[Homework can be handed in to me or to my mail box in the Math Lounge (opposite the Math main office). Please show your work to receive full credit.]

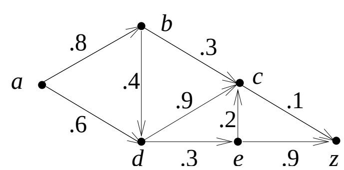
**A**. Consider a weighted digraph $G = (V, A)$ that has no cycle of negative weight. Prove that if $P$ is any path from $s \in V$ to $v \in V$, then there exists a simple path from $s$ to $v$ whose weight is less than or equal to that of $P$. [If your argument involves a cycle, then you should explain carefully why the vertices do not repeat (except at the start and end).] Since there is a finite number of simple paths from $s$ to $v$, one of these simple paths must have minimum weight and hence a SP.

**B**. [*Applied Combinatorics*, §13.2, #6.] A company wants to invest in a fleet of automobiles and is trying to decide on the best strategy for how long to keep a car over the next 5 years. After 5 years it will sell all remaining cars and let an outside firm provide transportation. The following matrix shows the estimated cost (in thousands of dollars) of buying a car at the beginning of year $i$ and sold at the beginning of year $j$.

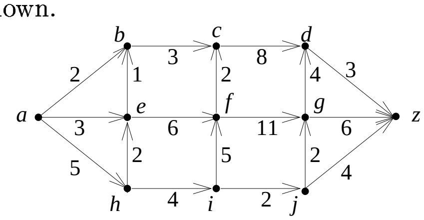
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 4 & 6 & 9 & 12 & 20 \\ & 5 & 7 & 11 & 16 \\ & & 6 & 8 & 13 \\ & & & 8 & 11 \\ & & & & 10 \end{pmatrix} \end{array}.$$

To determine the cheapest strategy for when to buy and sell cars, we consider a weighted digraph with vertices numbered 1, 2, 3, 4, 5, 6, with arcs $(i, j)$ for all $i < j$, and with weight $w_{ij}$ of arc $(i, j)$ being the $(i, j)$th matrix entry. The arc $(i, j)$ has the interpretation of buying a car at the beginning of year $i$ and selling it at the beginning of year $j$. Find the cheapest strategy.

**C**. [*Applied Combinatorics*, §13.2, #8.] A communication network is shown below. [arc $(i, j)$ corresponds to a link over which $i$ can send messages directly to $j$.] Also shown is the probability $p_{ij}$ that each link from $i$ to $j$ is operative. Assuming that the links fail independently of each other, the probability that all the links in a path are operative is the product of the link probabilities. Find the most reliable path from $a$ to $z$. [Hint: Consider $-\log p_{ij}$.]




**D**. Consider the weighted digraph shown.

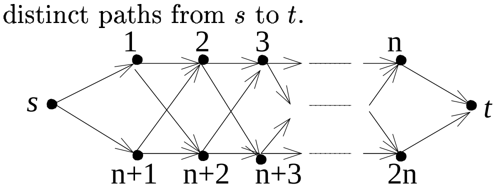



(a) Use SP-Dijkstra algorithm to find a shortest path (SP) from $a$ to each vertex. Indicate the order in which each SP is found and show your calculation of SP weight/distance.

(b) Use SP-Bellman-Ford algorithm to find a SP from $a$ to each vertex.

**E**. (a) If all arcs of a digraph have different nonnegative weights, is the SP from a vertex $s$ to another vertex $t$ necessarily unique? Explain.

(b) It is not efficient to consider all possible paths from $s$ to $t$ in search of a SP from $s$ to $t$. For the digraph shown below, find the number of distinct paths from $s$ to $t$.




**F.** Consider a digraph $G = (V, A)$ with nonnegative arc weights $\{\omega_{uv}\}_{(u,v)\in A}$ and $s \in V$. For any path $P:\ u_1, ..., u_{p+1}$ in $G$, define its *max-weight* to be

$$\omega_{\max}(P) = \max_{i=1,..,p} \omega_{u_i u_{i+1}}.$$

We wish to find, for each $v \in V$ reachable from $s$, a path from $s$ to $v$ whose max-weight is minimum.

(a) Show by example that a SP need not have minimum max-weight nor vice versa.

(b) How would you modify SP-Dijkstra to find a path of minimum max-weight?