Approximation Algorithms for Conic Programs with Extreme Ray Constraints

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Abstract

Consider a problem of minimizing a linear function subject to linear constraints and an additional constraint that the variables lie in the extreme rays of a nonempty closed convex cone K. Dropping the extreme ray constraint yields a convex conic program relaxation of this problem. We study approximation algorithms based on this relaxation. Examples with K being the semidefinite cone or the Cartesian product of second-order cones are studied.

Key words. Conic program relaxation, extreme rays, semidefinite program, second-order cone program, approximation algorithm.

1 Introduction

For solving difficult nonconvex optimization problems, one approach that has gained much attention in the last decade is based on relaxing certain constraints to obtain a convex optimization problem and then modifying a solution of the convex problem to obtain an approximately optimal solution of the original problem. This approach has been particularly successful for certain NP-hard combinatorial optimization problems and nonconvex quadratic optimization problems, which are formulated as a problem of minimizing a linear function subject to linear constraints and an additional constraint that the variables form a rank-1 symmetric positive semidefinite real matrix [3, 4, 5, 8, 9, 10, 12, 14, 15, 17, 18, 20, 21]. Relaxing the rank-1 constraint yields a convex optimization problem, called *semidefinite program* (SDP), which is known to be efficiently solvable [1, 11, 12]. Since rank-1 matrices form the extreme rays of the *semidefinite cone* $\mathcal{S}^n_+$ of $n \times n$ symmetric positive semidefinite real matrices, it is natural to consider a general conic program with extreme ray constraints and to seek analogous approximation algorithms based on relaxing the extreme ray constraint to obtain a convex conic program. This is the topic of our study.

Let $\mathcal{H}$ be a finite-dimensional real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Let K be a closed convex cone in $\mathcal{H}$. Consider the following conic program with extreme ray constraints:

$$\begin{array}{rll} v_{\text{ECP}} := & \min & \langle b^0, x \rangle \\ & \text{s.t.} & \langle b^k, x \rangle \le h^k, \ k = 1, ..., m, \\ & & x \in \text{extr}K, \end{array} \tag{1}$$

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where $h^k \geq 0$, $b^k \in \mathcal{H}$ for $k = 0, 1, ..., m$. By scaling the inequality constraints, we can assume that $h^k \in \{0, 1\}$. Here and throughout, $\text{extr}K$ denotes the set of elements in the extreme rays of K. This problem is NP-hard in general.

By relaxing the extreme ray constraints in (1), we obtain the following relaxation which is a convex conic program:

$$\begin{array}{rll} v_{\text{CP}} & := \min & \langle b^0, x \rangle \\ & \text{s.t.} & \langle b^k, x \rangle \leq h^k, \ k = 1, ..., m, \\ & & x \in K. \end{array} \tag{2}$$

Clearly $v_{\text{CP}} \leq v_{\text{ECP}}$. We are interested in upper bounds on performance of the form

$$v_{\text{ECP}} \leq C \, v_{\text{CP}},$$

where $C > 0$. Of particular interest is the case where K is a Cartesian product of second-order cones, also known as Lorentz cones, for which (2) is a *second-order cone program* (SOCP) and is known to be easier to solve than SDP; see [2] and references therein. Thus, approximation algorithms based on SOCP relaxation can handle larger problems than those based on SDP relaxation. Unlike SDP relaxation, SOCP relaxation has been little studied thus far. Kim and Kojima [6], Kim, Kojima, Yamashita [7] considered alternative approachs of relaxing SDP constraints to SOC constraints. However, it is unclear whether performance bounds can be derived in their approach.

A key to approximation algorithms is the construction of a feasible but approximately optimal solution of the original problem from an optimal solution of the relaxed problem. For SDP relaxation, two approaches have been used to make this construction. One is based on random separating hyperplane, as was proposed by Goemans and Williamson [5] and used by many others. Another is based on decomposing a symmetric positive semidefinite matrix into the sum of rank-1 matrices that lie on the same side of a given hyperplane, as was proposed recently by Sturm and Zhang [15]. We will follow the second approach of Sturm and Zhang. By Carathéodory's theorem [13], any $x \in K$ can be decomposed as the sum of a finite number of nonzero elements of $\text{extr}K$. We will call such a decomposition a *C-decomposition* and call the number of nonzero elements the *rank*. We make the following assumption on K.

Assumption 1 *There exists an integer* $\text{rank}K > 0$ *such that, for any* $x \in K$*, we can construct a C-decomposition of* x *whose rank is at most* $\text{rank}K$*.*

For $K = \mathcal{S}^n_+$, we have $\text{rank}K = n$. This is because any symmetric positive semidefinite real matrix of rank r can be decomposed into the sum of r matrices of rank 1. For K being the Cartesian product of Lorentz/second-order cones of dimension 2 or greater, we have $\text{rank}K = 2$. We consider a second assumption on K that in addition assumes that the elements of the C-decomposition can be chosen to lie in the same halfspace as x.

Assumption 2 *For any* $b \in \mathcal{H}$ *and any* $x \in K$ *satisfying* $\langle b, x\rangle \le 0$*, we can construct* $x^1, ..., x^p \in \mathrm{extr}K$*, for some* $p \ge 1$*, satisfying*

$$x = \sum_{j=1}^{p} x^j, \qquad \langle b, x^j\rangle \le 0, \quad j = 1, ..., p. \tag{3}$$

Not all closed convex cones satisfy Assumption 2. For example, the orthant $\Re^2_+$ does not. However, it was shown by Sturm and Zhang [15] that this assumption is satisfied by the semidefinite cone $\mathcal{S}^n_+$ $(n \ge 2)$. They proposed a construction procedure requiring $O(n^2)$ iterations, with each iteration executable in a polynomial number of arithmetic/root operations. We will see that Assumption 2 is also satisfied by the product of Lorentz/second-order cones of dimension 3 or greater.

We will further make the following assumptions:

Assumption 3 **(a)** b^k *belongs to the dual cone* $K^* := \{y \in \mathcal{H} : \langle y, x\rangle \ge 0 \ \forall x \in K\}$*, for* $k = 1, ..., m$*.*

(b) (2) *has an optimal solution* x^**.*

Assumption 3(a) is motivated by the SDP relaxation of minimizing a nonconvex quadratic function subject to ellipsoid constraints [9, 16, 20, 19]. In the case where the ellipsoids have a common center, the SDP relaxation has the form (2) with $b^k \in \mathcal{S}^n_+ = (\mathcal{S}^n_+)^*$, $k = 1, ..., m$. Assumption 3(b) is mild. If, in addition to Assumptions 2 and 3(a), $\{d \in \mathrm{extr}K : \langle b^k, d\rangle \le 0, k = 0, 1, ..., m\} = \{0\}$, then (feasible set of (2)) $\cap \{x : \langle b^0, x\rangle \le 0\}$ is nonempty and bounded so that Assumption 3(b) holds.[2]

Theorem 1 *Under Assumptions 1, 2 and 3(b), we can construct from an optimal solution* x^* *of* (2) *a C-decomposition of rank at most* $\min\{m-1, \mathrm{rank}K\}$*.*

Proof. The proof is nearly identical to the proof of Ye [19, Theorem 2] for the case of $K = \mathcal{S}^n_+$. ■

Theorem 2 *Under Assumptions 1, 2 and 3, we can construct from an optimal solution* x^* *of* (2) *a feasible solution* x *of* (1) *satisfying*

$$\langle b^0, x\rangle \le \frac{1}{r} v_{\mathrm{CP}} \le \frac{1}{r} v_{\mathrm{ECP}}, \tag{4}$$

where $r = \min\{m-1, \mathrm{rank}K\}$*.*

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[2] In particular, $x = 0$ is a feasible solution of (2) with $\langle b^0, x\rangle \le 0$. Thus, $\mathcal{X}^0 :=$ (feasible set of (2)) $\cap \{x : \langle b^0, x\rangle \le 0\}$ is nonempty. For any $x \in \mathcal{X}^0$, Assumption 1 yields that $x = \sum_{j=1}^p x^j$ for some $x^j \in \mathrm{extr}K$ satisfying $\langle b^0, x^j\rangle \le 0$. Since $\sum_{j=1}^p \langle b^k, x^j\rangle = \langle b^k, x\rangle \le h^k$, Assumption 3(a) implies $\langle b^k, x^j\rangle \le h^k$ for $j = 1, ..., p$, $k = 1, 2, ..., m$. If x^j is unbounded for some j, then dividing by $\|x^j\|_2$ and taking limit yields a cluster point $d \in \mathrm{extr}K$ of $x^j/\|x^j\|_2$ satisfying $\|d\|_2 = 1$, $\langle b^k, d\rangle \le 0$, $k = 0, 1, ..., m$. Thus, if $\{d \in \mathrm{extr}K : \langle b^k, d\rangle \le 0, k = 0, 1, ..., m\} = \{0\}$, then $\mathcal{X}^0$ is bounded.

Proof. The proof is nearly identical to the proof of Ye [19, Theorem 3] for the case of $K = \mathcal{S}^n_+$. ■

Throughout, $\Re^n$ denotes the space of n-dimensional column vectors, $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices, and ^{T} denotes transpose. For $x \in \Re^n$, x_j denotes jth component of x and $\|x\|_p = \left(\sum_j |x_j|^p\right)^{1/2}$. Also, ":=" means "define".

2 Semidefinite Cone

Suppose

$$\mathcal{H} = \mathcal{S}^n, \quad K = \mathcal{S}^n_+, \quad \langle x, y \rangle = \mathrm{tr}[x^T y]. \tag{5}$$

As is well known, $K^* = K$, $\mathrm{extr}K = \{x : x = uu^T \text{ for some } u \in \Re^n\}$, and $\mathrm{rank}K = n$.

Lemma 1 *If* (5) *holds, then Assumption 2 is satisfied.*

Proof. This is a result of Sturm and Zhang [15, Procedure 1, Proposition 3]. ■

3 Norm-Epigraph Cones

Suppose

$$\mathcal{H} = \Re^n, \quad K = K_1 \times \cdots \times K_J, \quad \langle x, y \rangle = x^T y. \tag{6}$$

where $n = n_1 + \cdots + n_J$,

$$K_j := \{(x_1, x_2) \in \Re \times \Re^{n_j - 1} : x_1 \geq \|x_2\|_{p_j}\},$$

and $n_j \geq 2$ and $1 < p_j < \infty$, for $j = 1, ..., J$. [$\|x\|_p$ denotes the p-norm of x.] Thus, K_j is the epigraph of the p_j-norm function. If $p_j = 2$ for all j, then K is the Cartesian product of Lorentz/second-order cones. It can be verified that $K^* = K_1^* \times \cdots \times K_J^*$ with

$$K_j^* = \{(x_1, x_2) \in \Re \times \Re^{n_j - 1} : x_1 \geq \|x_2\|_{q_j}\},$$

where $1/p_j + 1/q_j = 1$. Also, $\mathrm{extr}K = \mathrm{extr}K_1 \times \cdots \times \mathrm{extr}K_J$, with $\mathrm{extr}K_j = \{(x_1, x_2) \in \Re \times \Re^{n_j - 1} : x_1 = \|x_2\|_{p_j}\}$. It is not difficult to see that $\mathrm{rank}K^j = 2$ for all j, so that $\mathrm{rank}K = 2$. Also, the following result can be shown.

Lemma 2 *If* (6) *holds and* $n_j \geq 3$, $1 < p_1 < \infty$ *for all* j, *then Assumption 2 is satisfied.*

Proof. Fix any $b = (b^1, ..., b^J) \in \Re^{n_1} \times \cdots \times \Re^{n_J}$ and any $x = (x^1, ..., x^J) \in K_1 \times \cdots \times K_J$ satisfying $\langle b, x \rangle \leq 0$. For each $j \in \{1, ..., J\}$, write $x^j = (x_1^j, x_2^j) \in \Re \times \Re^{n_j - 1}$ and $b^j = (b_1^j, b_2^j) \in \Re \times \Re^{n_j - 1}$.

Case 1. Suppose $x_1^1 > \|x_2^1\|_{p_1}$. Since $n_1 \geq 3$, there exists nonzero $d_2 \in \Re^{n_1-1}$ such that ${b_2^1}^T d_2 = 0$. Then, there exists $\lambda_1 \in \Re_{++}$ satisfying $x_1^1 = \|x_2^1 + \lambda_1 d_2\|_{p_1}$ and there exists $\lambda_2 \in \Re_{++}$ satisfying $x_1^1 = \|x_2^1 - \lambda_2 d_2\|_{p_1}$. Then,

$$x_2^1 = \frac{\lambda_2}{\lambda_1 + \lambda_2}(x_2^1 + \lambda_1 d_2) + \frac{\lambda_1}{\lambda_1 + \lambda_2}(x_2^1 - \lambda_2 d_2),$$

implying $x^1 = y^1 + z^1$, where

$$y^1 := \bar{\lambda}(x_1^1, x_2^1 + \lambda_1 d_2), \quad z^1 := (1-\bar{\lambda})(x_1^1, x_2^1 - \lambda_2 d_2), \quad \bar{\lambda} := \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Moreover, $y^1 \in \text{extr}K_1$ and

$$(b^1)^T y^1 = \bar{\lambda}(b_1^1 x_1 + {b_2^1}^T(x_2^1 + \lambda_1 d_2)) = \bar{\lambda}(b_1^1 x_1 + {b_2^1}^T x_2) = \bar{\lambda}(b^1)^T x^1.$$

Similarly, we have that $z^1 \in \text{extr}K_1$ and $(b^1)^T z^1 = (1-\bar{\lambda})(b^1)^T x^1$.
Case 2. Suppose $x_1^1 = \|x_2^1\|_{p_1}$, then $x^1 \in \text{extr}K_1$. Let $y^1 := x^1/2$, $z^1 := x^1/2$, $\bar{\lambda} := 1/2$.

In both cases, we have $\bar{\lambda} \in (0,1)$ and

$$y^1 + z^1 = x^1, \quad (b^1)^T y^1 = \bar{\lambda}(b^1)^T x^1, \quad (b^1)^T z^1 = (1-\bar{\lambda})(b^1)^T x^1.$$

Let

$$y^j := \bar{\lambda} x^j,\ z^j := (1-\bar{\lambda})x^j\ \forall j \neq 1, \quad y := (y^1, ..., y^J),\ z := (z^1, ..., z^J).$$

Then y and z satisfy

$$x = y + z, \quad y^1, z^1 \in \text{extr}K_1, \quad \langle b, y\rangle = \bar{\lambda}\langle b, x\rangle \leq 0, \quad \langle b, z\rangle = (1-\bar{\lambda})\langle b, x\rangle \leq 0.$$

Now we repeat the above decomposition for y and z separately, working with $b^2 = (b_1^2, b_2^2)$ and K_2 instead. This yields $t^2, u^2, v^2, w^2 \in \text{extr}K_2$ and $\bar{\mu}, \bar{\nu} \in (0,1)$ satisfying

$$t^2 + u^2 = y^2, \quad (b^2)^T t^2 = \bar{\mu}(b^2)^T y^2, \quad (b^2)^T t^2 = (1-\bar{\mu})(b^2)^T y^2,$$

$$v^2 + w^2 = z^2, \quad (b^2)^T v^2 = \bar{\nu}(b^2)^T z^2, \quad (b^2)^T w^2 = (1-\bar{\nu})(b^2)^T z^2.$$

Let

$$t^j := \bar{\mu} y^j,\ u^j := (1-\bar{\mu})y^j\ \forall j \neq 2, \quad t := (t^1, ..., t^J),\ u := (u^1, ..., u^J).$$

$$v^j := \bar{\nu} z^j,\ w^j := (1-\bar{\nu})z^j\ \forall j \neq 2, \quad v := (v^1, ..., v^J),\ w := (w^1, ..., w^J).$$

Then t, u, v and w satisfy

$$y = t + u, \quad t^1, u^1 \in \text{extr}K_1,\ t^2, u^2 \in \text{extr}K_2, \quad \langle b, t\rangle = \bar{\mu}\langle b, y\rangle \leq 0, \quad \langle b, u\rangle = (1-\bar{\mu})\langle b, y\rangle \leq 0.$$

$$z = v + w, \quad v^1, w^1 \in \text{extr}K_1,\ v^2, w^2 \in \text{extr}K_2, \quad \langle b, v\rangle = \bar{\nu}\langle b, z\rangle \leq 0, \quad \langle b, w\rangle = (1-\bar{\nu})\langle b, z\rangle \leq 0.$$

Then we repeat the above decomposition for t, u, v, w separately, working with $b^3 = (b_1^3, b_2^3)$ and $x^3 = (x_1^3, x_2^3)$ instead, and so on. Continuing in this manner, we obtain a C-decomposition of x of rank at most 2^J, with the elements of the decomposition making an obtuse angle with b. ■

The proof of Lemma 2 is constructive and the construction procedure requires $O(2^J)$ iterations, with each iteration executable in a polynomial number of arithmetic operations. Thus, the procedure is polynomial time if $J = O(\log n + \log m)$. This lemma is illustrated in the figure below.

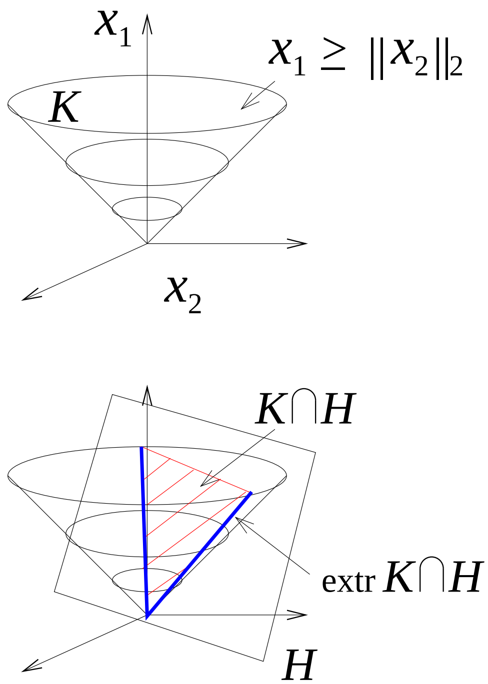

Figure 1: Illustration of Lemma 2 for $J = 1$, $p_1 = 2$, $n_1 = 3$. $H = \{x = (x_1, x_2) : \langle b, x \rangle = 0\}$. Any $x \in K$ can be expressed as the sum of some $y, z \in \text{extr}K$ lying on the same side of H as x.

4 SOCP Relaxation

Consider the following problem

$$\begin{array}{ll} \min & \sum_{j=1}^J c_1^j \|x_2^j\|_2 + (c_2^j)^T x_2^j \\ \text{s.t.} & \sum_{j=1}^J a_1^{ij} \|x_2^j\|_2 + (a_2^{ij})^T x_2^j \leq b_i, \qquad i = 1, ..., m, \end{array} \tag{7}$$

where $c_1^j, a_1^{ij} \in \Re$ and $c_2^j, a_2^{ij} \in \Re^{n_j-1}$ satisfy $a_1^{ij} \geq \|a_2^{ij}\|_2$ for all i, j. This problem is nonconvex. It can be written in the form (1) by substituting $x_1^j = \|x_2^j\|_2$:

$$\begin{array}{lll} \min & \sum_{j=1}^J (b^{0j})^T x^j & \\ \text{s.t.} & \sum_{j=1}^J (b^{ij})^T x^j \leq b_i, & i = 1, ..., m, \\ & x^j \in \text{extr} K^j, & j = 1, ..., J, \end{array} \tag{8}$$

where $b^{0j} = (c_1^j, c_2^j)$, $b^{ij} = (a_1^{ij}, a_2^{ij})$, $x^j = (x_1^j, x_2^j)$, $K^j = \{(x_1, x_2) \in \Re \times \Re^{n_j-1} : x_1 \geq \|x_2\|_2\}$. Moreover, $b^{ij} \in K^j = (K^j)^*$ for all $i = 1, ..., m$, $j = 1, ..., J$, so Assumption 3(a) is satisfied, where $K = K_1 \times \cdots \times K_J$. The corresponding relaxation (2) is a SOCP.

If $n_j \geq 3$ for all j, then Lemma 2 implies that Assumption 2 is satisfied. Thus, if the relaxed problem (2) has an optimal solution, by Theorem 2 and $\text{rank} K = 2$, we can construct a feasible solution x of (8) whose objective value is within a factor of

$$\frac{1}{\min\{m-1, 2\}}$$

of the optimal objective value of (8). If $n_j \geq 2$ for all j, then the performance bound worsens to $1/2$. Notice that the above result still holds if 2-norm is replaced by p-norm with $1 < p < \infty$. However, the relaxed problem may be more difficult to solve.

5 Further Extensions

An important variant of (1) entails the addition of a normalization constraint:

$$\begin{array}{rll} v_{\text{ECP}'} := & \min & \langle b^0, x \rangle \\ & \text{s.t.} & \langle b^k, x \rangle \leq h^k, \ k = 1, ..., m, \\ & & x \in \text{extr} K \cap L, \end{array} \tag{9}$$

where $b^{m+1} \in K^*$ and

$$L := \{x : \langle b^{m+1}, x \rangle = 1\}.$$

This class of problem arises, for example, in minimizing a nonconvex quadratic function subject to ellipsoid constraints, where the ellipsoids do not have a common center [16].

Suppose the relaxation of (9), namely

$$\begin{array}{rll} v_{\text{CP}'} := & \min & \langle b^0, x \rangle \\ & \text{s.t.} & \langle b^k, x \rangle \leq h^k, \ k = 1, ..., m, \\ & & x \in K \cap L, \end{array}$$

has an optimal solution x^*. We have from $\langle b^{m+1}, x^* \rangle = 1$ that $\langle b^0 - v_{\text{CP}'} b^{m+1}, x^* \rangle = 0$. Under Assumption 2, we can find $x^j \in \text{extr} K$, $j = 1, ..., p$, such that

$$x^* = \sum_{j=1}^p x^j \quad \text{and} \quad \langle b^0 - v_{\text{CP}'} b^{m+1}, x^j \rangle \leq 0, \ j = 1, ..., p.$$

[In fact, the inequalities are satisfied with equalities.]

As in [16], this yields the existence of $\bar{j} \in \{1, ..., p\}$ such that

$$\langle b^{m+1}, x^{\bar{j}} \rangle > 0 \quad \text{and} \quad \sum_{k:h^k=1} \frac{\langle b^k, x^{\bar{j}} \rangle}{\langle b^{m+1}, x^{\bar{j}} \rangle} \le \kappa,$$

where $\kappa := \mathrm{Card}\{k \in \{1, ..., m\} : h^k = 1\}$. Thus, letting $\hat{x} := x^{\bar{j}} / \langle b^{m+1}, x^{\bar{j}} \rangle$, we obtain that

$$\hat{x} \in \mathrm{extr} K \cap L, \quad \langle b^k, \hat{x} \rangle = 0 \ \text{ if } h^k = 0, \quad \sum_{k:h^k=1} \langle b^k, \hat{x} \rangle \le \kappa, \quad \langle b^0, \hat{x} \rangle \le \upsilon_{\mathrm{CP}'}. \tag{10}$$

Since Assumption 3(a) implies $\langle b^k, \hat{x} \rangle \ge 0$ for $k = 1, ..., m$, (10) yields $\langle b^k, \hat{x} \rangle \le \kappa$. Then $\hat{x}$ is a feasible solution of (9) except that $\langle b^k, \hat{x} \rangle$ might exceed 1 by a factor of κ for those k with $h^k = 1$. We need to modify $\hat{x}$ to obtain a feasible solution of (9) whose objective value is not too far from that of $\hat{x}$.

In the case of (5) and

$$b^{m+1} = e_n e_n^T, \qquad e_n = n\text{th coordinate vector in } \Re^n,$$

for which (9) corresponds to minimizing a nonconvex quadratic function subject to ellipsoid constraints, if there exists a feasible solution $\bar{x}$ of (9) with $\langle b^0, \bar{x} \rangle \le 0$ and $\langle b^k, \bar{x} \rangle < 1$ whenever $h^k = 1$, then we can construct from $\hat{x}$ and $\bar{x}$ a feasible solution x of (9) satisfying

$$\langle b^0, x \rangle \le \frac{(1 - \sqrt{\varpi})^2}{(\sqrt{\kappa} + \sqrt{\varpi})^2} \, \upsilon_{\mathrm{CP}'} \ ,$$

where $\varpi := \max_{k:h_k=1} \langle b^k, \bar{x} \rangle$ [16].

In the case of (6), we are currently studying ways to analogously construct a feasible solution of (9) from $\hat{x}$ and a known strictly feasible solution of (9).

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