February 1, 2009

Here are some type A and type B problems. More will be posted later.

A1. Show that the NP-hard integer linear feasibility problem:

Given $C \in \mathbb{Z}^{p\times q}$ and $d \in \mathbb{Z}^p$, is there an $x \in \{0,1\}^q$ satisfying $Cx = d$?

is equivalent to the following compressed sensing problem:

$$\begin{array}{ll} \min_x & \sharp(x) \\ \text{s.t.} & Ax = b \end{array}$$

where $A = \begin{pmatrix} C & 0 \\ I & I \end{pmatrix}$, $b = \begin{pmatrix} d \\ e \end{pmatrix}$, I is the identify matrix, and e is the vector of 1's. Here $\sharp(x)$ counts the number of nonzero components in $x \in \mathbb{R}^n$. (Hint: Show that the answer to the former is 'yes' if and only if the optimal value of the latter equals p.)

A2. Verify that the second-order cone

$$K = \left\{ x \in \mathbb{R}^n \,\middle|\, \sqrt{x_1^2 + \cdots + x_{n-1}^2} \le x_n \right\}$$

is self-dual. Also verify that its log-barrier function

$$f(x) = \begin{cases} -\log\left(x_n^2 - (x_1^2 + \cdots + x_{n-1}^2)\right) & \text{if } x \in \text{int}K \\ \infty & \text{else} \end{cases}$$

is strictly convex.

A3. Verify that the semidefinite cone $K = \{X \in \mathbb{R}^{n\times n} \mid X \succeq 0\}$ is self-dual. You can use the fact that, for any $Y \in \mathbb{S}^n$, $Y = V\text{diag}[\lambda_1, \ldots, \lambda_n]V^T$, where $\lambda_i \in \mathbb{R}$ and $V \in \mathbb{R}^{n\times n}$ is orthogonal, i.e., $V^TV = I$.

A4. Show that the gradient of $\log\det X$ over $X \succ 0$ is X^{-1}.

A5. Give an example of an SOCP that has an optimal solution and whose dual is infeasible.

A6. Consider the following SDP ($n \ge 2$):

$$\begin{array}{ll} \min_x & x_n \\ \text{s.t.} & \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & 0 & \cdots & 0 \\ x_2 & 0 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & 0 & 0 & \cdots & x_n \end{bmatrix} \succeq 0, \quad x_1 = 2. \end{array}$$

Show that the optimal value is at least $2^{2^{n-1}}$. Thus SDP (and SOCP) can have very large optimal solution relative to the size of the problem data. (Try $n = 2, 3$ first if you are not sure how to proceed.)

B1. Consider the convex quadratically constrained quadratic optimization problem

$$\begin{array}{ll} \min_x & q_0(x) \\ \text{s.t.} & q_i(x) \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

where $q_i(x) = x^T Q_i x + c_i^T x$, $i = 0, 1, \ldots, m$, and $Q_i \succeq 0$, $c_i \in \mathbb{R}^n$, $b_i \in \Re$. Show that this may be reformulated as an SOCP. (Hint: Write $Q_i = A_i^T A_i$ for some $A_i \in \mathbb{R}^{p_i \times n}$. Note that a linear term $t \in \mathbb{R}$ can be expressed as the difference of squares: $(0.5 + t)^2 - 0.5^2 - t^2$.)

B2. The Hadamard product of $A, B \in \mathbb{R}^{n \times n}$ is defined to be $A \odot B = [a_{ij} b_{ij}]_{i,j=1}^n$. Show that if $A \succeq 0, B \succeq 0$, then $A \odot B \succeq 0$. (The proof is short, but a bit tricky.)

B3. Give an example of an SOCP that is feasible and whose dual is also feasible, but their optimal values are unequal (i.e., nonzero duality gap). (Caution: This is not easy.)