $\bar{x}_{\bar{J}} \in \mathrm{VI}(\Sigma_{\bar{J}}, p_{\bar{J}})$ or $\bar{x}_{\bar{J}} \in \mathrm{VI}(\Sigma_{\bar{J}}, \zeta p_{\bar{J}} + G_{\bar{J}})$ for some $\zeta \in \Re_+$, where $\Sigma_{\bar{J}} = \{(t, 1-t) : 0 \le t \le 1\}$ and $p_{\bar{J}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} / \sqrt{5}$. In either case, we have $\bar{x}_2 > 0$, so that the third equation in (44) yields

$$x_3^\epsilon = G_3^{-1}(-(x_2^\epsilon + x_3^\epsilon)/\epsilon) = (\epsilon/(x_2^\epsilon + x_3^\epsilon))^{1/\beta} \approx (\epsilon/\bar{x}_2)^{1/\beta}.$$

Since $\beta > 1$, this shows $x_3^\epsilon/\epsilon \to \infty$ so we are in the case of $\bar{x}_{\bar{J}} \in \mathrm{VI}(\Sigma_{\bar{J}}, p_{\bar{J}})$, yielding $\bar{x} = (0, 1, 0)$ and $J = \{2\}$. Thus $\bar{J} \backslash J = \{1\}$. Now, subtracting the second equation in (44) from the first equation and using $G_2(t) = -1/t$ yields $x_3^\epsilon + \epsilon/x_2^\epsilon + \epsilon G_1(x_1^\epsilon) = 0$, so that (cf. (22))

$$x_1^\epsilon = G_1^{-1}(-(x_3^\epsilon + \epsilon/x_2^\epsilon)/\epsilon) \approx G_1^{-1}(-(\epsilon/\bar{x}_2)^{1/\beta}/\epsilon) = \epsilon^{1-1/\beta}/(\bar{x}_2)^{1/\beta}.$$

Finally, the second equation in (44) implies

$$x_2^\epsilon - 1 = -x_1^\epsilon - x_3^\epsilon + \epsilon/x_2^\epsilon \approx -\epsilon^{\min\{1/\beta, 1-1/\beta\}}/(\bar{x}_2)^{1/\beta}.$$

Notice that G_2 is locally Lipschitzian at $\bar{x}_2 = 1$ and satisfies (17) with some $\sigma > 0, \delta > 0, \varrho = 1$. This illustrates parts (a), (c3) and (c4) of Proposition 3. For part (c4), we have $H_1 = \{1, 2\}$, $H_2 = \{2\}$. Correspondingly, for $l = 1$, the decomposition (35) (with the subscript l restored) holds with $q_{1,1} = M_{N3} = [2\ 1\ 1]^T$, $\alpha_{1,1}^\epsilon = x_3^\epsilon$, yielding $\lambda_1^\epsilon = \epsilon/x_3^\epsilon$. For $l = 2$, (35) holds with $q_{1,2} = M_{N1} = [1\ 1\ 0]^T$, $\alpha_{1,1}^\epsilon = x_1^\epsilon$ if $\beta \in (1, 2)$; or $q_{1,2} = M_{N3} = [2\ 1\ 1]^T$, $\alpha_{1,1}^\epsilon = x_3^\epsilon$ if $\beta > 2$, etc.

(iv). Suppose $\beta > 1$ and G_1, G_2 are changed to $G_1(t) = G_2(t) = -1/t$ as in (iii). Suppose we also change M to $M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. It can be seen that this does not change Σ. Moreover, M is positive semidefinite, so $\Sigma = \bar{\Sigma}$ is convex and $\bar{J} = \{1, 2\}$. Using an argument analogous to that used in (iii), we have that x^ϵ exists and is unique for all $\epsilon > 0$, and x^ϵ is bounded as $\epsilon \to 0$. Also, M satisfies the assumptions in Proposition 3(c2) for any $I \subset N$, so it follows that any cluster point of x^ϵ is in $\mathrm{VI}(\Sigma, G) = \{(.5, .5, 0)\}$. Thus $x^\epsilon \to \bar{x} = (.5, .5, 0)$, with $J = \bar{J}$. Moreover, $F(x^\epsilon) + \epsilon G(x^\epsilon) = 0$ yields

$$x_3^\epsilon = h_3(\epsilon) = \epsilon^{1/(\beta+1)}$$

(recall $h_3(\epsilon)$ is the unique $\eta > 0$ satisfying $-\eta/\epsilon = G_3(\eta) = -1/\eta^\beta$) and, using symmetry, $x_1^\epsilon - .5 = x_2^\epsilon - .5 = O(x_3^\epsilon) = O(\epsilon^{1/(\beta+1)})$. This illustrates parts (c2) and (c3) of Proposition 3. Compared to (iii), we see that changing M changes both the limit point $\bar{x}$ and the convergence rate, even when G and the solution set Σ are unchanged!

3 Summary and Open Questions

We have considered regularizing the mapping F in a complementarity problem by another mapping and we studied properties of any limit point of the solution of the regularized problem. We have also derived error bounds on the distance from the solution of the regularized problem to its limit point. These error bounds are fairly complex, reflecting both the local growth rate of the regularization mapping and the linkage among solution components through the complementarity condition.

There remain many open questions to be answered. We list a few below.

Q1. Can parts (c3) and (c4) of Proposition 3 be simplified/strengthened in the case of $G_1 = \cdots = G_n$?

Q2. For the G given by (5), the convergence result of McLinden [42] requires F to be monotone and continuous, whereas our error bound result requires F to be affine and satisfying the assumptions of either part (c2) or part (c4) of Proposition 3. For this particular choice of G, can an error bound result analogous to Proposition 3(c2)–(c4) be obtained for non-affine F?

Q3. Consider higher-order regularization of the form $F^\epsilon(x) := F(x) + \epsilon G^1(x) + \epsilon^2 G^2(x) + \cdots + \epsilon^p G^p(x)$, where $p \geq 1$ and $G^1, ..., G^p$ are suitable mappings. What can we say about any limit point of $x^\epsilon \in \mathrm{VI}(\Re_+^n, F^\epsilon)$ as $\epsilon \to 0$? [See [1, Section 4] for discussions in the optimization setting.] What kind of error bounds can be derived?

Q4. Here we have considered the CP where the feasible set is $\Re_+^n$. Can our results be extended to variational inequality problems where the feasible set is a polyhedral set or, more generally, a nonempty closed convex set of $\Re^n$? How about extension to spaces other than $\Re^n$, such as the space of $n \times n$ symmetric matrices (with $\Re_+^n$ replaced by the convex cone of $n \times n$ symmetric positive semidefinite matrices) or an infinite-dimensional space?

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