

A STOCHASTIC AND DYNAMIC VEHICLE ROUTING PROBLEM IN THE EUCLIDEAN PLANE

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We propose and analyze a generic mathematical model for dynamic, stochastic vehicle routing problems, the dynamic traveling repairman problem (DTRP). The model is motivated by applications in which the objective is to minimize the wait for service in a stochastic and dynamically changing environment. This is a departure from classical vehicle routing problems where one seeks to minimize total travel time in a static, deterministic environment. Potential areas of application include repair, inventory, emergency service and scheduling problems. The DTRP is defined as follows: Demands for service arrive in time according to a Poisson process, are independent and uniformly distributed in a Euclidean service region, and require an independent and identically distributed amount of on-site service by a vehicle. The problem is to find a policy for routing the service vehicle that minimizes the average time demands spent in the system. We propose and analyze several policies for the DTRP. We find a provably optimal policy in light traffic and several policies with system times within a constant factor of the optimal policy in heavy traffic. We also show that the waiting time grows much faster than in traditional queues as the traffic intensity increases, yet the stability condition does not depend on the system geometry.

The traveling salesman problem (TSP) is one of the most studied problems in the operations research and applied mathematics literature. The attention it receives is due both to the problem's richness and inherent elegance and to its frequent occurrence in practical problems, both directly and as a subproblem. Yet, in many practical applications, the TSP is a deterministic, static approximation to a problem which is, in reality, both probabilistic and time varying (dynamic). In addition, there are often costs associated with the wait for delivery that are not captured in the objective of minimizing travel distance.

For example, a prototypical application of the TSP is the routing of a vehicle from a central depot to a set of dispersed demand points to minimize the total travel costs. In real distribution systems, however, orders (demands) arrive randomly in time, and the dispatching of vehicles is a continuous process of collecting demands, forming tours and dispatching vehicles. In such a dynamic setting, the wait for a delivery (service) may be a *more* important factor than the travel cost. Applications, in which the wait for service, rather than the total travel time, is a more suitable objective and the demand pattern is both dynamic and stochastic, include the following:

1. The demands are requests for replenishment of stock (raw materials, merchandise, etc.) from

- remote sites that must be delivered from a central depot. In this case, large waiting times mean that large inventories are needed at the remote sites to prevent stockout.

2. In managing a fleet of taxis, one would like to minimize the average waiting time of customers. Decision makers therefore need good dispatching policies, fleet sizing models and estimates of the level of service.
3. Demands represent requests for emergency service. The objective is therefore to reduce the wait for service rather than to minimize the travel cost of the emergency vehicle. In this case, we want real-time policies that can be applied in a stochastic environment.
4. The demands are geographically dispersed failures that must be serviced by a mobile repairman. The objective in this case is to minimize the downtime (wait plus service time) at the various locations. Examples in this category include servicing of geographically distributed communications or utility networks, automobile road service (AAA), or the dispatching of a roving expert to local sites.
5. Finally, for completeness, consider the problem in which a salesman receives leads randomly in time and wants to make sales calls to minimize the time customers spend contemplating their purchases!

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Motivated by these application areas, we propose and analyze a generic mathematical model which we call the dynamic traveling repairman problem (DTRP). The model has several important characteristics:

1. The objective is to minimize waiting time, not travel cost;
2. Information about future demands is stochastic;
3. The demands vary over time (i.e., they are dynamic);
4. Policies have to be implemented in real time;
5. The problem involves queueing phenomena.

In general, little is known about dynamic versions of the TSP. Psaraftis (1988) defines the dynamic traveling salesman problem (DTSP), which initially motivated our investigation: In a complete graph on n nodes, demands for service are independently generated at each node i according to a Poisson process with parameter λ_i . These demands are to be serviced by a salesman who takes a known time t_{ij} to go from i to j , and spends a stochastic time X , which has a known distribution, servicing each demand (on location). The goal is to find strategies that optimize over some performance measure (waiting time, throughput). By comparison, the DTRP is defined in the Euclidean plane and optimization is over the total system time. No general results were obtained for this problem, but useful insights and conjectures were made.

Some of the characteristics of the DTRP have been considered in isolation in the literature. The first is the objective of minimizing average system time rather than total travel time. In a deterministic setting, this idea appears in the traveling repairman (or delivery) problem (TRP), in which a repair unit has to service a set of demands V starting from a depot. If $d(i, j)$ denotes the travel time from i to j , the problem is to find a tour starting from the depot through the demands to minimize the total waiting time of the demands. As a result, if the sequence in which the repair unit travels is $t = (1, 2, \dots, n, 1)$, then the total waiting time is $W_t = \sum_{i=1}^n w_i$, where $w_i = \sum_{j=i}^n d(j, j+1)$ is the waiting time of the demand i . The problem closely resembles the TSP and can be thought of as the deterministic and static analog of the DTRP. As is the case with the TSP, the TRP is NP-complete both on a graph and in the Euclidean plane (Sahni and Gonzalez 1976). In contrast with the TSP, which is trivial on trees, the TRP seems difficult on trees. Minieka (1987) proposes an exponential $O(n^n)$ algorithm for the TRP on a tree $T = (V, E)$, where $|V| = n$ and p is the number of

leaves in T . Despite its interest and applicability, the problem has not received much attention from the research community. As a result, not much is known about the TRP.

Jaillet (1988), Bertsimas (1988b), and Bertsimas, Jaillet and Odoni (1990) address the second and fourth characteristics under the unifying framework of a priori optimization. They define and analyze the probabilistic traveling salesman problem (PTSP) and the probabilistic vehicle routing problem (PVRP) as follows: There are n known points, and on any given instance of the problem only a subset S consisting of $|S| = k$ out of n points ($0 \leq k \leq n$) must be visited. Suppose that the probability that instance S occurs is $p(S)$. We wish to find a priori a tour through all n points. On any given instance of the problem, the k points present will be visited in the same order as they appear in the a priori tour. The problem of finding such an a priori tour that is of minimum length in the expected value sense is defined as the PTSP. In the case where the vehicle has capacity Q , the corresponding problem is the probabilistic vehicle routing problem (Bertsimas 1988a). It is clear that a real-time policy is followed, but the problem is inherently static and is solved a priori using only probabilistic information.

An important characteristic of the DTRP is that it incorporates queueing phenomena. Queueing considerations in the context of location problems have been considered in Berman et al. (1989), and Batta, Larson and Odoni (1988). In this setting, the authors define the stochastic queue median problem (SQMP) in which the important decision is a strategic one; we would like to locate a server in a network which behaves like an M/G/1 queue. Arrivals occur in a dynamic manner according to a Poisson process, and a server (vehicle), following a first-come, first-served (FCFS) discipline, is dispatched from a central depot and returns to the depot again after service is completed. The problem is to locate the depot on a network so that the mean queueing delay plus travel time is minimized. The model is appropriate for analyzing emergency service systems (e.g., police, fire and ambulance service). In our setting, the SQMP can be seen as a network case of the DTRP in which the policy is to strategically locate the server and then follow an FCFS dispatching rule. The connection between location/queueing problems and dynamic vehicle routing problems was also recognized by Psaraftis, where he conjectures that for low arrival rates, the DTSP resembles the 1-median problem. We analyze the performance of this policy in Section 4.1 for the Euclidean case and show formally that this is indeed true.

Our strategy in analyzing the DTRP is the following: First, we establish some lower bounds on the average system time for all policies. Then, using a variety of techniques from combinatorial optimization, queueing theory, geometrical probability and simulation, we analyze several policies and compare their performance to the lower bounds. A variant of the FCFS policy is shown to be optimal in the case of light traffic. In heavy traffic, several policies are shown to be within a constant factor of the lower bounds and thus from the optimal policy. The policy with the best provable performance guarantee in heavy traffic is one based on forming TSP tours, while the best policy empirically is the nearest neighbor policy. Our results also show that the system time grows much more rapidly with traffic intensity than in traditional queues and the stability condition is independent of the system geometry.

The paper is organized as follows: Since we use a variety of results from several areas, we briefly describe them and give the appropriate references in Section 1. In Section 2, we formally describe the DTRP and introduce the notation. Lower bounds for the optimal system time are derived in Section 3. In Section 4, which is central to the paper, we introduce and analyze several policies for the DTRP. In Section 4.6, an example is given to illustrate the relative performance of the policies. Finally, in Section 5, we summarize the contributions of the paper and give some concluding remarks.

1. PROBABILISTIC AND QUEUEING BACKGROUND

In this section, we briefly describe the results used in the paper.

An Upper Bound for the Waiting Time in a GI/G/1 Queue. In a GI/G/1 queue let $1/\lambda$ be the expected interarrival time and \bar{s} be the expected service time. Let σ_a^2, σ_s^2 denote the variances of the interarrival and service time distribution, respectively. The traffic intensity is $\rho = \lambda\bar{s}$. There is no simple explicit expression for the expected waiting time W in this case. (The average system time T is simply $W + \bar{s}$.) However, Kingman (1962) (see also, Kleinrock 1976b) proves that

$$W \leq \frac{\lambda(\sigma_a^2 + \sigma_s^2)}{2(1 - \rho)}. \tag{1}$$

In addition, this upper bound is asymptotically exact as $\rho \rightarrow 1$. For M/G/1 it is well known (see

Kleinrock 1976a) that

$$W = \frac{\lambda \bar{s}^2}{2(1 - \rho)} \tag{2}$$

where $\bar{s}^2 = \sigma_s^2 + \bar{s}^2$ is the second moment of the service time.

Symmetric Cyclic Queues. Consider a queueing system that consists of k queues Q_1, Q_2, \dots, Q_k each with infinite capacity. Customers arrive at each queue according to independent Poisson processes with the same arrival intensity λ/k . The queues are served by a single server who visits the queues in a fixed cyclic order $Q_1, Q_2, \dots, Q_k, Q_1, Q_2, \dots$. The travel time around the cycle is a constant d . The service times at every queue are independent, identically distributed random variables with mean \bar{s} and second moment \bar{s}^2 . The traffic intensity is $\rho = \lambda\bar{s}$. The server uses the exhaustive service policy, i.e., servicing each queue i until the queue is empty before proceeding. The expected waiting time for this system is given by (see, Bertsekas and Gallager 1987, p. 156):

$$W = \frac{\lambda \bar{s}^2}{2(1 - \rho)} + \frac{1 - \rho/k}{2(1 - \rho)} d. \tag{3}$$

Note that in an asymmetric cyclic queue, in which arrival processes and service times are not identical, there are no closed form expressions for the waiting time (see Ferguson and Aminetzah 1985).

Jensen's Inequality. If f is a convex function and X is a random variable, then

$$E[f(X)] \geq f(E[X]) \tag{4}$$

provided that the expectations exist.

Markov's Inequality. If X is a nonnegative random variable and x is any nonnegative number, then

$$E[X] \geq xP\{X > x\}. \tag{5}$$

Wald's Equation. Let $\{X_i; i \geq 1\}$ be a sequence of i.i.d. random variables with $E[X] < \infty$ and N be a finite-mean random variable with the property that $P\{N = n\}$ is independent of $\{X_i; i > n\}$ for all n . (Such a random variable N is a *stopping time* for the sequence $\{X_i; i \geq 1\}$.) Then

$$E\left[\sum_{i=1}^N X_i\right] = E[N]E[X_i]. \tag{6}$$

Stochastically Larger (Definition). A random variable X is said to be *stochastically larger* than a random

variable Y , denoted $X \geq_{ST} Y$, if

$$P\{X > z\} \geq P\{Y > z\} \quad \text{for all } z. \quad (7)$$

Geometrical Probability. Given two uniformly and independently distributed points X_1, X_2 in a square of area A , then

$$\begin{aligned} E[\|X_1 - X_2\|] &= c_1 \sqrt{A}, \\ E[\|X_1 - X_2\|^2] &= c_2 A \end{aligned} \quad (8)$$

where $c_1 \approx 0.52$, $c_2 = 1/3$ (see, Larson and Odoni 1981, p. 135). If we let x^* denote the center of a square of area A , then it is known (Larson and Odoni) that the first and second moments of the distance to a uniformly chosen point X are given by

$$E[\|X - x^*\|] = c_3 \sqrt{A}, \quad E[\|X - x^*\|^2] = c_4 A \quad (9)$$

where $c_3 = (\sqrt{2} + \ln(1 + \sqrt{2}))/6 \approx 0.383$, $c_4 = 1/6$.

Asymptotic Properties of the TSP in the Euclidean Plane. Let $X_1 \dots X_n$ be independently and uniformly distributed points in a square of area A and L_n denotes the length of the optimal tour through the points. Then there exists a constant, β_{TSP} , such that

$$\lim_{n \rightarrow \infty} \frac{L_n}{\sqrt{n}} = \beta_{TSP} \sqrt{A} \quad (10)$$

with probability one (see, Beardwood, Halton and Hammersley 1959, Steele 1981, and Lawler et al. 1985). In his experimental work with very large-scale TSPs, Johnson (1988) estimates $\beta_{TSP} \approx 0.72$. In addition, it is also well known (see, Lawler et al., p. 189) that $\lim_{n \rightarrow \infty} \text{var}(L_n) = O(1)$, and therefore

$$\lim_{n \rightarrow \infty} \frac{\text{var}(L_n)}{n} = 0. \quad (11)$$

Space Filling Curves. The following results are due to Platzman and Bartholdi (1983). Let $\mathcal{C} = \{\theta \mid 0 \leq \theta < 1\}$ denote the unit circle and $\mathcal{S} = \{(x, y) \mid 0 \leq x < 1, 0 \leq y \leq 1\}$ the unit square. Then there exists a continuous mapping ψ from \mathcal{C} onto \mathcal{S} with the property that for any $\theta, \theta' \in \mathcal{C}$

$$\|\psi(\theta) - \psi(\theta')\| \leq 2\sqrt{|\theta - \theta'|}. \quad (12)$$

If $X_1 \dots X_n$ are any n points in \mathcal{S} and L_n is the length of a tour of these n points formed by visiting them in increasing order of their *preimages* in \mathcal{C} (i.e., increasing θ order), then

$$L_n \leq 2\sqrt{n}. \quad (13)$$

If the points $X_1 \dots X_n$ are independently and uniformly distributed in \mathcal{S} , then there exists a

constant, β_{SPC} , such that

$$\limsup_{n \rightarrow \infty} \frac{L_n}{\sqrt{n}} = \beta_{SFC} \quad (14)$$

with probability one. The value of β_{SFC} is approximately 0.956.

2. PROBLEM DEFINITION AND NOTATION

A convex, bounded region \mathcal{A} of area A contains a vehicle (server) that travels at a constant unit velocity between demand (or customer) locations. Demands for service arrive in time according to a Poisson process with rate λ , and their locations are independent and uniformly distributed in \mathcal{A} . Each demand i requires an independent and identically distributed amount of on-site service with mean duration \bar{s} and second moment \bar{s}^2 . No preemption of on-site service is allowed. We assume that $\bar{s} > 0$. The fraction of time the server spends in on-site service is denoted ρ , and for stable systems $\rho = \lambda \bar{s}$. To simplify the calculations and presentation, we often assume that the region \mathcal{A} is a square of area A . This restriction can usually be relaxed without affecting the results, though numerical calculations may be more difficult.

Let d_i be the travel time from the location of the demand served prior to i to demand i 's location. The quantity d_i can be considered the travel time component of demand i 's *total service requirement*. The steady-state expected value of d_i is denoted \bar{d} and is given by $\bar{d} \equiv \lim_{i \rightarrow \infty} E[d_i]$, where we assume the limit exists.

The system time of demand i , denoted T_i , is defined as the elapsed time between the arrival of demand i and the time the server completes the service of i . The waiting time of demand i , W_i , is defined by $W_i = T_i - s_i$. The steady-state system time, T , is defined by $T \equiv \lim_{i \rightarrow \infty} E[T_i]$ and $W \equiv T - \bar{s}$. The problem is to find a policy for servicing demands that minimizes T , and this optimal system time is denoted T^* . We use system time rather than waiting time because, in relating the DTRP to traditional queueing systems, d_i can mistakenly be interpreted as part of the "service time," which does not correspond to our definition.

A final remark concerning the difference between the DTRP and the M/G/1 queue: In the DTRP, the total service requirement has both a travel and on-site service component. Although the on-site service requirements are independent, the travel times generally are not. As a result, total service requirements are not i.i.d. random variables, and therefore the methodology of the M/G/1 queue is not applicable.

3. LOWER BOUNDS ON THE OPTIMAL DTRP POLICY

We first establish two simple but powerful lower bounds on the optimal expected system time, T^* . In Section 4, we then use these lower bounds to evaluate the performance of the proposed policies.

3.1. A Light Traffic Lower Bound

The first bound for the DTRP is established by dividing the system time of demand i , T_i , into three components: the waiting time due to the server's travel prior to serving i , denoted W_i^d ; the waiting time due to the on-site service times of demands served prior to i , denoted W_i^s ; and demand i 's own on-site service time, s_i . Thus

$$T_i = W_i^d + W_i^s + s_i.$$

Taking expectations and letting $i \rightarrow \infty$ gives

$$T = W^d + W^s + \bar{s} \quad (15)$$

where $W^d \equiv \lim_{i \rightarrow \infty} E[W_i^d]$ and $W^s \equiv \lim_{i \rightarrow \infty} E[W_i^s]$. Note that $W = W^d + W^s$.

To bound W^d , note that it is at least as large as the travel delay (distance) between the server's location at the time of a demand's arrival and its location. In general, the server is located in the region according to some (generally unknown) spatial distribution that depends on the server's policy. Thus, W^d is bounded below by the expected distance between a server location selected from this distribution and a uniform location. Suppose that we have the option of locating the server in the best a priori location, x^* ; that is, the location that minimizes the expected distance to a uniformly chosen location, X . This certainly yields a lower bound on the expected distance between the server and the arrival, so

$$W^d \geq \min_{x_0 \in \mathcal{A}} E[\|X - x_0\|]. \quad (16)$$

The location x^* that achieves this minimization is the *median* of the region \mathcal{A} . For the case where \mathcal{A} is a square, x^* is simply the center of the square, in which the lower bound is from (9)

$$W^d \geq c_3 \sqrt{A} \approx 0.383 \sqrt{A}. \quad (17)$$

To bound W^s , let N denote the expected number of demands served during a waiting time. Since service times are independent, we then have

$$W^s = \bar{s}N + \frac{\lambda \bar{s}^2}{2}$$

where the second term is the expected residual service time of the demand being served at the time of arrival.

Since in steady state the expected number of demands served during a wait is equal to the expected number that arrive, we can apply Little's law to get

$$W^s = \bar{s}\lambda W + \frac{\lambda \bar{s}^2}{2} = \rho W + \frac{\lambda \bar{s}^2}{2}.$$

Since $W = W^d + W^s$ we obtain

$$W^s = \frac{\rho}{1-\rho} (W^d) + \frac{\lambda \bar{s}^2}{2(1-\rho)}. \quad (18)$$

Combining (15), (16), and (18) and noting that these bounds are true for *all* policies we get the following theorem.

Theorem 1

$$T^* \geq \frac{E[\|X - x^*\|]}{1-\rho} + \frac{\lambda \bar{s}^2}{2(1-\rho)} + \bar{s}$$

where x^* is the median of region \mathcal{A} . For the special case where \mathcal{A} is a square

$$E[\|X - x^*\|] = c_3 \sqrt{A} \approx 0.383 \sqrt{A}.$$

As shown below, this bound is most useful in the case of light traffic ($\lambda \rightarrow 0$).

3.2. A Heavy Traffic Lower Bound

A lower bound that is most useful for $\rho \rightarrow 1$ is provided by the following theorem.

Theorem 2. *There exists a constant $\gamma = 2/3\sqrt{2}\pi \approx 0.266$ such that*

$$T^* \geq \gamma^2 \frac{\lambda A}{(1-\rho)^2} - \frac{1-\rho}{2\lambda}. \quad (20)$$

Proof. First, suppose that for *all* service policies the following bound is known:

$$\bar{d} \geq \gamma \frac{\sqrt{A}}{\sqrt{N + 1/2}} \quad (21)$$

where N is the average number of demands in queue. Then the stability condition

$$\bar{s} + \bar{d} \leq \frac{1}{\lambda} \quad (22)$$

implies

$$\bar{s} + \frac{\gamma \sqrt{A}}{\sqrt{N + 1/2}} \leq \frac{1}{\lambda}.$$

After rearranging, and noting that $T = W + \bar{s}$ and $N = \lambda W$, we obtain the bound of Theorem 2. So the theorem is established once (21) is proven.

To prove (21), consider a random “tagged” arrival and define:

- $\mathcal{S}_0 \equiv$ The set of locations of demands that are in queue at the time of the tagged demand’s arrival plus the server’s location;
- $\mathcal{S}_1 \equiv$ The set of locations of the demands that arrive during the tagged demand’s waiting time ordered by their time of arrival;
- $X_0 \equiv$ The tagged demand’s location;
- $N_i \equiv |\mathcal{S}_i| \ i = 0, 1;$
- $Z_0^* \equiv \min_{x \in \mathcal{S}_0} \|x - X_0\|.$

Furthermore, define $Z_i \equiv \|X_i - X_0\|$, where X_i is the location of the i th demand to arrive after the tagged demand (e.g., $\mathcal{S}_1 = \{X_1, X_2, \dots, X_{N_1}\}$). Note that $\{Z_i; i \geq 1\}$ are i.i.d. with

$$P\{Z_i \leq z\} \leq \frac{\pi z^2}{A} \tag{23}$$

and N_1 is a stopping time for the sequence $\{Z_i; i \geq 1\}$.

The set of locations from which the server can visit the tagged demand is at most $\mathcal{S}_0 \cup \mathcal{S}_1$; therefore, the travel time component of the tagged demand’s total service requirement is at least $\min\{Z_0^*, Z_1, \dots, Z_{N_1}\}$. Hence

$$\bar{d} \geq E[\min\{Z_0^*, Z_1, \dots, Z_{N_1}\}]. \tag{24}$$

We next bound the right-hand side of (24). To do so, define an indicator variable for the random variable X by

$$I_x = \begin{cases} 1 & \text{if } X \leq z \\ 0 & \text{if } X > z \end{cases}$$

where z is a positive constant to be determined. Then

$$\begin{aligned} &P\{\min\{Z_0^*, Z_1, \dots, Z_{N_1}\} > z\} \\ &= P\left\{I_{Z_0^*} + \sum_{i=1}^{N_1} I_{Z_i} = 0\right\} \\ &= 1 - P\left\{I_{Z_0^*} + \sum_{i=1}^{N_1} I_{Z_i} > 0\right\} \\ &\geq 1 - E\left[I_{Z_0^*} + \sum_{i=1}^{N_1} I_{Z_i}\right] \quad (I_x \geq 0, \text{ Integer}) \\ &= 1 - E[I_{Z_0^*}] - E[N_1]E[I_{Z_i}] \quad (\text{Wald’s equation}). \end{aligned}$$

Since $E[N_1] = P\{Z_i \leq z\}$ is bounded according to (23), and $E\{I_{Z_0^*}\} = P\{Z_0^* \leq z\}$ we obtain

$$\begin{aligned} &P\{\min\{Z_0^*, Z_1, \dots, Z_{N_1}\} > z\} \\ &\geq 1 - P\{Z_0^* \leq z\} - N \frac{\pi z^2}{A}. \end{aligned} \tag{25}$$

An upper bound on $P\{Z_0^* \leq z\}$ is provided by the following lemma.

Lemma 1

$$P\{Z_0^* \leq z\} \leq \frac{\pi z^2}{A} (N + 1).$$

Proof. The proof relies on the result due to Haimovich and Magnanti (1988) for the k -median problem: Let \mathcal{S} be any set of points in \mathcal{A} with $|\mathcal{S}| = k$, X be a uniformly distributed location in \mathcal{A} independent of \mathcal{S} and define

$$Z^* \equiv \min_{x \in \mathcal{S}} \| \mathcal{S} \| X - x \|.$$

Define the random variable Y to be the distance from the center of a circle of area A/k to a uniformly distributed point within the circle. Then for all nondecreasing functions f

$$E[f(Z^*)] \geq E[f(Y)].$$

An immediate consequence of this (see, for example Ross 1983) is that $Z^* \geq_{ST} Y$. As a result

$$P\{Z^* \leq z\} \leq P\{Y \leq z\} \leq \frac{\pi z^2}{A} k$$

where the last inequality follows from the definition of Y .

Consider conditioning on N_0 , and note that X_0 is independent of \mathcal{S}_0 under any condition on \mathcal{S}_0 . Therefore, from the above result

$$P\{Z_0^* \leq z \mid N_0\} \leq \frac{\pi z^2}{A} N_0.$$

Unconditioning and observing that $E[N_0] = N + 1$ establishes the lemma.

Using the result of Lemma 1 in (25) and the trivial bound $P\{\cdot\} > 0$ we obtain

$$\begin{aligned} &E[\min\{Z_0^*, Z_1, \dots, Z_{N_1}\}] \\ &\geq \int_0^{(A/\pi(2N+1))^{1/2}} \left(1 - \frac{\pi z^2}{A} (2N + 1)\right) dz \\ &= \frac{2}{2\sqrt{2\pi}} \frac{\sqrt{A}}{\sqrt{N + Y_2}}, \end{aligned}$$

which establishes (21) with $\gamma = 2/3\sqrt{2\pi} \approx 0.266$, and thus Theorem 2 is proven.

A few comments on the lower bound of Theorem 2 are in order. First, it shows that the waiting time grows at least as fast as $(1 - \rho)^{-2}$ rather than $(1 - \rho)^{-1}$, as is the case for a classical queueing system. Also, it is only

a function of the first moment of the on-site service time, which again is a significant departure from traditional queueing system behavior (e.g., the M/G/1 system). The explanation lies in the geometry of the system. The bound of Theorem 2 gives (via Little's theorem) the minimum average number of demands that must be maintained in the system to ensure that the average travel distance, \bar{d} , satisfies the stability condition (22). This number, however, grows much more rapidly than the average number in the system due to traditional queueing delays.

Because several loose assumptions were used in the proof (e.g., \mathcal{S}_0 is the set of N_0 -median locations, etc.), it is likely that the value $\gamma \approx 0.266$ is not tight. For example, if one assumes that locations of demands at service completion epochs are approximately uniform, then by a modified argument a value of $\gamma = 1/2$ is obtained. We conjecture that Theorem 2 remains true even for this larger value of γ .

Finally, we have also developed a slightly simpler proof of (21) that does not require any stopping time arguments. We do not include it, however, because the resulting constant value is weaker.

4. SOME PROPOSED POLICIES FOR THE DTRP

In this section, we propose and analyze several policies for the DTRP. The first class of policies is based on variants of the FCFS discipline. We show that one such policy is optimal in light traffic, in the sense that it asymptotically achieves the light traffic lower bound of the last section for $\lambda \rightarrow 0$. These policies, however, are unstable for high utilizations; therefore, we turn next to a partitioning policy based on subdividing the large square \mathcal{A} into smaller squares, each of which is served locally using an FCFS discipline. Using results on cyclic queues, we show that this policy is within a constant factor of the lower bounds for all values of $\rho < 1$. This also establishes $\rho < 1$ as a sufficient (as well as, obviously, necessary) condition for stability in the sense that there exist stable policies for every $\rho < 1$. We next introduce a more sophisticated policy based on forming successive TSP tours. Its average system time is nearly half that of the partitioning policy. Next, we examine a policy based on space filling curves. It too has a constant factor performance guarantee and is shown via simulation to perform about 15% better than the TSP policy. Finally, we examine the policy of serving the nearest neighbor. Because of analytical difficulties, we simulate it and show that the average system time is about 10% lower than the SFC policy.

4.1. FCFS Policies

The simplest policy for the DTRP is to service demands in the order in which they arrive (FCFS). The first policy we examine of this type is defined as: 1) when demands are present, the server travels directly from one demand location to the next following an FCFS order, and 2) when no unserved demands are present following a service completion, the server waits until a new demand arrives before moving.

Because demand locations are independent of the order of arrivals and the number of demands in queue, the system behaves like an M/G/1 queue. Note that the travel times d_i are not strictly independent (e.g., consider the case $d_i = \sqrt{2A}$); however, it is true that they are identically distributed, because each d_i is simply the distance between two independent, uniformly distributed locations in \mathcal{A} . Therefore, the Pollaczek-Khinchin (P-K) formula (2) still holds. (See Bertsekas and Gallager 1987, pages 142–143 for a proof of the P-K formula under these assumptions.)

The first and second moments of the total service requirement are, by (8), $\bar{s} + c_1\sqrt{A}$ and $\bar{s}^2 + 2c_1\sqrt{A}\bar{s} + c_2A$, respectively, where $c_1 \approx 0.52$, $c_2 = 1/3$. The average system time is therefore, by the P-K formula (2):

$$T_{\text{FCFS}} = \frac{\lambda(\bar{s}^2 + 2c_1\sqrt{A}\bar{s} + c_2A)}{2(1 - \lambda c_1\sqrt{A} - \rho)} + \bar{s} + c_1\sqrt{A}. \quad (26)$$

The stability condition for this policy is $\rho + \lambda c_1\sqrt{A} < 1$; therefore, this policy is unstable for values of ρ approaching 1. For $\lambda \rightarrow 0$, the first term in (26) approaches zero. Likewise, the second term in Theorem 1 also approaches zero as $\lambda \rightarrow 0$. So for the light traffic case we have

$$\frac{T_{\text{FCFS}}}{T^*} \leq \frac{\bar{s} + c_1\sqrt{A}}{\bar{s} + c_3\sqrt{A}} \quad \text{as } \lambda \rightarrow 0.$$

Since \bar{s} could be arbitrarily small, the worst case relative performance for this policy in light traffic is $T_{\text{FCFS}}/T^* \leq c_1/c_3 \approx 1.36$.

The FCFS policy can be modified to yield asymptotically optimal performance in light traffic as follows: Consider the policy of locating the server at the median of \mathcal{A} and following an FCFS policy, where the server travels directly to the service site from the median, services the demand, and then returns to the median after service is completed. We call this policy the stochastic queue median policy (SQM). As before, the server waits at the median if no demands are present in the system. Again, since locations are independent of the order of arrival and the number in queue, the system behaves as an M/G/1 queue;

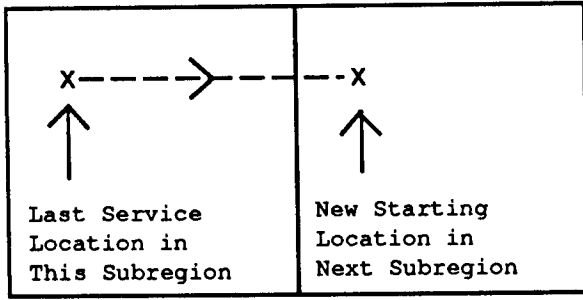


Figure 2. PART projection policy for moving to adjacent subregion.

To simplify the analysis, we use only the expression for even m . As shown below, m must be large in heavy traffic, so for $\rho \rightarrow 1$ the relative error in total travel distance is negligible.

Each subregion behaves as an M/G/1 queue with an arrival rate of λ/m^2 , and first and second moments of $\bar{s} + c_1(\sqrt{A}/m)$ and $\bar{s}^2 + 2c_1\bar{s}(\sqrt{A}/m^2) + c_2(A/m^2)$, respectively, ($c_1 \approx 0.52, c_3 = 1/3$). The policy as a whole behaves as a cyclic queue with $k = m^2$ queues and exhaustive service, where the total travel time around the cycle is $m\sqrt{A}$ and the queue parameters are those given before. Again, as with the FCFS policy, the travel times are not mutually independent. However, they are identically distributed and independent of the number in queue. Therefore, the analysis in Bertsekas and Gallager still holds. Recalling that the expression in (3) is for the waiting time in queue only, the average system time for this policy is given by

$$T_{\text{PART}} = \frac{\lambda(\bar{s}^2 + 2c_1\bar{s}(\sqrt{A}/m) + c_2(A/m^2))}{2(1 - \lambda(\bar{s} + c_1(\sqrt{A}/m))} + \frac{1 - (\lambda/m^2)(\bar{s} + c_1(\sqrt{A}/m))}{2(1 - \lambda(\bar{s} + c_1(\sqrt{A}/m))} m\sqrt{A} + c_1 \frac{\sqrt{A}}{m} + \bar{s}. \tag{29}$$

The stability condition is

$$\lambda\left(\bar{s} + c_1 \frac{\sqrt{A}}{m}\right) < 1 \Leftrightarrow m > \frac{c_1 \lambda \sqrt{A}}{1 - \rho}.$$

Defining the critical value m_c by

$$m_c \equiv \frac{c_1 \lambda \sqrt{A}}{1 - \rho} \tag{30}$$

the stability condition becomes $m > m_c$. Note that for any $\rho < 1$ we can find an $m > m_c$, such that this policy is stable. Since the optimal policy has a waiting time no greater than the PART policy, we have the following theorem.

Theorem 4. *There exists a DTRP policy that has a finite waiting time for all $\rho < 1$ (the PART policy) and, hence, there exists an optimal policy for all $\rho < 1$.*

This establishes $\rho < 1$ as a sufficient condition for stability. Furthermore, since ρ is determined only by the on-site service mean and the arrival rate, we see that the service region characteristics (size, shape, etc.) do not affect the amount of traffic the system can support (provided the service region is bounded, of course).

For given system parameters, $\lambda, \bar{s}, \bar{s}^2$ and A , one could perform a one-dimensional optimization over $m > 1$ using (29) to get the optimum number of partitions; however, since (29) is quite complicated, we concentrate on finding the optimal value, m^* , for the heavy traffic case.

For ($\rho \rightarrow 1$), (30) implies that any feasible m is large ($m > m_c$). Therefore, ignoring the $O(1/m)$ and smaller terms in the numerators of (29) we obtain

$$T_{\text{PART}} \approx \frac{\lambda\bar{s}^2 + m\sqrt{A}}{2(1 - \rho - \lambda c_1 \sqrt{A}/m)} = \frac{m^2\sqrt{A} + m\lambda\bar{s}^2}{2(m(1 - \rho) - \lambda c_1 \sqrt{A})}. \tag{31}$$

Differentiating (31) with respect to m and setting the result equal to zero, we get the critical points:

$$\frac{\lambda c_1 \sqrt{A} \pm \sqrt{\lambda^2 c_1^2 A + (1 - \rho)\lambda^2 c_1 \bar{s}^2}}{1 - \rho}.$$

Only the positive root is feasible. For $\rho \rightarrow 1$ the second term under the radical approaches zero; therefore

$$m^* \approx \frac{2\lambda c_1 \sqrt{A}}{1 - \rho} = 2m_c.$$

If we substitute this value into (31), then in heavy traffic

$$T_{\text{PART}} \approx 2c_1 \frac{\lambda A}{(1 - \rho)^2} + \frac{\lambda\bar{s}^2}{1 - \rho}. \tag{32}$$

For $\rho \rightarrow 1$, the first term in (32) dominates; therefore (recalling the bound in Theorem 2) we have

$$\frac{T_{\text{PART1}}}{T^*} \leq \frac{2c_1}{\gamma^2} \text{ as } \rho \rightarrow 1. \tag{33}$$

This says that the PART system time is within a constant factor of the optimum in heavy traffic, though the provable factor is indeed quite large (about 15). If we use the conjectured value of $\gamma = 1/2$, the factor is a more reasonable 4.2.

4.3. The Traveling Salesman Policy

The traveling salesman policy (TSP for short) is based on collecting demands into sets that can then be served using an optimal TSP tour. Let \mathcal{N}_k denote the k th set of n demands to arrive, where n is a given constant that parameterizes the policy, e.g., \mathcal{N}_1 is the set of demands $1, \dots, n$, \mathcal{N}_2 is the set of demands $n + 1, \dots, 2n$, etc. Assume that the server operates out of a depot at a random location in \mathcal{A} . When all demands in \mathcal{N}_1 have arrived, we form a TSP tour on these demands that starts and ends at the depot. Demands are then serviced by following the tour. If all demands in \mathcal{N}_2 have arrived when the tour of \mathcal{N}_1 is completed, they are serviced using a TSP tour; otherwise, the server waits until all \mathcal{N}_2 demands arrive before serving it. In this manner, sets are serviced in an FCFS order. Observe that queuing of sets can occur.

Suppose that one considers the set \mathcal{N}_k to be the k th "customer." Since the interarrival time (the time for n new demands to arrive) and service time (n on-site services plus the travel time around the tour) of sets are i.i.d., the service of sets forms a GI/G/1 queue, where the interarrival distribution is an Erlang of order n . The mean and variance of the interarrival times for sets are n/λ and n/λ^2 , respectively. The service time of sets is the sum of the travel time around the tour, which we denote L_n , and the n on-site service times. If we let $E[L_n]$ and $\text{var}[L_n]$ denote, respectively, the mean and variance of L_n , then the expected value of the service time of a set is $E[L_n] + n\bar{s}$ and the variance is $\text{var}(L_n) + n\sigma_s^2$, where $\sigma_s^2 = \bar{s}^2 - \bar{s}^2$ is the variance of the on-site service time.

We are now in a position to apply the GI/G/1 upper bound (1) for the averaging waiting time of sets, W_{set} . This gives

$$W_{\text{set}} \leq \frac{\frac{\lambda}{n} \left(\frac{n}{\lambda^2} + \text{var}[L_n] + n\sigma_s^2 \right)}{2 \left(1 - \frac{\lambda}{n} (E[L_n] + n\bar{s}) \right)} \quad (34)$$

$$= \frac{\lambda \left(\frac{1}{\lambda^2} + \frac{\text{var}[L_n]}{n} + \sigma_s^2 \right)}{2 \left(1 - \rho - \lambda \frac{E[L_n]}{n} \right)}. \quad (35)$$

As we show below, in order for the policy to be stable in heavy traffic n has to be large. Thus, because the locations of points are uniform and i.i.d. in the region, we have from the asymptotic results for

the TSP (10) and (11) that

$$\frac{E[L_n]}{n} \approx \beta_{\text{TSP}} \frac{\sqrt{A}}{\sqrt{n}} \quad (36)$$

and

$$\frac{\text{var}[L_n]}{n} \approx 0 \quad (37)$$

where the approximations become exact for $n \rightarrow \infty$. In order to simplify the final expressions, we neglect the difference between $n + 1$ and n in the above expressions. (The tour includes n points plus the depot.) Since n is large, the difference is negligible. Therefore, for large n

$$W_{\text{set}} \leq \frac{\lambda(1/\lambda^2 + \sigma_s^2)}{2(1 - \rho - \lambda\beta_{\text{TSP}}(\sqrt{A}/\sqrt{n}))}. \quad (38)$$

For stability, we require $\rho + \lambda\beta_{\text{TSP}}(\sqrt{A}/\sqrt{n}) < 1$, which implies

$$n > \frac{\lambda^2 \beta_{\text{TSP}}^2 A}{(1 - \rho)^2}. \quad (39)$$

For $\rho \rightarrow 1$ this implies that n must be large, and thus, our use of asymptotic TSP results is indeed justified.

The waiting time given in (38) is not itself an upper bound on the wait for service of an individual demand; it is the wait in queue for a set. The time of arrival of a set is actually the time of arrival of the last demand in that set. Therefore, we must add to (38) the time a demand waits for its set to form and the time it takes to complete service of the demand once its set enters service. By conditioning on the position that a given demand takes within its set, it is easy to show that the average wait for a demand's set to form is $(n - 1)/2\lambda \leq n/2\lambda$. By doing the same conditioning and noting that the travel time around the tour is no more than the length of the tour itself, the expected wait for service once a demand's set enters service is no more than $\beta_{\text{TSP}}\sqrt{nA} + (1/n) \sum_{k=1}^n k\bar{s} \leq \beta_{\text{TSP}}\sqrt{nA} + (n/2)\bar{s}$. Therefore, if the total system time is denoted T_{TSP} :

$$T_{\text{TSP}} \leq \frac{\lambda(1/\lambda^2 + \sigma_s^2)}{2 \left(1 - \rho - \lambda\beta_{\text{TSP}} \frac{\sqrt{A}}{\sqrt{n}} \right)} + \frac{n(1 + \rho)}{2\lambda} + \beta_{\text{TSP}}\sqrt{nA}. \quad (40)$$

We would like to minimize (40) with respect to n to get the least upper bound. (One can verify that (40) is convex, so there is indeed a minimum.) First,

however, consider a change of variable

$$y = \frac{\lambda\beta\sqrt{A}}{(1-\rho)\sqrt{n}}.$$

Physically, y represents a ratio of the average distance $\bar{d} = \beta_{\text{TSP}}\sqrt{A}/\sqrt{n}$ to its critical value $(1-\rho)/\lambda$. With this change:

$$T_{\text{TSP}} \leq \frac{\lambda(1/\lambda^2 + \sigma_s^2)}{2(1-\rho)(1-y)} + \frac{\lambda\beta_{\text{TSP}}^2 A(1+\rho)}{2(1-\rho)^2 y^2} + \frac{\lambda\beta_{\text{TSP}}^2 A}{(1-\rho)y}. \quad (41)$$

For $\rho \rightarrow 1$, one can verify that the optimum y approaches 1. Therefore, by linearizing the last two terms about $y = 1$, an approximate optimum value, y^* , is

$$y^* \approx 1 - \frac{\sqrt{(1/\lambda^2 + \sigma_s^2)(1-\rho)}}{2\beta_{\text{TSP}}\sqrt{A}}.$$

Substituting this approximation into (41) and noting that for $\rho \rightarrow 1$ the approximate y^* approaches 1 we have

$$T_{\text{TSP}} \leq \beta_{\text{TSP}}^2 \frac{\lambda A}{(1-\rho)^2} + \frac{\beta_{\text{TSP}}\lambda(\sqrt{A(1/\lambda^2 + \sigma_s^2)})}{(1-\rho)^{3/2}} + \frac{\beta_{\text{TSP}}^2 \lambda A}{(1-\rho)} \rho \rightarrow 1.$$

Again, the leading term is proportional to $\lambda A/(1-\rho)^2$. Therefore, using Theorem 2

$$\frac{T_{\text{TSP}}}{T^*} \leq \frac{\beta_{\text{TSP}}^2}{\gamma^2} \rho \rightarrow 1.$$

The best estimate to date of β_{TSP} is approximately 0.72 (Johnson), so the TSP policy has a system time in heavy traffic about half that of the partitioning policy. (In practice, heuristic rather than optimal tours would be used to reduce the computational burden, which would produce slightly higher system times.) These results suggest that the policy of forming successive TSP tours, which is reasonable in practice, is quite good theoretically. In addition to providing a theoretical guarantee, the results give a practical means of optimally sizing routes for such policies by either minimizing the right-hand side of (40) or using the approximate y^* .

4.4. The Space Filling Curve Policy

We next analyze a policy based on space filling curves which we call the SFC policy. It was first proposed by Bartholdi and Platzman (1988). The reader is encouraged to re-examine Section 1 for notation and basic

results related to space filling curves. Let \mathcal{E} and ψ be defined as in Section 1, and the DTRP service region, \mathcal{A} , be a square of area A . Suppose that we maintain the preimages of all demands in the system (i.e., their positions in \mathcal{E}). Then the SFC policy is to service demands as they are encountered in repeated clockwise sweeps of the circle \mathcal{E} . (Note that one could treat a depot as a permanent “demand” and visit it once per sweep.)

We now analyze this policy. Consider a randomly tagged arrival and let W_0 denote the waiting time of the tagged arrival, \mathcal{N}_0 denote the set of locations of the $N_0 \equiv |\mathcal{N}_0|$ demands served prior to the tagged demand, and L denote the length of the path from the server’s location through the points in \mathcal{N}_0 to the tagged demand’s location which is induced by the SFC rule. Finally, let s_i be the on-site service time of demand $i \in \mathcal{N}_0$, and R be the residual service time of the demand under service. Then

$$W_0 = \sum_{i=1}^{N_0} s_i + L + R.$$

Taking expectation on both sides gives

$$W = E[N_0]\bar{s} + E[L] + \frac{\lambda\bar{s}^2}{2}. \quad (42)$$

Since in steady state the expected number of demands served during a wait equals the expected number who arrive, $E[N_0] = N = \lambda W$. Also, since L is the length of a path through $N_0 + 2$ points in \mathcal{A} , $L \leq$

$2\sqrt{(N_0 + 2)A}$. Therefore

$$\begin{aligned} E[L] &\leq 2E[\sqrt{(N_0 + 2)A}] \\ &\leq 2\sqrt{(N + 2)A} \quad (\text{Jensen's inequality}) \\ &\leq 2\sqrt{\lambda WA} + 2\sqrt{2A}. \end{aligned} \quad (43)$$

Substituting these results into (42) we obtain the quadratic inequality:

$$W - \frac{2\sqrt{\lambda A}}{1-\rho}\sqrt{W} - \frac{\lambda\bar{s}^2 + 4\sqrt{2A}}{2(1-\rho)} \leq 0.$$

Solving for W and recalling that $T = W + \bar{s}$ it is straightforward to show that

$$T_{\text{SFC}} \leq \gamma_{\text{SFC}}^2 \frac{\lambda A}{(1-\rho)^2} + o((1-\rho)^{-2}) \quad (44)$$

where $\gamma_{\text{SFC}} \leq 2$ and $o((1-\rho)^{-2})$ denote terms that increase more slowly than $(1-\rho)^{-2}$ as $\rho \rightarrow 1$. This shows that the SFC policy is within a constant factor of optimal. The constant γ_{SFC} obtained by this argument, however, is based on worst case tours and is

probably too large. If one assumes that the clockwise interval between the preimages of the server and the tagged demand is a uniform $[0, 1]$ random variable and the \mathcal{N}_0 points are approximately uniformly distributed on this interval, then a constant of $\gamma_{\text{SFC}} \approx \frac{2}{3} \beta_{\text{SFC}} \approx 0.64$ is obtained.

To estimate γ_{SFC} more precisely, we performed simulation experiments. The method of *batch means* (see Law and Kelton 1982) was used to estimate the steady-state value of T_{SFC} . In this method, demands are grouped into batches of a fixed size. If the batch size is large enough, the sample means from each batch are approximately uncorrelated and normally distributed (Law and Carson 1979). (We use 200 times the minimum average number in the system given by Theorem 2 as our batch size.) The sample mean and variance of the individual batch means were then used in *t*-test to estimate T_{SFC} . The simulation was terminated when the 99% confidence interval about the estimate reached a width less than 10% of the value of the estimate. This method was selected because the busy periods of the SFC policy were quite long (indeed, almost nonterminating) at high utilization values, which precluded the use of techniques based on regeneration points.

The simulation was run for $A = 1$ and a range of parameter values ρ , \bar{s} and \bar{s}^2 . Figure 3 shows one example of the simulation estimate of T_{SFC} plotted against $\lambda A / (1 - \rho)^2$ for the case $A = 1$, $\bar{s} = 0.1$ and $\bar{s}^2 = 0.01$ (zero variance). Each point is a different value of ρ in the range 0.5–0.8. The results show that γ_{SFC} is approximately 0.66, which is very close to the

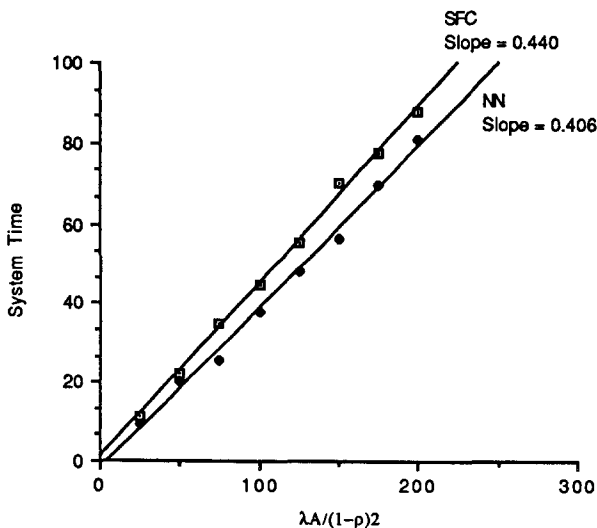


Figure 3. Simulation results: T_{SFC} and T_{NN} versus $\lambda A / (1 - \rho)^2$.

approximate value of $\frac{2}{3} \beta_{\text{SFC}}$. The system time for this policy is therefore about 15% lower than that of the TSP policy. It is also much more computationally efficient.

4.5. The Nearest Neighbor Policy

The last policy we consider is to serve the closest available demand after every service completion (the nearest neighbor (NN) policy). The motivations for considering such a policy are: 1) the nearest neighbor was used in the heavy traffic lower bound on Theorem 2, and 2) the shortest processing time (SPT) rule is known to be optimal for the classical M/G/1 queue (Conway, Maxwell and Miller 1967). As mentioned, however, the travel component of service times in the DTRP depends on the service sequence, so the classical M/G/1 results are not directly applicable; they are only suggestive.

Because of the dependencies among the travel distances d_i , we were unable to obtain rigorous analytical results for the NN policy. However, if one assumes there exists a constant γ_{NN} such that

$$E[d_i | N_T] \leq \gamma_{\text{NN}} \frac{\sqrt{A}}{\sqrt{N_T}} \quad (45)$$

where N_T is the number of demands in the system at a completion epoch, then by using a modification of the argument in (Kleinrock 1976b) Section 5.5, it is possible to show that

$$T_{\text{NN}} \leq \gamma_{\text{NN}}^2 \frac{\lambda A}{(1 - \rho)^2} \quad \rho \rightarrow 1$$

where T_{NN} denotes the system time of the NN policy. Assumption (45) is analogous to (21), but unlike (21), has not been established formally.

We therefore perform simulation experiments identical to those for the SFC policy to verify the asymptotic behavior of T_{NN} and estimate γ_{NN} . The results show that γ_{NN} is approximately 0.64. (See Figure 3.) This means that T_{NN} is about 10% lower than T_{SFC} and about 20% lower than T_{TSP} .

The results again confirm that the system time T_{NN} follows the $\lambda A / (1 - \rho)^2$ growth predicted by the lower bound in Theorem 2. Figure 3 clearly shows this highly linear relationship.

4.6. A Numerical Example

To illustrate the relative performance of the various DTRP policies, the system time of each policy was calculated (simulated in the case of the SFC and NN policies) for the case $A = 1$, $\bar{s} = 0.1$ and $\bar{s}^2 = 0.01$ (zero variance) for a range of values of ρ . For the

parameterized policies (PART and TSP), we perform numerical optimization to find the best parameter for each value of ρ . The results show that the FCFS, SQM, SFC and NN policies perform well in light traffic, but the FCFS and SQM policies were unstable for $\rho > 0.2$. The PART, TSP, SFC and NN policies perform best in heavy traffic. Results for each group are graphed separately.

Figure 4 shows system times as a function of ρ for the light traffic case. The lower bound is also included. Note that although the SQM policy is asymptotically optimal as $\rho \rightarrow 0$, it is quickly surpassed by the FCFS policy as ρ increases. This is due to the extra travel distance of the SQM policy, which hinders the policy as queuing sets in. Also note that both policies reach their saturation points for relatively low values of ρ . The SFC and NN policies were comparable to the FCFS policy in very light traffic, which is to be expected because they essentially behave like the FCFS policy in this case. For $\rho > 0.05$, the SFC and NN policies quickly surpass the FCFS and SQM policies. Notice that the NN policy consistently performs better than the SFC policy even in the light traffic cases.

The heavy traffic results are shown in Figure 5. Note that the curves have nearly identical shapes as one would expect from the $\lambda A / (1 - \rho)^2$ asymptotic behavior of each policy. (Only the constant of proportionality differs.) The graphs show the sharp increase in system time as the traffic intensity increases. The NN policy is the best in this case with the SFC a close second best. The TSP and especially the PART policy are less effective.

This example suggests that both the SFC and NN policies are effective over a wide range of traffic intensities. Indeed, if one locates a depot at the median of the region \mathcal{A} and treats it as a permanent "demand,"

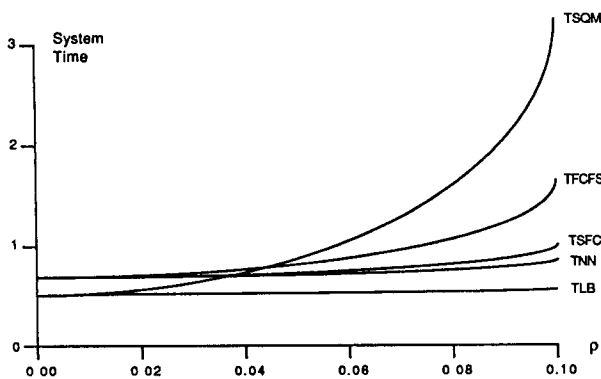


Figure 4. System times for light traffic case: numerical example.

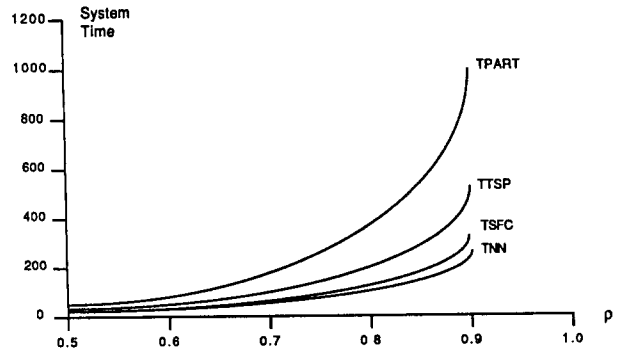


Figure 5. System times for heavy traffic policies: numerical example.

then both these policies can be made to behave like the SQM policy as $\rho \rightarrow 1$. These policies also have the advantage of being nonparametric (i.e., the system parameters are not needed to implement them as is the case for the TSP and PART policies), and they are therefore self-regulating. This feature is especially desirable for systems that operate under highly variable and/or unpredictable traffic conditions.

5. CONCLUDING REMARKS

We present a new model for dynamic vehicle routing problems that attempts to capture the dynamic and stochastic environment in which real-world systems operate. It constitutes a significant departure from traditional static and deterministic models. We suggest several application areas for which this model is appropriate. We derive lower bounds on the optimal system time and characterize the performance of several diverse policies.

The stochastic queue median policy, in which we strategically locate a depot and then follow an FCFS service order, was shown to be optimal in light traffic. As the traffic intensity increases, however, FCFS policies become unstable. We then show that the partitioning policy behaves reasonably well in heavy traffic because it has a constant factor performance guarantee and a finite system time for all values of $\rho < 1$.

In heavy traffic, the best policies were the TSP, SFC and NN. The SFC and NN policies have a desirable self-regulating behavior, while the TSP policy has the advantage of returning regularly to the depot. The TSP and SFC would appear to be more "fair" than the NN policy because they partially obey an FCFS discipline (i.e., sets are served in FCFS order in the case of the TSP policy and for the SFC policy, the entire region is periodically "swept" by the server).

In addition, they have provable performance guarantees. The NN policy, on the other hand, has system times about 10% lower than the SFC policy and 20% lower than the TSP strategy according to our simulation study. It does not, however, have a provable performance guarantee.

These policies, though quite diverse, have identical asymptotic behavior in heavy traffic. Their asymptotic system time is proportional to $(1 - \rho)^{-2}$ and does not depend on the service time variation (\bar{s}^2). This is in stark contrast to the behavior of traditional queues, and it illustrates the unique insights that can be obtained by considering combined queueing/routing models.

We believe that this class of dynamic vehicle routing problems constitutes a very interesting and realistic class of models, and as such deserves additional attention. An obvious extension is to multiple server (m -vehicle) models. This is a topic we recently investigated in Bertsimas and Van Ryzin (1990), where similar bounds and policies are established. In particular, the system time is shown to have a $\lambda A/m^2(1 - \rho)^2$ behavior in heavy traffic. We have also investigated (cf. Bertsimas and Van Ryzin) the effect of vehicle capacity for both the single and multiple vehicle cases. Our results suggest that the stability condition is no longer independent of the service region size in the capacitated case. Finally, one could certainly construct other DTRP policies and analyze them using the technique of Section 1.

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