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ESTIMATES OF THE DUALITY GAP FOR LARGE-SCALE SEPARABLE NONCONVEX OPTIMIZATION PROBLEMS*

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ABSTRACT

We derive some estimates of the duality gap for separable constrained optimization problems involving nonconvex, possibly discontinuous, objective functions, and nonconvex, possibly discrete, constraint sets. The main result is that as the number of separable terms increases to infinity the duality gap as a fraction of the optimal cost decreases to zero. The analysis is related to the one of Aubin and Ekeland [1], and is based on the Shapley-Folkman theorem. Our assumptions are different and our estimates are sharper and more convenient for integer programming problems.

1. INTRODUCTION

Dual methods are often useful for large-scale optimization problems particularly when the objective function and the constraints have a separable and decomposable structure. However, when applying dual methods to nonconvex problems one must find a way to deal with the potential discrepancy between the optimal values of the dual problem and the original primal problem (i.e., the duality gap). Fortunately for many large problems the duality gap tends to be small as has been observed for linear programming problems by Lasdon [2], and established in a more general context by Aubin and Ekeland [3]. It is thus possible to solve many nonconvex problems to within satisfactory accuracy by essentially treating them as if they were convex and without having to resort to time consuming branch and bound techniques.

The present paper is related to our recent work on solution of large-scale power system scheduling problems [4],[5], where the small relative size of the duality gap played a key role in the solution methodology. Our purpose is to quantify the size of the duality gap in a manner which is similar to Aubin and Ekeland, but is sharper and more convenient for problems where the constraint set is nonconvex and discrete. Our sharper analysis is based on a new and often satisfied in practice assumption (Assumption A3, in the next section). Without this assumption the results of Aubin and Ekeland are very cumbersome to use for the case where the constraint functions are nonlinear and/or the constraint set is nonconvex as they involve the solution of certain perturbed optimization problems (see page 231 of that reference). Even for the case where the constraint functions are linear and the constraint set is convex there are problems where our estimate is sharper than that of Aubin and Ekeland [1]. For example, for the simple scalar problem $\min\{f(x) \mid x \le b,\ x \in [0,b]\}$ where f is a strictly concave monotonically increasing function on the interval $[0,b]$, Theorem A of Aubin and Ekeland [1] estimates a positive duality gap but our estimate of the next section shows that there is no duality gap.

2. AN ESTIMATE OF THE DUALITY GAP

Consider the following problem:

$$\text{(P)} \qquad \text{minimize} \quad \sum_{i=1}^{I} f_i(x_i)$$

$$\text{subject to} \quad x_i \in X_i \ , \quad \sum_{i=1}^{I} h_i(x_i) \le b \ ,$$

where I is a positive integer, b is a given vector in R^m (m is a positive integer), X_i is a subset of R^{p_i} (p_i is a positive integer for each i), and $f_i:\mathrm{conv}(X_i)\to R$ and $h_i:\mathrm{conv}(X_i)\to R^m$ are given functions defined on the convex hull of X_i denoted $\mathrm{conv}(X_i)$. The vector inequality in the constraint of (P) is meant to be for each component, and all subsequent vector inequalities should be interpreted likewise. We assume the following.

(P) Assumption A1: There exists at least one feasible solution of problem (P).

Assumption A2: For each i, the subset of R^{p_i+m+1}

$$\{(x_i, h_i(x_i), f_i(x_i)) \mid x_i \in X_i\}$$

is compact.

Assumption A2 implies that X_i is compact. It is satisfied whenever X_i is compact and both f_i and h_i are continuous on X_i. Note that no convexity assumptions are made on f_i, h_i, or X_i.

For each i, define the function $\tilde{f}_i:\mathrm{conv}(X_i)\to R$ by

$$\tilde{f}_i(\tilde{x}) = \inf\left\{ \sum_{j=1}^{p_i+1} a^j f_i(x^j) \;\middle|\; \tilde{x} = \sum_{j=1}^{p_i+1} a^j x^j,\ x^j \in X_i,\ \sum_{j=1}^{p_i+1} a^j = 1,\ a^j \ge 0 \right\}$$

$$\text{for all } \tilde{x} \in \mathrm{conv}(X_i) \ . \tag{1}$$

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The function $\tilde{f}_i$ may be viewed as a "convexified" version of f_i on $\text{conv}(X_i)$. Figure 1 shows an example of f_i and the corresponding $\tilde{f}_i$, where X_i consists of the union of an interval and a single point. Similarly, define the function $\tilde{h}_i:\text{conv}(X_i)\to R^m$ by

$$\tilde{h}_i(\tilde{x})$$

$$= \inf\left\{\sum_{j=1}^{p_i+1} a^j h_i(x^j) \,\middle|\, \tilde{x} = \sum_{j=1}^{p_i+1} a^j x^j, x^j \varepsilon X_i, \sum_{j=1}^{p_i+1} a^j = 1, a^j \geq 0\right\}$$

$$\text{for all } \tilde{x} \ \varepsilon \ \text{conv}(X_i) \quad , \tag{2}$$

where the infimum is taken separately for each of the m coordinates of the function h_i. Note that if f_i is a convex function on $\text{conv}(X_i)$, then $f_i = \tilde{f}_i$. A similar statement can be made concerning h_i and $\tilde{h}_i$.

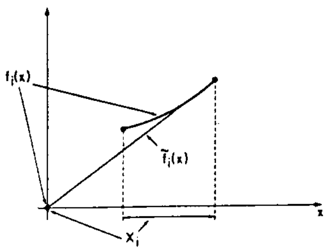

Figure 1

Our third assumption is:

Assumption A3: For each i, given any vector $\tilde{x}$ in $\text{conv}(X_i)$, there exists $x\varepsilon X_i$ such that $h_i(x) \leq \tilde{h}_i(\tilde{x}_i)$.

Note that Assumption A3 is satisfied if X_i is convex and each component of H_i is convex on X_i for in this case we have $h_i(x)=\tilde{h}_i(x)$ for all $x\varepsilon X_i$. On the other hand, Assumption A3 can be expected to be satisfied for many other problems of practical interest — for example, if h_i is a linear real-valued function (compare with Figs. 2 through 5 that follow and refs. [4],[5]).

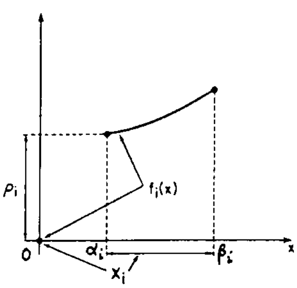

Figure 2. $h_i(x) = -x, p_i > 0$

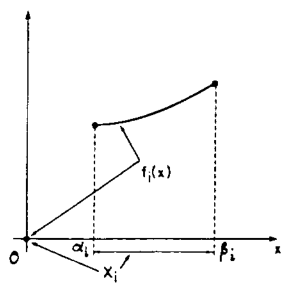

Figure 3. $h_i(x) = x, p_i = 0$

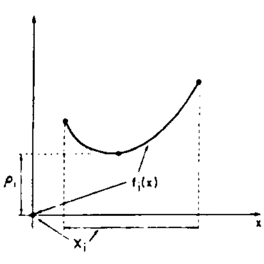

Figure 4. $h_i(x) = -x, p_i > 0$

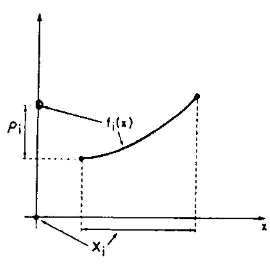

Figure 5. $h_i(x) = x, p_i > 0$

Define for each i the function $\hat{f}_i:\text{conv}(X_i)\to R$ by

$$\hat{f}_i(\tilde{x}) = \inf\{f_i(x) \mid h_i(x) \leq \tilde{h}_i(\tilde{x}), x\varepsilon X_i\}$$

$$\text{for all } \tilde{x} \ \varepsilon \ \text{conv}(X_i) \quad . \tag{3}$$

Note that, by Assumption A3, the constraint set for the minimization indicated in Eq. 3 is nonempty. Our estimate of the duality gap is given in terms of the scalars

$$\rho_i = \sup\{\hat{f}_i(x) - \tilde{f}_i(x) \mid x\varepsilon\text{conv}(X_i)\} \quad . \tag{4}$$

Since we have, for all $x\varepsilon\text{conv}(X_i)$,

$$\hat{f}_i(\tilde{x}) \leq \sup\{f_i(x_i) \mid x_i\varepsilon X_i\} \quad , \quad \tilde{f}_i(\tilde{x}) \geq \inf\{f_i(x_i) \mid x_i \varepsilon X_i\} \ ,$$

it follows that an easily obtainable overestimate of ρ_i is

$$\rho_i \leq \sup\{f_i(x_i) \mid x_i\varepsilon X_i\} - \inf\{f_i(x_i) \mid x_i\varepsilon X_i\} \quad .$$

Figures 2 through 5 show the scalar ρ_i for the X_i consisting of the union of an interval and a single point, and for specific cases of f_i and h_i. For example, for the cases shown in both Figs. 2 and 3 the function $\tilde{f}_i$ has the form shown in Fig. 1. In the case of Fig. 2 where

$$X_i = \{0\} \cup [\alpha_i, \beta_i] \quad , \quad \text{conv}(X_i) = [0, \beta_i] \quad , \quad h_i(x) = -x$$

we have using Eq. 3

$$\hat{f}_i(\tilde{x}) = \begin{cases} f_i(\tilde{x}) & \text{if} \quad \alpha_i \leq \tilde{x} \leq \beta_i \\ f_i(\alpha_i) & \text{if} \quad 0 < \tilde{x} < \alpha_i \\ f_i(0) & \text{if} \quad \tilde{x} = 0 \end{cases} \quad .$$

Therefore the scalar ρ_i of Eq. 4 is positive and equal to

$$p_i = f_i(\alpha_i) - f_i(0)$$

as shown in Fig. 2. In the case of Fig. 3, $\tilde{f}_i$, X_i, and $\text{conv}(X_i)$ are the same as in Fig. 2 but now $h_i(x) = x$. This changes the form of $\hat{f}_i$ which, using Eq. 3, is now given by

$$\hat{f}_i(\tilde{x}) = f_i(0) \quad , \quad \text{for all } \tilde{x}\varepsilon[0, \beta_i] \quad .$$

Therefore, the scalar ρ_i of Eq. 4 is now zero.

Consider now the dual problem

(D) maximize

$$d(\mu) = \inf_{\substack{x_i \,\varepsilon\, X_i \\ i=1,\ldots,I}} \sum_{i=1}^{I} [f_i(x_i) + \mu' h_i(x_i)] - \mu' b$$

subject to $\mu \geq 0$.

Let inf(P) and sup(D) denote the optimal values of the primal and dual problems respectively. We have the following result.

Proposition: Under Assumption A1 through A3 there holds

$$\inf(P) - \sup(D) \leq (m+1)E \quad , \tag{5}$$

where

$$E = \max\{\rho_i \mid i=1,\ldots,I\} \quad . \tag{6}$$

Proof: Consider the subsets of R^{m+1}

$$Y_i = \{y_i | y_i=[h_i(x_i), f_i(x_i)], x_i \varepsilon X_i\} \quad , \quad i=1,\ldots,I \quad , \tag{7}$$

and their vector sum

$$Y = Y_1 + Y_2 + \ldots + Y_I \quad . \tag{8}$$

In view of Assumption A2, Y, conv(Y), and Y_i, conv(Y_i), i=1,...,I, are all compact sets. By definition of Y, we have

$$\inf(P) = \min\{w | \text{there exists}(z,w)\varepsilon Y \text{ with } z\leq b\} \quad . \tag{9}$$

By using Assumptions A1 and A2, and a standard duality argument [6],[7] we can also show that

$$\sup(D) = \min\{w | \text{there exists}(z,w)\varepsilon \text{conv}(Y) \text{ with } z\leq b\} \quad . \tag{10}$$

We now use the following theorem ([3] Appendix I).

Shapley-Folkman Theorem: Let Y_i, i=1,...,I, be a collection of subsets of R^{m+1}. Then for every $y\varepsilon\text{conv}(\Sigma_{i=1}^I Y_i)$, there exists a subset $I(y)\subset\{1,\ldots,I\}$ containing at most (m+1) indices such that

$$y \ \varepsilon \left[\sum_{i\notin I(y)} Y_i + \sum_{i\varepsilon I(y)} \text{conv}(Y_i) \right] .$$

Now let $(\bar{z},\bar{w})\varepsilon\text{conv}(Y)$ be such that (compare with Eq. 10)

$$\bar{w} = \sup(D) \quad , \quad \bar{z} \leq b \quad . \tag{11}$$

By applying the Shapley-Folkman theorem to the set $Y=\Sigma_{i=1}^I Y_i$ given by Eqs. 7 and 8, we have that there exists a subset $\bar{I}\subset\{1,\ldots,I\}$, with at most (m+1) indices, and vectors

$$(\bar{b}_i,\bar{w}_i)\varepsilon\text{conv}(Y_i) \quad , \quad i\varepsilon\bar{I} \quad ,$$

$$\bar{x}_i\varepsilon X_i \quad , \quad i\notin\bar{I} \quad ,$$

such that (compare with Eq. 11)

$$\sum_{i\notin\bar{I}} h_i(\bar{x}_i) + \sum_{i\varepsilon\bar{I}} \bar{b}_i = \bar{z} \leq b \quad , \tag{12}$$

$$\sum_{i\notin\bar{I}} f_i(\bar{x}_i) + \sum_{i\varepsilon\bar{I}} \bar{w}_i = \sup(D) \quad . \tag{13}$$

Using the Caratheodory theorem for representing elements of the convex hull of a set, we have that, for each $i\varepsilon\bar{I}$, there must exist vectors $x_i^1,\ldots,x_i^{m+2}\varepsilon X_i$, and scalars $a_i^1,\ldots,a_i^{m+2}$ such that

$$\sum_{j=1}^{m+2} a_i^j = 1 \quad , \quad a_i^j \geq 0 \quad , \quad j=1,\ldots,m+2$$

$$\bar{b}_i = \sum_{j=1}^{m+2} a_i^j h_i(x_i^j) \quad , \quad \bar{w}_i = \sum_{j=1}^{m+2} a_i^j f_i(x_i^j) \quad .$$

Using the definition of $\tilde{f}_i$, $\tilde{h}_i$, and ρ_i (compare with Eqs. 1 through 4), we have

$$\bar{b}_i \geq \tilde{h}_i\left(\sum_{j=1}^{m+2} a_i^j x_i^j\right) \quad , \tag{14}$$

$$\bar{w}_i \geq \tilde{f}_i\left(\sum_{j=1}^{m+2} a_i^j x_i^j\right) \geq \hat{f}_i\left(\sum_{j=1}^{m+2} a_i^j x_i^j\right) - \rho_i \quad . \tag{15}$$

By combining Eqs. 12 through 15, we obtain

$$\sum_{i\notin\bar{I}} h_i(\bar{x}_i) + \sum_{i\varepsilon\bar{I}} \tilde{h}_i\left(\sum_{j=1}^{m+2} \alpha_i^j x_i^j\right) \leq b \tag{16}$$

$$\sum_{i\notin\bar{I}} f_i(\bar{x}_i) + \sum_{i\varepsilon\bar{I}} \hat{f}_i\left(\sum_{j=1}^{m+2} a_i^j x_i^j\right) \leq \sup(D) + \sum_{i\varepsilon\bar{I}} \rho_i \quad . \tag{17}$$

Given any $\varepsilon>0$ and $i\varepsilon\bar{I}$, we can find (using Assumption A3) a vector $\bar{x}_i\varepsilon X_i$ such that (compare with Eq. 13)

$$f_i(\bar{x}_i) \leq \hat{f}\left(\sum_{j=1}^{m+2} \alpha_i^j x_i^j\right) + \varepsilon \quad , \quad h_i(\bar{x}_i) \leq \tilde{h}_i\left(\sum_{j=1}^{m+2} \alpha_i^j x_i^j\right) \quad .$$

These relations together with Eqs. 16 and 17 yield

$$\sum_{i=1}^{I} h_i(\bar{x}_i) \leq b \quad , \tag{18}$$

$$\sum_{i=1}^{I} f_i(\bar{x}_i) \leq \sup(D) + \sum_{i\varepsilon\bar{I}} (\rho_i+\varepsilon) \quad , \tag{19}$$

Since by Eq. 18, $(\bar{x}_1,\ldots,\bar{x}_I)$ is a feasible vector for (P), we have $\inf(P)\leq\Sigma_{i=1}^I f_i(\bar{x}_i)$, and Eq. 19 yields

$$\inf(P) \leq \sup(D) + \sum_{i\varepsilon\bar{I}} (\rho_i+\varepsilon) \quad . \tag{20}$$

Since ε is arbitrary, $\bar{I}$ contains at most (m+1) elements, and $E = \max\{\rho_i | i = 1,\ldots,I\}$, Eq. 20 proves the desired estimate. Q.E.D.

The significance of the proposition lies in the fact that the estimate (m+1)E depends only on m and E but not on I. Thus if we consider instead of problem (P), the problem

$$\text{minimize} \quad \frac{1}{I} \sum_{i=1}^{I} f_i(x_i)$$

$$\text{subject to} \quad x_i\varepsilon X_i \quad , \quad \sum_{i=1}^{I} h_i(x_i) \leq b \quad ,$$

the objective function of which represents "average cost per term," the duality gap estimate becomes

$$\inf(P) - \sup(D) \leq \frac{m+1}{I} E \quad .$$

Thus the duality gap goes to zero as $I\to\infty$. Otherwise stated, if the optimal value of problem (P) is proportional to I, the ratio of the duality gap over the optimal value goes to zero as $I\to\infty$.

Experimental validation of the results of the Proposition may be found in refs. [4],[5], and [8].

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