# Existence of Global Minima for Constrained Optimization

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**Abstract**

We present a unified approach to establishing the existence of global minima of a (non)convex constrained optimization problem. Our results unify and generalize previous existence results for convex and nonconvex programs, including the Frank-Wolfe theorem, and for (quasi-)convex quadratically constrained quadratic programs and convex polynomial programs. For example, instead of requiring the objective/constraint functions to be constant along certain recession directions, we only require them to linearly recede along these directions. Instead of requiring the objective/constraint functions to be convex polynomials, we only require the objective function to be (quasi-)convex polynomial over a polyhedral set and the constraint functions to be convex polynomials or the composition of coercive functions with linear mappings.

**Key words.** Solution existence, global minima, constrained optimization, recession directions, convex polynomial functions.

## 1 Introduction

Consider the constrained optimization problem

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \ i = 1, ..., r, \end{array} \tag{P}$$

where $f_i : \Re^n \to (-\infty, \infty]$, $i = 0, 1, ..., r$, are proper lower semicontinuous (lsc) functions. We are interested in conditions on $f_0, f_1, ..., f_r$ under which a global minimum of (P) exists. In what follows, we denote by $D$ the feasible set of (P), i.e.,

$$D = \text{dom} f_0 \cap C, \qquad C = \cap_{i=1}^r C_i, \qquad C_i = \{x \mid f_i(x) \le 0\}, \ i = 0, 1, ..., r, \tag{1}$$

with $\text{dom} f = \{x \mid f(x) < \infty\}$ for any $f : \Re^n \to (-\infty, \infty]$.

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The existence question has been studied extensively in the case where (P) is a convex program, i.e., $f_0, f_1, ..., f_r$ are convex [2], [6], [7], [10, Sec. 2.3.1], [13]. In particular, it was shown by Luo and Zhang [13, Thm. 3] that (P) has a global minimum whenever $f_1, ..., f_r$ are convex quadratic and $f_0$ is quasi-convex quadratic over a polyhedral set. By adapting the proof idea of Luo and Zhang, Auslender [6, Thm. 2], proved existence of global minimum of (P) when $f_0, f_1, ..., f_r$ are convex, have the same effective domain, and each belongs to the class $\mathcal{F}$ defined in [6]. This result is further expanded in the books [7], [10]. The result of Luo and Zhang was also extended by Belousov and Klatte [9] to the case where $f_0, f_1, ..., f_r$ are convex polynomials. However, for nonconvex programs, the results have been less complete [3], [5], [6], [8]; also see [7, Sec. 3.4]. Auslender [6, Thm. 3] (also see [7, Cor. 3.4.3]) showed that if $f_0$ belongs to $\mathcal{F}$ and the feasible set $C$ is *asymptotically linear* and the horizon function of $f_0$ is nonnegative on the horizon cone of $C$, then (P) has a global minimum. However, the assumption of $C$ being asymptotically linear excludes cases such as when $f_1, ..., f_r$ are convex quadratic. Thus, these existence results for nonconvex programs do not generalize the aforementioned results for convex programs.

In this paper, we present an approach to establishing the existence of global minimum of (P) that unifies and generalizes some of the approaches in [6], [9], [10, Sec. 2.3], [13] for convex programs and the approaches in [3], [5], [6], [7] for nonconvex programs. In particular, in Sec. 3, we prove our main result, which asserts the existence of a global minimum under key properties about the asymptotic behavior of the functions $f_i$ along "directions of unboundedness" of the corresponding level sets. In the following sections, we use this result to prove the existence of global minima for many different classes of nonconvex and convex programs. In particular, we consider nonconvex programs in which the functions $f_i$ have the form

$$f_i(x) = h_i(A_i x) + b_i^T x + c_i, \tag{2}$$

where $h_i$ is an lsc function having certain coercivity properties. The above functions may be viewed as nonconvex generalizations of convex quadratic functions. We next consider several problem classes studied in [5], [9], [13] and prove existence under more general assumptions. In particular, instead of requiring $f_0$ to be constant along certain recession directions as in [5, Thm. 21], we only require $f_0$ to linearly recede along these directions. Instead of requiring $f_0$ to be quadratic over a polyhedral set and $f_1, ..., f_r$ to be quadratic as in [13], we only require $f_0$ to be quasi-convex polynomial over a polyhedral set and $f_1, ..., f_r$ to have the form (2); see Sec. 5. Instead of requiring $f_0$ to be a convex polynomial as in [9], we only require $f_0$ to be quasi-convex polynomial over a polyhedral set; see Secs. 4 and 5. We also use our main result to deduce, as a direct consequence, the Frank-Wolfe theorem [11];

see Sec. 6. The notion of a function retracting along "asymptotically nonpositive" directions will play a key role in our analysis.

Regarding notation, all vectors are viewed as column vectors, and $x^T y$ denotes the inner product of the vectors $x$ and $y$. We denote the 2-norm $\|x\| = (x^T x)^{1/2}$ and $\infty$-norm $\|x\|_\infty = \max_i |x_i|$, where $x_i$ denotes the $i$th component of $x$. For any proper lsc function $f : \Re^n \to (-\infty, \infty]$, we denote, for each $\gamma \in \Re$, the $\gamma$-level set as

$$\mathrm{lev}_f(\gamma) = \{x \mid f(x) \leq \gamma\}.$$

We write $\gamma^k \downarrow 0$ when the sequence $\gamma^k$ approaches 0 monotonically from above.

## 2 Functions Retracting Along Asymptotically Nonpositive Directions

The existence of a global minimum of (P) is closely related to properties of the level sets of the functions $f_i$ and "directions of unboundedness" for these sets. In particular, when (P) has a finite minimum value which we assume without loss of generality to be zero, there exists a sequence $x^k \in C$, $k = 1, 2, ....$, with $f_0(x^k) \downarrow 0$. Existence of global minima amounts to $f_0$ having certain asymptotic linearity properties along each direction $d$ that is a cluster point of $\{x^k/\|x^k\|\}$ when $\|x^k\| \to \infty$. We define these notions below. The first two definitions play a key role in the analysis of Auslender, among others [3], [5], [6], [7].

**Definition 1** For any set $S \subseteq \Re^n$, its *horizon cone* $S_\infty$ [16, Sec. 3B], also called *asymptotic cone* in [7], is defined to be

$$S_\infty = \left\{ d \mid \exists\, x^k \in S,\ t^k \to \infty,\ \text{with } \frac{x^k}{t^k} \to d \right\}.$$

When $S$ is a closed convex set, we have

$$S_\infty = \{d \mid \exists\, x \in S,\ \text{with } x + \alpha d \in S\ \forall\ \alpha \geq 0\},$$

and moreover $S_\infty$ is convex.

**Definition 2** For any function $f : \Re^n \to (-\infty, \infty]$, let $f_\infty$ be the *horizon function* [16, p. 88], also called the *asymptotic function* in [7], defined by

$$f_\infty(d) = \liminf_{\substack{h \to d \\ t \to \infty}} \frac{f(th)}{t}.$$

We say that $d \in \Re^n$ is a *recession direction* of $f$ [5, p. 7] if

$$f_\infty(d) \leq 0.$$

The *recession cone* of $f$ is defined to be $R_f = \{d \mid f_\infty(d) \leq 0\}$.

We next introduce the notion of a direction along which a function asymptotically approaches below zero. Such directions will play a key role in our analysis.

**Definition 3** For any function $f : \Re^n \to (-\infty, \infty]$, we say that $d \in \Re^n$ is an *asymptotically nonpositive direction* (AND) of $f$ if there exists a sequence $x^k \in \mathrm{dom} f$, $k = 1, 2, ...$, satisfying

$$\lim_{k\to\infty} \sup f(x^k) \leq 0, \quad \|x^k\| \to \infty, \quad \text{and} \quad \frac{x^k}{\|x^k\|} \to d.$$

We refer to $\{x^k\}$ as a *unbounded sequence* (US) associated with $d$.

Notice that every AND is a recession direction of $f$ with unity norm. If $f$ is convex and $\inf_x f(x) \leq 0$, then the converse is also true. This is because there exists $\{y^k\} \subseteq \Re^n$ with $\lim_{k\to\infty} \sup f(y^k) \leq 0$, so that, for any recession direction $d$ of $f$ with unity norm, the sequence $x^k = y^k + t^k d$, with $t^k = \|y^k\|^2 + k$, is a US associated with $d$.

We will develop conditions for each $f_i$, based on its behavior along any AND $d$, which collectively will ensure existence of global minima for (P). Thus, the question of existence is decomposed into questions about whether each $f_i$ has certain properties along an AND $d$. In particular, the following two properties along an AND will play a key role in our analysis.

**Definition 4** For any function $f : \Re^n \to (-\infty, \infty]$ and any $d \in \Re^n$, we say that $f$ *recedes below 0* along $d$ on a set $S \subseteq \mathrm{dom} f$ if, for each $x \in S$, there exists $\bar{\alpha} \geq 0$ such that

$$f(x + \alpha d) \leq 0 \quad \forall\, \alpha \geq \bar{\alpha}.$$

**Definition 5** For any function $f : \Re^n \to (-\infty, \infty]$ and any AND $d$ of $f$, we say that $f$ *retracts* along $d$ on a set $S \subseteq \mathrm{dom} f$ if, for any US $\{x^k\} \subseteq S$ associated with $d$, there exists $\bar{k}$ such that

$$f(x^k - d) \leq \max\{0, f(x^k)\} \quad \forall\, k \geq \bar{k}. \tag{3}$$

In the case of $S = \mathrm{dom} f$, we say that $f$ *retracts* along $d$. We define $f$ to *retract strongly* along $d$ in the same way except that "$\max\{0, f(x^k)\}$" is replaced by "$f(x^k)$" in (3).

Roughly speaking, the above definition says that one can retract from $x^k$ along $d$ to stay in the level set whenever $x^k$ is sufficiently far from the origin. Notice that any AND $d$ of $f$ is an AND of $\max\{0, f\}$ and vice versa. Moreover,

$$\begin{array}{lcl} f \text{ retracts strongly along } d & \Rightarrow & f \text{ retracts along } d \\ & \Leftrightarrow & \max\{0, f\} \text{ retracts strongly along } d. \end{array}$$

The next two examples show that neither of the preceding two notions implies the other.

**Example 1** Let

$$f(x,y) = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } y = 0, \\ 0 & \text{if } x = 0 \text{ and } y = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f$ is proper, lsc, and $d = (1,0)$ is an AND. Moreover, $f$ retracts along $d$, but $f$ does not recede below 0 along $d$ on $\text{lev}_f(0)$.

**Example 2** Let

$$f(x,y) = \begin{cases} 0 & \text{if } x \geq 0 \text{ and } |y| \leq \sqrt{x}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f$ is proper, lsc, and $d = (1,0)$ is the only AND of $f$. $f$ recedes below 0 along $d$ on $\text{dom} f$; however, $f$ does not retract along $d$. In particular, the sequence $\{(k, \sqrt{k})\}$ is a US associated with $(1,0)$ for which the relation (3) does not hold for any $\bar{k}$.

The notion of function retraction is closely related to the class $\mathcal{F}$ of functions defined in [6, Def. 7], also called *asymptotically level stable* functions in [7, p. 94]. Specifically, a proper lsc function $f : \Re^n \to (-\infty, \infty]$ belongs to $\mathcal{F}$ if, for any $\alpha > 0$, any convergent sequence $\{\epsilon^k\} \subset \Re$, any sequence $\{x^k\} \subseteq \Re^n$, and any $d \in \Re^n$ satisfying

$$x^k \in \text{lev}_f(\epsilon^k), \qquad \|x^k\| \to \infty, \qquad \frac{x^k}{\|x^k\|} \to d, \qquad f_\infty(d) = 0,$$

there exists $\bar{k}$ such that

$$x^k - \alpha d \in \text{lev}_f(\epsilon^k) \quad \forall\, k \geq \bar{k}.$$

Various examples of functions in $\mathcal{F}$, such as convex piecewise linear-quadratic functions and asymptotically linear functions, are given in [6]. The following lemma shows that $f \in \mathcal{F}$ implies $f$ retracts along any AND whose horizon function value is 0.

**Lemma 1** If $f \in \mathcal{F}$, then, for any AND $d$ of $f$, either $f$ retracts along $d$ or $f_\infty(d) < 0$.

**Proof.** Fix any AND $d$ of $f$. Then $d \neq 0$ and $f_\infty(d) \leq 0$. If $f_\infty(d) < 0$, the proof is complete. Suppose that $f_\infty(d) = 0$. For any US $\{x^k\}$ associated with $d$, let $\epsilon^k = \max\{0, f(x^k)\}$. Then $x^k \in \text{lev}_f(\epsilon^k)$ for all $k$ and $\epsilon^k \to 0$, so $f \in \mathcal{F}$ implies there exists $\bar{k}$ such that, for all $k \geq \bar{k}$, $x^k - d \in \text{lev}_f(\epsilon^k)$ or, equivalently, $f(x^k - d) \leq \max\{0, f(x^k)\}$. ■

In general, $f_\infty(d) < 0$ does not imply $f$ retracts along $d$. In his earlier work [3, p. 777], [5, Thm. 13], Auslender considered a larger class of functions which, roughly speaking, allow different $\alpha$ for different $k$. However, this class is too large to apply to the problem (P) since one needs a common $\alpha$ to work for the objective function and the constraint functions. Hence the class $\mathcal{F}$ was introduced in [6] and, analogously, we introduce the notion of a function retracting along an AND (which uses a common $\alpha = 1$). We will discuss the class $\mathcal{F}$ further in Sec. 7.

# 3 Main Existence Result

Define

$$L(\gamma) = \text{lev}_{f_0}(\gamma) \cap C,$$

where $C$ is given by (1). Below, we prove our main result on the existence of a global minimum of (P). This amounts to showing that $L(\gamma) \neq \emptyset$ for all $\gamma > 0$ implies $L(0) \neq \emptyset$. The proof, by induction on $r$, uses similar ideas as in the proof of [13, Thm. 3] and [6, Thm. 1] (also see [10, Prop. 1.5.7]), but we deal with more general functions than convex quadratic or convex functions belonging to $\mathcal{F}$. The following key assumption on the functions will be used:

**Assumption 1**

(a) For each $i \in \{0, 1, ..., r\}$ and each AND $d$ of $f_i$,

either (i) $f_i$ recedes below 0 along $d$ on $\text{dom} f_i$,

or (ii) $f_i$ retracts along $d$ and, in case $r \neq 0$, $f_i$ recedes below 0 along $d$ on $C_i$.

(b) In case $r \neq 0$, $C_0 \subseteq \text{dom} f_i$ for $i = 1, ..., r$.

**Proposition 1 (Existence of Global Minima of (P))** Suppose that $f_i : \Re^n \to (-\infty, \infty]$, $i = 0, 1, ..., r$, are proper, lsc, and satisfy Assumption 1. Also, suppose that $L(\gamma) \neq \emptyset$ for all $\gamma > 0$. Then $L(0) \neq \emptyset$.

**Proof.** Take any sequence of scalars $\{\gamma^k\}_{k=1,2,...} \downarrow 0$. By assumption, $L(\gamma^k) \neq \emptyset$ for all $k$. Since $f_i$ is lsc for all $i$, $L(\gamma^k)$ is closed. Let

$$x^k \in \arg\min\{\|x\| \mid x \in L(\gamma^k)\}.$$

If $x^k$ has a convergent subsequence, then the limit point of this subsequence would, by lsc of $f_i$ for all $i$, be in $L(0)$. Thus, it suffices to consider the case of $\|x^k\| \to \infty$.

Since $\{x^k/\|x^k\|\}$ is bounded, it has a subsequence converging to some limit $d \neq 0$. By passing to a subsequence if necessary, we can assume that $x^k/\|x^k\| \to d$. Since $f_0(x^k) \leq \gamma^k$ while $f_i(x^k) \leq 0$ for $i = 1, ..., r$, this implies that $d$ is an AND of $f_i$ for $i = 0, 1, ..., r$. Then, by Assumption 1(a), for each $i \in \{0, 1, ..., r\}$, either $f_i$ recedes below 0 along $d$ on $\text{dom} f_i$ or $f_i$ retracts along $d$. Moreover, if $r \neq 0$, then $f_i$ recedes below 0 along $d$ on $C_i$ for $i \in \{0, 1, ..., r\}$.

If $f_0$ recedes below 0 along $d$ on $\text{dom} f_0$, then for any $\bar{x} \in D \neq \emptyset$, we have $\bar{x} \in \text{dom} f_0$ so that $f_0(\bar{x} + \alpha d) \leq 0$ for all $\alpha$ sufficiently large. In case $r \neq 0$, for each $i \in \{1, ..., r\}$ we have $\bar{x} \in C_i$ and hence $f_i(\bar{x} + \alpha d) \leq 0$ for all $\alpha$ sufficiently large. Thus, $\bar{x} + \alpha d \in L(0)$ for all $\alpha$ sufficiently large, implying $L(0) \neq \emptyset$. Thus, it remains to consider the case of $f_0$ retracting along $d$. If $f_i$ retracts along $d$ for all $i = 1, ..., r$, then for each $i \in \{0, 1, ..., r\}$, there exist $\bar{k}_i$ such that

$$x^k - d \in \text{lev}_{f_i}(\gamma^{k,i}) \quad \forall\, k \geq \bar{k}_i,$$

where $\gamma^{k,0} = \gamma^k$ and $\gamma^{k,i} = 0$ for $i = 1, ..., r$. Then, for all $k \geq \max_{i=0,1,...,r} \bar{k}_i$,

$$\tilde{x}^k = x^k - d$$

would satisfy $\tilde{x}^k \in L(\gamma^k)$. Then, by $x^k/\|x^k\| \to d$, we would have $d^T x^k/\|x^k\| \to \|d\|^2$, which implies $d^T x^k \to \infty$ and hence

$$\|\tilde{x}^k\|^2 = \|x^k\|^2 - 2d^T x^k + 1 < \|x^k\|^2$$

for all $k$ sufficiently large, contradicting $x^k$ being an element of $L(\gamma^k)$ of least 2-norm. Thus, either $L(0) \neq \emptyset$ or else $r \geq 1$ and $f_{\bar{i}}$ recedes below 0 along $d$ on $\text{dom} f_{\bar{i}}$ for some $\bar{i} \in \{1, ..., r\}$.

We now complete the proof by induction on $r$. The above argument shows that Prop. 1 is true when $r = 0$. Suppose Prop. 1 is true for $r = 0, 1, ..., \bar{r}$ for some $\bar{r} \geq 0$. Consider $r = \bar{r} + 1$. We showed above that either $L(0) \neq \emptyset$ or there exists some AND $d$ of $f_0, f_1, ..., f_r$ such that $f_{\bar{i}}$ recedes below 0 along $d$ on $\mathrm{dom} f_{\bar{i}}$ for some $\bar{i} \in \{1, ..., r\}$. Let us drop $f_{\bar{i}}$ from $L(\gamma)$ to obtain

$$\tilde{L}(\gamma) = \mathrm{lev}_{f_0}(\gamma) \bigcap \left( \bigcap_{\substack{i=1 \\ i \neq \bar{i}}}^{r} C_i \right).$$

Then $\tilde{L}(\gamma^k) \supset L(\gamma^k) \neq \emptyset$ for all $k$. Also, Assumption 1 is still satisfied upon dropping $f_{\bar{i}}$. Thus, by the induction hypothesis, $\tilde{L}(0) \neq \emptyset$. For any $\tilde{x} \in \tilde{L}(0)$, since $f_i$ recedes below 0 along $d$ on $C_i$ for all $i \in \{0, 1, ..., r\} \setminus \{\bar{i}\}$, we have $\tilde{x} + \alpha d \in \tilde{L}(0)$ for all $\alpha \geq 0$ sufficiently large. Moreover, Assumption 1(b) implies $\tilde{x} \in \mathrm{lev}_{f_0}(0) = C_0 \subseteq \mathrm{dom} f_{\bar{i}}$ so $f_{\bar{i}}$ receding below 0 along $d$ on $\mathrm{dom} f_{\bar{i}}$ implies $f_{\bar{i}}(\tilde{x} + \alpha d) \leq 0$ for all $\alpha$ sufficiently large, yielding $\tilde{x} + \alpha d \in L(0)$. ∎

Prop. 1 shows that when $f_0, f_1, ..., f_r$ satisfy Assumption 1 and the minimum value of (P) is finite which we assume without loss of generality to be zero, a global minimum of (P) exists. We discuss Prop. 1 and its assumptions in more detail below.

1. It can be seen from its proof that Prop. 1 still holds if we relax Assumption 1(b) so that $C_0 \subseteq \mathrm{dom} f_i$ holds only for those $i \in \{1, \ldots, r\}$ for which there exists some AND $d$ such that $f_i$ does not retract along $d$. For instance, if every constraint function $f_i$ is polyhedral, then it can be shown that $f_i$ retracts along every AND of $f_i$, so that Assumption 1(b) can be dropped altogether. In general, Assumption 1(b) cannot be dropped as we will show in Example 5.

2. Prop. 1 still holds if we relax the assumption on $f_0$ within Assumption 1(a) to

   For each AND $d$ of $f_0$, there exists $F \subseteq \mathrm{dom} f_0$ such that

   (i) $f_0$ recedes below 0 along $d$ on $\mathrm{dom} f_0 \setminus F$,
   (ii) $f_0$ retracts along $d$ on $F$,
   (iii) In case $r \neq 0$, $f_0$ recedes below 0 along $d$ on $C_0$.

   [The assumption on $f_0$ within Assumption 1(a) corresponds to $F = \emptyset$ or $F = \mathrm{dom} f_0$.] This is because in the proof we can analogously divide into two cases: If $D \setminus F \neq \emptyset$, then for any $\bar{x} \in D \setminus F$ we have $f_0(\bar{x} + \alpha d) \leq 0$ for all $\alpha$ sufficiently large. Otherwise $D \subseteq F$, and the proof proceeds as before. This seemingly technical generalization of Prop. 1 is needed for Secs. 5 and 6.

   The following example shows that the assumption on the constraint functions $f_1, ..., f_r$ cannot be relaxed in the same way.

**Example 3** Let

$$f_0(x,y) = \begin{cases} \frac{1}{|y|+1} - \frac{1}{2} & \text{if } |y| < 1 \text{ and } x \geq 0, \\ 0 & \text{if } |y| \geq 1 \text{ and } x \geq 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$f_1(x,y) = \begin{cases} 0 & \text{if } |y| \leq 1 - e^{-x} \text{ and } x \geq 0, \\ 1 & \text{if } |y| > 1 - e^{-x} \text{ and } x \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Then, both $f_0$ and $f_1$ are proper, lsc. Any $d$ with $\|d\| = 1$ and $d_1 \geq 0$ is an AND of $f_0$. It can be seen that for any AND $d$ of $f_0$, $f_0$ retracts along $d$, and also $f_0$ recedes along $d$ on $C_0$. The only AND of $f_1$ is $d = (0,1)$, and $f_1$ does not recede below 0 along $d$ on $\text{dom} f_1$. Moreover, the sequence $\{(k, 1-e^{-k})\}$ is a US associated with $d = (0,1)$, which does not satisfy (3), hence $f_1$ does not retract along $d$. But $f_1$ does satisfy the following relaxed assumption (which is analogous to the relaxed assumption on $f_0$): Let $F_1 = \{(x,y) \mid x \geq 0, |y| \geq 1\}$. Then $f_1$ recedes below 0 along $d$ on $\text{dom} f_1 \setminus F_1$, retracts along $d$ on $F_1$, and $f_1$ recedes below 0 along $d$ on $C_0$. Although the point $(x^k, y^k) = (k, 1 - e^{-k})$ satisfies $f_1(x^k, y^k) = 0$ and $f_0(x^k, y^k) \to 0$, there is no solution to $f_1(x,y) \leq 0$ and $f_0(x,y) \leq 0$.

3. If, for each $i = 1, ..., r$ and each AND $d$ of the constraint function $f_i$, $f_i$ retracts along $d$ and recedes below 0 along $d$ on $C_i$, then Prop. 1 still holds if the second part of Assumption 1(a)(ii) on the objective function $f_0$, namely "in case $r \neq 0$, $f_0$ recedes below 0 along $d$ on $C_0$", is dropped. In fact, in this case, (P) can be reduced to the case of $r = 0$ by working with the extended objective function

$$\tilde{f}_0(x) = \begin{cases} f_0(x) & \text{if } x \in D, \\ \infty & \text{otherwise} \end{cases}$$

It can be shown that, for this equivalent problem, Assumption 1(a) is satisfied. This trick will be used in the proof of Lemma 7.

The following example shows that the second part of Assumption 1(a)(ii) on $f_0$ cannot be dropped in general.

**Example 4** Let

$$f_0(x,y) = \begin{cases} \frac{1}{|y|+1} & \text{if } x > 0 \text{ and } |y| < 2\sqrt{x}, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_1(x,y) = \begin{cases} 0 & \text{if } x \ge 1 \text{ and } |y| \le \sqrt{x}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, both $f_0$ and $f_1$ are proper, lsc. Moreover, any $d$ with $\|d\| = 1$ is an AND of $f_0$ and $f_0$ retracts along $d$. The only AND of $f_1$ is $d = (1,0)$ and $f_1$ recedes below 0 along $d$ on $\mathrm{dom} f_1$. Also, the point $x^k = k, y^k = \sqrt{k}$ satisfies $f_1(x^k, y^k) = 0$ and $f_0(x^k, y^k) = 1/(\sqrt{k}+1) \to 0$. However, there is no solution to $f_1(x,y) \le 0$ and $f_0(x,y) \le 0$. The problem is that $f_0$ does not recede below 0 along $d = (1,0)$ on $C_0$.

If the objective function $f_0$ retracts strongly along each AND $d$ of $f_0, f_1, ... f_r$ and each constraint function $f_i$ retracts along $d$, then the existence of a global minimum of (P) can be deduced without assuming its minimum value is finite. This result generalizes an existence result of Auslender [3, Thm. 2.4]; see Prop. 5.

**Proposition 2** Consider the problem (P). Suppose that $D \neq \emptyset$ and, for each AND $d$ of $f_0, f_1, ... f_r$, we have that $f_0$ retracts strongly along $d$ and $f_i$ retracts along $d$ for $i = 1, ..., r$. Then (P) has a global minimum.

**Proof.** Let $\gamma^*$ denote the minimum value of (P), which we assume without loss of generality to be either 0 or $-\infty$. Then, for any sequence $\gamma^k \downarrow \gamma^*$, $L(\gamma^k) \neq \emptyset$ for all $k$. Let $x^k$ be an element of $L(\gamma^k)$ of minimum 2-norm. We claim that $\{x^k\}$ has a cluster point, which would be a global minimum of (P). We argue this by contradiction. Suppose that $\|x^k\| \to \infty$. By passing to a subsequence if necessary, we assume that $x^k/\|x^k\|$ converges to some $d$. Then $d$ is an AND of $f_0, f_1, ..., f_r$. Since $f_0$ retracts strongly along $d$, $f_0(x^k - d) \le f_0(x^k) \le \gamma^k$ for all $k$. For each $i \in \{1, ..., r\}$, since $f_i$ retracts along $d$ and $x^k \in C_i$, $f_i(x^k - d) \le \max\{0, f_i(x^k)\} \le 0$ for all $k$ sufficiently large. Thus $x^k - d \in L(\gamma^k)$ for all $k$ sufficiently large. But also $\|x^k - d\|^2 < \|x^k\|^2$ for all $k$ sufficiently large (see the proof of Prop. 1), contradicting $x^k$ being of minimum 2-norm. ∎

# 4 Linearly Receding Functions

In this section we introduce an important class of functions that satisfy Assumption 1(a). We say that $f$ *linearly recedes* along $d$ if there exists $\theta \in \Re$ (depending on $d$) such that $\theta \le 0$ and

$$f(x + \alpha d) = f(x) + \alpha\theta \quad \forall\, \alpha \in \Re,\ \forall\, x \in \mathrm{dom} f. \tag{4}$$

We say that $f$ is *constant* along $d$ if we can take $\theta = 0$ in the above expression. Consider the following assumption on a proper lsc $f : \Re^n \to (-\infty, \infty]$:

**Assumption 2** For any AND $d$ of $f$, $f$ linearly recedes along $d$.

Assumption 2 says that $f$ is affine on each line intersecting dom$f$ and parallel to an AND of $f$. This assumption is weaker than $f$ belonging to the class $\mathcal{F}_1$ defined in [6, Def. 3], namely, $f$ is convex and $f$ linearly recedes along any recession direction of $f$. [Recall that any AND of $f$ is a recession direction of $f$.] This assumption is also weaker than $f$ being *asymptotically directionally constant* [7, p. 86], namely, $f$ is constant along any recession direction of $f$. We will show in Lemma 4 that there is a large class of nonconvex functions $f$ that satisfy Assumption 2 but are not asymptotically directionally constant nor in $\mathcal{F}_1$.

The following lemma shows that it suffices to verify Assumption 2 for each $f_i$ in lieu of Assumption 1(a).

**Lemma 2** If $f_0, f_1, ..., f_r$ each satisfies Assumption 2, then they satisfy Assumption 1(a).

**Proof.** Consider any proper lsc $f : \Re^n \to (-\infty, \infty]$ satisfying Assumption 2. Fix any $d \in \Re^n$ that is an AND of $f$. By Assumption 2, $f$ linearly recedes along $d$, i.e., there exists $\theta \in \Re$ such that $\theta \leq 0$ and (4). If $\theta < 0$, then for any $x \in \mathrm{dom} f$, we have from (4) that $f(x + \alpha d) \to -\infty$ as $\alpha \to \infty$. Thus $f$ recedes below 0 along $d$ on dom$f$. If $\theta = 0$, then for any $x \in \mathrm{dom} f$, we have from (4) that $f(x - \alpha d) = f(x)$ for all $\alpha \in \Re$. It follows that $f$ retracts along $d$ and $f$ recedes below 0 along $d$ on $\mathrm{lev}_f(0)$.

Applying the above result to $f_0, f_1, ..., f_r$, we see that they satisfy Assumption 1(a). ■

Using Lemma 2, we give below an example showing that Prop. 1 is false if Assumption 1(b) is dropped.

**Example 5** Define

$$\phi(t) = \begin{cases} -\ln t - \ln(1-t) & \text{if } 0 < t < 1, \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$f_0(x_1, x_2) = x_1, \qquad f_1(x_1, x_2) = \phi(x_1) - x_2.$$

Then $f_0$ is linear, so $f_0$ satisfies Assumption 2. Since $\phi$ is lsc, then $f_1$ is lsc and, in fact, convex. Since $\phi$ has bounded support, it is not difficult to see that, for any $d = (d_1, d_2) \in \Re^2$ that is an AND of $f_1$, we have $d_1 = 0$ and $d_2 \geq 0$. Then $f_1$ linearly recedes along $d$. Thus $f_0, f_1$ each satisfies Assumption 2. By Lemma 2, they satisfy Assumption 1(a).

As $x_1 \to 0^+$ and setting $x_2 = \phi(x_1)$, we have that $f_0(x_1, x_2) \to 0$ and $f_1(x_1, x_2) \le 0$. However, there is no $x \in \Re^2$ satisfying $f_0(x) = 0$ and $f_1(x) \le 0$ (since $f_1(x) \le 0$ implies $x_1 > 0$). Here, the problem is that $\mathrm{dom} f_1$ is not contained in the 0-level set of $f_0$ (in fact, the two sets are disjoint), thus violating Assumption 1(b).[3]

The following lemma shows that the assumption $\theta \le 0$, implicit in Assumption 2, is redundant whenever $f$ does not tend to $-\infty$ at a superlinear rate. In particular, convex functions have the latter property.

**Lemma 3** Suppose that $f : \Re^n \to (-\infty, \infty]$ is proper, and

$$f_\infty(d) > -\infty \quad \forall\, d \in \Re^n. \tag{5}$$

If an AND $d$ of $f$ and $\theta \in \Re$ satisfies (4), then $\theta \le 0$.

**Proof.** Fix any AND $d$ of $f$ and $\theta \in \Re$ that satisfies (4). If $\theta = 0$, then clearly $\theta \le 0$. If $\theta \ne 0$, then fix any $\bar{x} \in \mathrm{dom} f$, which exists since $f$ is proper. Since $d$ is an AND of $f$, there exists an associated US $\{x^k\}$. For each $k$, let $t^k = \|x^k - \bar{x}\|$ and $d^k = (x^k - \bar{x})/\|x^k - \bar{x}\|$. Then $d^k \to d$ and

$$x^k = y^k + t^k d, \quad \text{with} \quad y^k = \bar{x} + t^k(d^k - d).$$

For each $k$, since $x^k \in \mathrm{dom} f$, (4) implies $y^k \in \mathrm{dom} f$ and

$$f(x^k) = f(y^k + t^k d) = f(y^k) + t^k \theta.$$

Thus

$$\theta = \frac{f(x^k) - f(y^k)}{t^k}. \tag{6}$$

Notice that $\{f(x^k)\}$ is bounded above and, by $\|d^k - d\| \to 0$ and $f$ satisfying (5), we have

$$\liminf_{k\to\infty} \frac{f(y^k)}{t^k} = \liminf_{\substack{k\to\infty \\ d^k \ne d}} \frac{f(\bar{x} + t^k(d^k - d))}{t^k \|d^k - d\|} \|d^k - d\| \ge 0.$$

If $d^k = d$ for $k$ along a subsequence, then $y^k = \bar{x}$ so $f(y^k)/t^k \to 0$ along this subsequence. Then the limit superior of the right-hand side of (6) as $k \to \infty$ is nonpositive, implying $\theta \le 0$. ■

[3] We can make these two sets have nonempty intersection by redefining $\phi(t)$ to have the value 1 (instead of $\infty$) for all $t \le -1$, say. However, the resulting $f_1$ is no longer convex. If $f_0, f_1, ..., f_r$ are convex, can Assumption 1(b) be relaxed to $C_0 \cap (\cap_{i=1}^r \mathrm{dom} f_i) \ne \emptyset$? This is an open question.

If $f : \Re^n \to (-\infty, \infty]$ is convex and $\inf_x f(x) \leq 0$, then we saw in Sec. 2 that

$$d \text{ is an AND of } f \quad \Leftrightarrow \quad d \in R_f.$$

In this case, the assumption that $f$ is constant along any AND $d$ of $f$ (i.e., $\theta$ can be taken equal to 0 in (4)) is equivalent to $R_f \subseteq L_f$, where $L_f = R_f \cap (-R_f)$ denotes the constancy space of $f$. Since $L_f \subseteq R_f$, this is equivalent to

$$R_f \;=\; L_f.$$

Similarly, it can be seen that $f$ linearly recedes along $d$ at rate $\theta$ if and only if $(d, \theta) \in L_{\text{epi}f}$, where epi$f$ denotes the epigraph of $f$ and $L_S = S_\infty \cap (-S_\infty)$ denotes the lineality space of a set $S \subseteq \Re^{n+1}$. Thus, Assumption 2 is equivalent to

$$R_f \;\subseteq\; \text{Proj}_{\Re^n} L_{\text{epi}f},$$

where $\text{Proj}_{\Re^n}(x, \zeta) = x$ for all $(x, \zeta) \in \Re^n \times \Re$. Some related conditions are given in [10, Prop. 2.3.4].

We now give examples of functions that satisfy Assumption 2. Consider the following assumption on $f$.

**Assumption 3**

$$f(x) = h(Ax) + b^T x + c, \tag{7}$$

where $A \in \Re^{m \times n}, b \in \Re^n, c \in \Re$ and $h : \Re^m \to (-\infty, \infty]$ is a function satisfying (5) and

$$\text{either} \quad \text{(i) } b = 0 \text{ and } \lim_{\|y\| \to \infty} \inf h(y) = \infty \quad \text{or} \quad \text{(ii) } b \neq 0 \text{ and } \lim_{\|y\| \to \infty} \inf \frac{h(y)}{\|y\|} = \infty.$$

Assumption 3 is satisfied by, for example, convex quadratic functions and the function $g(Ax - b)$ in [5, p. 13] with $g$ coercive (i.e., $g_\infty(d) > 0$ for all $d \in \Re^n$). We show below that Assumption 3 implies Assumption 2.

**Lemma 4** Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper lsc function that satisfies Assumption 3. Then $f$ satisfies Assumption 2.

**Proof.** First, we claim that

$$d \text{ is an AND of } f \quad \Rightarrow \quad Ad = 0, \quad b^T d \leq 0. \tag{8}$$

To see this, fix any AND $d$ of $f$ and let $\{x^k\}$ be an associated US.

If $b = 0$, then $f(x^k) = h(Ax^k)$ is bounded above, so that $\|Ax^k\|$ is bounded. Hence $\{\|Ax^k\|/\|x^k\|\} \to 0$, implying $Ad = 0$. Since $b = 0$, we also have $b^T d \leq 0$.

If $b \neq 0$, then we have

$$\lim_{k\to\infty} \sup \frac{h(Ax^k)}{\|x^k\|} + \frac{b^T x^k + c}{\|x^k\|} = \lim_{k\to\infty} \sup \frac{f(x^k)}{\|x^k\|} \leq 0.$$

Thus $\dfrac{h(Ax^k)}{\|x^k\|}$ is bounded from above. If $\|Ax^k\| \to \infty$ along a subsequence, then boundedness from above of $\dfrac{h(Ax^k)}{\|x^k\|} = \dfrac{h(Ax^k)}{\|Ax^k\|}\dfrac{\|Ax^k\|}{\|x^k\|}$ and $\dfrac{h(Ax^k)}{\|Ax^k\|} \to \infty$ imply $\dfrac{\|Ax^k\|}{\|x^k\|} \to 0$ along this subsequence. If $\|Ax^k\|$ is bounded along a subsequence, then $\|x^k\| \to \infty$ implies $\|Ax^k\|/\|x^k\| \to 0$ along this subsequence. In either case, we see that $\|Ax^k\|/\|x^k\| \to 0$ and hence $Ad = 0$. Also, we have from $\{f(x^k)\}$ being bounded above (by, say, $\kappa$) that

$$\frac{b^T x^k}{\|x^k\|} \leq -\frac{h(Ax^k)}{\|x^k\|} + \frac{\kappa - c}{\|x^k\|},$$

which, together with $\lim_{k\to\infty} \inf h(Ax^k)/\|x^k\| \geq 0$,[4] yields $\lim_{k\to\infty} \sup b^T x^k/\|x^k\| \leq 0$, so that $b^T d \leq 0$. Thus $d$ satisfies

$$Ad = 0, \quad b^T d \leq 0.$$

It follows from (7) and (8) that $f$ satisfies Assumption 2. ∎

Thus, if each of $f_0, f_1, ..., f_r$ satisfies Assumption 3, then it satisfies Assumption 2. By Lemma 2, they collectively satisfy Assumption 1(a). So, if they in addition satisfy Assumption 1(b), then Prop. 1 yields that a global minimum of (P) exists. To our knowledge, this existence result is new.

[4] Why? If $\|Ax^k\|$ is bounded along a subsequence, then $h(Ax^k)$ is bounded below along this subsequence (since $h$ is lsc and nowhere equal to $-\infty$) and hence $\dfrac{h(Ax^k)}{\|x^k\|} \to 0$. If $\|Ax^k\| \to \infty$ along a subsequence, then the assumption of $h_\infty(d) > -\infty$ for all $d \in \Re^m$ implies $\dfrac{h(Ax^k)}{\|Ax^k\|}$ is bounded below along this subsequence, which together with $\dfrac{\|Ax^k\|}{\|x^k\|} \to 0$ implies $\lim_{k\to\infty} \inf \dfrac{h(Ax^k)}{\|x^k\|} = \lim_{k\to\infty} \inf \dfrac{h(Ax^k)}{\|Ax^k\|}\dfrac{\|Ax^k\|}{\|x^k\|} \geq 0$.

# 5 Quasi-Convex Polynomial Functions over Polyhedral Sets

Consider the following polynomial assumption on $f$, generalizing the quadratic assumption of Luo and Zhang [13, Sec. 4].

**Assumption 4**

$$f(x) = \begin{cases} g(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases}$$

with $g : \Re^n \to \Re$ a polynomial function and

$$X = \{x \mid Bx \leq b\},$$

for some $B \in \Re^{m \times n}, b \in \Re^m$. Moreover, $g$ is assumed to be quasi-convex on $X$ [14, Chap. 9].

Any convex polynomial $f$ satisfies this assumption. A nonconvex example is $g(x_1, x_2) = x_1 x_2$, $X = [0, \infty) \times (-\infty, 0]$. The following lemma shows that $f$ satisfies the relaxed assumption in Remark 2 following Prop. 1.

**Lemma 5** Let $f : \Re^n \to (-\infty, \infty]$ satisfy Assumption 4. Then, for any AND $d$ of $f$, there exists $F \subseteq X$ such that (i) $f$ recedes below 0 along $d$ on $X \setminus F$, (ii) $f$ retracts along $d$ on $F$, (iii) $f$ recedes below 0 along $d$ on $\text{lev}_f(0)$.

**Proof.** Fix any AND $d$ of $f$. Let $\{x^k\}$ be a US associated with $d$. Let

$$\gamma^k = \sup_{\ell \geq k} \max\{0, f(x^\ell)\} \quad \forall\, k.$$

Then $\{\gamma^k\} \downarrow 0$ and, for each $k$, $x^\ell \in \text{lev}_f(\gamma^k)$ for all $\ell \geq k$. Since $g$ is lsc quasi-convex on $X$, the level sets of $f$ are closed convex. Since $\text{lev}_f(\gamma^k)$ is closed convex and $x^\ell/\|x^\ell\| \to d$, $d$ is in the horizon cone of $\text{lev}_f(\gamma^k)$ [15]. Thus, $d$ is in the intersection of the horizon cones of $\text{lev}_f(\gamma^k)$ for all $k$. Since the horizon cones of $\text{lev}_f(\gamma^k)$, $k = 1, 2, \ldots$, are nested, this implies that $d$ is in the horizon cone of $\text{lev}_f(\gamma)$ for all $\gamma > 0$.

Fix any $x \in X$. Then $x \in \text{lev}_f(\gamma)$ for some $\gamma > 0$, so that $x + \alpha d \in \text{lev}_f(\gamma)$ for all $\alpha \geq 0$. Thus, $B(x + \alpha d) \leq c$ and $g(x + \alpha d) \leq \gamma$ for all $\alpha \geq 0$. The former implies $Bd \leq 0$, i.e., $d \in X_\infty$. Since $g(x + \alpha d)$ is a polynomial function of $\alpha$, the latter implies that

$$\text{either} \quad g(x + \alpha d) \to -\infty \ \text{ as } \alpha \to \infty \quad \text{or} \quad g(x + \alpha d) = g(x) \ \ \forall\, \alpha \in \Re. \tag{9}$$

Define

$$F = \{x \in X \mid g(x + \alpha d) = g(x) \ \forall \ \alpha \in \Re\}.$$

Then, for any $x \in X \setminus F$, we have from $Bd \leq 0$ that $x + \alpha d \in X$ for all $\alpha \geq 0$, so that (9) yields

$$f(x + \alpha d) = g(x + \alpha d) \to -\infty \quad \text{as} \quad \alpha \to \infty.$$

This shows that $f$ recedes below 0 along $d$ on $X \setminus F$. Fix any US $\{x^k\} \subseteq F$ associated with $d$. For each $i \in \{1, ..., m\}$, let $B_i$ and $b_i$ denote the $i$th row of $B$ and $c$. If $B_i d = 0$, then $B_i(x^k - d) = B_i x^k \leq b_i$ for all $k$. If $B_i d < 0$, then it follows from $B_i x^k / \|x^k\| \to B_i d$ and $\|x^k\| \to \infty$ that there exists $\bar{k}_i$ such that

$$B_i(x^k - d) \leq b_i \quad \forall \ k \geq \bar{k}_i.$$

Let $\bar{k} = \max_{\{i|B_i d<0\}} \bar{k}_i$. Since $x^k \in F$, this yields

$$f(x^k - d) = g(x^k - d) = g(x^k) = f(x^k) \quad \forall \ k \geq \bar{k},$$

implying that $f$ retracts along $d$ on $F$. By $Bd \leq 0$ and (9), for any $x \in \text{lev}_f(0)$, there exists $\bar{\alpha} \geq 0$ such that

$$x + \alpha d \in X, \qquad g(x + \alpha d) \leq g(x) \leq 0 \quad \forall \ \alpha \geq \bar{\alpha}.$$

Thus $f(x + \alpha d) \leq 0$ for all $\alpha \geq \bar{\alpha}$, so $f$ recedes below 0 on $\text{lev}_f(0)$. ■

We note that the above proof generalizes to any continuous function $g$ that is either constant or tends to $\infty$ or $-\infty$ on each line. In the case where $g$ is quadratic, i.e., $g(x) = \frac{1}{2}x^T Q x + q^T x$ for some symmetric $Q \in \Re^{n \times n}$ and $q \in \Re^n$, it can be seen that, for any AND $d$ of $f$, either $d^T Q d < 0$, in which case $F = \emptyset$, or else $d^T Q d = 0, (Qx + q)^T d \leq 0$, in which case $F$ is the set of maxima of the linear program

$$\max \quad (Qx + q)^T d \quad \text{subject to} \quad x \in X.$$

Thus, $F$ is a face of $X$ in this case. In the special case where $f$ is a convex polynomial function on $\Re^n$, Lemma 5 can be further sharpened as is shown below.

**Lemma 6** Let $f : \Re^n \to \Re$ be a convex polynomial function. Then, for any AND $d$ of $f$, either (i) $f$ recedes below 0 along $d$ on $\Re^n$ or (ii) $f$ retracts along $d$ on $\Re^n$ and $f$ recedes below 0 along $d$ on $\text{lev}_f(0)$.

**Proof.** Fix any AND $d$ of $f$. Then $d$ is a recession direction of $f$. Since $f$ is convex, this implies that, for any $x \in \Re^n$, $f(x + \alpha d) \leq f(x)$ for all $\alpha \geq 0$ [10, 15]. Since $f(x + \alpha d)$ is a polynomial function of $\alpha$, this implies that

$$\text{either} \quad \lim_{\alpha \to \infty} f(x + \alpha d) = -\infty \quad \text{or} \quad f(x + \alpha d) = f(x) \ \ \forall \ \alpha \in \Re.$$

If $\lim_{\alpha\to\infty} f(x+\alpha d) = -\infty$ for every $x \in \Re^n$, then clearly $f$ recedes below 0 along $d$ on $\Re^n$. Thus, it remains to consider the case where, for some $\bar{x} \in \Re^n$, $f(\bar{x}+\alpha d) = f(\bar{x})$ for all $\alpha \in \Re$. Since $f$ is convex, this implies $d \in L_f$ and hence, for *every* $x \in \Re^n$,

$$f(x+\alpha d) = f(x) \quad \forall\, \alpha \in \Re.$$

Thus, $f$ retracts along $d$ on $\Re^n$ and $f$ recedes below 0 along $d$ on $\mathrm{lev}_f(0)$ (which may be empty). ■

Suppose that $f_0$ satisfies Assumption 4 and $f_1, ..., f_r$ each satisfies Assumption 3. By Lemma 5, $f_0$ satisfies the relaxed assumption in Remark 2 following Prop. 1. By Lemmas 4 and 2, $f_0, f_1, ..., f_r$ satisfy Assumption 1(a), subject to the above modification. Thus, if they also satisfy Assumption 1(b), then by Prop. 1 and Remark 2 following it, (P) has a global minimum whenever its minimum value is finite. This result generalizes Cor. 2 and Thm. 3 in [13] which further assume $g$ and $f_1, ..., f_r$ are quadratic. Even in the special case of $f_0$ being convex polynomial and $f_1, ..., f_r$ being convex quadratic, our result appears to be new. Unlike the proof in [13], our proof does not rely on the canonical form of a quasi-convex quadratic function over a convex set.

Suppose that $f_0$ satisfies Assumption 4 and $f_1, ..., f_r$ are convex polynomial functions on $\Re^n$. By Lemma 5, $f_0$ satisfies the relaxed assumption in Remark 2 following Prop. 1. By Lemma 6, $f_0, f_1, ..., f_r$ satisfy Assumption 1, subject to the above modification. Thus, by Prop. 1 and Remark 2 following it, (P) has a global minimum whenever its minimum value is finite. This result generalizes Thm. 3 in [9], which further assumes $f_0$ to be convex on $\Re^n$.

Notice that, because $g$ is quasi-convex only on $X$, we cannot treat $Bx \le c$ as constraints but, rather, must incorporate it into the objective function. Then we exploit the fact that a recession direction for a polyhedral set can be retracted from points in the set that are sufficiently far out.

# 6 The Frank-Wolfe Theorem

In this section we assume that $f_0$ is a quadratic function over a polyhedral set, as was studied by Frank and Wolfe [11] and many others; see [9] and [13] for more detailed discussions.

**Assumption 5**

$$f(x) = \begin{cases} g(x) & \text{if } x \in X, \\ \infty & \text{otherwise,} \end{cases}$$

with $g(x) = \frac{1}{2}x^TQx + q^Tx$ for some symmetric $Q \in \Re^{n\times n}$ and $q \in \Re^n$, and

$$X = \{x \mid Bx \le b\},$$

for some $B \in \Re^{m\times n}, b \in \Re^m$.

The following lemma shows that $f_0$ satisfies the relaxed assumption in Remark 2 following Prop. 1 for $r = 0$.

**Lemma 7** Let $f : \Re^n \to (-\infty, \infty]$ satisfy Assumption 5. Then, for any AND $d$ of $f$, there exists $F \subseteq X$ such that (i) $f$ recedes below 0 along $d$ on $X \setminus F$, (ii) $f$ retracts along $d$ on $F$.

**Proof.** Fix any AND $d$ of $f$. Let $\{x^k\}$ be a US associated with $d$. Then $x^k \in X$ for all $k$ and $\lim_{k\to\infty} \sup g(x^k) \le 0$. Then

$$0 \ge \limsup_{k\to\infty} \frac{g(x^k)}{\|x^k\|^2} = \limsup_{k\to\infty} \frac{\frac{1}{2}(x^k)^TQx^k + q^Tx^k}{\|x^k\|^2} = \frac{1}{2}d^TQd.$$

Thus $d^TQd \le 0$. Also, $d \in X_\infty$ (see the proof of Lemma 5).

Define

$$F = \{x \in X \mid \nabla g(x)^Td \ge 0\}.$$

For any $x \in X \setminus F$, since $d^TQd \le 0$, we have

$$g(x + \alpha d) = g(x) + \alpha\nabla g(x)^Td + \frac{1}{2}\alpha^2 d^TQd \to -\infty \quad \text{as} \quad \alpha \to \infty.$$

Since $d \in X_\infty$, we also have $x + \alpha d \in X$ for all $\alpha \ge 0$. Thus $f$ recedes below 0 along $d$ on $X \setminus F$.

Also, we have using $d^TQd \le 0$ that

$$g(x - d) = g(x) - \nabla g(x)^Td + \frac{1}{2}d^TQd \le g(x) \quad \forall\, x \in F.$$

Since $X$ is polyhedral, we have that $x^k - d \in X$ for all $k$ sufficiently large (see the proof of Lemma 5). This implies that if $\{x^k\} \subseteq F$, then $f(x^k - d) \le f(x^k)$ for all $k$ sufficiently large. Thus, $f$ retracts along $d$ on $F$. ∎

Suppose that $f_0$ satisfies Assumption 5 and $r = 0$. By Lemma 7, $f_0$ satisfies the relaxed assumption in Remark 2 following Prop. 1. Thus, by Prop. 1 and Remark 2 following it, (P) has a global minimum whenever its minimum value is finite. This is the classical Frank-Wolfe theorem [11].

In Sec. 1 of [9], it is mentioned that Andronov et al. [1] had extended the Frank-Wolfe theorem to the case of a cubic function over a polyhedral set. [It is known that the result does not extend to polynomial of degree 4 or higher [11].] Can this result be deduced from Prop. 1 similarly as the Frank-Wolfe theorem?

# 7 Further Applications

In this section, we present further applications of Prop. 1 and Lemma 2 and indicate their connection to existing results.

Following [6], [7], we say that a closed set $S \subseteq \Re^n$ is *asymptotically linear* if

$$\delta_S \in \mathcal{F},$$

where $\delta_S$ is the indicator function for $S$, i.e.,

$$\delta_S(x) = \begin{cases} 0 & \text{if } x \in S, \\ \infty & \text{otherwise.} \end{cases}$$

An example of such a set $S$ is the Minkowski sum of a compact set with a finite collection of polyhedral sets. The level set of a convex quadratic function is generally not asymptotically linear. We have as a corollary of Prop. 1 and Lemma 1 the following refinement of [6, Thm. 3] and [7, Cor. 3.4.3].

**Proposition 3** Consider any proper lsc $f : \Re^n \to (-\infty, \infty]$ and any closed asymptotically linear set $S \subseteq \Re^n$ such that $S \cap \text{dom} f \neq \emptyset$. Suppose that $\inf_{x \in S} f(x) = 0$ and

$$f_\infty(d) \geq 0 \quad \forall\, d \in S_\infty.$$

Suppose also that either $f \in \mathcal{F}$ or $f$ is constant along any AND of $f$. Then there exists an $x^* \in S$ with $f(x^*) = 0$.

**Proof.** Define

$$f_0(x) = f(x) + \delta_S(x) \quad \forall\, x \in \Re^n.$$

Then $f_0$ is proper, lsc, and $\inf_x f_0(x) = 0$. Moreover, $(f_0)_\infty(d) \geq f_\infty(d)$ for $d \in S_\infty$ and $(f_0)_\infty(d) = \infty$ otherwise. Thus, our assumption on $f_\infty(d)$ implies

$$(f_0)_\infty(d) \geq 0 \quad \forall\, d \in \Re^n. \tag{10}$$

Also, for any $\epsilon \in \Re$, $x \in \text{lev}_{f_0}(\epsilon)$ if and only if $x \in \text{lev}_f(\epsilon)$ and $x \in S$. For any nonzero $d \in \Re^n$, $(f_0)_\infty(d) = 0$ implies $f_\infty(d) = (\delta_S)_\infty(d) = 0$.

Suppose that $f \in \mathcal{F}$. Since $f$ and $\delta_S$ are in $\mathcal{F}$, the above observations show that $f_0$ is in $\mathcal{F}$. Then, by Lemma 1, (10), and Prop. 1 with $r = 0$, there exists an $x^* \in \Re^n$ with $f_0(x^*) = 0$.

Suppose instead that $f$ is constant along any AND of $f$. Fix any AND $d$ of $f_0$ and any associated US $\{x^k\}$. Then $d$ is an AND of $f$, so that $f$ is constant along $d$. Also, $x^k \in S$ for all $k$, implying $(\delta_S)_\infty(d) = 0$. Since $\delta_S \in \mathcal{F}$, this implies that there exists a $\bar{k}$ such that $x^k - d \in S$ for all $k \geq \bar{k}$. Also, $f(x^k - d) = f(x^k)$ for all $k$. Thus

$f_0(x^k - d) = f_0(x^k)$ for all $k \geq \bar{k}$. This shows that $f_0$ retracts along any AND of $f_0$. By Prop. 1 with $r = 0$, there exists an $x^* \in \Re^n$ with $f_0(x^*) = 0$. ■

If $S$ is further assumed to be asymptotically multipolyhedral, then instead of $f$ being constant along any AND of $f$, it suffices that $f$ linearly recedes along any AND of $f$ (see Sec. 4). The next lemma shows that if $f_0, f_1, ..., f_r$ are convex and in $\mathcal{F}$, then they satisfy Assumption 1(a).

**Lemma 8** Consider any proper lsc $f : \Re^n \to (-\infty, \infty]$ such that $f \in \mathcal{F}$ and $f$ is convex. Then, for each AND $d$ of $f$,

either (i) $f$ recedes below 0 along $d$ on $\mathrm{dom} f$

or (ii) $f$ retracts along $d$ and $f$ recedes below 0 along $d$ on $\mathrm{lev}_f(0)$.

**Proof.** Fix any AND $d$ of $f$. Then $f_\infty(d) \leq 0$. Since $f$ is convex, this implies that $f$ recedes below 0 along $d$ on $\mathrm{lev}_f(0)$. If $f_\infty(d) < 0$, then the convexity of $f$ implies that $f$ recedes below 0 along $d$ on $\mathrm{dom} f$. If $f_\infty(d) = 0$, then Lemma 1 implies that $f$ retracts along $d$. ■

Lemma 8 and Prop. 1 together yield the following existence result of Auslender [6, Thm. 2].

**Proposition 4** Suppose that $f_i : \Re^n \to (-\infty, \infty]$, $i = 0, 1, ..., r$, are proper, lsc, convex, and belong to $\mathcal{F}$. Also, suppose that $\mathrm{dom} f_0 = \mathrm{dom} f_i$, $i = 1, ..., r$, and $L(\gamma) \neq \emptyset$ for all $\gamma > 0$. Then $L(0) \neq \emptyset$.

If a proper lsc function $f : \Re^n \to (-\infty, \infty]$ satisfies Assumption 2 and (5), then the indicator of its 0-level set,

$$f^\circ(x) = \begin{cases} 0 & \text{if } f(x) \leq 0, \\ \infty & \text{otherwise,} \end{cases}$$

satisfies (5) (since $(f^\circ)_\infty(d) \geq 0$ for all $d \in \Re^n$) but may not satisfy Assumption 2. In particular, take $f(x) = \|Ax\|^2 + b^T x$ for some $A, b$ with $b \neq 0$. Then, we know from Sec. 4 that $f$ satisfies Assumption 2 and (5). However, for any $d \in \Re^n$ such that $Ad = 0$ and $b^T d < 0$, it is easily seen that $d$ is an AND of $f^\circ$, but $f^\circ$ does not linearly recede along $d$ (since $f^\circ(x + \alpha d) = 0$ for all $\alpha \geq 0$ while $f^\circ(x + \alpha d) = \infty$ for all $\alpha$ sufficiently negative). Thus, we cannot incorporate constraints into the objective by means of an indicator function and still satisfy Assumption 2 for the objective function $f_0$.

In [3, Thm. 2.4] (also see [5, Thm. 21]), Auslender considered the following problem

$$\text{minimize} \quad g(x) \quad \text{subject to} \quad x \in X, \tag{11}$$

where $X$ is an asymptotically multipolyhedral set in $\Re^n$ and $g : \Re^n \to (-\infty, \infty]$ is a proper lsc function. Auslender showed that if

$$\text{dom} g \cap X \neq \emptyset, \qquad g_\infty(d) \geq 0 \quad \forall\, d \in X_\infty, \tag{12}$$

and either (i) $g$ is constant along each $d \in X_\infty \cap R_g$ or (ii) epi$g$ is an asymptotically multipolyhedral set or (iii) $g$ is *weakly coercive*, then (11) has a global minimum. Notice that (iii) implies (i). This is because $g$ being weakly coercive means that $g_\infty(d) \geq 0$ for all $d \in \Re^n$ and, for any $d \in R_g$, $g$ is constant along $d$.

Auslender's existence result generalizes one of Rockafellar [15, Thm. 27.3] for the case where $X$ is a polyhedral set, $g$ is convex, and $g$ is constant along each $d \in X_\infty \cap R_g$. In the case where $g$ and $X$ are convex, it can be seen that (11) having finite minimum value implies (12), but not conversely (e.g., $g(x) = -\ln(x)$ if $x > 0$ and $g(x) = \infty$ otherwise satisfies $g_\infty(d) \geq 0$ for all $d \in \Re$). Thus, in this case, the assumption (12) is weaker than assuming (11) has a finite minimum value. In general, neither implies the other. We show below that Auslender's result may be viewed as a special case of Prop. 2.

**Proposition 5** Consider the problem (11). Suppose that (12) holds and either (i) $g$ linearly recedes along each $d \in X_\infty \cap R_g$ or (ii) epi$g$ is an asymptotically multipolyhedral set. Then (11) may be reformulated in the form of (P) while satisfying the assumptions of Prop. 2. Hence (11) has a global minimum.

**Proof.** Suppose that $g$ linearly recedes along each $d \in X_\infty \cap R_g$. Since $X$ is asymptotically multipolyhedral, $X = S + K$, where $S$ is a compact set in $\Re^n$ and $K = \cup_{j=1}^{\ell} K^j$ with each $K^j$ being a polyhedral cone. Then, $x \in X$ if and only if $x = x_1 + x_2$ with $x_1 \in S$ and $x_2 \in K$, so (11) is equivalent to

$$\text{minimize} \quad f_0(x_1, x_2) \quad \text{subject to} \quad f_1(x_1, x_2) \leq 0,$$

where

$$f_0(x_1, x_2) = \begin{cases} g(x_1 + x_2) & \text{if } x_1 \in S, \\ \infty & \text{otherwise,} \end{cases}$$

$$f_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \in K, \\ \infty & \text{otherwise.} \end{cases}$$

Notice that $f_0$ and $f_1$ are proper, lsc. We now verify that this problem satisfies the assumptions of Prop. 2, and hence it has a global minimum. By (12), dom$g \cap X \neq \emptyset$, so that $D = \mathrm{dom} f_0 \cap C_1 \neq \emptyset$. Fix any AND $(d_1, d_2)$ of $f_0, f_1$. Then $d$ is a recession direction of $f_i$, i.e., $(f_i)_\infty(d_1, d_2) \leq 0$, for $i = 1, 2$. We have

$$\begin{aligned}
(f_0)_\infty(d_1, d_2) &= \liminf_{\substack{d_1' \to d_1, d_2' \to d_2 \\ t \to \infty}} \frac{f_0(td_1', td_2')}{t} \\
&= \begin{cases} \displaystyle\liminf_{\substack{d_1' \to 0, d_2' \to d_2 \\ t \to \infty}} \frac{g(t(d_1' + d_2'))}{t} & \text{if } d_1 = 0, \\ \infty & \text{otherwise,} \end{cases} \\
&= \begin{cases} g_\infty(d_2) & \text{if } d_1 = 0, \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Similarly,

$$(f_1)_\infty(d_1, d_2) = \begin{cases} 0 & \text{if } d_2 \in K, \\ \infty & \text{otherwise.} \end{cases}$$

This implies $d_1 = 0$, $d_2 \in K$, and $g_\infty(d_2) \leq 0$. Thus, $d_2 \in X_\infty \cap R_g$ and our assumption on $g$ implies $g_\infty(d_2) \geq 0$ and $g$ linearly recedes along $d_2$. The latter implies there exists $\theta \leq 0$ such that

$$g(x + \alpha d_2) = g(x) + \alpha\theta \quad \forall\, \alpha \in \Re,\ \forall\, x \in \mathrm{dom} g.$$

Since $g_\infty(d_2) \geq 0$, then $\theta = 0$. This and $d_1 = 0$ imply that

$$\begin{aligned}
f_0((x_1, x_2) + \alpha(d_1, d_2)) &= g(x_1 + x_2 + \alpha d_2) \\
&= g(x_1 + x_2) \\
&= f_0(x_1, x_2) \qquad \forall\, \alpha \in \Re,\ \forall\, (x_1, x_2) \in \mathrm{dom} f_0.
\end{aligned}$$

Thus $f_0$ is constant (and hence retracts strongly) along $(d_1, d_2)$. Since $(d_1, d_2)$ is an AND of $f_1$, there exists a US $\{(x_1^k, x_2^k)\}$ such that $x_2^k \in K$ for all $k$, $\|(x_1^k, x_2^k)\| \to \infty$, and $(x_1^k, x_2^k)/\|(x_1^k, x_2^k)\| \to (d_1, d_2)$. Since $d_1 = 0$, then $\|x_2^k\| \to \infty$ and $x_2^k/\|x_2^k\| \to d_2$. Since $d_2 \in K$ and $K$ is the union of polyhedral cones, there exists $\bar{k} \geq 0$ such that

$$x_2^k - d_2 \in K \quad \forall\, k \geq \bar{k}$$

(see the proof of Lemma 5). Then

$$f_1((x_1^k, x_2^k) - (d_1, d_2)) = f_1(x_1^k, x_2^k - d_2) = 0 \quad \forall\, k \geq \bar{k}.$$

Thus $f_1$ retracts along $(d_1, d_2)$.

Suppose that epi$g$ is an asymptotically multipolyhedral set. Then (11) is equivalent to

$$\text{minimize} \quad h(x, \mu) \quad \text{subject to} \quad (x, \mu) \in Y,$$

where $h(x,\mu) = \mu$ and $Y = \text{epi}g \cap (X \times \Re)$. The assumption (12) implies $\text{dom}h \cap Y \neq \emptyset$ and $h_\infty(d,\delta) = \delta \geq 0$ for all $(d,\delta) \in Y_\infty$. Also, $\delta \leq 0$ for any $(d,\delta) \in R_h$, which implies $\delta = 0$ so that $h$ is constant along $(d,\delta)$. It can be shown that the intersection of two asymptotically multipolyhedral sets is also asymptotically multipolyhedral. Thus case (ii) reduces to case (i). ∎

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