Efficient computation of multiple solutions in quasibrittle fracture analysis

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Abstract

We propose a novel and efficient computational scheme for capturing the multiplicity of solutions that may exist at any loading step of a quasibrittle fracture analysis formulated as a linear complementarity problem (LCP). The algorithm proposed is based on successively augmenting the current LCP by complementarity contraints that effectively remove previously found solutions from the solution set. We focus on the single-step or so-called holonomic analysis for mode I behavior. The performance of the proposed approach is illustated by means of physically meaningful benchmark problems.

1 Introduction

The computational analysis of structures made of quasibrittle materials is not only an important task but also a challenging one in view of constitutive instabilities that are present. In particular, such materials exhibit unstable local softening which can result in such mechanically crucial and well-known global (i.e., at the structural level) phenomena as bifurcation and snap-back, including lack of uniqueness of response to a given loading history (see e.g. [1]-[5]).

A widely-used simulation for fracture processes is the one based on kinematic discontinuities endowed with an interface softening law. A simple, yet powerful, instance

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of such an idealization is the discrete fictitious or cohesive crack model which has its genesis in the work of Dugdale [6], Barenblatt [7], Hillerborg *et al.* [8] and Carpinteri [9]. This model has the following key features: a locus of potential displacement discontinuity (referred to as "craze" or "process zone") ahead of the fully formed crack; a softening law relating tractions to relative displacements across the discontinuity locus; and linear elasticity outside the process zone.

Complementarity is a particularly elegant and powerful mathematical structure that can be used to describe such softening laws [10, 11, 12]. The implied perpendicularity of two sign-constrained vectors is not surprising since contact-like conditions apply at the crack interfaces. For instance, the stress is zero at an open crack and nonnegative (compression is assumed positive) at a closed crack. In mode I fracture, as is essentially assumed in the present work, piecewise linearity of the softening laws is a good assumption that is corroborated by experimental evidence (e.g. [13]).

One of two types of fracture analyses needs to be performed: holonomic or nonholonomic. Holonomic behavior (analogous to the deformation theory of plasticity), it must be recalled, implies path-independence and reversibility. This applies, for instance, to engineering situations where the main actions are monotonically increasing in time and manifestations of irreversibility (i.e., local elastic unloading) can be *a priori* deemed to play a minor role on the essential behavior of the structure. Nonholonomic behavior (analogous to the flow theory of plasticity), on the other hand, involves significant elastic unloading phenomena so that a time-stepping scheme based on a rate formulation is required, instead of a formulation in total quantities.

Notwithstanding whether holonomy or nonholonomy is more appropriate, one (single-step holonomic analysis) or more (multi-step nonholonomic analysis) linear complementarity problems (LCPs) must be solved if piecewise linear softening laws are adopted. We recall that the LCP is a mathematical programming problem [14] which consists of finding a vector, in a finite-dimensional real vector space, that satisfies a system of inequalities and certain complementarity conditions. In standard form, this is represented as

$$0 \le f = Mx + q \quad \perp \quad x \ge 0 \tag{1}$$

where the vector $q \in \Re^n$ and the matrix $M \in \Re^{n\times n}$ are given. It is required to find a vector $x \in \Re^n$ satisfying (1) or to show that no such x exists. We adopt the symbol $\perp$ to denote complementarity; for nonnegative vectors f and x this implies the componentwise condition $f_i x_i = 0$ for all i; and, for brevity, we will henceforth refer to (1) as LCP(M, q).

After more than three decades of research, the study of the LCP by the mathematical programming community [14] has become in its own right a well-established and fruitful discipline within mathematical programming. In addition to the expected desire by mathematical programmers to develop robust and efficient algorithms for solving such problems, much of the research effort has been concentrated on the theoretical study of various matrix classes to which M belongs. Indeed, almost all the proposed algorithms for numerical solution are based on the assumption that M belongs to a particular matrix class.

Unfortunately, in the presence of constitutive instabilities as would be the case for our fracture problem, the LCP which needs to be solved is computationally challenging for two reasons. Firstly, M turns out to be in general nonsymmetric and indefinite with the consequence that there are currently no known algorithm guaranteed to process successfully this class of LCPs. Secondly, for a typical step of either a holonomic or nonholonomic analysis, we would like to numerically capture *all* solutions or to show that none exists. This largely open problem of obtaining efficiently and robustly all solutions to the often large-size LCPs that arise in fracture analyses forms the focus of the present paper. In essence, we propose a novel scheme to achieve this. We further note that the solutions to the LCP are expected to be isolated and we assume in our algorithm that they are indeed so.

Essential to our approach is the availability of an LCP solver capable of providing a solution to the often large-size LCPs arising in fracture problems. In spite of the above statement regarding the lack of algorithm guaranteed to solve our particular LCP, we have had complete success with the industry-strength mixed complementarity problem solver PATH [15] which, moreover, is only designed to solve for one solution. We have found PATH to be extremely fast and robust, even for large-size LCPs—an essential requirement for our purposes.

Our previous attempts to capture solution multiplicity have been partially successful [16, 17]. For instance, our implementation of the enumerative method of Judice and Mitra [18] could only process LCPs of sizes up to about $n = 50$. This is not unexpected since it requires exhaustive exploration of a binary tree with $2^{n+1} - 1$ nodes, albeit with the provision of various heuristics to fathom nodes (leading to a termination of the relevant branches). Larger size problems (up to about $n = 150$) were solved by simply starting PATH from different initial vectors x, judiciously chosen from a knowledge of expected crack patterns (and number of solutions). For problems with unknown crack patterns, this approach is not satisfactory as solutions can easily be left out. We have also observed that, with an increase in robustness of the PATH solver after each revision, there is more tendency for the same solution to be found with the different trial vectors, with some solutions again being missed. Interestingly, a recent paper by von Stengel *et al.* [19], in the context of Nash equilibria of two-person games, suggests precisely the same technique of varying the starting points of the adopted LCP solver to capture the several equilibria that may exist. A third scheme we have attempted but without any success on LCPs of more than about $n = 20$ is the method proposed by De Moor *et al.* [20] who implicitly generate all solutions to LCPs arising in piecewise linear resistive electrical circuits. Their method essentially alternates the construction of vertices of the polytope describing the constraints with a verification of their complementarity. A partial enumeration method, based on solving a series of set covering problems, was proposed by Murty to find all solutions of a certain type of LCP [21, Section 11.3]. This method has not been computationally tested.

The idea underpinning the method we propose in this paper is simple: by adding suitable constraints we successively eliminate each solution from the original LCP solution set as it is detected. We thus need to solve a series of LCPs of increasing sizes, until no solution is found. The requirement of a robust, fast and large-scale solver is thus evident.

This paper is organized as follows. In the next section, we review key concepts related to the discrete fracture model leading to a standard LCP(M, q) formulation; without any loss of generality, as discussed previously, we deal with the holonomic case. Section 3 then describes and provides a proof for the proposed algorithm. We present, in Section 4, some computational results to highlight its efficiency and robustness, before concluding in Section 5.

2 The discrete holonomic fracture problem

Consider the 2-dimensional body shown in Fig. 1a which provides a generic definition of our fracture problem. The domain is subdivided into two homogeneous, linear-elastic and isotropic zones by a known locus or interface Γ along which the assumed pure mode I crack will propagate. As usual, the boundary of the body consists of a constrained part and of an unconstrained part for which displacements and tractions are prescribed, respectively. The loading p is assumed to be governed by a monotonically increasing parameter.

We adopt a cohesive crack model which assumes that Γ is the locus of potential displacement discontinuities characterized by a softening constitutive law relating tractions t to displacement jumps w across the interface. Thus Γ can be conceived as the union of cracks Γ_c (where $t = 0$), process zone Γ_z (where $t \neq 0$ and $w \neq 0$) and undamaged material Γ_e (where $w = 0$). Clearly, in the fracture process zone, the damaged material is still able to transfer stresses, albeit at a reduced level as compared to the intact material. These stresses are, as expected, decreasing functions of the relative displacement discontinuities (cracks) and become zero when a crack is fully developed.

A (two-zone) discrete model of the structure can easily be constructed from either finite elements [10] or boundary elements [22, 23]. As is well known, a boundary element approach is computationally more attractive than a finite element approach as it involves variables on the boundary and on the locus only. However, in either case, the formulation simply consists of condensing all variables to the c node pairs on Γ.

We first state the mathematical description of a piecewise linear holonomic cohesive law for mode I processes as it would apply to a generic node i on the interface. Assuming a three-branch softening law (as in Fig. 1b), the relevant constitutive relationship can then be expressed in the following complementarity format [12]:

$$f^i = t_a v_1 + t_b v_2 + t_c v_3 + (h_1 M_1 + h_2 M_2 + h_3 M_3) z^i + n t^i, \quad (2)$$

$$0 \leq f^i \quad \perp \quad z^i \geq 0 \quad (3)$$

where $f^i \in \Re^4$ and $z^i \in \Re^4$ are nonnegative vectors; f^i actually collects the softening functions in the order: horizontal branch, first softening branch, second softening branch and third softening branch. More explicitly, $z^{iT} = [z_1^i, z_2^i, z_3^i, z_4^i]$ with z_1^i physically representing the final crack width (Fig. 1b).

The model is characterized by six constant (nonnegative) parameters t_a, t_b, t_c, h_1, h_2, h_3 (which obviously represent key values of tractions and absolute softening slopes at all c nodes), and by the following constant vectors and matrices:

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad n = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \quad (4)$$

It is easy to verify, for instance, that, for any given traction $0 < t^i < t_c$, precisely two solutions for w^i exists (one corresponding to the elastic $w^i = 0$ vertical branch and the other due to activation of a softening mode) and their values are as predicted by simple geometry of the piecewise linear model. Similarly, for, say, any given crack width $0 < w^i < w_c$ there exists only one solution to the LCP.

Next, we can conceive (see [23] for details) the nonlinear response of the structure as being governed by the following discrete counterpart (appropriately condensed to cover only the nodes pertaining to the crack locus) of an integral equation:

$$t = t^e + Zw \quad (5)$$

where, for c node pairs on the interface, $t^e \in \Re^c$ represents the elastic normal tractions of the uncracked structure (i.e., in the absence of displacement discontinuities) caused by the applied load parameter, and $Z \in \Re^{c \times c}$ is a matrix of influence coefficients which operates on $w \in \Re^c$ to give the normal tractions caused by unit crack openings in the otherwise unloaded structure.

All elements of the vector $t \in \Re^c$ of total tractions must in turn satisfy the softening laws given by (3) collected for all points i on the interface.

We choose to assemble the various quantities as follows: $f_1^T = [f_1^1 \dots f_1^c]$, $z_1^T = [z_1^1 \dots z_1^c]$, etc. so that $f_1 \in \Re^c$ gathers the softening functions complementary to the real crack widths $w = z_1 \in \Re^c$ (see Fig. 1b). Then, for the entire interface, we collect variables $f \in \Re^{4c}$ and $z \in \Re^{4c}$ as $f^T = [f_1^T \dots f_4^T]$ and $z^T = [z_1^T \dots z_4^T]$, respectively.

Finally, after some simple manipulations, we arrive at the following LCP:

$$0 \leq f = [h_1 \hat{M}_1 + h_2 \hat{M}_2 + h_3 \hat{M}_3 - \hat{Z}]z + t_a \hat{V}_1 + t_b \hat{V}_2 + t_c \hat{V}_3 - \hat{t}^e \quad \perp \quad z \geq 0, \quad (6)$$

which is clearly equivalent to a standard LCP(M, q) by setting

$$M = h_1 \hat{M}_1 + h_2 \hat{M}_2 + h_3 \hat{M}_3 - \hat{Z}, \quad (7)$$

$$q = t_a \hat{V}_1 + t_b \hat{V}_2 + t_c \hat{V}_3 - \hat{t}^e \quad (8)$$

where matrices $\hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{Z} \in \Re^{4c \times 4c}$ and vectors $\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{t}^e \in \Re^{4c}$ are given by

$$\hat{M}_1 = \begin{bmatrix} -I & I & 0 & 0 \\ -I & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{M}_2 = \begin{bmatrix} 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{M}_3 = \begin{bmatrix} 0 & 0 & -I & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & I \end{bmatrix},$$

$$\hat{V}_1 = \begin{bmatrix} 0 \\ 0 \\ -e \\ e \end{bmatrix}, \quad \hat{V}_2 = \begin{bmatrix} 0 \\ -e \\ e \\ 0 \end{bmatrix}, \quad \hat{V}_3 = \begin{bmatrix} e \\ e \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{Z} = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{t}^e = \begin{bmatrix} t^e \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (9)$$

with I being an identity matrix, e a vector of ones, and 0 a null vector or matrix, all of appropriate sizes.

3 Algorithm to find multiple solutions to LCP

Let LCP(M, q) be the original problem, with $M \in \Re^{n \times n}$, $q \in \Re^n$. At each iteration, we consider the following LCP:

$$\text{LCP}\left(\begin{bmatrix} M & 0 \\ N & D \end{bmatrix}, \begin{bmatrix} q \\ r \end{bmatrix} \right) \quad (10)$$

for some constructed matrices N, D and vector r. Initially, N, D, r are null matrices and vectors.

Let $(\bar{x}, \bar{y})$ be a solution found; if no solution is found, then we quit. It is readily seen that $\bar{x}$ is a solution of LCP(M, q). Now we will construct a new LCP that excludes $\bar{x}$ from its solution.

Let $J = \{i \in \{1, ..., n\} : \bar{x}_i > 0\}$. For any $x \geq 0$, we have that $x = \bar{x}$ if and only if x satisfies

$$x_i \geq \bar{x}_i \ \forall i \in J, \quad \sum_{i \in J} x_i \leq \sum_{i \in J} \bar{x}_i, \quad \sum_{i \notin J} x_i \leq 0. \quad (11)$$

This system of linear inequalities can be written in the form

$$Ax + b \geq 0 \tag{12}$$

for suitable $A \in \Re^{m \times n}$ and $b \in \Re^m$, where $m = \min\{|J| + 2, n + 1\}$. If $|J| = n$, the last inequality $\sum_{i \notin J} x_i \leq 0$ would hold vacuously and hence can be dropped. Thus any $x \neq \bar{x}$ would satisfy $\min(Ax + b) < 0$. Since the solutions of LCP(M, q) are assumed to be isolated, there exists $\epsilon > 0$ such that $\min(Ax + b) \leq -\epsilon$ for all solutions x of LCP(M, q) that are not equal to $\bar{x}$. In general, ϵ can be chosen to be the radius of the ℓ_1-ball around $\bar{x}$ from which future solutions are to be excluded.

Then we consider the new LCP given by

$$\text{LCP}\left(\begin{bmatrix} M & 0 & 0 & 0 \\ N & D & 0 & 0 \\ A & 0 & I & 0 \\ 0 & 0 & e^T & -1 \end{bmatrix}, \begin{bmatrix} q \\ r \\ b \\ -\epsilon \end{bmatrix}\right) \tag{13}$$

and we reiterate. In other words, we replace N, D, r by

$$N^{new} = \begin{bmatrix} N \\ A \\ 0 \end{bmatrix}, \quad D^{new} = \begin{bmatrix} D & 0 & 0 \\ 0 & I & 0 \\ 0 & e^T & -1 \end{bmatrix}, \quad r^{new} = \begin{bmatrix} r \\ b \\ -\epsilon \end{bmatrix}. \tag{14}$$

Now we show by induction that this works.

Suppose that x^* solves LCP(M, q), x^* does not equal any previously found solutions (including $\bar{x}$). By inductive hypothesis, there is some y^* such that (x^*, y^*) is a solution of LCP (10). Since x^* solves LCP(M, q) and $x^* \neq \bar{x}$, then $\min(Ax^* + b) \leq -\epsilon$. Set $z^* \in \Re^{m+1}$ according to

$$z_i^* = \max\{0, -[Ax^* + b]_i\}, \ i = 1, ..., m, \quad z_{m+1}^* = 0. \tag{15}$$

Then

$$z_i^* \geq 0, \quad z_i^* + [Ax^* + b]_i = 0 \quad \text{if } [Ax^* + b]_i \leq 0, \tag{16}$$

$$z_i^* = 0, \quad z_i^* + [Ax^* + b]_i > 0 \quad \text{if } [Ax^* + b]_i > 0, \tag{17}$$

$$z_{m+1}^* = 0, \quad z_1^* + \cdots + z_m^* - z_{m+1}^* - \epsilon \geq 0, \tag{18}$$

so that (x^*, y^*, z^*) is a solution of the new LCP.

Conversely, suppose that (x^*, y^*, z^*) solves the new LCP. Then (x^*, y^*) solves LCP (10) and x^* solves LCP(M, q). By the inductive hypothesis, x^* does not equal any previously found solutions (not including $\bar{x}$). We claim that $x^* \neq \bar{x}$. This is because

$$z_1^* + \cdots + z_m^* - z_{m+1}^* - \epsilon \geq 0, \tag{19}$$

implying $z_1^* + \cdots + z_m^* \geq z_{m+1}^* + \epsilon \geq \epsilon$, so $z_{i^*}^* \geq \epsilon/m$ for some $i^* \in \{1, ..., m\}$. Then the complementarity condition yields

$$z_{i^*}^* + [Ax^* + b]_{i^*} = 0 \tag{20}$$

so that $[Ax^* + b]_{i^*} = -z_{i^*}^* \leq -\epsilon/m$ and hence $\min(Ax^* + b) \leq -\epsilon/m < 0$. This shows that $x^* \neq \bar{x}$. Q.E.D.

To summarize, the overall algorithm is:

- *Step 0*: Initialize N, D, r to null matrices and vectors.
- *Step 1*: Solve LCP (10). If no solution is found, quit. Else let $(\bar{x}, \bar{y})$ be the solution found. Construct A, b as in (12). Choose a sufficiently small $\epsilon > 0$ as discussed earlier. Update N, D, r according to (14). Return to *Step 1*.

4 Computational results

Various computational results are provided in this section with the aim of highlighting the performance of the approach described in the previous section.

The algorithm was easily implemented as a MATLAB (ver 6.1) code, albeit without any special provisions for efficiency. For fracture problems, the code simply: (a) forms the data for LCP(M, q) precisely as detailed in (9); (b) writes to a text file the data in GAMS [24] format (GAMS is a mathematical programming modeling environment and stands for General Algebraic Modeling System); (c) invokes a MATLAB-GAMS link [25] to solve the LCP using the PATH solver; (d) exits if no solution is found, or reforms a new LCP in accordance to (13) and reiterates. There are only two initial parameters that are required: the value of ϵ and the initial vector x for the LCP(M, q).

All runs were carried out on a 1 GHz Pentium IV running Win NT4. GAMS/PATH (ver 4.6.03) was used for all MCP solves from within GAMS (Rev 133) and was run with default settings except with preprocessing turned "off" and the proximal perturbation factor set to 0.1 [15]. Approximate timings were obtained using the MATLAB "tic" and "toc" functions and include all file reads and writes, GAMS/PATH solves as well as MATLAB processing.

In the following, we report on three sets of examples. Each example consists of a whole series of (single-step) holonomic analyses carried out for different load levels, in effect solving a series of LCPs for the same M matrix but different q vectors. The holonomic runs for each example, it must be stressed, are independent runs but were carried out simultaneously in a single MATLAB run.

4.1 Example 1: three-bar softening truss

The simple textbook three-bar truss with softening elements [16], shown in Fig. 2a, can be considered a prototype of more meaningful quasibrittle fracture situations. Data (kN, mm units) are as follows: area of vertical bar = 800; areas of inclined bars = 500;

yield limit = 0.5; elastic modulus = 200; softening modulus = −50. The interested reader should refer to Bolzon *et al.* [16] for basic details of how M and q are formed (although, for the present work, we use a slightly reduced but equivalent LCP).

For the single-step analyses, the load p was set at 42 levels (increasing from 0 to 800 in steps of 20 and at the known maximum level of 815.68). We used $\epsilon = 0.1$ and $x^T = [10 \dots 10]$ for all LCPs.

Manifestations of overall instability and bifurcations of the equilibrium path, which are typical consequences of softening constitutive laws, are to expected to occur with both symmetric (S) and nonsymmetric (NS) responses (as indicated in Fig. 2b). Indeed the responses obtained confirm this. We show in Fig. 3, the load p versus vertical displacement u solutions obtained: one symmetric path and two nonsymmetric branches were identified correctly (although the two nonsymmetric solutions cannot be distinguished since their vertical deflections are identical). As expected, the nonsymmetric branch leads to a lower limit load.

The analyses took 41 secs, and successfully identified precisely (without any spurious or repeated solutions) all expected 159 solutions; 201 LCPs (viz. 159 + 42, since for each of the 42 single-step analyses an additional LCP had to be processed to ensure no further solution in each case) were solved with sizes ranging from $n = 3$ to $n = 21$.

4.2 Example 2: two-notch tensile (2NT) test

This second example concerns the well-studied two-notch tensile (2NT) test for which multiple equilibrium paths and unstable responses are expected. The loaded specimen is shown in Fig. 4a, while the expected deformed configurations consisting of one symmetric (S) and two nonsymmetric (NS) modes are drawn in Fig. 4b.

We adopted the same geometrical specimen as in [17, 23], but used a three-branch softening law, instead of a single branch (the properties of which were especially selected to induce a severe snap-back instability). Details of geometric and material properties are: length = 250 mm; width = 60 mm; thickness = 1 mm; notch depth = 5 mm (on each side); Young's modulus = 18000 MPa; Poisson ratio = 0.2; three-branch softening with $t_a = 1$, $t_b = 2$, $t_c = 3.4$, $h_1 = 350$, $h_2 = 250$, $h_3 = 200$ in N, mm units. A distributed load of $p = 48\alpha$ N was applied to each end of the specimen and the midside deflection u over half of the length determined.

We used a simple collocation boundary element model with a total of 78 quadratic elements, of which 10 element pairs (or 21 node pairs) were located on the interface. Thirty two single-step analyses were carried out within the same run, with load factor α increasing from 0.1 to 3.2 in steps of 0.1. We used $\epsilon = 10^{-5}$ and $x^T = [10 \dots 10]$ for all LCPs.

Figure 5 shows the α versus u solutions obtained. As in the previous example, symmetric and nonsymmetric softening deformations are evident, with the nonsymmetric path providing the lower limit load. We again note that there are in fact two nonsymmetric configurations (left crack and right crack).

The analyses took 65 secs, and again successfully identified all 118 expected solutions as well as the fact that the last two load levels had no solutions; 2 solutions were obtained twice; 152 LCPs were solved in all with sizes ranging from $n = 84$ to $n = 363$.

4.3 Example 3: wedge-splitting (WS) test

The wedge splitting (WS) and three-point bending (3PB) tests are standard experimental setups primarily aimed at obtained key material data from quasibrittle brittle fracture specimens. The respective idealized setups are as shown in Fig. 6a and Fig. 6b. In order to be able to trace the softening response, a displacement (u), rather than load control (p), is applied in experiments. In the following WST example, we, however, adopted a load control so that multiplicity of solutions exist.

We used data related to an actual, normal-strength concrete specimen from a sophisticated WS test carried out at the Swiss Federal Institute of Technology [26]. Details of the specimens used (geometry and elastic properties) are provided in Tin-Loi and Que [27]. A realistic three-branch law with $t_a = 0.255$, $t_b = 1.129$, $t_c = 4.138$, $h_1 = 173.04$, $h_2 = 14.34$, $h_3 = 1.17$ was adopted.

The discretized models of the WS test was constructed from the same direct two-zone, collocation boundary element approach used for the previous example. The discretization involved 192 quadratic elements, with 17 element pairs (35 node pairs) on the crack interface. The applied load is $p = 97\alpha$. Thirty single-step analyses were carried out within the same run, with α increasing from 1 to 30 in steps of 1. We set $x^T = [0 \ldots 0]$ for all LCPs but varied ϵ as follows: $\epsilon = 1$ for $\alpha < 10$, $\epsilon = 10^{-2}$ for $\alpha \geq 10$. The choice of ϵ appears to be related to the distance between the solutions; obviously if solutions are close together, a smaller ϵ needs to be used to ensure previous solutions are constrained out.

As is well known, that particular test (as also for the 3PB test) for well-designed experiments provides two solutions at the most. This is corroborated by the $p-u$ results obtained and displayed in Fig. 7; the response consists of a symmetric softening path only.

The analyses took 65 secs, and again successfully identified all 60 expected solutions; 99 LCPs were solved with sizes ranging from $n = 140$ to $n = 397$. However, in spite of increasing the PATH tolerance option to 5×10^{-8}, some (9 in all) spurious solutions (which were clearly physically unacceptable although they satisfied complementarity to the provided tolerance) were obtained.

5 Concluding remarks

Key features of quasibrittle fracture analyses are multiplicity or lack of solutions. This is expected from both mechanical and mathematical considerations. A fracture analysis can be elegantly formulated as a complementarity problem which is linear when the well-accepted piecewise linear softening laws are adopted. The problem, however, is to compute numerically all solutions to the LCP, or to show that none exits.

This paper proposes a novel scheme to achieve this. In essence, the algorithm simply keeps setting up a new LCP that does not contain any previously found solutions until no further solution is identified. It requires the availability of a good LCP solver such as the industry-standard PATH code.

It was conclusively found, from our numerical tests, that the indicated method is eminently feasible—in fact better than any other proposed so far. Firstly, it is simple as it involves minimal coding and requires essentially the setting of a single parameter ϵ representing the radius of the ℓ_1-ball from which future solutions are to be excluded. Secondly, it is extremely efficient since the core LCP solver, PATH, is so. In fact, it can be made more efficient if the scheme either makes use entirely of MATLAB or of GAMS, rather than having to rely on writing the MATLAB generated data to a GAMS input file, as done in this paper. Thirdly, it is surprisingly robust. And, lastly, it has the capability to solve large-scale problems.

Also worthy of mention is the fact that the proposed algorithm is not specific to fracture problems; it can obviously solve LCPs that arise in various other fields such as in the computation of Nash equilibrium points, transport equilibria and multiplicity of electrical networks. Nor is it specific to LCPs only; it can be used to process any nonlinear mixed complementarity problem. Finally, even in cases where the solutions of the LCP are not isolated, we believe that the method could still be employed to build a finite solution subset that spans, in some fashion, the solution set of the LCP.

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FIGURE CAPTIONS

Fig. 1. Fracture problem: (a) structure; (b) piecewise linear cohesive crack model.

Fig. 2. Three-bar softening truss: (a) structure; (b) possible deformed configurations.

Fig. 3. Three-bar softening truss: $p - u$ solutions.

Fig. 4. Two-notch tensile test: (a) specimen; (b) possible deformed configurations.

Fig. 5. Two-notch tensile test: $\alpha - u$ solutions.

Fig. 6. Standard fracture tests: (a) wedge splitting; (b) three-point bending.

Fig. 7. Wedge-splitting test: $p - u$ solutions.

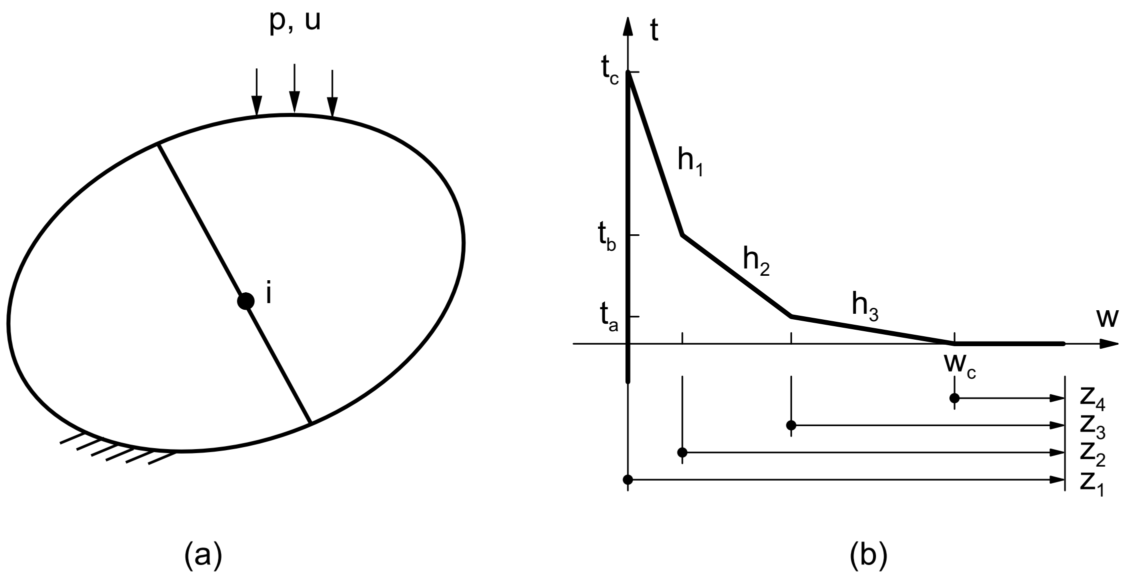


Figure 1

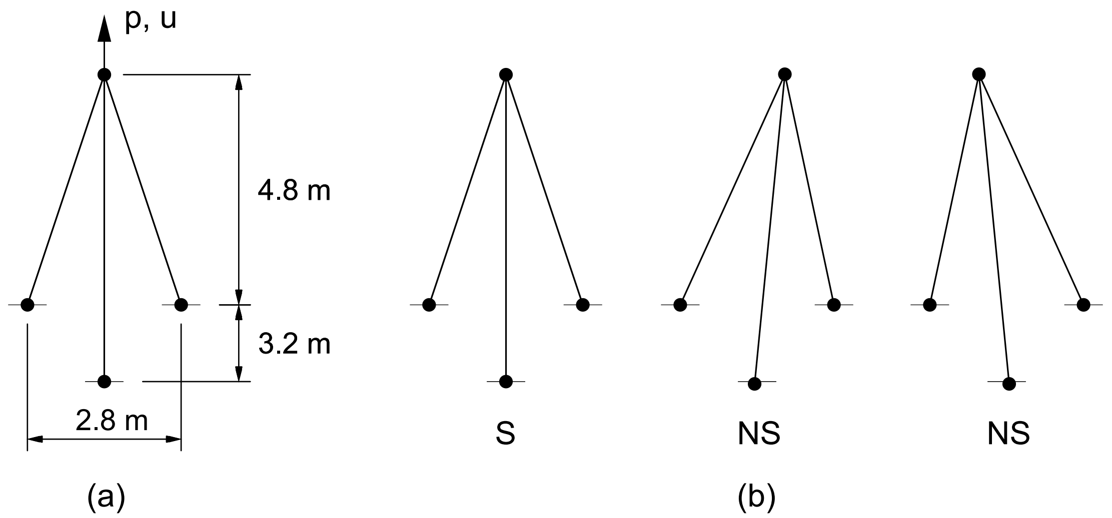


Figure 2

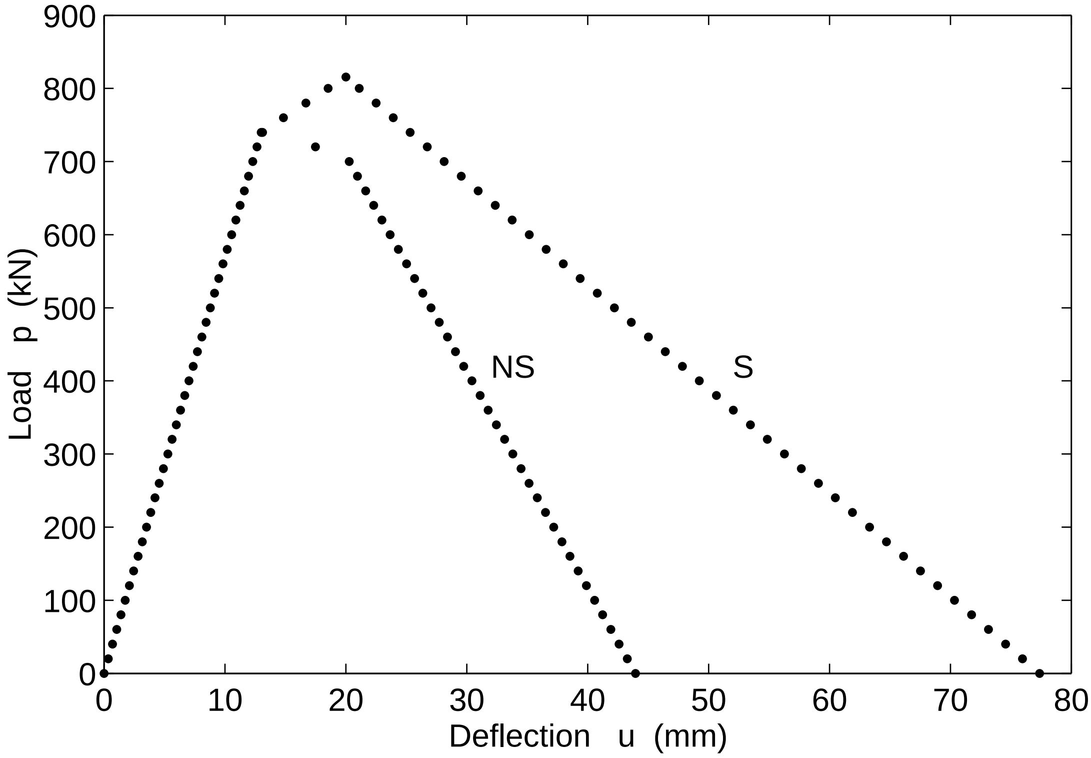
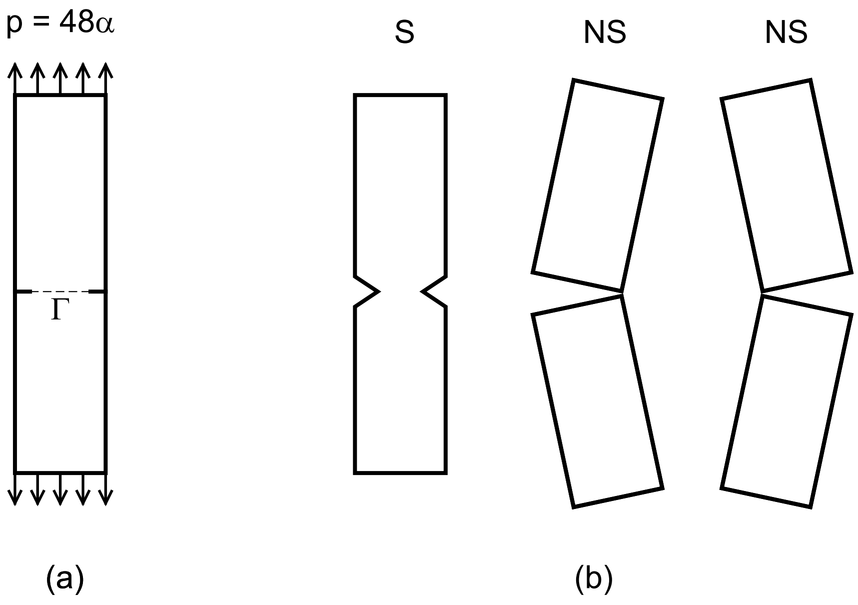


Figure 3



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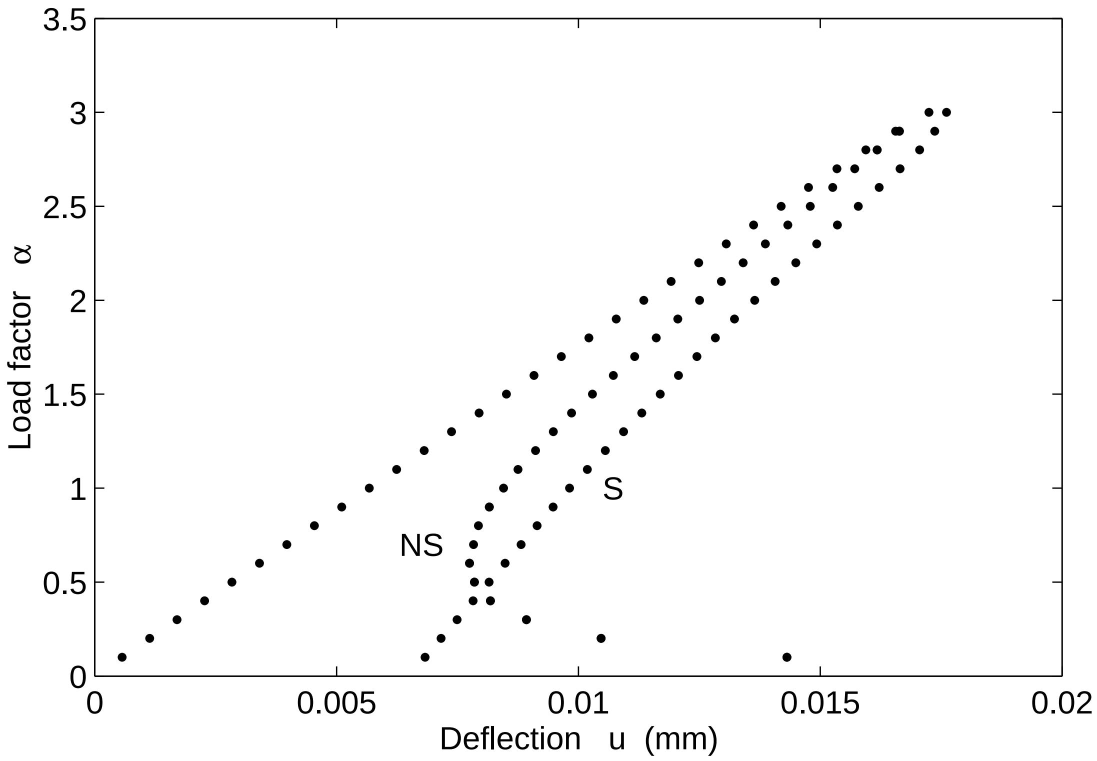

Figure 5

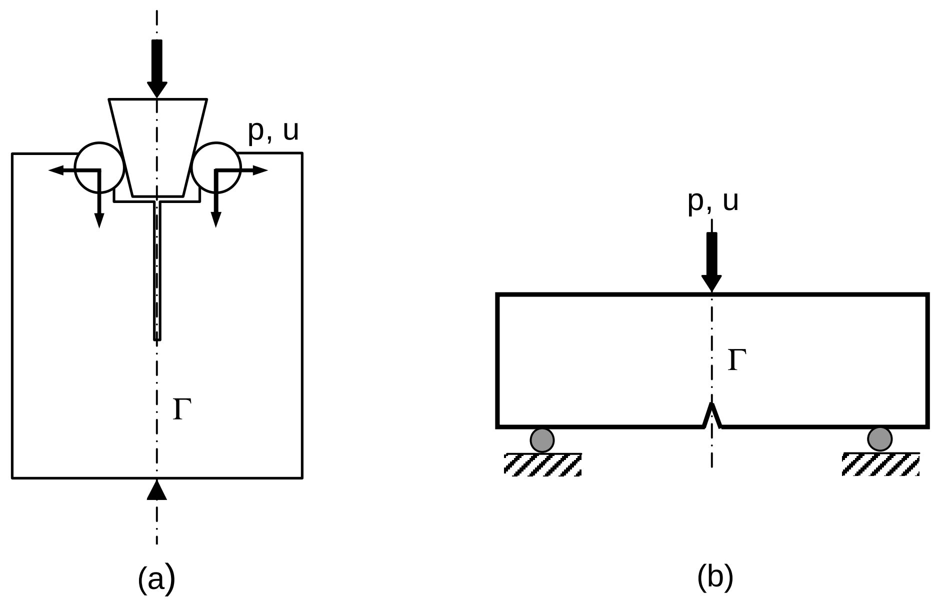

Figure 6

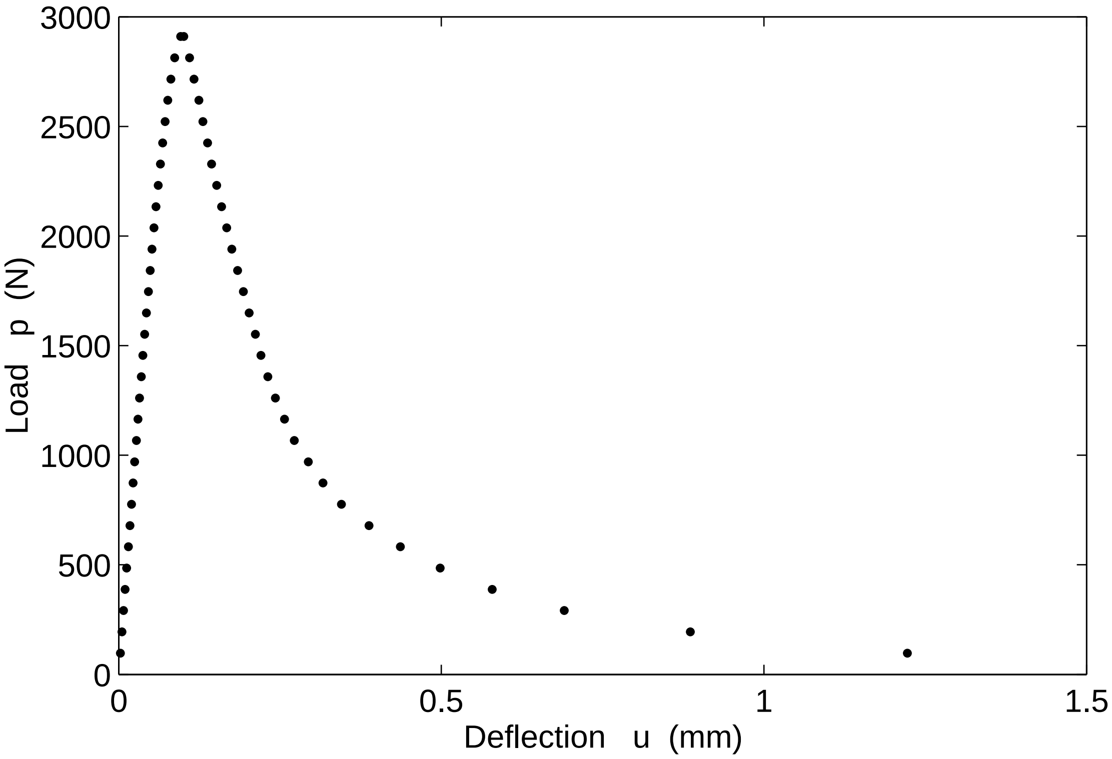

Figure 7