FURTHER NOTES ON THE EXISTENCE OF SEARCH DIRECTIONS IN INTERIOR-POINT METHODS FOR THE MONOTONE SEMI-DEFINITE LCP[1]

August 15, 1996

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ABSTRACT

Following the work of Shida, Shindoh, and Kojima and of Monteiro and Zanjacomo, we study the issue of existence and uniqueness of search directions in interior-point methods for the monotone LCP over the space of $n \times n$ symmetric block-diagonal matrices, with particular focus on the direction of Alizadeh, Haeberly, and Overton (AHO). We give an analog of a result of Shida et al., characterizing existence and uniqueness in terms of a certain composite linear mapping being strictly monotone. This characterization yields a short and unified derivation of existing sufficient conditions for existence and uniqueness of the AHO direction. In the $n = 2$ case, this characterization may be expressed by two algebraic inequalities and be used to show that the "1/2" in the sufficient condition of Monteiro and Zanjacomo cannot be improved beyond a certain number less than 1.

1 Introduction

There recently have been a number of work on the existence and uniqueness of search directions used in interior-point methods for the semi-definite linear programming problem (SDLP) and the semi-definite linear complementarity problem (SDLCP). For most of the search directions proposed, including ones proposed by Nesterov and Todd, by Monteiro, and the family proposed by Shida, Shindoh, and Kojima, existence and uniqueness require only that the iterates lie in the cone of positive definite matrices (see [8] for a nice survey; also see [5, 10] for subsequent work). An exception is a search direction proposed by Alizadeh, Haeberly, and Overton (AHO) [1, 2], whose existence, as is shown in [7, 8, 9], requires additional assumptions (such as approximate centering) on the iterates that may not be satisfied in practice. Yet, interior-point methods using the AHO direction have performed well in practice [1, 2, 9]. Motivated by the aforementioned work and, in particular, the two papers [7, 8], we further study the issue of existence and uniqueness of search

[1]This research is supported by National Science Foundation Grant CCR-9311621.

directions used in interior-point methods for SDLP and monotone SDLCP, with particular focus on the AHO direction. We describe the problem setting and our results in more detail below.

Let $\mathcal{X}$ denote the space of $n \times n$ block-diagonal real matrices with m blocks of sizes $n_1, ..., n_m$, respectively (the blocks are fixed). Thus, $\mathcal{X}$ is closed under matrix addition $x+y$, multiplication xy, transposition x^T, and inversion x^{-1}, where $x, y \in \mathcal{X}$. We endow $\mathcal{X}$ with the inner product $\langle x, y\rangle := \mathrm{tr}[x^T y]$ and the norm $\|x\| := \sqrt{\langle x, x\rangle}$, where $x, y \in \mathcal{X}$ and $\mathrm{tr}[\cdot]$ denotes the matrix trace. [":=" means "define".] Let $\mathcal{S}$ denote the subspace comprising those $x \in \mathcal{X}$ that are symmetric, i.e., $x^T = x$. Let $\mathcal{S}_{++}$ denote the convex cone comprising those elements of $\mathcal{S}$ that are positive definite. The search direction used in interior-point methods for solving SDLP and monotone SDLCP is the solution to a system of linear equations in $(u, v) \in \mathcal{S} \times \mathcal{S}$ of the form:

$$Au + Bv = r, \qquad Mu - v = -s, \tag{1}$$

with $M : \mathcal{S} \mapsto \mathcal{S}$ a given monotone linear mapping (so $\langle x, Mx\rangle \geq 0$ for all $x \in \mathcal{S}$) and $A, B : \mathcal{S} \mapsto \mathcal{S}$ two user-chosen invertible linear mappings, and with $(r, s) \in \mathcal{S} \times \mathcal{S}$ given. [For feasible methods, $s = 0$ always.] There are many choices for A, B (see [5, 8, 10, 9, 11]) and, as is mentioned earlier, we will focus on a particular choice of A and B proposed by Alizadeh, Haeberly and Overton [1, 2]:

$$Au := \frac{1}{2}(yu + uy), \qquad Bv := \frac{1}{2}(xv + vx), \tag{2}$$

where $(x, y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$. First, we give a short and constructive proof that AB^* being strictly monotone is in some sense necessary and sufficient for (1) to have a unique solution (see Cor. 1), where B^* denotes the adjoint of B (i.e., $\langle d, Bv\rangle = \langle B^* d, v\rangle$ for all $d, v \in \mathcal{S}$). This result is an analog of [8, Lemma 2.1], given in the general context of finite-dimensional real vector space and in terms of strictly monotone affine subspaces. Second, we show that the existence and uniqueness results for the AHO direction given in [7, 8, 9] can be derived in a unified and simple way by showing that the corresponding AB^* is strictly monotone (see Prop. 2). Third, for the special case of $n = m = 2$ (i.e., $\mathcal{S}$ is the space of 2×2 symmetric matrices), we derive a necessary and sufficient condition for AB^* corresponding to the AHO choice (2) to be strictly monotone (see Prop. 3). We also show that the "1/2" in the Monteiro-Zanjacomo condition [7, Eq. (4)] cannot be improved beyond a certain number less than 1 while still ensuring AB^* corresponding to the AHO choice is strictly monotone. This indicates some difficulties in proving polynomial-time complexity for long-step path-following methods using the AHO direction.

In what follows, for any $x \in \mathcal{S}_{++}$, $x^{1/2} \in \mathcal{S}_{++}$ denotes the square root of x, i.e., $x^{1/2}x^{1/2} = x$. We denote by e the identity matrix in $\Re^{n\times n}$. For any $x \in \mathcal{S}$ with eigenvalues $\lambda_1, ..., \lambda_n$, we have $\|x\| = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$ and we denote $\|x\|_\infty := \max\{|\lambda_1|, \cdots, |\lambda_n|\}$ and $\|x\|_1 := |\lambda_1| + \cdots + |\lambda_n|$. [$\|x\|$ and $\|x\|_\infty$ are, respectively, the Frobenius-norm and the 2-norm of x, with $\|x\|_\infty \leq \|x\|$.] We will freely use the following facts about trace [3]: For any $x, y \in \mathcal{X}$, $\mathrm{tr}[x] = \mathrm{tr}[x^T]$, $\mathrm{tr}[xy] = \mathrm{tr}[yx]$, and $\mathrm{tr}[x + y] = \mathrm{tr}[x] + \mathrm{tr}[y]$.

2 The Results

By using the second equation in (1) to eliminate v from the first equation and making the substitution $u = B^*d$ for $d \in \mathcal{S}$, we can rewrite (1) in the reduced form

$$(AB^* + BMB^*)d = r - Bs. \tag{3}$$

The following proposition and its corollary show that (3) has a (unique) solution for all monotone M and all s if and only if AB^* is strictly monotone (i.e., $\langle d, AB^*d\rangle > 0$ whenever $d \neq 0$). This result is an analog of Lemma 2.1 in [8], given in terms of monotone and antitone affine subspaces representing the two equations in (1). The proof is short and constructive, using only elementary facts about monotone linear mapping on a finite-dimensional real vector space.

Proposition 1 *Let $R : \mathcal{S} \mapsto \mathcal{S}$ be any invertible linear mapping. If R is strictly monotone, then, for every monotone linear mapping $S : \mathcal{S} \mapsto \mathcal{S}$ and every $b \in \mathcal{S}$, the equation*

$$(R + S)d = b \tag{4}$$

has a unique solution $d \in \mathcal{S}$. Conversely, if R is not strictly monotone, then there exist monotone linear mapping $S : \mathcal{S} \mapsto \mathcal{S}$ and $b \in \mathcal{S}$ such that (4) has no solution.

Proof. Assume R is strictly monotone. Then, for every monotone linear mapping $S : \mathcal{S} \mapsto \mathcal{S}$ and every $b \in \mathcal{S}$, we have that $R + S$ is strictly monotone linear, so it is invertible and (4) has the unique solution $d = (R + S)^{-1}b$.

Assume R is not strictly monotone. Let $R_1 := \frac{1}{2}(R + R^*)$ and $R_2 := \frac{1}{2}(R - R^*)$, so that $R = R_1 + R_2$. Then R_1 is self-adjoint (i.e., $R_1^* = R_1$) and not strictly monotone (since $\langle d, Rd\rangle = \langle d, R_1d\rangle$ for all $d \in \mathcal{S}$). This implies that R_1, when represented as a $\nu \times \nu$ real matrix with $\nu := \sum_{k=1}^m \frac{n_k(n_k+1)}{2}$, is symmetric and has at least one non-positive eigenvalue. Thus,

$$R_1 = P\mathrm{diag}[\lambda_1, ..., \lambda_\nu]P^T$$

for some $\nu \times \nu$ orthogonal matrix P and some scalars $\lambda_1, ..., \lambda_\nu$, with $\lambda_1 \leq 0, ..., \lambda_k \leq 0$ for some $k \geq 1$. Let

$$R_3 := -P\mathrm{diag}[\lambda_1, ..., \lambda_k, 0, ..., 0]P^T, \qquad S := R_3 - R_2.$$

Then, S is monotone (since R_3 and $-R_2$ are monotone) and $R + S = P\mathrm{diag}[0, ..., 0, \lambda_{k+1}, ..., \lambda_\nu]P^T$ is singular, implying there exists a $b \in \mathcal{S}$ not in the range of $R + S$ (e.g., b corresponding to the first column of P), for which (4) has no solution. ∎

Corollary 1 *Let $A : \mathcal{S} \mapsto \mathcal{S}$ and $B : \mathcal{S} \mapsto \mathcal{S}$ be any invertible linear mappings and let $r \in \mathcal{S}$. If AB^* is strictly monotone, then, for every monotone linear mapping $M : \mathcal{S} \mapsto \mathcal{S}$ and every $s \in \mathcal{S}$, the equation (3) has a unique solution $d \in \mathcal{S}$. Conversely, if AB^* is not strictly monotone, then there exists a monotone linear mapping $M : \mathcal{S} \mapsto \mathcal{S}$ and an $s \in \mathcal{S}$ such that (3) has no solution.*

Proof. Since A and B are invertible, then B^* and hence AB^* are invertible, so we can apply Prop. 1 with $R := AB^*$, $S := BMB^*$ and $b := r - Bs$. Since B is invertible, we have a one-to-one correspondence between S and M and between b and s. ∎

The proposition below shows that the existence and uniqueness results given by Shida et al. [8, Thm. 3.1] (also see [9, Thm. 3.1]) and by Monteiro and Zanjacomo [7, Thm. 2.1] for the AHO direction can be derived in a unified way by showing that the corresponding AB^* is strictly monotone and then applying Prop. 1. The same approach was used by the author in an earlier work [10] to show the existence and uniqueness of other search directions.

Proposition 2 *For any $(x,y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$, if either (i) $xy + yx$ is positive semidefinite or (ii) $\|x^{1/2}yx^{1/2} - \omega e\|_\infty/\omega \leq 1/2$ for some $\omega \in (0,\infty)$, then the linear mappings A, B given by (2) are invertible and AB^* is strictly monotone.*

Proof. Since $y \in \mathcal{S}_{++}$, we have $\langle d, Ad\rangle = \mathrm{tr}[d(yd+dy)/2] = \mathrm{tr}[dyd] = \|y^{1/2}d\| > 0$ for all nonzero $d \in \mathcal{S}$, so A is strictly monotone and hence invertible. Similarly, B is invertible. It is easily checked that $B^* = B$, so for any $d \in \mathcal{S}$, we have

$$\begin{aligned} 2\langle d, AB^*d\rangle &= 2\langle d, \frac{1}{4}\left(y(xd+dx) + (xd+dx)y\right)\rangle \\ &= \frac{1}{2}\mathrm{tr}\left[d(y(xd+dx) + (xd+dx)y)\right] \\ &= \mathrm{tr}\left[dyxd + dydx\right]. \end{aligned} \tag{5}$$

If (i) $yx + xy$ is psd, then $\mathrm{tr}\left[dyxd\right] = \mathrm{tr}\left[d(yx+xy)d\right]/2 \geq 0$ and (5) would yield

$$2\langle d, AB^*d\rangle \geq \mathrm{tr}\left[dydx\right] = \mathrm{tr}\left[x^{1/2}dy^{1/2}y^{1/2}dx^{1/2}\right] = \|y^{1/2}dx^{1/2}\|^2 > 0$$

whenever $d \neq 0$. If (ii) $\|x^{1/2}yx^{1/2} - \omega e\|_\infty \leq \omega/2$ for some $\omega \in (0,\infty)$, then, by letting $c := x^{1/2}dx^{-1/2}$, (5) would yield

$$\begin{aligned} 2\langle d, AB^*d\rangle &= \mathrm{tr}\left[x^{1/2}yx^{1/2}(c^2 + c^Tc)\right] \\ &= \mathrm{tr}\left[(x^{1/2}yx^{1/2} - \omega e)(c^2 + c^Tc) + \omega(c^2 + c^Tc)\right] \\ &= \mathrm{tr}\left[(x^{1/2}yx^{1/2} - \omega e)(c^2 + c^Tc)\right] + \omega\|d\|^2 + \omega\|c\|^2 \\ &\geq -2\|x^{1/2}yx^{1/2} - \omega e\|_\infty\|c\|^2 + \omega\|d\|^2 + \omega\|c\|^2 \\ &\geq \omega\|d\|^2, \end{aligned}$$

where the first inequality follows from $\mathrm{tr}[zc^2] \leq \|z\|_\infty\|c\|^2$ and $\mathrm{tr}[zc^Tc] \leq \|z\|_\infty\|c\|^2$ for any $z \in \mathcal{S}$. Thus, in this case we also have $\langle d, AB^*d\rangle > 0$ whenever $d \neq 0$. ∎

Prop. 2 readily extends to the Monteiro-Zhang (MZ) family of directions [11], corresponding to $Au := H_p[uy]$ and $Bv := H_p[xv]$ for some invertible $p \in \mathcal{X}$, where

$H_p[w] := \frac{1}{2}pwp^{-1} + \frac{1}{2}(pwp^{-1})^T$. This is because, as noted in [6, Lem. 2.1], (1) with the MZ choice of A, B is equivalent to (1) with the AHO choice (2) of A, B via the substitution $\tilde{x} := pxp^T$, $\tilde{u} := pup^T$, $\tilde{y} := (p^T)^{-1}yp^{-1}$, $\tilde{v} := (p^T)^{-1}vp^{-1}$ (with M, s transformed similarly). In particular, condition (i) translates to a condition of Todd et al. [9, Thm. 3.1], i.e., $\tilde{x}\tilde{y} + \tilde{y}\tilde{x} = 2H_p[xy]$ be positive semidefinite.

As is shown in [7] with an example, condition (ii) does not imply (i). In particular, the example of (x, y) given in [7] satisfies $\|x^{1/2}yx^{1/2} - \omega e\|/\omega = 1/2$ with $\omega = 8$ while $xy + yx$ is not psd. We sharpen this observation below by showing that there exist (x, y) and ω with $\|x^{1/2}yx^{1/2} - \omega e\|/\omega$ arbitrarily small while $xy + yx$ is not psd. Consider the following example of x and y, parameterized by $\omega \in (4, \infty)$:

$$x := \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}, \qquad y := \begin{bmatrix} \omega & 2 \\ 2 & 1 \end{bmatrix}.$$

Straightforward calculation yields $\|x^{1/2}yx^{1/2} - \omega e\|/\omega = \sqrt{8}/\sqrt{\omega}$, which tends to zero as $\omega \to \infty$, while $xy + yx = 2\begin{bmatrix} \omega & \omega + 1 \\ \omega + 1 & \omega \end{bmatrix}$ which is not psd for any ω.

Although it is not mentioned in [7], it is also true that condition (i) does not imply (ii). To see this, we use the fact that, for any $z \in \mathcal{S}_{++}$,

$$\min_{\omega > 0}\{\|z - \omega e\|/\omega\} = (n - \|z\|_1^2/\|z\|^2)^{1/2}. \tag{6}$$

[Proof: Let $f(\omega) := (\|z - \omega e\|/\omega)^2$. Then, $f(\omega) = \sum_{i=1}^n (\lambda_i/\omega - 1)^2 = a/\omega^2 - 2b/\omega + n$, where $a := \sum_{i=1}^n \lambda_i^2$, $b := \sum_{i=1}^n \lambda_i$, and $\lambda_1, ..., \lambda_n$ are positive numbers denoting the eigenvalues of z. Setting $f'(\omega) = 0$ gives $-2a/\omega^3 + 2b/\omega^2 = 0$, implying $\omega = a/b$ and $f(\omega) = n - b^2/a$. Since $f(\omega) \to \infty$ as $\omega \to 0$ and $f(\omega) \to n$ as $\omega \to \infty$, and f is continuously differentiable on $(-\infty, \infty)$, this implies $f(\omega)$ attains its minimum at $\omega = a/b$.] Now, consider the following example of x and y, parameterized by $\epsilon \in (0, 1)$:

$$x := \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, \qquad y := \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}.$$

Applying (6) with $z := x^{1/2}yx^{1/2}$ and $n := 2$ yields

$$\min_{\omega > 0}\{\|x^{1/2}yx^{1/2} - \omega e\|/\omega\} = \left(2 - \frac{\|x^{1/2}yx^{1/2}\|_1^2}{\|x^{1/2}yx^{1/2}\|^2}\right)^{1/2} = \left(2 - \frac{(1+\epsilon^2)^2}{1+\epsilon^4}\right)^{1/2},$$

whose right-hand side tends to 1 as $\epsilon \to 0$. Thus, for all ϵ sufficiently small, there does not exist any $\omega \in (0, \infty)$ with $\|x^{1/2}yx^{1/2} - \omega e\|/\omega \le \frac{1}{2}$. On the other hand, $(xy + yx)/2$ is clearly positive semi-definite for all ϵ.

Although the two conditions (i) and (ii) in Prop. 2, as we just saw, are unrelated in that neither implies the other, there nevertheless is a connection between them. In particular, we note that, for any $(x, y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ and any $\omega \in (0, \infty)$, the following relations hold:

$$\|x^{1/2}yx^{1/2} - \omega e\|^2 \le \frac{1}{2}\|x^{1/2}yx^{1/2} - \omega e\|^2 + \frac{1}{2}\|xy - \omega e\|^2 = \left\|\frac{xy + yx}{2} - \omega e\right\|^2 \le \|xy - \omega e\|^2, \tag{7}$$

where the inequalities follow from the fact $\|x^{1/2}yx^{1/2}-\omega e\|^2 = \mathrm{tr}\,[xyxy - 2\omega xy + \omega^2 e] \leq \mathrm{tr}\,[yxxy - 2\omega xy + \omega^2 e] = \|xy - \omega e\|^2$ (since $\mathrm{tr}[xyxy] = \langle yx, xy\rangle \leq \|yx\|\|xy\| = \|xy\|^2 = \mathrm{tr}[yxxy]$); the equality can be checked by straightforward calculation. Combining (7) with Prop. 2 and we obtain that, for any $(x,y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$, the following implications hold:

$$\begin{array}{ccc}
\min_{\omega>0}\{\left\|\frac{xy+yx}{2} - \omega e\right\|/\omega\} \leq \frac{1}{2} & \Longrightarrow & \min_{\omega>0}\{\left\|\frac{xy+yx}{2} - \omega e\right\|_\infty/\omega\} \leq 1 \\
\Downarrow & & \Downarrow \\
\min_{\omega>0}\{\|x^{1/2}yx^{1/2} - \omega e\|/\omega\} \leq \frac{1}{2} & & xy + yx \text{ is positive semi-definite} \\
\Downarrow & & \Downarrow \\
\min_{\omega>0}\{\|x^{1/2}yx^{1/2} - \omega e\|_\infty/\omega\} \leq \frac{1}{2} & \Longrightarrow & \begin{array}{c} AB^* \text{ given by (2)} \\ \text{is strictly monotone} \end{array}
\end{array}$$

Prop. 2 only gives sufficient conditions for AB^* given by the AHO choice (2) to be strictly monotone. In the case where $m = n$ (i.e., all elements of $\mathcal{S}$ are $n \times n$ diagonal matrices), the condition $(x,y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$ is both necessary and sufficient. The following proposition gives a necessary and sufficient condition for the case of $n = 2$ and $m = 2$. It also shows that the number "1/2" in condition (ii) of Prop. 2 (also see [7, Thm. 1]) cannot be improved beyond a certain number less than 1 (specifically, the minimum of the middle expression in (9) over all a, b, c satisfying $a > 0$ and $c > b^2$ but not satisfying (10)).

Proposition 3 *For any $(x,y) \in \mathcal{S}_{++} \times \mathcal{S}_{++}$, normalized so that they have the form*

$$x = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \qquad y = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \tag{8}$$

for some scalars a, b, c satisfying $a > 0$ and $c > b^2$, we have that

$$\min_{\omega>0}\{\|x^{1/2}yx^{1/2} - \omega e\|/\omega\} = \left(2 - \frac{(1+ac)^2}{(1+ac)^2 - 2a(c-b^2)}\right)^{1/2} < 1 \tag{9}$$

and that AB^* given by (2) is strictly monotone if and only if a, b, c in addition satisfy

$$\begin{aligned} c(1-a)^2 &< 8(1+a) + (3+a)^2(c-b^2), \\ (1+ac)c(1-a)^2 &< \left((1+3a)^2 + (3+a)^2 ac\right)(c-b^2). \end{aligned} \tag{10}$$

Proof. Straightforward calculation yields $x^{1/2}yx^{1/2} = \begin{bmatrix} 1 & b\sqrt{a} \\ b\sqrt{a} & ac \end{bmatrix}$ with characteristic polynomial $s^2 - (1+ac)s + a(c-b^2)$. This polynomial has roots $\lambda_1, \lambda_2 := \left((1+ac)\pm\sqrt{(1+ac)^2 - 4a(c-b^2)}\right)/2$, so that $\|x^{1/2}yx^{1/2}\|_1 = \lambda_1 + \lambda_2 = 1 + ac$ and $\|x^{1/2}yx^{1/2}\|^2 = \lambda_1^2 + \lambda_2^2 = (1+ac)^2 - 2a(c-b^2)$. Plugging these quantities into the formula (6) and the equality in (9) follows. The inequality in (9) follows from $a(c-b^2) > 0$.

For any $d = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}$, where d_1, d_2, d_3 are scalars, straightforward calculation using (8) yields

$$2\langle d, AB^*d\rangle = \mathrm{tr}[dyxd+dydx] = 2d_1^2+(3+a)bd_1d_2+(1+a)(1+c)d_2^2+(1+3a)bd_2d_3+2acd_3^2,$$

so AB^* is strictly monotone if and only if the Hessian of the right-hand side, viewed as a quadratic function of (d_1, d_2, d_3), is positive definite. This Hessian is computed to be

$$\begin{bmatrix} 4 & (3+a)b & 0 \\ (3+a)b & 2(1+a)(1+c) & (1+3a)b \\ 0 & (1+3a)b & 4ac \end{bmatrix}$$

which is positive definite if and only if all its principal minors are positive or, equivalently,

$$\begin{aligned} 0 &< 8(1+a)(1+c) - (3+a)^2b^2 \;=\; 8(1+a) + (3+a)^2(c-b^2) - c(1-a)^2, \\ 0 &< 8(1+a)(1+c)ac - (1+3a)^2b^2 - (3+a)^2b^2ac \\ &= \left((1+3a)^2 + (3+a)^2ac\right)(c-b^2) - (1+ac)c(1-a)^2. \end{aligned}$$

The above inequalities are exactly (10). ■

The assumption that x is diagonal in (8) can be made without loss of generality by choosing an orthogonal $p \in \mathcal{X}$ so that $\tilde{x} = pxp^T$ is diagonal and accordingly make the substitution $\tilde{y} = pyp^T$. The condition (10) roughly says that, for each a and b, c should be "small" relative to $c - b^2$. For example, it can be checked that, for $a = 15$ and $b = 1$, the condition (10) is equivalent to $c > 2.0649...$ Alternatively, the condition (10) is violated when, for each a and b, c is sufficiently larger than $c - b^2$. Thus, for x, y of the form (8) with $a = 15$ and $b = 1$ and $1 < c \leq 2.0649...$, we have from Prop. 3 that AB^* given by (2) is not strictly monotone, so, by Cor. 1, for any symmetric $r \in \Re^{2\times 2}$, there exists a symmetric $s \in \Re^{2\times 2}$ such that (3) (or, equivalently, (1)) has no solution. For $a = 15, b = 1, c = 2.0649...$, the middle expression in (9) works out to be 0.9837..., so the number "1/2" in condition (ii) of Prop. 2 cannot be improved beyond 0.9837...

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