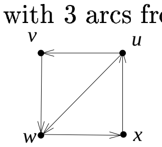
MATH 409, SPRING 2009 HOMEWORK 2 due **Wed, April 15, 11:30 a.m.**

[Homework can be handed in to me or to my mail box in the Math Lounge (opposite the Math main office). Please show your work to receive full credit.]

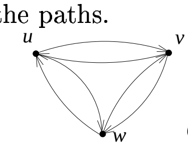
A. For the two digraphs shown below, use the fact

$$((i,j)\text{th entry of } (\text{Adj})^k) = (\#\text{ paths with } k \text{ arcs from } i \text{ to } j) \quad 1 \le i,j \le |V|,\; k = 1,2,..$$

to find the number of paths with 3 arcs from u to v. Identify the paths.



G_1



G_2

B. Let G be a graph such that any two vertices are joined by exactly one simple path. Prove that G is a tree.

C. Explain your answer to the following question. Is there a tree of seven vertices:

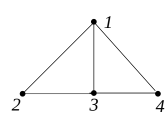
(a) With each vertex having degree 1?

(b) With two vertices having degree 1 and five vertices having degree 2?

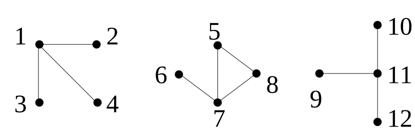
(c) With five vertices having degree 1 and two vertices having degree 2?

(d) With vertices having degrees 2, 2, 2, 3, 1, 1, 1?

D. (a) For the following graph, write down its adjacency matrix and use it to find the number of paths with 8 arcs joining vertices 1 and 2. [Do the matrix multiplications by hand and notice the special structures. Only 3 matrix mult. are needed.]

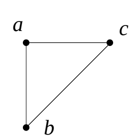


(b) For the graph below



find all connected components.

(c) Identify all subgraphs of the graph:



E. Exercise 2.4.1 on page 12 of course notes.

F. (a) In a connected graph with n $(n \ge 1)$ vertices, what is the maximum possible number of arcs? What is the minimum possible number of arcs?

(b) In a connected graph with 15 arcs, what is the the maximum possible number of vertices? What is the minimum possible number of vertices? [Hint: Use your answer to (a).]

G. Exercise 2.4.2 on page 13 of course notes.

Bonus: Prove that, for any graph/digraph $G = (V, A)$ with adjacency matrix Adj,

$$((i,j)\text{th entry of } (\text{Adj})^k) = (\#\text{ paths with } k \text{ arcs from } i \text{ to } j) \quad 1 \leq i,j \leq |V|,\ k = 1, 2, ..$$

[For the induction proof, you should first show that the claim is true for $k = 1$ and then show that if the claim is true for any integer $k \geq 1$, then the claim is true for $k + 1$.]