Individual Path Recommendation Under Public Transit Service Disruptions Considering Behavior Uncertainty and Equity

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Abstract

This study proposes a mixed-integer programming (MIP) formulation to model the individual-based path recommendation problem during PT service disruptions with the objective of minimizing system travel time and respecting passengers’ path choice preferences. Passengers’ behavior uncertainty in path choices given recommendations and their travel time equity are also considered. We model the behavior uncertainty based on passenger’s prior preferences and posterior path choice probability distribution with two new concepts: $\epsilon$-feasibility and $\Gamma$-concentration, which control the mean and variance of path flows in the optimization problem. The IPR problem with behavior uncertainty is solved efficiently with Benders decomposition. A post-adjustment heuristic is used to address the equity requirement. The proposed approach is implemented in the Chicago Transit Authority (CTA) system with a real-world urban rail disruption as the case study. Results show that the proposed IPR model significantly reduces the average travel times compared to the status quo and outperforms the capacity-based benchmark path recommendation strategy. We also show that incorporating behavior uncertainty with respect to responses to information achieves lower system travel times than assuming that all passengers would follow the recommendations. The post-adjustment heuristic effectively reduces the difference in passengers’ travel times and increases equity. The equity requirement slightly increases the system travel time, showing the trade-off between efficiency and equity. We also show that it is possible to make recommendations so that most of the passengers (e.g., more than 70%) use their preferred paths while only increasing the system travel time by 0.51%.

Keywords: Transportation; Individualized recommendation; Mixed-integer programming; Behavior uncertainty.

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1. Introduction

1.1. Background and challenges

With aging systems and near-capacity operations, service disruptions often occur in urban public transit (PT) systems. These incidents may result in passengers delays, cancellation of trips, and economic losses (Cox et al., 2011).

During a significant disruption where the service is interrupted for a relatively long period of time (e.g., 1 hour), affected passengers usually need to find an alternative path or use other travel modes (such as transfer to another bus route). However, due to a lack of knowledge of the system (especially during incidents), the routes chosen by passengers may not be optimal or even cause more congestion (Mo et al., 2022b). For example, during a rail disruption, most of the passengers may choose bus routes that are parallel to the interrupted rail line as an alternative. However, given limited bus capacity, parallel bus lines may become oversaturated and passengers have to wait for a long time to board due to being denied boarding (or left behind).

One of the strategies to better guide passengers is to provide path recommendations so that passenger flows are re-distributed in a better way and the system travel times are minimized. This can be seen as solving an optimal passenger flow distribution (or assignment) problem over a public transit network. However, different from the typical flow redistribution problem, there are several unique characteristics and challenges for the path recommendation problem under PT service disruptions.

- Passengers may have different preferences on different alternative paths. This heterogeneity suggests that we cannot treat a group of passengers simply as flows. Individualization is needed in the path recommendation design.

- Passengers may not follow the recommendation. When providing a specific path recommendation to a passenger, their actual path choice is uncertain (though the recommendation may change their preferences). This behavior uncertainty brings challenges in the recommendation system design and has not been considered in the path recommendation literature. In the context of individualization, the behavior uncertainty is also individual-specific, which requires a more granular modeling approach.

- From the operator’s point of view, the objective of path recommendations is to reduce system congestion by better utilizing available capacity. However, under system optimal flow patterns, passengers with the same origin, destination (OD), and departure time may end up with very different travel times due to being recommended different paths where some paths are shorter and some are longer (Sheffi, 1985), resulting in equity issues. Therefore, we do not want the path recommendations to result in passengers having large differences in travel times.

1.2. Organization and contributions

To tackle these challenges, this study proposes an individual-based path recommendation model to reduce the system congestion during public transit disruptions, considering behavior uncertainty and passenger travel time equity. We first formulate an optimal flow problem as a linear program based on the model of Bertsimas et al. (2020), which solves the optimal path flows for each OD pair and time interval that minimize the system travel time. Then, we add the recommendation decision variables, \( x_{p,r} \) (binary variable indicating whether path \( r \) is recommended passenger \( p \)) and associated constraints to capture the behavior uncertainty. The behavior uncertainty is modeled with a conditional path choice probability distribution for
each passenger given their received path recommendation. We introduce two new concepts: \( \epsilon \)-feasible flows and \( \Gamma \)-concentrated flows, to connect the optimal flow problem with the conditional path choice probabilities. The individual path recommendation problem with behavior uncertainty is a mixed-integer program. We solve it efficiently with Benders decomposition. Finally, we use a post-adjustment heuristic to address the equity requirement. The proposed approach is implemented in the Chicago Transit Authority (CTA) system with a real-world urban rail disruption as the case study.

The main contributions of this paper are as follows:

- To the best of the authors’ knowledge, this is the first article dealing with individual path recommendations under public transit service disruptions considering behavior uncertainty and equity. Previous studies only considered uncertainty in demand (Mo et al., 2022a) or incident duration (Tan et al., 2020). And for the objective function, they either focus on minimizing travel time or maximizing individual preferences (Roelofsen et al., 2018). Equity has not been considered in the literature.

- To model behavior uncertainty, this paper proposes a framework with prior path utility and posterior path choice distribution given recommendations. We use two new concepts: \( \epsilon \)-feasibility and \( \Gamma \)-concentration, to control the mean and variance of path flows due to behavior uncertainty and transform these two requirements to linear constraints in the optimization model using Chebyshev’s inequality.

- Benders decomposition (BD) is used to solve the mixed-integer individual path recommendation problem efficiently. Under BD, the master problem becomes a small-scale integer program and the sub-problem reduces to a linear program.

- This paper mathematically defines the equity requirement in the individual path recommendations, and proposes a post-adjustment heuristic method to solve it. We also propose an integrated mixed-integer programming formulation with both behavior uncertainty and equity requirement, discuss the difficulty in solving the corresponding problem, and highlight the importance of the post-adjustment heuristic.

The remainder of the paper is organized as follows. Literature review is discussed in Section 2. In Section 3, we describe the problem conceptually and analytically. Section 4 develops the solution methods, including the optimal flow problem formulation, modeling of the behavior uncertainty, Benders decomposition, and the post-adjustment heuristic for equity. Section 5 discusses model extensions and generalizability. In Section 6, we apply the proposed model on the CTA system as a case study. The model results are analyzed in Section 7. Finally, we conclude the paper and summarize the main findings in Section 8.

2. Literature review

2.1. Individualized recommendations system

Individualized recommendations design is a popular topic in the field of computer science and operations research, with many real-world implications such as Ads ranking (Richardson et al., 2007; Khalid et al., 2013), mobile news recommendations (Yeung and Yang, 2010), travel recommendations (Majid et al., 2013), etc. Most of these recommendation systems focus on individual preference maximization, which, in return, can increase indicators of interest such as click-through rate (CTR) and conversion rate. However, in the context of path recommendations under disruptions, though respecting passenger’s preferences is important, the ultimate goal is to minimize the system travel time and mitigate the impact of disruptions, which is different from typical recommendation design literature. Another difference is that, the typical recommendation
systems are usually designed with machine learning algorithms trained with the real-world user and system interaction data because they have to learn users’ preferences based on their interaction histories. However, in this study, the system travel time can be evaluated using a network loading model. This implies that, instead of using machine learning models, we can use an optimization formulation to determine the individualized path recommendations that minimizes system travel time.

In summary, different from the typical individualized recommendation system literature, this study focuses on system-level objectives instead of individual-level preferences. It leverages an optimization model to design the recommendation, rather than machine learning models.

2.2. Path recommendations during disruptions

Most previous studies on path recommendation under incidents are like designing a “trip planner”. That is, the main objective is to find available routes or the shortest path given an OD pair when the network is interrupted by incidents. For example, Bruglieri et al. (2015) designed a trip planner to find the fastest path in the public transit network during service disruptions based on real-time mobility information. Böhmová et al. (2013) developed a routing algorithm in urban public transportation to find reliable journeys that are robust for system delays. Roelofsen et al. (2018) provided a framework for generating and assessing alternative routes in case of disruptions in urban public transport systems. To the best of the authors’ knowledge, none of the previous studies have considered path recommendations aiming to minimize the system-wide travel time, given equity constraints.

Providing path recommendations during disruptions is similar to the topic of passenger evacuation under emergencies. The objective of evacuation is usually to minimize the total evacuation time. For example, Abdelgawad and Abdulhai (2012) developed an evacuation model with routing and scheduling of subway and bus transit to alleviate congestion during the evacuation of busy urban areas. Wang et al. (2019) proposed an optimal bus bridging design method under operational disruptions on a single metro line. Tan et al. (2020) propose an evacuation model with urban bus networks as alternatives in the case of common metro service disruptions by jointly designing the bus lines and frequencies.

However, although these passenger evacuation papers focus on minimizing the system travel time, there are several differences from this paper. First, in our paper, the service disruption is not as severe as the emergency situation. We assume the service will recover after a period of time and passengers are allowed to wait. They do not necessarily need to cancel trips or follow the evacuation plan as assumed in previous evacuation studies. Second, in this article, we do not adjust the operations on the supply side. Instead, we focus on providing information to the passengers to better utilize the existing resources/capacities of the system. Third, as mentioned before, this paper considers passenger heterogeneity and focuses on individual-level path recommendations, while previous evacuation papers simply model passengers as flows. Besides, we also assume that passengers may not follow the recommendation (i.e., behavior uncertainty) and incorporate equity into consideration, which has not been considered in any evacuation paper before.

2.3. Behavior uncertainty

Behavior uncertainty is a well-known challenge in transportation modeling (Mahmassani, 1984). Typically, passenger’s behavior is modeled using various econometrics approaches (Ben-Akiva et al., 1985; Train, 2009; Mo et al., 2021) or machine learning models (Mirchevska, 2013; Wang et al., 2020). These models output the probability distribution for the passenger’s possible behavior. At the aggregate level, there are numerous studies using the predicted demand for different transportation applications taking demand
uncertainty into consideration, such as ride-sharing (Guo et al., 2021), transit route planning (Yoon and Chow, 2020), and supply chain management (Jung et al., 2004).

However, at the individual level, the number of studies is limited. The main reason is that, individual-level decision-making is usually discrete, it is challenging to use typical robust optimization to address discrete uncertain variables (Subramanyam et al., 2021). In terms of stochastic optimization, the number of possible scenarios increases exponentially with the number of individuals in the system. Some studies use simulation to incorporate individual-level behavior uncertainty. For example, Horne et al. (2005) use a discrete choice model to simulate how different hybrid energy-economy policies can motivate users’ responses. However, to incorporate behavior uncertainty in an optimization model (such as the individual path recommendation model in this paper), new modeling techniques are needed.

Another difference in this study compared to previous literature is that the behavior uncertainty (i.e., passenger’s response to the recommendation) makes the decision variables (i.e., passenger flow) random variables. Typical robust optimization or stochastic optimization usually assumes the parameters of constraints are random variables, but not the decision variable.

2.4. Equity in travel times

Equity has been an important topic in the transportation field (Litman, 2002; Ramjerdi, 2006; Martens, 2011; Zheng et al., 2021) and the design of recommendation systems (Vargas and Castells, 2011; Adomavicius and Kwon, 2011; Dwork et al., 2012). The motivation for considering equity in this study comes from the well-known trade-off between efficiency and equity in the network flow problems. As known from the canonical static traffic assignment literature, under system optimal solutions (best efficiency), passengers with the same OD pair may have different travel times due to using different paths. Though the total travel time of all passengers is minimized, those passengers who use longer paths are worse-off in the system. On the contrary, under user equilibrium conditions (best equity\(^1\)), all passengers with the same OD pair have the same travel times\(^2\). User equilibrium is assumed to represent real-world flow patterns without interventions (i.e., the Nash equilibrium from the game theory perspective). When considering controls to reduce the system travel time, equity issues arise. Two well-known tools to increase efficiency and ensure equity is through congestion pricing (Button and Hensher, 2001; Yang and Huang, 2005) and tradable credits (Nie and Yin, 2013; Tian et al., 2013). The former focuses on charging drivers with advantages in travel time (i.e., those who use shorter paths) and the latter compensates drivers in longer paths.

This study also falls into the topic of using control strategies (i.e., path recommendation) to reduce the system travel time (though in disruption scenarios). Therefore, it is likely that the system optimal recommendations would suggest some passengers use longer paths in order to increase efficiency, resulting in equity issues. None of the previous studies in transit systems have considered equity in the individual path recommendations. For the evacuation literature, some studies have pointed out that there are equity issues in evacuation planning (Renne, 2006; Sanchez and Brenman, 2008; Bish, 2011; Renne and Mayorga, 2018; Wong et al., 2021). However, these papers focus on the equity of opportunities in evacuation, aiming to help careless or vulnerable populations who cannot receive services during incidents. In this study, the focus is on travel time equity in the context of using the information as a control mechanism under transit disruptions.

\(^1\) Here we refer to the simplest horizontal equity and we understand that there are more discussions on the definition of equity.

\(^2\) More precisely, the travel time of all paths with passengers are the same in this status.
3. Problem description

3.1. Conceptual description

Consider a service disruption in an urban rail system. During the disruption, some stations in the incident line (or the whole line) are blocked. Passengers in the blocked trains are usually offloaded to the nearest platforms. To respond to the incident, some operating changes are made, such as dispatching shuttle buses, rerouting existing services, short-turning in the incident line, headway adjustment, etc. Assume that all information about the operating changes is available. These changes define a new PT service network and available path sets. Our objective is to develop an individual-based path recommendation model that, when an incident happens, provides a recommended path to every passenger who uses their phones, websites, or electronic boards at stations to enter their origin, destination, and departure time. The recommendation considers the individual’s preferences and behavioral histories. Hence, passengers with the same origin, destination, and departure time may get different recommended paths. The overall system aims to minimize the total travel time for all passengers, including passengers in nearby lines or bus routes without incidents (note that these passengers may experience additional crowding due to transfer passengers from the incident line).

Figure 1 shows a simple example of the path recommendation problem. In this example, Rail Line 1 has an incident and cannot provide service for a period of time. Both of the two passengers at station A want to go to station C. Assuming that they request path recommendations. The alternative paths include using the bus route (blue dashed line), using the Rail Line 2 (green dashed line), or waiting for the system to recover (i.e., still using Rail Line 1). Note that using either the bus route or Rail Line 2 will take away capacity from passengers who originally use these two services (i.e., the orange passengers in the figure). Hence, the model should consider the total travel time of all four passengers in the system to design recommendation strategies.

Moreover, as mentioned in the introduction, behavior uncertainty and equity need to be considered. In this example, if we recommend a passenger to use a bus route, he/she may not follow the recommendation and choose Rail Line 2 instead. Although these two passengers share the same origin, destination, and departure time, they may receive different recommended paths. For example, one of them is recommended to use a bus route and another Rail Line 2. The model should ensure that the travel times of these two paths are similar, otherwise, there will be equity issues.

3.2. Analytical description

Let us divide the analysis period into several time intervals with equal length \( \tau \) (e.g., \( \tau = 5 \) min). Let \( t \) be the integer time index. \( t = 1 \) is the start of the incident and \( t < 0 \) indicates the time before the incident.
Let $\mathcal{P}$ be the set of passengers that will receive path recommendations. We assume $\mathcal{P}$ is known as we can obtain passengers’ requests before running the model. Given the revised operation during the incident, let $\mathcal{R}_p$ be the feasible path set for each passenger $p \in \mathcal{P}$. Note that $\mathcal{R}_p$ includes all feasible services that are provided by the PT operator. A path $r \in \mathcal{R}_p$ may be waiting for the system to recover (i.e., using the incident line), or transfer to nearby bus lines, using shuttle services, etc. We do not consider non-PT modes such as TNC or driving for the following reasons: 1) This study aims to design a path recommendation system used by PT operators. The major audience should be all PT users. Considering non-PT modes needs the supply information of all other travel modes and even consider non-PT users (such as the impact of traffic congestion on drivers), which is beyond the scope of this study. Future research may consider a multi-modal path recommendation system. 2) Passengers using non-PT modes can be simply treated as demand reduction for the PT system. So their impact on the PT system can still be captured.

Given a passenger $p \in \mathcal{P}$, we aim to determine $x_{p,r}$ for each $p$, where $x_{p,r}$ indicates whether path $r \in \mathcal{R}_p$ is recommended to passenger $p$ or not. Assume only one path is recommended to each passenger, we have

$$\sum_{r \in \mathcal{R}_p} x_{p,r} = 1 \quad \forall p \in \mathcal{P}$$

(1)

Note that we can relax this assumption by designing the recommendation to a passenger as a “composition” including multiple paths or travel times. This generalization is discussed in Section 5.1.

$\mathcal{P}$ includes passengers with different origins, destinations, and departure times. If an incident ends at $t^{\text{end}}$, the recommendation should consider a time horizon after $t^{\text{end}}$ because there is remaining congestion in the system. Hence, we provide recommendations until time $T^D > t^{\text{end}}$ (e.g., $T^D$ can be one hour after $t^{\text{end}}$). Therefore, the departure times for passenger $p \in \mathcal{P}$ range from $[1, T^D]$ ($T^D$ and $t^{\text{end}}$ are both time indices).

The recommendation model will be solved at $t = 1$ and will generate the recommendation strategies $x = (x_{p,r})_{p \in \mathcal{P}, r \in \mathcal{R}_p}$ for passengers who depart at time $t \in [1, T^D]$. In reality, the model can be implemented in a rolling horizon manner. Specifically, at each time interval $t \geq 1$, we first update the demand and supply information in the system, including new demand estimates, new to-be-recommended passenger set $\mathcal{P}$, new available path sets $\mathcal{R}_p$, new service routes and frequencies, new incident duration estimates, new onboard passenger estimates, etc. Based on this information, we solve the model to obtain recommendations for passengers with departure time in $[t, T^D]$. But we only implement the recommendation strategies for passengers who depart at the current time $t$.

Therefore, in the following formulation, we only focus on solving the model at $t = 1$, which is the start of the incident. The whole analysis period includes warm-up and cool-down periods to better estimate the system states (e.g., vehicle loads, passenger travel times, etc.). Therefore, the analysis period is defined as $[t^{\min}, T]$, where $t^{\min} < 1$ (time before the incident) and $T > T^D$. For example, $t^{\min}$ and $T$ can be one hour before and after $t = 1$ and $T^D$, respectively. And we define all time intervals in the analysis period as $\mathcal{T} = \{t^{\min}, t^{\min} + 1, \ldots, T\}$. The overall path recommendation framework can be summarized in Figure 2.
4. Formulation

In this section, we elaborate on the detailed formulation of the individual path recommendation model. Section 4.1 develops an optimization model to solve the optimal flow distribution over a public transit network with disruptions. Section 4.2 describes how passenger’s behavior uncertainty (i.e., non-compliance to recommendation) is modeled based on a random utility maximization framework. Section 4.3 provides the overall formulation of the individual path recommendation model by combining the optimal flow model in Section 4.1 and the behavior uncertainty component in Section 4.2. Section 4.4 shows how the individual path recommendation model can be solved efficiently using Benders decomposition. Section 4.5 proposes a post-adjustment method to obtain the final recommendation strategy with equity (i.e., the travel times of passengers with the same origin, destination, and departure time do not differ a lot).

The notations used in the paper are summarized in Table A.4 (Appendix A).

4.1. Optimal flow during disruptions

In this section, we formulate a linear programming (LP) model to solve the optimal flow distribution in a public transit system with service disruptions. Consider an OD pair \((u, v)\) and departure time \(t\). Let \(\mathcal{R}^{u,v}\) be the set of feasible paths for OD pair \((u, v)\). Define \(q_{t}^{u,v,r}\) (resp. \(f_{t}^{u,v,r}\)) as the number of passengers in (resp. not in) \(\mathcal{P}\) with OD pair \((u, v)\) and departure time \(t\), who use path \(r \in \mathcal{R}^{u,v}\). Specifically, \(q_{t}^{u,v,r}\) represents the passenger flows that receive recommendations while \(f_{t}^{u,v,r}\) those do not. Hence, the total path flow in \(r \in \mathcal{R}^{u,v}\) is \(q_{t}^{u,v,r} + f_{t}^{u,v,r}\). Let \(d_{t}^{u,v}\) be the total demand of OD pair \((u, v)\) at time \(t\), we have

\[
q_{t}^{u,v,r} + f_{t}^{u,v,r} = d_{t}^{u,v} \quad \forall (u, v) \in \mathcal{W}, \ t \in \mathcal{T}
\]

where \(\mathcal{W}\) is the set of all OD pairs. As we focus on path recommendations for \(\mathcal{P}\), in this study, \(q_{t}^{u,v,r}\) is the decision variable while \(f_{t}^{u,v,r}\) is a known constant (i.e., the estimated demand information). For mathematical convenience, we define \(\mathcal{F}\) as the set of all triplets \((u, v, r)\) in the system. And the objective in this section is to find the optimal flows \(q_{t}^{u,v,r}\) \((\forall (u, v, r) \in \mathcal{F}, \ t \in \mathcal{T})\) that minimize the total system travel time.

The LP-based optimal flow model is adapted from Bertsimas et al. (2020) with the following differences:

1) Bertsimas et al. (2020)’s model only considers waiting time in the system while ignoring in-vehicle time. In this study, we extend their formulation to include the in-vehicle times. 2) In Bertsimas et al. (2020), the capacity constraints are formulated for a vehicle without time indices, which neglects the fact that alighting passengers can release capacity. In this study, we modify the capacity constraints to a vehicle-time-based constraints.
formulation, which considers the occupancy of only onboard passengers in each time interval. 3) We adapt the model from normal scenarios to incident scenarios by pre-processing the supply of incident lines and pre-defining the system state before the incident. These two operations will be described later.

Consider a path $r$ for OD pair $(u,v)$. A path may include multiple legs, where each leg is associated with the service in a rail or a bus line. For example, the path in Figure 3 (indicated by green arrows) has two legs: the first one in the rail line and the second in the bus line. Every leg has a boarding and an alighting station. For example, Leg 1 (resp. 2) in this example has boarding station A (resp. C) and alighting station B (resp. D). Let $I_{u,v,r} = \{1,\ldots,|I_{u,v,r}|\}$ be the set of legs for path $r$. We use a four-element tuple $(u,v,r,i)$ to represent a leg $i$ of path $r$ for OD pair $(u,v)$, where $i \in I_{u,v,r}$.

![Figure 3: Definition of paths and legs](image)

Let $\Delta_t^{u,v,r,i}$ (resp. $\delta_t^{u,v,r,i}$) be the travel time between the terminal and the boarding (resp. alighting) station of leg $(u,v,r,i)$ for a vehicle departing from the terminal at time $t$. Hence, the vehicle’s arrival time at the boarding (resp. alighting) station of leg $(u,v,r,i)$ is $t + \Delta_t^{u,v,r,i}$ (resp. $t + \delta_t^{u,v,r,i}$). $\delta_t^{u,v,r,i} - \Delta_t^{u,v,r,i}$ represents the total in-vehicle time of leg $(u,v,r,i)$ for the vehicle. Define $z_t^{u,v,r,i}$ (decision variable) as the total number of on-board passengers in leg $(u,v,r,i)$ who board a vehicle that had departed from the terminal at time $t$.

There are three types of constraints for the network flow description: 1) existing flows constraints, 2) vehicle capacity constraints, and 3) flow conservation constraints.

**Existing flows constraints:** Although the path recommendations starts at time $t = 1$, there are passengers that already boarded the vehicles. Ignoring these existing flows may lead to overestimation of the system’s available capacity. To capture the existing onboard flows at $t = 1$, we define the set of onboard flow indices as

$$\Omega_1 = \{(u,v,r,i,t) : t + \Delta_t^{u,v,r,i} \leq 1 \leq t + \delta_t^{u,v,r,i}\} \quad (3)$$

And the existing flow constraints can be expressed as

$$z_t^{u,v,r,i} = z_t^{u,v,r,i} \quad \forall (u,v,r,i,t) \in \Omega_1 \quad (4)$$

where $z_t^{u,v,r,i}$ are constants that capture the existing onboard flows when the incident happens. These flows can be directly obtained from a simulation model or real-time passenger counting data.

**Capacity constraints:** Transit vehicles have limited capacity. Consider a vehicle departing at time $t$ on line $l$ (referred to as vehicle $(l,t)$). We denote its total number of onboard passengers at time $t'$ as $O_{l,t,t'}$. 


Specifically, $O_{l,t,t'}(z)$ can be expressed as

$$O_{l,t,t'}(z) = \sum_{\{(u,v,r,i,t)\in\text{Onboard}(l,t')\}} z_{l}^{u,v,r,i} \quad \forall l \in L, \forall t \in T, t' = t, t+1, ..., T_{l,t}$$  \hspace{1cm} (5)

where $T_{l,t}$ is the time index that vehicle $(l, t)$ arrives at the last station of line $l$. \text{Onboard}(l, t')$ is the set of onboard flow indices for vehicle $(l, t)$, defined as

$$\text{Onboard}(l, t') = \{(u, v, r, i, t) : \text{Leg} (u, v, r, i) \text{ on line } l, \text{ and } t + \Delta_{i}^{u,v,r,i} \leq t' \leq t + \delta_{i}^{u,v,r,i}\}$$  \hspace{1cm} (6)

Then the capacity constraint is:

$$O_{l,t,t'}(z) \leq K_{l,t} \quad \forall l \in L, t \in T, t' = t, t+1, ..., T_{l,t}$$  \hspace{1cm} (7)

where $K_{l,t}$ is the capacity of the vehicle $(l, t)$. $L$ is the set of all lines.

**Flow conservation constraint:** There are two different flow conservation constraints: 1) flow conservation at origin stations and 2) at transfer stations. To ensure the origin flow conservation, the cumulative number of arrival passengers should be larger than the cumulative number of boarding passengers at an origin at any time. This indicates that not all arrival passengers can board due to potentially being left behind because of capacity constraints.

The number of arriving passengers (i.e., demand) for $(u, v, r)$ at time $t$ is $d_{t}^{u,v,r} + f_{t}^{u,v,r}$. And the number of boarding passengers at the origin station (i.e., $u$) at time $t$ is $z_{u}^{u,v,r,1}$ (i.e., the first leg) with $t' + \Delta_{i}^{u,v,r,1} = t$. $t'$ is the vehicle departure time from the terminal and $t' + \Delta_{i}^{u,v,r,1}$ is the time when the vehicle arrives at the boarding station. Therefore, the origin flow conservation constraint can be written as:

$$\sum_{\{t',t_{\text{min}} \leq t' \leq t + \Delta_{i}^{u,v,r,1} \leq t\}} z_{t'}^{u,v,r,1} \leq \sum_{t'=t_{\text{min}}}^{t} (f_{t'}^{u,v,r} + q_{t'}^{u,v,r}) \quad \forall (u, v, r) \in F, t \in T$$  \hspace{1cm} (8)

Now consider the flow conservation at a transfer station. All arrival passengers at a transfer station of a path are the onboard passengers from the last leg. Therefore, we use a similar way to define the transfer flow conservation: the cumulative number of onboard passengers from the last leg should be larger than the cumulative number of boarding passengers at the transfer station. And the number of boarding passengers at the transfer station is simply $z_{t}^{u,v,r,i}$ with $i \geq 2$. Hence, flow conservation constraints at a transfer station are:

$$\sum_{\{t',t_{\text{min}} \leq t' \leq t + \Delta_{i}^{u,v,r,i} \leq t\}} z_{t'}^{u,v,r,i} \leq \sum_{\{t',t_{\text{min}} \leq t' \leq t + \delta_{i}^{u,v,r,i-1} \leq t\}} z_{t'}^{u,v,r,i-1} \quad \forall (u, v, r) \in F, i \in I^{(u,v,r)} \setminus \{1\}, t \in T$$  \hspace{1cm} (9)

Note that $z_{t'}^{u,v,r,i}$ is defined as the onboard passengers for vehicles departing at time $t'$. Therefore, $t' + \delta_{i}^{u,v,r,i-1}$ is the alighting time for passengers at leg $i-1$ (which is also the transfer demand arrival time at leg $i$ as we assume transfer walk time is within a time interval $\tau$ and is negligible). $t' + \Delta_{i}^{u,v,r,i}$ is the boarding time for passengers at leg $i$.

The objective is to minimize the total travel time for all passengers in the system. Total travel time can be decomposed into waiting time and in-vehicle time.
**In-vehicle time:** Total in-vehicle time is simply the onboard flow multiplied by the travel time on each leg:

$$ IVT(z) = \sum_{(u,v,r) \in F} \sum_{i \in T^{u,v,r}} \sum_{t \in T} z_{t}^{u,v,r,i} \cdot T_{u,v,r,i,t}^{IVT} $$

where $T_{u,v,r,i,t}^{IVT}$ is the in-vehicle time of leg $(u, v, r, i)$ of the vehicle departing at time $t$.

**Waiting time:** There are two causes of waiting time: 1) waiting time because of vehicle headways, and 2) waiting time resulting from being left behind. During a specific time interval $t$, all left behind passengers would have a waiting time of $\tau$. All boarding passengers, assuming uniform arrival, have an average waiting time that is half of the time interval (i.e., $\frac{\tau}{2}$). Therefore, the total waiting time for passengers at station $s$ and time $t$ can be formulated as

$$ WT_{s,t} = \tau (AD_{s,t} + XD_{s,t} - BD_{s,t}) + \frac{\tau}{2} (BD_{s,t+1} - BD_{s,t}) $$

where $AD_{s,t}$ represents the cumulative arriving demand at station $s$ up to time $t$, $XD_{s,t}$ represents the cumulative transferring demand at station $s$ up to time $t$, and $BD_{s,t}$ represents the cumulative boarded demand at station $s$ up to time $t$. Hence, $(BD_{s,t+1} - BD_{s,t})$ represents the total number of boarding passengers at time $t$ and station $s$, and $(AD_{s,t} + XD_{s,t} - BD_{s,t})$ represents the total number of left behind passengers at station $s$ and time $t$. Finally, the total system waiting time is

$$ WT(q, z) = \sum_{s \in S} \sum_{t = 1}^{T} WT_{s,t} $$

The cumulative arriving demand $AD_{s,t}$ is simply all arriving passengers with origin $s$ up to time $t$:

$$ AD_{s,t} = \sum_{\{(u,v,r): u = s\}} \sum_{t' = t_{\text{min}}}^{t} (f_{t'}^{u,v,r} + q_{t'}^{u,v,r}) \quad \forall s \in S, t \in T $$

where $S$ is the set of all stations.

The cumulative transferring demand is all passengers alighting at station $s$ from their previous leg $i - 1$ for their next leg $i$:

$$ XD_{s,t} = \sum_{\{(u,v,r,i) \in X^{s}(s)\}} \sum_{t' = t_{\text{min}}}^{t} \sum_{t'' = \Delta_{t',t''}}^{t'} z_{t''}^{u,v,r,i-1} \quad \forall t' = t_{\text{min}}, ..., T $$

where $X^{s}(s)$ is the set of legs that transfer at station $s$.

The cumulative boarded demand is all passengers that successfully board a vehicle at station $s$ at time $t$. Define $B^{s}(s)$ as the set of all legs with boarding station $s$, we have

$$ BD_{s,t} = \sum_{\{(u,v,r,i) \in B^{s}(s)\}} \sum_{t' = t_{\text{min}}}^{t} z_{t'}^{u,v,r,i} \quad \forall t' = t_{\text{min}}, ..., T $$

Taking everything into consideration, the total travel time in the system is $WT(x, z) + IVT(z)$. The
optimal flow problem is:

\[
(OF) \quad \min \quad WT(q, z) + IVT(z) \tag{16a}
\]

\[
s.t. \quad O_{l,t,t'}(z) \leq K_{l,t} \quad \forall l \in \mathcal{L}, t \in \mathcal{T}, t' = t, t+1, \ldots, T_{l,t} \tag{16b}
\]

\[
\sum_{\{t': t_{\min} \leq t' \leq \delta_{t'}^u,v,r,i \leq t\}} z_{t'}^{u,v,r,i} \leq \sum_{\{t': t_{\min} \leq t' + \delta_{t'}^u,v,r,i-1 \leq t\}} z_{t'}^{u,v,r,i-1} \quad \forall (u, v, r) \in \mathcal{F}, t \in \mathcal{T} \tag{16c}
\]

\[
\sum_{r \in \mathcal{R}_{u,v}} q_t^{u,v,r} + f_t^{u,v,r} = d_t^{u,v} \quad \forall (u, v) \in \mathcal{W}, t \in \mathcal{T} \tag{16d}
\]

\[
\forall (u, v, r, i, t) \in \Omega_1 \tag{16e}
\]

\[
z_t^{u,v,r,i} = z_t^{u,v,r,i-1} \quad \forall (u, v, r, i, t) \in \Omega_1 \tag{16f}
\]

\[
z_t^{u,v,r,i} \geq 0 \quad \forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F}, i \in \mathcal{I}^{u,v,r} \tag{16g}
\]

\[
q_t^{u,v,r} \geq 0 \quad \forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F} \tag{16h}
\]

As the objective function is minimizing the system travel time, this formulation will automatically load passengers to a train as long as there is available capacity (Bertsimas et al., 2020).

**Path travel time calculation:** It is worth noting that Eq. 16 does not explicitly output the travel time of different paths. The travel time of a path \((u, v, r)\) for trips departs at time \(t\) (denoted as \(TT_t^{u,v,r}\)) has to be obtained from the network flow patterns after solving Eq. 16. Specifically, consider the group of passengers using path \((u, v, r)\) and departing at time \(t\). Their arrival time at the destination (denoted as \(AT_t^{u,v,r}\)) can be calculated as

\[
AT_t^{u,v,r} = \min \left\{ \tilde{t} \in \mathcal{T}_t^{u,v,r} : \sum_{t'=t_{\min}}^{t} (f_{t'}^{u,v,r} + q_{t'}^{u,v,r}) \leq \sum_{t_{\min} \leq t' + \delta_{t'}^{u,v,r,i} \leq t} z_{t'}^{u,v,r,i} \right\} \tag{17}
\]

\[
\forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F}
\]

where \(\mathcal{T}_t^{u,v,r}\) is the set of possible arrival time indices, defined as \(\mathcal{T}_t^{u,v,r} = \{t' : t \leq t' \leq T\}\). Eq. 17 represents the travel time calculation with cumulative demand curves at origins and destinations. \(\sum_{t'=t_{\min}}^{t} (f_{t'}^{u,v,r} + q_{t'}^{u,v,r})\) is the cumulative demand up to time \(t\) at the origin, \(\sum_{t_{\min} \leq t' + \delta_{t'}^{u,v,r,i} \leq t} z_{t'}^{u,v,r,i} \) is the cumulative passengers arriving at the destination up to time \(t'\). When the cumulative arrivals at the destination are greater or equal to the cumulative demand at the origin (up to time \(t\)), all passengers finish the trip. So taking the minimum over \(t'\) gives the arrival time for passengers departing at \(t\). The path travel time is then simply:

\[
TT_t^{u,v,r} = AT_t^{u,v,r} - t \quad \forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F} \tag{18}
\]

Figure 4 illustrates the travel time calculation.
Incident specification: Eq. 16 is a general formulation of the optimal flow problem. Now we will introduce how the incident-specific information is incorporated into this problem. We assume the incident causes a service disruption in a specific line (if only several stations are interrupted, we can separate the line into multiple lines so that the assumption always holds). The service disruption in a line can be seen as stops of vehicles for a period of time. The vehicle stopping can be captured by the parameters $\Delta_t^{u,v,r,i}$, $\delta_t^{u,v,r,i}$, and $K_{t,t}$. Specifically, a long stop due to an incident can be seen as an increase in travel time from the terminal to downstream stations (i.e., increase in $\Delta_t^{u,v,r,i}$ and $\delta_t^{u,v,r,i}$). Moreover, since there is no vehicle dispatching during the incident, we set $K_{t,t} = 0$ for the corresponding time and line. In this way, we can model the incident without changing the formulation.

4.2. Behavior uncertainty

Consider a passenger $p$ with a path set $R_p$. Their inherent preference (utility) of using path $r$ is denoted as $V^r_p$. If path $r'$ was recommended, the impact of the recommendation on the utility of path $r$ is denoted as $I^r_{p,r'}$. Hence, his/her overall utility of using path $r$ can be represented as

$$U^r_p = V^r_p + \sum_{r' \in R_p} x_{p,r'} \cdot I^r_{p,r'} + \xi^r_p \quad \forall r \in R_p, \ p \in P.$$  \hspace{1cm} (19)

where $\xi^r_p$ is the random error. $x_{p,r'} = 1$ if passenger $p$ is recommended path $r'$, otherwise $x_{p,r'} = 0$. Let $\pi^r_{p,r'}$ be the conditional probability that passenger $p$ chooses path $r$ given that the recommended path is $r'$. Assuming a utility maximizing behavior, we have

$$\pi^r_{p,r'} = \mathbb{P}(V^r_p + I^r_{p,r'} + \xi^r_p \geq V^{r''}_p + I^{r''}_{p,r'} + \xi^{r''}_p \mid \forall r'' \in R_p)$$  \hspace{1cm} (20)

Different assumptions for the distribution of $\xi^r_p$ can lead to different expression. For example, if $\xi^r_p$ are i.i.d. Gumbel distributed, the choice probability reduces to multinomial logit model (Train, 2009) and we have

$$\pi^r_{p,r'} = \frac{\exp(V^r_p + I^r_{p,r'})}{\sum_{r'' \in R_p} \exp(V^{r''}_p + I^{r''}_{p,r'})}$$  \hspace{1cm} (21)

The value of $V^r_p$ and $I^r_{p,r'}$ can be calibrated using data from individual-level survey or smart card, which deserves separate research. When developing the individual path recommendation model, we assume $\pi^r_{p,r'}$.\pagebreak
is known. Figure 5 shows an example for the conditional probability matrix. The specific values assume that paths with recommendations are more likely to be chosen.

![Figure 5: Example of conditional path choice probability](image)

The conditional probability $\pi_{p,r'}^r$ captures the individual’s inherent preference for different paths as well as the response to the recommendation system. It varies across individuals and reflects their behavioral uncertainties. This study focuses on designing path recommendation systems based on the value of $\pi_{p,r'}^r$.

### 4.3. Individual path recommendation

Let $\mathbbm{1}_{p,r'}$ be the indicator random variable representing whether passenger $p$ actually chooses path $r$ or not given that he/she is recommended path $r'$. By definition, $\mathbbm{1}_{p,r'}^r$ is a Bernoulli random variable with $E[\mathbbm{1}_{p,r'}^r] = \pi_{p,r'}^r$ and $\text{Var}[\mathbbm{1}_{p,r'}^r] = \pi_{p,r'}^r \cdot (1 - \pi_{p,r'})$.

Therefore, the actual flow for path $(u, v, r)$ at time $t$ is

$$Q_{t}^{u,v,r} = \sum_{p \in \mathcal{P}_{t}^{u,v}} \sum_{r' \in \mathcal{R}_{u,v}} x_{p,r'} \cdot \mathbbm{1}_{p,r'}^r$$

(22)

$Q_{t}^{u,v,r}$ is also a random variable. $\mathcal{P}_{t}^{u,v} \subseteq \mathcal{P}$ is the set of passengers with OD pair $(u, v)$ arriving at the system at time interval $t$ that receive path recommendations. $\mathcal{R}_{u,v}$ is the set of paths of OD pair $(u, v)$. The mean and variance of the actual flow is

$$\mu_{t}^{u,v,r}(x) := E[Q_{t}^{u,v,r}] = \sum_{p \in \mathcal{P}_{t}^{u,v}} \sum_{r' \in \mathcal{R}_{u,v}} x_{p,r'} \cdot \pi_{p,r'}^r$$

(23)

$$\sigma_{t}^{u,v,r}(x)^2 := \text{Var}[Q_{t}^{u,v,r}] = \sum_{p \in \mathcal{P}_{t}^{u,v}} \sum_{r' \in \mathcal{R}_{u,v}} x_{p,r'} \cdot \pi_{p,r'}^r \cdot (1 - \pi_{p,r'})$$

(24)

Note that Eqs. 24 is based on the fact that $x_{p,r'}^2 = x_{p,r'}$ and $\text{Cov}[\mathbbm{1}_{p,r'}^r, \mathbbm{1}_{p,r''}^r] = 0$ if $r' \neq r''$.

In an optimization model, we cannot use a random variable (e.g., actual flow) as the decision variable. Therefore, let us treat $q$ in Eq. 16 as a realization of the random variable flow $Q$. To make $q$ a reasonable realization, some constraints need to be considered between the value of $q$ and the distribution of $Q$. We define two new concepts: “$\epsilon$-feasibility” and “$\Gamma$-concentration”.

**Definition 1 (\(\epsilon\)-feasible flows).** A flow $q_{t}^{u,v,r}$ is $\epsilon$-feasible if and only if

$$|q_{t}^{u,v,r} - \mu_{t}^{u,v,r}(x)| \leq \epsilon_{t}^{u,v,r}, \quad \forall (u, v, r) \in \mathcal{F}, t = t^\text{min}, \ldots, T$$

(25)
where $\epsilon_{u,v,r}^t$ is a small positive constant. This means that $q$ is close to the expectation of the actual flow under recommendation strategy $x$.

**Definition 2 (\(\Gamma\)-concentrated flows).** A flow $q_{t}^{u,v,r}$ is $\Gamma$-concentrated if and only if it is $\epsilon$-feasible and for any constant $a > \epsilon_{u,v,r}^t$, we have

$$P[|Q_{t}^{u,v,r} - q_{t}^{u,v,r}| \geq a] \leq \left(\frac{\Gamma_{u,v,r}^t}{a - \epsilon_{u,v,r}^t}\right)^2 \forall (u,v,r) \in F, t = t^{\text{min}}, ..., T$$

(26)

where $\Gamma_{u,v,r}^t$ is a small positive constant. This means that the probability that $Q_{t}^{u,v,r}$ and $q_{t}^{u,v,r}$ are very different (i.e., with difference greater than $a$) is bounded from above, suggesting that $Q_{t}^{u,v,r}$ is concentrated around $q_{t}^{u,v,r}$.

**Remark 1.** The logic of using $q$ as the decision variable and defining the above two concepts is as follows. The objective of this study is to find the best recommendation strategy $x$ that minimizes the system travel time. The system travel time is a function of network flows. Given a recommendation strategy $x$, the actual flow $Q$ is a random variable, which cannot be directly used in the optimization model (as decision variable) to evaluate the system travel time. Hence, we assume that $q$ in Eq. 16 is a realization of the actual flow (deterministic variable). We also add two constraints to $q$ so that $q$ is close to the mean of the actual flow, and the distribution of the actual flow is concentrated around $q$. Then, using $q$ to evaluate the system travel time is similar to that of using the actual flows.

Note that one may argue that we can directly use $\mu_{u,v,r}^t(x)$ as decision variables to represent network flows and eliminate $q$. This idea is essentially equivalent to setting $\epsilon_{u,v,r}^t = 0$ and does not consider the concentration property (i.e., $\Gamma_{u,v,r}^t = +\infty$), which is a special case of our framework. Our framework has more advantages in controlling the variance. Specifically, ignoring the $\Gamma$-concentration may make the model recommendation strategies meaningless. Consider an extreme scenario that there is a recommendation strategy $x$, under which the actual flow is uniformly distributed in $[0, 1]$. Further assume that the system travel time is simply a linear function of the actual flow, say the factor is 1 (i.e., the system travel time is also uniformly distributed in $[0, 1]$). Suppose that the recommendation strategy $x$ minimizes the expected system travel time (now the system travel time is $1 \times \mu_{u,v,r}^t(x) = 0.5$). However, as the actual flow can be any value between 0 and 1 with equal probability, we know that the actual system travel time can also be any value between 0 and 1. Hence, the recommendation strategy $x$, though minimizing the expected system travel time, is meaningless because there are too many variations for the actual system travel time under this recommendation. $\Gamma$-concentration is an important property to ensure that the distribution of actual flows is not too dispersed so that the recommendation strategy $x$ is solved based on a reliable estimate of the system travel time.

We will therefore incorporate $\epsilon$-feasibility and $\Gamma$-concentration as constraints into the optimization formulation (Eq. 16). It turns out that both of them can be modeled as linear constraints. $\epsilon$-feasibility (Eq. 25) can be easily transformed into a linear constraint by eliminating the absolute value. To incorporate $\Gamma$-concentration (Eq. 26), the following Proposition is used:

**Proposition 1.** The $\Gamma$-concentration inequality (Eq. 26) holds if the variance of $Q_{t}^{u,v,r}$ is bounded from

\[\text{In reality, as the actual flow is the summation of many Bernoulli random variables, the coefficient of variation will shrink with the increase in passenger size. So in the case of a large number of passengers, the $\Gamma$-concentration should be naturally satisfied} \]
above by \((\Gamma_{^u,v,r}^t)^2\). Mathematically:

\[
\sum_{p \in P_{^u,v,r}^t} \sum_{r' \in R_{^u,v}} x_{p,r'} \cdot \pi_{p,r'} \cdot (1 - \pi_{p,r'}) \leq (\Gamma_{^u,v,r}^t)^2
\]  

(27)

**Proof.** From the triangular inequality, we have:

\[
\begin{align*}
|Q_{^u,v,r}^t - q_{^u,v,r}^t| & \leq |Q_{^u,v,r}^t - \mu_{^u,v,r}^t(x)| + |\mu_{^u,v,r}^t(x) - q_{^u,v,r}^t| \\
& \leq |Q_{^u,v,r}^t - \mu_{^u,v,r}^t(x)| + \epsilon_{^u,v,r}^t
\end{align*}
\]

(28)

As LHS \(\leq\) RHS, the probability measure satisfies (for all \(a > \epsilon_{^u,v,r}^t\)):

\[
P[\text{LHS} \geq a] \leq P[\text{RHS} \geq a]
\]

(29)

Notice that

\[
P[\text{RHS} \geq a] = P[|Q_{^u,v,r}^t - \mu_{^u,v,r}^t(x)| \geq a - \epsilon_{^u,v,r}^t] \leq \left(\frac{\sigma_{^u,v,r}^t(x)}{a - \epsilon_{^u,v,r}^t}\right)^2
\]

(30)

Eq. 30 is based on Chebyshev’s inequality. Therefore,

\[
P[\text{LHS} \geq a] = P[|Q_{^u,v,r}^t - \mu_{^u,v,r}^t(x)| \geq a - \epsilon_{^u,v,r}^t] \leq \left(\frac{\sigma_{^u,v,r}^t(x)}{a - \epsilon_{^u,v,r}^t}\right)^2
\]

(31)

Comparing Eqs. 31 and 26, we know that to satisfy Eq. 26, we only need \(\sigma_{^u,v,r}^t(x) \leq \Gamma_{^u,v,r}^t\), which completes the proof.

For modeling convenience, we set \(\epsilon_{^u,v,r}^t = \epsilon \cdot \mu_{^u,v,r}^t(x)\) and \(\Gamma_{^u,v,r}^t = \Gamma \cdot \delta_{^u,v}^t\), where \(\epsilon\) and \(\Gamma\) are hyper-parameters determining how close and concentrated the actual flow should be. Then the final constraint becomes:

\[
(1 - \epsilon) \sum_{p \in P_{^u,v,r}^t} \sum_{r' \in R_{^u,v}} x_{p,r'} \cdot \pi_{p,r'} \leq q_{^u,v,r}^t \leq (1 + \epsilon) \sum_{p \in P_{^u,v,r}^t} \sum_{r' \in R_{^u,v}} x_{p,r'} \cdot \pi_{p,r'}
\]

(32)

and

\[
\sum_{p \in P_{^u,v,r}^t} \sum_{r' \in R_{^u,v}} x_{p,r'} \cdot \pi_{p,r'} \cdot (1 - \pi_{p,r'}) \leq (\Gamma \cdot \delta_{^u,v}^t)^2
\]

(33)

Both constraints are linear and can be added into Eq. 16.

Besides the total system travel time, many recommendation systems also aim to respect passenger’s preferences. That is, if possible, a path with high inherent utility \(V_p^r\) should be recommended. Hence the following term is added into the objective function.

\[
\max_{p \in P} \sum_{r \in R_p} x_{p,r} \cdot V_p^r \iff \min_{p \in P} \sum_{r \in R_p} -x_{p,r} \cdot V_p^r
\]

(34)
The final individual path recommendation (IPR) model can be formulated as:

\[
(IPR) \quad \min_{x, q, z} \quad W_T(q, z) + IV_T(z) + \Psi \sum_{p \in P} \sum_{r \in R_p} -x_{p, r} \cdot V_{p, r}
\]  

subject to Constraints \((16b) - (16h)\)

\[
(1 - \epsilon) \sum_{p \in P_{t, u, v}^w} \sum_{r' \in R_{u, v}} x_{p, r'} \cdot \pi_{p, r'}^{u, v} \leq q_{t, u, v}^{u, v} \leq (1 + \epsilon) \sum_{p \in P_{t, u, v}^w} \sum_{r' \in R_{u, v}} x_{p, r'} \cdot \pi_{p, r'}^{u, v} \\
\forall t \in T, (u, v, r) \in F
\]

\[
\sum_{p \in P_{t, u, v}^w} \sum_{r' \in R_{u, v}} x_{p, r'} \cdot \pi_{p, r'}^{u, v} \cdot (1 - \pi_{p, r'}^{u, v}) \leq (\Gamma \cdot d_{t, u, v}^{u, v})^2 \quad \forall t \in T, (u, v, r) \in F
\]

\[
\sum_{r \in R_p} x_{p, r} = 1 \quad \forall p \in P
\]

\[
x_{p, r} \in \{0, 1\} \quad \forall p \in P, r \in R_p
\]

where \(\Psi\) is a hyper-parameter to adjust the scale and balance the trade-off between system efficiency and passenger preferences.

4.4. Benders decomposition

Eq. 35 is a mixed-integer linear programming (MILP). The structure of Eq. 35 allows us to efficiently solve it by Benders decomposition (BD) (Benders, 1962). The basic idea of BD is to decompose the problem into a master problem and a subproblem and solve these problems iteratively. The decision variables are divided into difficult variables, which in our case are the binary variables \(x\), and a set of easier variables, the continuous \(q\) and \(z\). At each iteration, the master problem determines one possible leader decision \(x\). This solution is used in the subproblem to generate optimality-cuts or feasibility-cuts, which are added to the master problem.

Interestingly, in this study, the master problem decides the recommendation strategies, which is a MILP of a smaller scale and can be solved efficiently using existing solvers. The subproblem reduces to the optimal flow problem (Eq. 16) with one more linear constraint (still linear programming). This format makes the BD an appropriate algorithm for the original problem.

4.4.1. Subproblem

The subproblem is derived by fixing the decision variables \(x\), and only considering the components including \(q\) and \(z\).

\[
[SP(x)] \quad \min_{q, z} \quad W_T(q, z) + IV_T(z)
\]

subject to Constraints \((16b) - (16h)\)

Constraint \((35c)\)
The objective of the dual problem of Eq. 36 is

\[
D(\alpha, \beta, \gamma, \eta, \kappa, \rho; x) = \sum_{l \in \mathcal{L}} \sum_{t \in \mathcal{T}} K_{l,t} \alpha_{l,t,t'} + \sum_{(u,v,r) \in \mathcal{F}} \sum_{t \in \mathcal{T}} f_{t'}^{u,v,r} \beta_{t}^{u,v,r} \\
+ \sum_{(u,v,r,t) \in \Omega} \sum_{t' \in \mathcal{T}} K_{l,t}^{u,v} \gamma_{l,t}^{u,v,r,t'} + \sum_{(u,v) \in \mathcal{W}} \sum_{r \in \mathcal{R}} \sum_{t \in \mathcal{T}} d_{l}^{u,v} \eta_{l}^{u,v} \\
+ \sum_{(u,v,r) \in \mathcal{F}} \sum_{t \in \mathcal{T}} \sum_{(u,v) \in \mathcal{W}} \sum_{r \in \mathcal{R}} x_{p,r'} \pi_{p,r'}^{r} \\
+ \sum_{(u,v) \in \mathcal{W}} \sum_{t \in \mathcal{T}} \sum_{(u,v,r) \in \mathcal{F}} \sum_{t' \in \mathcal{T}} \kappa_{l,t}^{u,v} \cdot (1 - \epsilon) \sum_{r \in \mathcal{R}} x_{p,r'} \pi_{p,r'}^{r} \\
+ \sum_{(u,v) \in \mathcal{W}} \sum_{t \in \mathcal{T}} \sum_{(u,v,r) \in \mathcal{F}} \sum_{t' \in \mathcal{T}} \rho_{l,t}^{u,v} \cdot (1 + \epsilon) \sum_{r \in \mathcal{R}} x_{p,r'} \pi_{p,r'}^{r} \\
\tag{37}
\]

where \(\alpha, \beta, \gamma, \eta\) are the dual variables associated with constraints 16b, 16c, 16f, 16e, respectively. \(\kappa, \rho\) are the dual variables associated with constraint 35c. Let \(\Theta := (\alpha, \beta, \gamma, \eta, \kappa, \rho)\). If the dual problem of Eq. 36 is feasible and bounded with a solution \(\Theta^*\), the following optimality cut is added to the master problem:

\[
Y \geq D(\Theta^*; x) \\
\tag{38}
\]

where \(Y\) is a decision variable in the master problem. If the dual problem of Eq. 36 is unbounded, and \(\Theta^*\) is an optimal extreme ray of the dual, the following feasibility cut is added to the master problem:

\[
D(\Theta^*; x) \leq 0 \\
\tag{39}
\]

4.4.2. Master problem

Let \(\mathcal{A}^o\) be the set of solutions \(\Theta^*\) of optimality cuts and \(\mathcal{A}^f\) be the set of solutions \(\Theta^*\) of feasibility cuts.

At each iteration of the BD, a cut based on the solution of the subproblem is added to the respective set, and the corresponding master problem is defined as follows:

\[
[MP(\mathcal{A}^o, \mathcal{A}^f)] \min_{x,Y} \Psi \sum_{p \in \mathcal{P}} \sum_{r \in \mathcal{R}_p} -x_{p,r} \cdot V_{p,r} + Y \\
\text{s.t.} \quad Y \geq D(\Theta^*; x) \quad \forall \Theta^* \in \mathcal{A}^o \\
D(\Theta^*; x) \leq 0 \quad \forall \Theta^* \in \mathcal{A}^f \\
\text{Constraints (35d) – (35f)} \\
\tag{40a}
\]

Note that the master problem has a smaller scale compared to the original problem (because there are no \(z\) and \(q\)), which can be solved efficiently.
4.4.3. Convergence

Let \((x^{(k)}, Y^{(k)})\) and \((q^{(k)}, z^{(k)})\) be the solutions of the master problem and subproblem, respectively, in the \(k\)-th iteration. Then, the upper \((UB^{(k)})\) and lower \((LB^{(k)})\) bounds at the \(k\)-th iteration are given by:

\[
UB^{(k)} = \Psi \sum_{p \in P} \sum_{r \in R_p} -x^{(k)}_{p,r} \cdot V_{p,r} + WT(q^{(k)}, z^{(k)}) + IVT(z^{(k)})
\]

\[
LB^{(k)} = \Psi \sum_{p \in P} \sum_{r \in R_p} -x^{(k)}_{p,r} \cdot V_{p,r} + Y^{(k)}
\]

\(LB^{(k)}\) will keep increasing as \(k\) increases because more cuts are added into the master problem. \(UB^{(k)}\) does not necessarily decrease at every iteration. The convergence criterion is

\[
Gap^{(k)} = \frac{UB^{(k)} - LB^{(k)}}{LB^{(k)}} \leq \text{Predetermined threshold}
\]  

4.5. Post-adjustment for equity

In this study, we define the equity requirement of a recommendation strategy as \(\text{passengers with the same OD (u, v) and departure time t should not have too much difference in travel time if they follow the recommendations.}\) Mathematically:

\[
w_p := \sum_{r \in R_p} TT_{p,r} \cdot x_{p,r} - \min_{p' \in P_p} \{ \sum_{r \in R_{p'}} TT_{p',r} \cdot x_{p',r} \} \leq E \cdot \tau \quad \forall p \in P
\]

where \(TT_{p,r}\) is the travel time of path \(r \in R_p\) for passenger \(p\), which can be obtained from Eq. 18. \(\sum_{r \in R_p} TT_{p,r} \cdot x_{p,r}\) is the travel time on the recommended path. \(w_p\) represents the difference between passenger \(p\)’s travel time (if he/she follows the recommendation) and the minimum travel time of all passengers with the same \((u, v, t)\). \(E \cdot \tau\) is a predetermined threshold for the travel time difference (where \(E \in \mathbb{N}\)).

As mentioned in Section 4.1, we cannot formulate path travel time in the optimal flow model. Hence, Eq. 44 cannot be added into Eq. 35 as a constraint directly. In this section, we propose a post-adjustment heuristic method to address the equity constraint. The basic idea of the post-adjustment is to change the recommendations of high-cost paths to low-cost ones after solving Eq. 35. Notice that if \(E = 0\) in Eq. 44, we obtain a user equilibrium recommendation pattern. This motivates us to direct the current system optimal recommendations toward user equilibrium recommendations. The user equilibrium solution is usually obtained by the method of successive averages (MSA):

\[
(q^{u,v,r}_{t})^{(k+1)}(k+1) = (1 - \lambda_k) \cdot (q^{u,v,r}_{t})^{(k)} + \lambda_k \cdot (q^{u,v,r}_{t})^{(k)}
\]

where \((q^{u,v,r}_{t})^{(k)}\) is the flow of all-or-nothing assignment to the shortest paths at iteration \(k\). \((q^{u,v,r}_{t})^{(k)}\) is the current flow at iteration \(k\), and \(\lambda_k\) is the step size at iteration \(k\). Eq. 45 means at each iteration, some of the flows are moved to the shortest path so that we expect that ultimately all paths have similar travel time.

However, directly using MSA may violate the \(\epsilon\)-feasibility and \(\Gamma\)-concentration because Eq. 45 does not guarantee that \((q^{u,v,r}_{t})^{(k+1)}(k+1)\) satisfies these two constraints. Therefore, we propose a new MILP formulation that works similar to the MSA-based flow redistribution but guarantees \(\epsilon\)-feasibility and \(\Gamma\)-concentration at each iteration.

Note that the post-adjustment is initialized at \((q^{u,v,r}_{t})^{(0)} = (q^{u,v,r}_{t})^{*}\), where \((q^{u,v,r}_{t})^{*}\) is the optimal flow
solution from Eq. 35. By definition, \((q^{u,v,r}_t)^{(0)}\) satisfies \(\epsilon\)-feasibility and \(\Gamma\)-concentration. Now we need to develop a method that satisfies \((q^{u,v,r}_t)^{(k)}\) is \(\epsilon\)-feasible and \(\Gamma\)-concentrated for all \(k \geq 1\).

Let the \(w_p\) at iteration \(k\) be:

\[
u_p^{(k)} := \sum_{r \in \mathcal{R}_p} TT_{p,r}^{(k)} \cdot x_p^{(k)} - \min_{p' \in \mathcal{P}_{t}} \left\{ \sum_{r \in \mathcal{R}_{p'}} TT_{p',r}^{(k)} \cdot x_{p'}^{(k)} \right\}
\]

where \(w_p^{(k)}\) represents the “degree of unfairness” for passenger \(p\) at iteration \(k\) because the larger the value, the longer the travel time of his/her recommended path compared to the shortest one. Let \(r_{p,\text{min}}^{(k)} = \arg \min_{r \in \mathcal{R}_p} \{TT_{p,r}\}\) be the shortest path ID for passenger \(p\) at iteration \(k\). We formulate the post-adjustment MILP at iteration \(k\) as:

\[
(PA(x^{(k)}, q^{(k)})) \max_{x^{(k+1)}, q^{(k+1)}} \sum_{p \in \mathcal{P}} w_p^{(k)} \cdot \frac{1}{\mathcal{R}_p} \cdot x_p^{(k)} (k+1) \quad \text{s.t.} \quad (q^{u,v,r}_t)^{(k)} \leq (q^{u,v,r}_t)^{(k+1)} \leq (1 - \lambda_k) \cdot (q^{u,v,r}_t)^{(k)} + \lambda_k \cdot (q^{u,v,r}_t)^{(k)}
\]

\[
\forall (u, v, r, t) \in \mathcal{U}_p^{(k)} \quad (1 - \lambda_k) \cdot (q^{u,v,r}_t)^{(k)} + \lambda_k \cdot (q^{u,v,r}_t)^{(k)} \leq (q^{u,v,r}_t)^{(k+1)} \leq (q^{u,v,r}_t)^{(k)}
\]

\[
\forall (u, v, r, t) \in \mathcal{U}_p^{(k)} \quad \sum_{r \in \mathcal{R}_{u,v}} (q^{u,v,r}_t)^{(k+1)} = d_t^{u,v} \quad \forall (u, v, t) \in \mathcal{W}
\]

\[
(q^{u,v,r}_t)^{(k+1)} \geq 0 \quad \forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F}
\]

\[
(x_{p,r})^{(k+1)} = x_{p,r}^{(k)} \quad \forall p \in \mathcal{P}_{\text{Eq}}^{(k)}
\]

Constraints 47b and 47c mean that, at each step, the new flows at iteration \(k + 1\) (i.e., \(q^{(k+1)}\)) are between \(q^{(k)}\) and the values obtained by MSA, where \(\mathcal{U}_p^{(k)}\) (resp. \(\mathcal{U}_p^{(k)}\)) is the set of (\(u, v, r, t\)) indices that make \((q^{u,v,r}_t)^{(k)} > 0\) (resp. \((q^{u,v,r}_t)^{(k)} = 0\)). With constraints 47g, the solution \(q^{(k+1)}\) and \(x^{(k+1)}\) satisfy \(\epsilon\)-feasibility and \(\Gamma\)-concentration.

The objective function is maximized (without constraints) if \(x_{p,r}^{(k+1)} = 1\) for all \(r = r_{p,\text{min}}^{(k)}\) which means we should change as many of the recommendations as possible to the shortest path. However, 47b and 47c do not allow to change all recommendations to the shortest path. The unfairness weight \(w_p^{(k)}\) suggests that the recommendations to the passengers with the largest degree of unfairness should be changed first.

\(\mathcal{P}_{\text{Eq}}^{(k)}\) is the set of passengers whose recommendations already satisfy the equity requirement (i.e., \(\mathcal{P}_{\text{Eq}}^{(k)} = \{p \in \mathcal{P} : w_p^{(k)} \leq E \cdot \tau\}\)). Constraint 47f helps to fix the recommendations that already satisfy the equity requirement, which reduces the scale of the problem.

**Remark 2.** The post-adjustment problem (Eq. 47) aims to find a new recommendation strategy \(x^{(k+1)}\) and flow patterns \(q^{(k+1)}\) closer to the user equilibrium solutions (i.e., with more equity). And the solutions are guaranteed to satisfy \(\epsilon\)-feasibility and \(\Gamma\)-concentration. The post-adjustment problem is always feasible because \((x^{(k)}, q^{(k)})\) is a feasible solution (associated with no flow and recommendation changes). And it can be solved efficiently using branch-and-cut algorithms supported by many off-the-shelf solvers because the size of the problem is much smaller given that many integer variables are fixed (due to constraint 47f).
Moreover, we can use \((x^{(k)}, q^{(k)})\) as a warm-start to further accelerate the solution.

Eq. 47 presents the formulation at iteration \(k\). The complete algorithm for the post-adjustment problem is shown in Algorithm 1. There are two stopping criteria: 1) \((x^{(k)}, q^{(k)}) = (x^{(k-1)}, q^{(k-1)})\) or 2) \(\mathcal{P}_{\text{Eq}}^{(k)} = \mathcal{P}\). The first one means that we cannot find a new recommendation strategy and flow pattern that satisfies equity, \(\epsilon\)-feasibility, and \(\Gamma\)-concentration requirements. And the second criterion means that all passenger’s recommendations satisfy the equity requirement.

**Algorithm 1 Solution procedure of the post-adjustment approach for equity**

1: Initialize \((x^{(0)}, q^{(0)})\) as the optimal solution of Eq. 35.
2: Set the iteration counter \(k = 0\).
3: do
4: Solve the post-adjustment problem (Eq. 47) with \((x^{(k)}, q^{(k)})\) as inputs, and return \((x^{(k+1)}, q^{(k+1)})\)
5: \(k = k + 1\)
6: while \((x^{(k)}, q^{(k)}) = (x^{(k-1)}, q^{(k-1)})\) and \(\mathcal{P}_{\text{Eq}}^{(k)} \neq \mathcal{P}\)
7: return \((x^{(k)}, q^{(k)})\)

**Remark 3.** Post-adjustment is a heuristic method to find a recommendation strategy that satisfies the equity requirements. Ideally, we also want the system travel time to be minimized under the equity constraint. But the proposed heuristic method cannot guarantee travel time optimality. With the updating by the post-adjustment procedure, the flow patterns are closer to the user equilibrium, and the system travel time will be increased. We may control the step size \(\lambda_k\) to make sure the new flows are not too different from the system optimal one, so that the new travel time is not too bad. However, a too small value of \(\lambda_k\) may limit the flow updating range and lead to a situation where not all passengers with \(w_p \leq E \cdot \tau\) (i.e., Algorithm 1 stops with the first criterion). Hence, the step size \(\lambda_k\) plays a trade-off role between satisfying equity and controlling the increase in system travel time.

5. Model extension

In this section, we discuss several extensions of the model to accommodate more realistic/general scenarios.

5.1. Generalization of recommendations

In this study, we assume the information given to passengers is a recommended path. In reality, the recommendation system may provide a bundle of recommended paths with information like estimated in-vehicle time, waiting time, travel cost, etc. The proposed framework can be extended to handle different recommendation typologies. Figure 6 shows an example where the recommendation system will provide a composition of path and travel time information, where each composition can include different paths, different estimated waiting/in-vehicle times, etc. Then, we can change \(x_{p,r}\) to \(x_{p,c}\), where \(x_{p,c}\) indicates whether we will present composition \(c\) to passenger \(p\). Similarly, each \(c\) is associated with a conditional probability \(\pi_{p,c}\) as shown in Figure 6 (the probability for passenger \(p\) to choose path \(r\) given that he/she is recommended composition \(c\)).
In this way, we only need to calibrate $\pi_{p,c}^r$ and predetermine the composition set $C_p$ for each passenger $p$. The overall framework proposed above can be easily adapted to the new recommendation typology by replacing $x_{p,r}$ and $\pi_{p,r}^r$ with $x_{p,c}$ and $\pi_{p,c}^r$, respectively.

5.2. Feedback and rolling-horizon

As mentioned in Section 3.2, the whole path recommendation problem should be solved in a rolling-horizon manner. At each time interval $t \geq 1$, we update the demand, supply, and system state information, and solve the proposed framework above to get a recommendation strategy $x$. But we only implement the $x_{p,r}$ for $p \in \mathcal{P}_{u,v}^t$, $\forall (u, v)$ (i.e., passengers departing at current time $t$).

The rolling horizon requires updating the estimated demand and system state information. The recommendation system can ask for passenger feedback to facilitate the estimation. For example, after providing a recommendation, we can ask the passenger to respond whether he/she will actually use it or not. This feedback can be used to update the demand predictions.

5.3. Integrated formulation with behavior uncertainty and equity

As mentioned in Section 4.5, the equity requirement (Eq. 44) cannot be incorporated into the recommendation model (Eq. 35) due to the fact that it is hard to formulate path travel time directly. In this section, we show a possible MILP formulation that incorporates the equity requirement. However, this formulation is far more complicated than Eq. 35 and is hard to solve, representing a future research activity.

Let $y_{i}^{u,v,r,t}$ represent a binary variable indicating whether the arrival time of a passenger departing at time $t$ using $(u, v, r)$ is $i$ or not. By definition,

$$\sum_{i \in T_{u,v,r}^{t}} y_{i}^{u,v,r,t} = 1 \quad \forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F}$$

(48)

$$y_{i}^{u,v,r,t} \in \{0, 1\} \quad \forall t \in \mathcal{T}, (u, v, r) \in \mathcal{F}, \tilde{t} \in T_{u,v}^{t}$$

(49)

where $T_{u,v,r}^{t}$ is the set of possible arrival time indices (see Eq. 17). From the relationship between arrival
time and cumulative flows (Eq. 17), we have
\[
\sum_{t' = t_{\text{min}}}^{t} (f_{t'}^{u,v,r} + q_{t'}^{u,v,r}) \leq \sum_{t_{\text{min}} \leq t' \leq t_{\text{min}} + \delta_{t'}^{u,v,r} + |E_{t'}^{u,v,r}|} z_{t'}^{u,v,r} + (1 - y_{t'}^{u,v,r,t}) M
\]
\forall t \in T, (u, v, r) \in \mathcal{F}, \bar{t} \in T_{\bar{t}}^{u,v,r}
\]
where \(M\) is a big positive constant. When \(y_{t'}^{u,v,r,t} = 0\), \(\bar{t}\) is not the arrival time, and Eq. 50 is not binding. As the arrival time is the minimum \(\bar{t}\) that satisfies Eq. 50, the following component should be added into the objective function
\[
\min \sum_{t} \sum_{(u, v, r) \in \mathcal{F}} \sum_{\bar{t} \in T_{\bar{t}}^{u,v,r}} y_{t}^{u,v,r,t} \cdot \bar{t}
\]
Eqs 48 \sim 51 provide a possible way to model path travel times by defining the binary variable \(y\). To incorporate the equity constraint, let the earliest arrival time over all paths of OD pair \((u, v)\) and time \(t\) be \(EAT_{t}^{u,v}\):
\[
EAT_{t}^{u,v} = \min_{r \in \mathcal{R}_{u,v}} \{ \sum_{\bar{t} \in T_{\bar{t}}^{u,v,r}} y_{t}^{u,v,r,t} \cdot \bar{t} \} \quad \forall t \in T, (u, v) \in \mathcal{W}
\]
As the objective is minimizing \(\sum_{t} y_{t}^{u,v,r,t} \cdot \bar{t}\), Eq. 52 can be transformed to a linear constraint:
\[
EAT_{t}^{u,v} \leq \sum_{\bar{t} \in T_{\bar{t}}^{u,v,r}} y_{t}^{u,v,r,t} \cdot \bar{t} \quad \forall t \in T, (u, v) \in \mathcal{W}, r \in \mathcal{R}_{u,v}
\]
And the equity requirement can be formulated as
\[
\sum_{\bar{t} \in T_{\bar{t}}^{u,v,r}} y_{t}^{u,v,r,t} \cdot \bar{t} - EAT_{t}^{u,v} \leq E \cdot \tau + (1 - x_{p,r}) . M \quad \forall t \in T, (u, v) \in \mathcal{W}, p \in \mathcal{P}_{t}^{u,v}, r \in \mathcal{R}_{u,v}
\]
If \(x_{p,r} = 0\) for all \(p \in \mathcal{P}_{t}^{u,v}\), path \(r\) is not recommended and Eq. 54 becomes ineffective.

Adding all these constraints and the objective component into Eq. 35 results in the integrated formulation. However, it is extremely hard to solve due to a large number of integer variables \(y\) and the complicated constraint interactions. We show this formulation to demonstrate the difficulty of incorporating equity requirements and highlight the importance of the heuristic post-adjustment approach in Section 4.5.

6. Case study

6.1. Case study design

6.1.1. CTA Blue Line disruption

For the case study, we consider an actual incident in the Blue Line of the Chicago Transit Authority (CTA) urban rail system (Figure 7). The incident starts at 8:14 AM and ends at 9:13 AM on Feb 1st, 2019 due to infrastructure issues between Harlem and Jefferson Park stations (the red X in the figure) that led to a whole Blue Line suspension. During the disruption (morning hours), the destination for most of the passengers is the “Loop” in the CBD area in Chicago. There are four alternative paths to the Loop: 1) using
the Blue Line (i.e., waiting for the system to recover), 2) using the parallel bus lines, 3) using the North-South (NS) bus lines to transfer to the Green Line, and 4) using the West-East (WE) bus lines to transfer to the Brown Line. Based on the service structure, the route sets \( R^{(u,v)} \) for each OD pair \((u,v)\) can be constructed.

![Figure 7: Case study network](image)

In the case study, we divide the time into \( \tau = 5 \) mins equal-length intervals, and focus on solving the problem at \( t = 1 \) (i.e., beginning of the incident). We assume that the set of passengers to receive recommendations \( (P) \) consists of all passengers with their intended origins at the Blue Line and destinations in the Loop. A simulation model (Mo et al., 2020) is used to get the system state up to time \( t = 1 \) (i.e., the incident time 8:14 AM) and generate \( z_{t}^{u,v,r,i} \) and \( \Omega_1 \). The recommendation strategy covers passengers departing between \( t = 1 \) and \( T^D = 23 \), approximately one hour after the end of the incident (9:13 AM). The analysis period is set as \( t_{\text{min}} = -13 \) and \( T = 34 \), approximately one hour before \( t = 1 \) and after \( T^D \), providing enough buffer (warm-up and cool-down time) for passengers in \( P \) to finish their trips. As demand and incident duration predictions are out of the scope of this paper, we simply use the actual demand and incident duration for all experiments. Our other work (Mo et al., 2022a) proposes to use robust and stochastic optimization to address demand and incident duration uncertainty, respectively.

6.1.2. **Conditional probability matrix \( \pi \)**

In this section, we describe how to generate the synthetic conditional probability matrix \( \pi \) used for the case study. During the incident, CTA does not provide specific path recommendation information. For every individual, we assume that their actual path choices (referred to as the “status quo” choices) reflect their inherent preferences. Appendix B presents the method of inferring passengers’ status quo choices during the disruption using smart card data (Mo et al., 2022c). The basic idea is to track their tap-in records when entering the Blue Line and nearby bus routes, and compare them with their historical travel histories to get the transfer information.

Given the status quo choices, we assume that the “true” passenger \( p \)'s inherent preference for path \( r \) is given by

\[
V^r_p = \begin{cases} 
1 + v^r_p & \text{if } r \text{'s actual path choice} \\
v^r_p & \text{otherwise,} 
\end{cases} \quad \forall p \in P, r \in R_p 
\]
where \( v_r^p \) is drawn uniformly from \( U[0,1] \). Eq. 55 indicates every path has a random utility \( v_r^p \) normalized to \( 0 \sim 1 \). And the chosen path has an additional utility value of 1. We assume that the impact of the recommendation of \( r' \) on the utility of path \( r \) is

\[
I_{p,r,r'}^r = \begin{cases} 
\text{Drawn from } U[0,5] & \text{if } r = r' \\
0 & \text{otherwise, } \forall p \in \mathcal{P}, r, r' \in \mathcal{R}_p 
\end{cases}
\]  

(56)

Eq. 56 means that the utility of the path recommended (i.e., \( r = r' \)) has an additional positive impact drawn uniformly from \( U[0,5] \). The utilities of paths not being recommended (\( r \neq r' \)) do not change. Given Eqs. 55 and 56, we can generate the conditional probability \( \pi \) using Eq. 21. It is worth mentioning that the above assumptions for generating synthetic passenger prior preferences are based on two reasonable principles: 1) Passenger’s actual chosen path has a higher inherent utility. 2) Recommendations of a path can increase its probability of being chosen.

6.2. Parameter settings

The \( \epsilon \)-feasibility and \( \Gamma \)-concentration parameters are set as \( \epsilon = 0.05 \) and \( \Gamma = 0.3 \), indicating 5% and 30% variation constraints in mean and variance. The equity requirement parameter is set as \( E = 2 \), meaning that we allow at most 10 minutes difference in travel time for passengers with the same OD and departure time. The convergence gap threshold for Benders decomposition is set as \( 1 \times 10^{-8} \). The post-adjustment updating step is set as \( \lambda_k = \frac{1}{4} \) based on numerical tests.

6.3. Benchmark models

There are two benchmark path choice scenarios we use for comparison purposes:

- **Status-quo path choices.** This scenario provides the status quo situation which does not include any recommendations. It represents the worst case. In this scenario, no behavior uncertainty is considered because this is based on the actual path choices realized by passengers.

- **Capacity-based path recommendations.** The capacity-based path recommendations aim to recommend passengers to different paths according to the available capacity of paths. Specifically, for a path in OD pair \((u,v)\) and time \(t\), its capacity is the total available capacity of all vehicles passing through the first boarding station of the path during the time period. For example, for a path consisting of an NS bus route and the Green Line, the path capacity is the total available capacity of all buses at the boarding station of the NS bus route during time interval \(t\). The available capacity can be obtained from a simulation model using historical demand as the input or using historical passenger counting data. The available capacity for the Blue line (the incident line) depends on modified operations during the incident (i.e., the service suspension is considered). When no vehicles operate in the Blue line during time interval \(t\), the path capacity is zero.

6.4. System travel time evaluation

Given a recommendation strategy \( x \), as mentioned above, the actual system travel time is a random variable because of the passenger behavior uncertainty. To obtain the mean and standard deviation of the system travel time, we generate multiple passenger choice realizations based on \( \pi \) and \( x \). For each generated passenger choice \( (\pi^r_{p,r'}) \), the realized path flows are

\[
q_{t}^{u,v,r} = \sum_{p \in \mathcal{P}_t^{u,v}} \sum_{r' \in \mathcal{R}_{u,v}} x_{p,r'} \cdot \mathbb{1}_{p,r'}^{r'} \quad \forall (u,v) \in \mathcal{F}, t \in \mathcal{T}.
\]  

(57)
The system travel time for the above passenger choice realization is calculated by solving the optimal flow problem (Eq. 16) with the constraints \( q_{t,u,v,r} = q_{t}^{u,v,r} \) for all \((u, v, r) \in \mathcal{F} \) and \( t \in \mathcal{T} \). This process is repeated with multiple realizations, providing the sample mean and standard deviation of the system travel time under recommendation strategy \( x \).

6.5. Experimental design

As this paper considers various components (such as the optimal flow optimization, passengers’ path preferences, behavior uncertainty, equity, etc.), it is useful to test different components separately to identify the impact of each one. Hence, we design the following test cases, each one with specific parameter settings to systematically evaluate the impacts of each component.

**Model performance compared to benchmark models.** The most straightforward model validation is to evaluate the effect on reducing system travel time. In this test case, we set \( \Psi = 0 \), meaning that we ignore the passengers’ preference and focus only on minimizing system travel time. We also solve the individual path recommendation model (Eq. 35) without the post-adjustment process. The impact of the equity adjustments will be evaluated in another experiment introduced later. The results of this test case are discussed in Section 7.2.

**The benefit of considering behavior uncertainty.** In this test case, we evaluate the importance of incorporating behavior uncertainty in the model. The model without behavior uncertainty assumes that passengers take the recommended path. The recommendation strategy is obtained by solving Eq. 35 with \( \pi_{p,r'} = 1 \) if \( r = r' \). Similarly, we set \( \Psi = 0 \) and ignore the post-adjustment process. Note that, when we evaluate the recommendation strategy, the behavior uncertainty is still considered in generating the system travel time (see Section 6.4). The results of this test case are shown in Section 7.3.

**Impact of considering travel time equity.** In this test case, the post-adjustment method is used to obtain “equity”-constrained path recommendations. The results before and after the post-adjustment are compared. The results of this test case are shown in Section 7.4.

**Impact of considering passenger preference.** In all the above tests, \( \Psi = 0 \) is used, focusing on the system travel time. In this test case, we evaluate the model performance under different values of \( \Psi \) in order to assess the impact of considering passenger preferences. The results of this test case are discussed in Section 7.5.

7. Results

7.1. Model convergence

**Convergence of Benders decomposition and computational time**

Figure 8 shows the convergence of the BD algorithm. As expected, the lower bound of the model keeps increasing, while the upper bound, after dropping significantly in early iterations, exhibits some fluctuations. The model converges after 28 iterations with a relative gap of less than \( 1 \times 10^{-8} \). The number of optimality cuts was 28 and no feasibility cut was generated.
Figure 8: Convergence of the Benders decomposition

Table 1 compares the computational time of the Benders decomposition and off-the-shelf solvers. The BD algorithm was implemented using Julia 1.6 with the Gurobi 9.1 solver (Gurobi Optimization, LLC, 2021) on a personal computer with the I9-9900K CPU. The total computational time is 17.8 seconds (master problem 8.2 seconds + subproblem 9.6 seconds), which is more efficient than directly using the Mixed integer programming (MIP) solvers, including Gurobi (Gurobi Optimization, LLC, 2021), CPLEX (Cplex, 2009), GLPK (GNU Linear Programming Kit) (Makhorin, 2008), and CBC (Coin-or branch and cut) (Forrest and Lougee-Heimer, 2005).

<table>
<thead>
<tr>
<th>Solver</th>
<th>CPU time (sec)</th>
<th>Gap</th>
<th>Solver</th>
<th>CPU time (sec)</th>
<th>Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>BD</td>
<td>17.8</td>
<td>0.000%</td>
<td>Gurobi</td>
<td>55.1</td>
<td>0.000%</td>
</tr>
<tr>
<td>CPLEX</td>
<td>65.7</td>
<td>0.000%</td>
<td>CBC</td>
<td>425.4</td>
<td>0.000%</td>
</tr>
<tr>
<td>GLPK</td>
<td>562.6</td>
<td>0.000%</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7.1.2. Convergence of post-adjustment algorithm

Figure 9 shows the convergence process of the post-adjustment algorithm. Specifically, we show the number of non-equity passengers (i.e., \(|P| - |P_{Eq}^{(k)}|\)) and the average value of \(w_p\) (i.e., \(\sum_{p \in P} w_p / |P|\)). Iteration 0 indicates the system optimal state (i.e., the solution of Eq. 35). We observe that, after only three iterations, the post-adjustment algorithm terminates and all passengers in \(P\) have a travel time difference relative to the shortest path (i.e., \(w_p\)) smaller than 10 minutes, satisfying the equity requirement. The average \(w_p\) keeps decreasing with the post-adjustment process, meaning that more and more passengers have similar travel times.
7.2. Model performance compared to benchmark models

In this section, we compare the system travel time under the proposed individual path recommendations (without post-adjustment) and two benchmark models. All travel times (except for the status quo that is deterministic) are calculated based on 10 replications using the randomly sampled actual path choices based on the given recommendation (see Section 6.4).

Table 2 shows that the proposed model (IPR) significantly reduces the average travel time in the system compared to the status quo. Specifically, there is a 6.6% reduction in travel times of all passengers in the system. And for passengers in the incident line (i.e., passengers who received the recommendation, $P$), the average travel time reduction is 19.0%. Our model also outperforms the capacity-based benchmark path recommendation strategy, which reduces the travel time of all passengers by 2.5% and incident line passengers by 15.9%. It is also worth noting that the standard deviation is small, meaning that variations due to behavior uncertainty are not significant.

### Table 2: Average travel time comparison for different models

<table>
<thead>
<tr>
<th>Models</th>
<th>Average travel time (all passengers)</th>
<th>Average travel time (incident line passengers)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (min)</td>
<td>Std. (min)</td>
</tr>
<tr>
<td>Status quo</td>
<td>28.318</td>
<td>N.A.</td>
</tr>
<tr>
<td>Capacity-based</td>
<td>27.609 (-2.5%)</td>
<td>0.033</td>
</tr>
<tr>
<td>IPR model</td>
<td>26.457 (-6.6%)</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Numbers in parentheses represent percentage travel time reduction compared to the status quo.

7.3. Benefits of considering behavior uncertainty

In this section, we aim to compare the model with and without considering the behavior uncertainty. The model without behavior uncertainty assumes that all passengers follow the recommended path when designing the recommendation (but they may not in reality).

Table 3 shows the comparison of average travel time for the two models. As expected, considering behavior uncertainty in the path recommendation design achieves smaller travel time for all passengers and incident line passengers. Note that, though the 0.93% reduction (around 15 seconds saving per passenger) is
relatively small, considering the large number of passengers in the system, the total travel time savings are still significant.

<table>
<thead>
<tr>
<th>Models</th>
<th>Average travel time (all passengers)</th>
<th>Average travel time (incident line passengers)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (min)</td>
<td>Std. (min)</td>
</tr>
<tr>
<td>IPR model (w.o. BU)</td>
<td>26.706</td>
<td>0.026</td>
</tr>
<tr>
<td>IPR model (w. BU)</td>
<td>26.457 (-0.93%)</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Numbers in parentheses represent percentage travel time reduction compared to the IPR model w.o. BU

7.4. Impact of considering travel time equity

Figure 10 shows the comparison before and after the post-adjustment, which reflects the impact of considering passenger equity. Figure 10a shows that before the post-adjustment, there are hundreds of passengers with more than 10 minutes longer travel time than the shortest path travel time, showing equity issues. After the post-adjustment, \( w_p \) is less than 10 minutes for all passengers in \( P \). Furthermore, the distribution of \( w_p \) is shifted to smaller values. Note that \( |P| \) is around 5,800, so most passengers are recommended a path with the shortest travel time (i.e., \( w_p = 0 \))

Figure 10b compares the system travel time before and after the post-adjustment as a function of the number of iterations. Since the system optimal solution has the smallest travel time, the post-adjustment, as expected, slightly increases the average travel time of all passengers by 0.45% (from 26.457 to 26.576 minutes), suggesting that a small sacrifice is enough to satisfy the equity requirement. Interestingly, the average travel time of the incident line passengers decreased (from 32.626 to 32.475 minutes), which implies that the increase in travel time mainly happens to passengers in nearby lines who are indirectly affected by the incidents, rather than the incident-line passengers.

Figure 10: Comparison before and after the post-adjustment

\(^4 w_p = 0 \) is not shown in the distribution because it is too large and will distort the figure
7.5. Impact of respecting passenger’s prior preferences

In this section, we evaluate the impact of different values of $\Psi$ in terms of respecting passenger’s prior preferences. Besides the system travel time, we also evaluate the total utility, defined as the sum of the prior utilities of the recommended path:

$$TU(x) = \sum_{p \in \mathcal{P}} \sum_{r \in \mathcal{R}_p} x_{p,r} \cdot V_{p,r}.$$  \hspace{1cm} (58)

Note that the maximum value of $TU(x)$ is achieved when every passenger is recommended their preferred path (i.e., the path with the highest prior utility, $V_{p,r}$). Denote this maximum value as $TU^{\text{max}}$. The relative ratio of total utility, $\frac{TU(x)}{TU^{\text{max}}}$, represents the fraction of the total (prior) utility that the recommendation has achieved.

Another indicator is the number of passengers recommended their preferred path (denoted as $NP(x)$). Similarly, we also define the proportion of passengers recommended their preferred path (i.e., $\frac{NP(x)}{|\mathcal{P}|}$, where $|\mathcal{P}| = 5,827$ in the case study).

Figure 11 shows the results for different value of $\Psi$. The x-axis is plotted in log-scale. In Figure 11a, the average travel time for all passengers and incident-line passengers increases with the increase of $\Psi$, which is as expected because the larger value of $\Psi$ means that the recommendation generation focuses more on satisfying passenger’s inherent preferences rather than minimizing the system travel time. Similarly, in Figure 11b, as expected, both $TU(x)$ and $NP(x)$ increase with the increase in $\Psi$. When $\Psi = 10^5$, the average travel time of the incident line passengers increased by 21.3%, which is close to the status quo scenario. This is because we generate passengers’ prior utilities based on the status quo choices. Figure 11b shows that nearly all passengers in $\mathcal{P}$ are recommended with their preferred path when $\Psi = 10^5$.

Figure 11 illustrates the trade-off between respecting passenger’s preferences and reducing system congestion. When the value of $\Psi$ is relatively small (e.g., less than $10^3$), increasing $\Psi$ can effectively increase the total utility and number of passengers recommended their preferred path. Meanwhile, the system travel time only slightly increases. But when $\Psi$ is large (e.g., greater than $10^4$), increasing $\Psi$ significantly increases the system travel time, but the impact on increasing passenger’s utility is limited. The reason may be that, in the system, there are some passengers whose preferred paths are not at the capacity bottlenecks. Hence, when $\Psi$ is small, the optimal solution recommends those passengers use their preferred paths without significantly impacting the system travel time. When $\Psi$ is large, passengers are recommended to use their preferred paths even if these paths are highly congested, causing a significant increase in the system travel time. The results imply that a reasonable value of $\Psi$ should be relatively small. With small $\Psi$, most of the passengers (e.g., more than 70%) are recommended to use their preferred paths without significantly reducing the system efficiency.
8. Conclusion and discussion

This study proposes a mixed-integer programming (MIP) formulation to model the individual-based path (IPR) recommendation problem during PT service disruptions with the objective of minimizing total system travel time and respecting passengers’ path choice preference. Passengers’ behavior uncertainty in path choices given recommendations and their travel time equity are also considered in the formulation. We first formulate the optimal flow distribution problem in PT systems as a linear programming, which outputs the optimal path flows for each OD pair and time interval that minimize the total system travel time. Then, we model the behavior uncertainty based on passenger’s prior preferences and posterior path choice probability distribution with two new concepts: $\epsilon$-feasible flows and $\Gamma$-concentrated flows, which control the mean and variance of path flows in the optimization problem. We show that these two concepts can be transformed into linear constraints using Chebyshev’s inequality. The individual path recommendation problem with behavior uncertainty is solved using Benders decomposition (BD) efficiently. The master problem of BD is a
small-scale integer programming and the subproblem of BD reduces the optimal flow problem that is a linear program. The BD is more efficient than many off-the-shelf MIP solvers. Finally, we use a post-adjustment heuristic to address equity requirements.

The proposed approach is demonstrated in a case study using data from a real-world urban rail disruption in the CTA system. The results show that the proposed IPR model significantly reduces the average travel times in the system compared to the status quo. Specifically, there is a 6.6% reduction in travel times for all passengers in the system. Passengers in the incident line (i.e., passengers who received the recommendation), experience a 19.0% average travel time reduction. Our model also outperforms the capacity-based benchmark path recommendation strategy. Compared to the model that assumes all passengers would follow the recommendations, considering behavior uncertainty in the path recommendation design can achieve smaller system travel time. Post-adjustment effectively reduces the difference in passengers’ travel times and increases equity. After the post-adjustment, the travel time difference to the shortest path is within 10 minutes for all passengers. The equity requirement slightly increases the system travel time by 0.46%, showing the trade-off between efficiency and equity. In terms of respecting passenger’s preferences, we show that it is possible that most of the passengers (e.g., more than 70%) are recommended their preferred paths while only increasing the system travel time by 0.51%.

Following the discussion in Section 5, future studies can be pursued in the following directions. First, as shown in Section 5.1, it is possible to extend the current framework with more complex recommendation compositions. The challenges in implementing the more general framework stem from the quantification of the posterior path choice probabilities. Future studies may conduct corresponding surveys to calibrate passengers’ responses to the recommendations. Second, the integrated formulation with both behavior uncertainty and equity is hard to solve (Section 5.3). Though the post-adjustment heuristic works well, it is still a methodological challenge to develop a direct solution algorithm for the integrated formulation. Finally, future studies may consider different sources of uncertainty (including incident duration, in-vehicle time, demand, etc.) for a more realistic modeling framework.

9. Author contribution statement

Baichuan Mo: Conceptualization, Methodology, Software, Formal analysis, Data Curation, Writing - Original Draft, Writing - Review & Editing, Visualization. Haris N. Koutsopoulos: Conceptualization, Writing - Review & Editing, Supervision. Jinhua Zhao: Conceptualization, Supervision, Project administration, Funding acquisition.

10. Acknowledgement

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References


## Appendices

### Appendix A. Notation

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(u, v, r, i)$</td>
<td>The $i$-th leg of path $r$ for OD pair $(u, v)$</td>
</tr>
<tr>
<td>$t$</td>
<td>Integer time index, $t = 1$ represents the start of the incident. Non-positive time indices indicate time before the incident.</td>
</tr>
<tr>
<td>$T^{D}$</td>
<td>Time index at which the recommendation system stops working</td>
</tr>
<tr>
<td>$t_{\text{min}}$</td>
<td>Start time index of the whole analysis period (negative by definition)</td>
</tr>
<tr>
<td>$t_{\text{end}}$</td>
<td>End time index of the incident</td>
</tr>
<tr>
<td>$T$</td>
<td>End time index of the whole analysis period (greater than $T^{D}$)</td>
</tr>
<tr>
<td>$\mathcal{T}$</td>
<td>The set of time indices of analysis and $\mathcal{T} = {t_{\text{min}}, t_{\text{min}} + 1, \ldots, T}$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>The time duration that each time index represents</td>
</tr>
<tr>
<td>$\mathcal{P}$</td>
<td>The set of passengers that will receive the path recommendation</td>
</tr>
<tr>
<td>$\mathcal{R}_{p}$</td>
<td>The set of feasible paths for passenger $p \in \mathcal{P}$</td>
</tr>
<tr>
<td>$\mathcal{R}_{u,v}$</td>
<td>The set of feasible paths for OD pair $(u, v)$</td>
</tr>
<tr>
<td>$\Delta_{t}^{u,v,r,i}$</td>
<td>Travel time between the terminal station and the boarding station of leg $((u, v, r, i))$ for vehicle departing at time $t$</td>
</tr>
<tr>
<td>$\delta_{t}^{u,v,r,i}$</td>
<td>Travel time between the terminal station and the alighting station of leg $((u, v, r, i))$ for vehicle departing at time $t$</td>
</tr>
<tr>
<td>$f_{t}^{u,v,r}$</td>
<td>Number of passengers with OD pair $(u, v)$ and departure time $t$ and using path $r$ who are not provided path recommendations</td>
</tr>
<tr>
<td>$d_{t}^{u,v}$</td>
<td>Total number of passengers with OD pair $(u, v)$ and departure time $t$</td>
</tr>
<tr>
<td>$\Omega_{1}$</td>
<td>The set of onboard flow indices at time $t = 1$</td>
</tr>
<tr>
<td>$z_{t}^{u,v,r,i}$</td>
<td>Number of onboard passengers in the vehicle departing at time $t$ in leg $(u, v, r, i)$</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mathcal{P}_{t}^{u,v}$</td>
<td>The set of passengers with OD pair ($u, v$) and departure time $t$ that will receive the path recommendation (a subset of $\mathcal{P}$)</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>The set of all ($u, v, r$) indices</td>
</tr>
<tr>
<td>$\mathcal{S}$</td>
<td>The set of all stations (stops)</td>
</tr>
<tr>
<td>$\mathcal{W}$</td>
<td>The set of all OD pairs</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>The set of all transit lines (routes) in the system</td>
</tr>
<tr>
<td>$T^{u,v,r}_{t}$</td>
<td>The set of legs for path $r$ of OD pair ($u, v$)</td>
</tr>
<tr>
<td>Vehicle ($l, t$)</td>
<td>The vehicle departing at time $t$ on line $l$</td>
</tr>
<tr>
<td>$T_{l,t}$</td>
<td>The time that vehicle ($l, t$) arrives at the last station of line $l$</td>
</tr>
<tr>
<td>$O_{l,t,u,v}$</td>
<td>Total number of onboard passengers at time $t'$ for vehicle ($l, t$)</td>
</tr>
<tr>
<td>$K_{l,t}$</td>
<td>The capacity of vehicles ($l, t$)</td>
</tr>
<tr>
<td>$T^{\text{INT}}_{u,v,r,i,t}$</td>
<td>In-vehicle time of leg ($u, v, r, i$) of vehicle departing at time $t$</td>
</tr>
<tr>
<td>$AD_{s,t}$</td>
<td>Cumulative arriving demand at station $s$ up to time $t$</td>
</tr>
<tr>
<td>$XD_{s,t}$</td>
<td>Cumulative transferring demand at station $s$ up to time $t$</td>
</tr>
<tr>
<td>$BD_{s,t}$</td>
<td>Cumulative boarded demand at station $s$ up to time $t$</td>
</tr>
<tr>
<td>$TT_{t}^{u,v,r}$</td>
<td>Travel time of a path ($u, v, r$) at time $t$</td>
</tr>
<tr>
<td>$T^{\text{Arrival}}_{t}^{u,v,r}$</td>
<td>Arrival time at the destination for the group of passengers using path ($u, v, r$) and departing at time $t$</td>
</tr>
<tr>
<td>$V^r_{p}$</td>
<td>Passenger $p$'s inherent preference (utility) of using path $r$</td>
</tr>
<tr>
<td>$I^r_{p,r'}$</td>
<td>The impact of the recommendation of path $r'$ for passenger $p$ on his/her utility of path $r$</td>
</tr>
<tr>
<td>$\pi^r_{p,r'}$</td>
<td>Conditional probability for passenger $p$ to choose path $r$ given that he/she is recommended with path $r'$</td>
</tr>
<tr>
<td>$\mu^{u,v,r}_{t}$</td>
<td>Expectation of $Q^{u,v,r}_{t}$</td>
</tr>
<tr>
<td>$\sigma^{u,v,r}_{t}$</td>
<td>Standard deviation of $Q^{u,v,r}_{t}$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Threshold parameter for $\epsilon$-feasibility</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Threshold parameter for $\Gamma$-concentration</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>The hyper-parameter to adjust the scale and balance the trade-off between system efficiency and passenger preference</td>
</tr>
<tr>
<td>$w_{p}$</td>
<td>Degree of unfairness for passenger $p$</td>
</tr>
<tr>
<td>$E$</td>
<td>A predetermined integer threshold for the travel time difference</td>
</tr>
<tr>
<td>$\lambda_{k}$</td>
<td>Step size for post-adjustment at iteration $k$</td>
</tr>
<tr>
<td>$\mathcal{P}_{\text{Eq}}^{(k)}$</td>
<td>The set of passengers whose recommendations already satisfy the equity requirement at iteration $k$</td>
</tr>
</tbody>
</table>

**Random Variables**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{1}_{p,r}$</td>
<td>Binary variable indicating whether passenger $p$ has chosen route $r$ or not</td>
</tr>
<tr>
<td>$Q^{u,v,r}_{t}$</td>
<td>Actual flow for path ($u, v, r$) at time $t$</td>
</tr>
</tbody>
</table>

**Decision Variables for Optimization Models**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{p,r}$</td>
<td>Binary variable indicating whether recommending passenger $p$ to use route $r$ or not</td>
</tr>
<tr>
<td>$q^{u,v,r}_{t}$</td>
<td>Number of passengers in $\mathcal{P}$ with OD pair ($u, v$) and departure time $t$ and using path $r$</td>
</tr>
<tr>
<td>$z^{u,v,r,i}_{t}$</td>
<td>Number of passengers boarding a vehicle that had started at time $t$ on leg ($u, v, r, i$)</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>----------------------</td>
<td>------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$y_t^{u,v,r,t}$</td>
<td>Binary decision variable indicating whether the arrival time of passenger departing at time $t$ using $(u, v, r)$ is $\tilde{t}$ or not.</td>
</tr>
<tr>
<td>$Y$</td>
<td>Decision variable in the master problem of the BD, representing the tentative objective function of the subproblem.</td>
</tr>
<tr>
<td>$(\alpha, \beta, \gamma, \eta, \kappa, \rho)$</td>
<td>Dual variables in the subproblem of the BD.</td>
</tr>
</tbody>
</table>

**Appendix B. Inference of status quo choices**

The status quo path choice inference method is based on our previous study (Mo et al., 2022c), which is also similar to the trip-train method used for destination inference in open public transit systems (i.e., no tap-out).

**[In the system when the incident happens]**: Consider a passenger $p \in \mathcal{P}$ with an incident line tap-in record before the end of the incident, meaning that he/she were in the transit system when the incident happens. We then track his/her next tap-in record. If he/she next tap-in is a transfer at nearby bus or rail stations, we can identify his/her chosen path based on the transfer station. We can also identify the waiting passenger if he/she continues to use the incident line to his/her intended destinations inferred by his/her next tap-in records.

**[Out of the system when the incident happens]**: For a passenger $p \in \mathcal{P}$ with only a tap-in record in nearby bus or rail stations. He/she may be affected by the incident to change the tap-in station, or just use the service as a normal commute. To identify whether he/she was affected, we extract his/her travel histories on previous days without incidents to get the normal commute trajectories. If his/her tap-in time and location on the incident day has never appeared in the historical records before, we treat him/her as a passenger affected by the incident and identify his/her chosen path based on the tap-in station.

For passengers in $\mathcal{P}$ without next tap-in records or travel histories, we randomly assign him/her a status quo path based on the proportion of inferred passengers.