Black Magic

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> Harry, I can't believe it.... You conjured up a Patronus that drove away all those dementors! That's very, very advanced magic.
>
> —Hermione Granger, *Harry Potter and the Prisoner of Azkaban*

1 Warm-Ups

Exercise 1 (2016 C1). In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B to the right of A. We say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town that cannot be swept away by any other one.

Example 2 (2018 TST 4). Let n be a positive integer and let $S \subseteq \{0,1\}^n$ be a set of binary strings of length n. Given an odd number $x_1, \ldots, x_{2k+1} \in S$ of binary strings (not necessarily distinct), their majority is defined as the binary string $y \in \{0,1\}^n$ for which the i^{th} bit of y is the most common bit among the i^{th} bits of $x_1, \ldots, x_{2k+1}$. (For example, if $n = 4$ the majority of 0000, 0000, 1101, 1100, 0101 is 0100.)

Suppose that for some positive integer k, S has the property P_k that the majority of any $2k+1$ binary strings in S (possibly with repetition) is also in S. Prove that S has the same property P_k for all positive integers k.

2 Throwing Pebbles

Problem 3 (EGMO 2021/5). A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that no three points in P lie on a line and no two points in P lie on a line through the origin.

A triangle with vertices in P is *fat* if O is strictly inside the triangle. Find the maximum number of fat triangles.

Problem 4 (ELMO 2017/5). The edges of K_{2017} are each labeled with $1, 2$, or 3 such that any triangle has sum of labels at least 5. Determine the minimum possible average of all $\binom{2017}{2}$ labels.

Problem 5 (CMC 2020/4 Generalized, William Wang). Let n be an odd positive integer. Some of the unit squares of an $n \times n$ unit-square board are colored green. It turns out the green squares can be partitioned into k connected components $C_1, \dots, C_k$, where two vertices are considered connected if and only if they share a vertex. Let $d(C_i)$ denote the diameter (maximum number of king moves needed to get from a to b for $a, b \in C_i$) of C_i. Show that

$$d(C_1) + \cdots + d(C_k) \leq \frac{n^2 - 1}{2}.$$

Problem 6 (2018 C5). Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

3 Problems

Problem 7 (2009 C3). Let n be a positive integer. Given a sequence $\varepsilon_1, \dots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \dots, n-1$, the sequences $a_0, \dots, a_n$ and $b_0, \dots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \qquad \text{for each } i = 1, \dots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \qquad \text{for each } i = 1, \dots, n-1.$$

Prove that $a_n = b_n$.

Problem 8 (2017 N7). An ordered pair (x, y) of integers is a primitive point if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \dots, a_n$ such that, for each (x, y) in S, we have:

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n = 1.$$

Problem 9 (2009 C7). Let $a_1, a_2, \dots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \dots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.