Classical and quantum algorithmic phase transitions

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arXiv:2510.08497 with Bobak Kiani, Eric Anschuetz
arXiv:2602.22619 with Bobak Kiani

Thermal phase transitions

For some k and temperature $1/\beta$, let $\rho_\beta = e^{-\beta H}/\mathrm{Tr}\, e^{-\beta H}$ for

$$H = \sum_{|P|=k} J_P P, \qquad J_P \sim \mathcal{N}\left(0, \frac{(k-1)!}{n^{k-1}}\right).$$

Ising k-spin: $P = Z_{i_1} \cdots Z_{i_k}$

Random k-local: P is k-local Pauli string $\sigma_{i_1}^{\mu_1} \otimes \cdots \otimes \sigma_{i_k}^{\mu_k}$ for $\mu_j \in \{X, Y, Z\}$

SYK: $P = i^{k/2} \psi_{i_1} \cdots \psi_{i_k}$ for Majorana fermions ($\{\psi_i, \psi_j\} = \delta_{ij}$ and $\psi_i = \psi_i^\dagger$)

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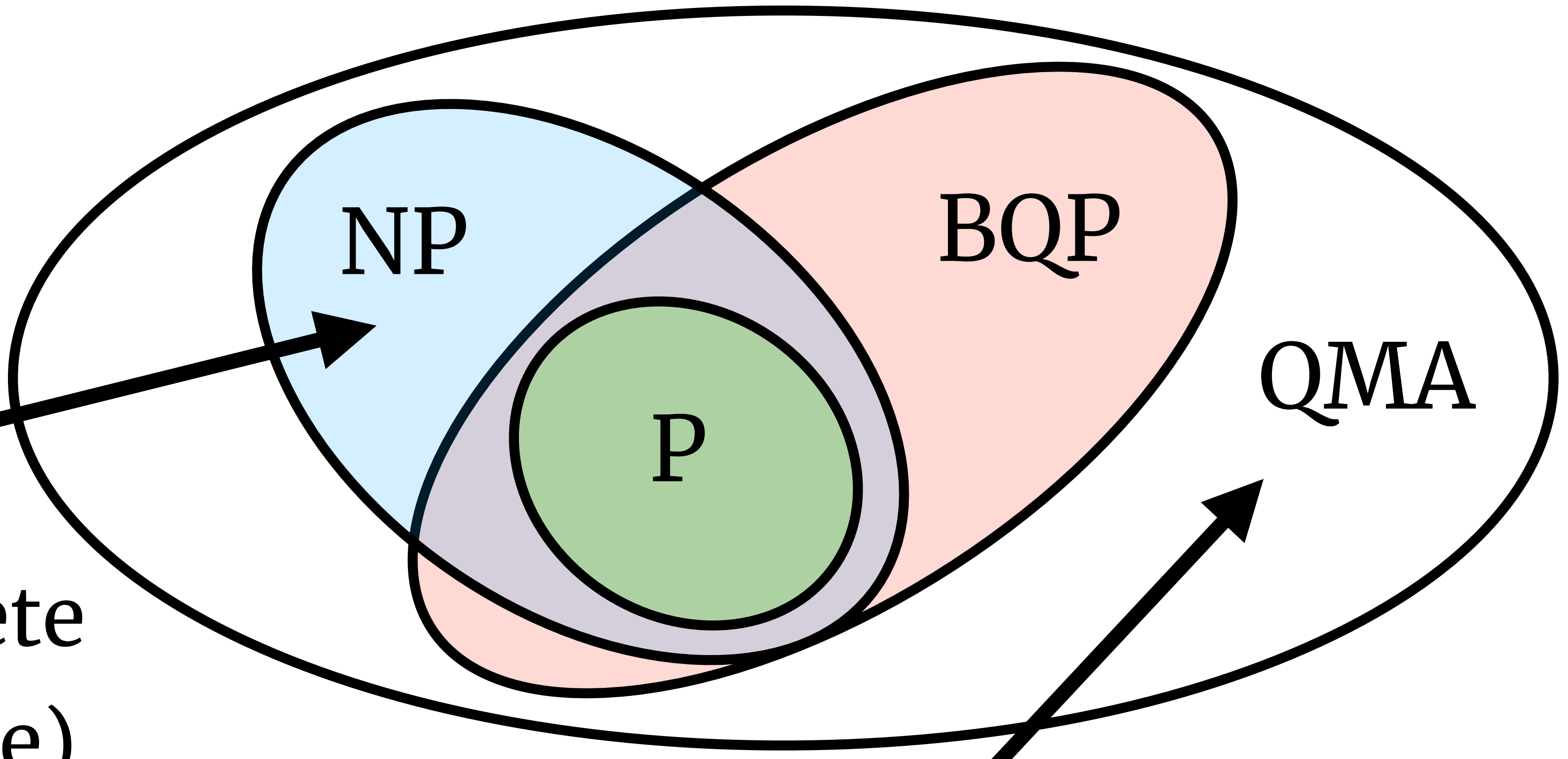

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Thermal *algorithmic* phase transitions

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Sampling: is preparing ρ_β quantumly easy/hard?

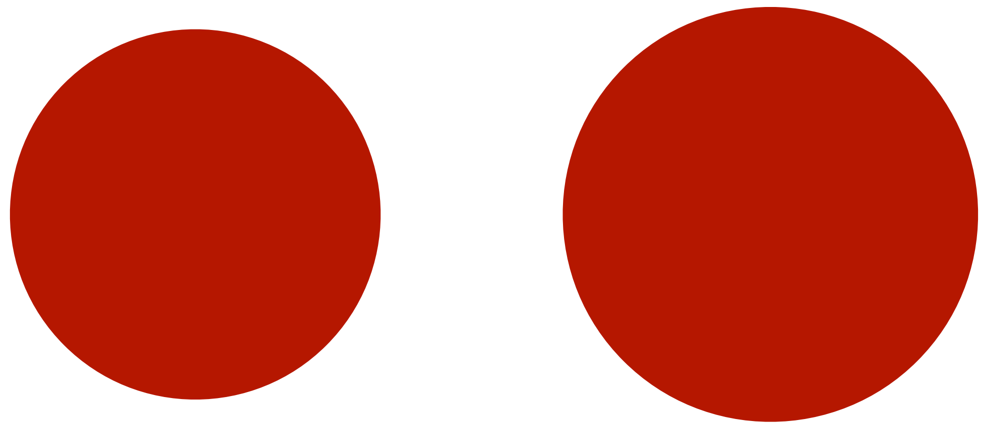
Observables: is estimating $\operatorname{Tr}(O\rho_\beta)$ classically/quantumly easy/hard?

Classical glasses

Classical glasses

Gibbs distribution μ_β of $H = \sum_{i_1,\ldots,i_k=1}^{n} J_{i_1\cdots i_k} Z_{i_1} \cdots Z_{i_k}$ decomposes into clusters:

$$\mathrm{supp}(\mu_\beta) \approx \bigcup_{i=1}^{m} C_i, \quad C_i \subset \{\pm 1\}^n, \quad d_H(C_i, C_j) \geq cn$$



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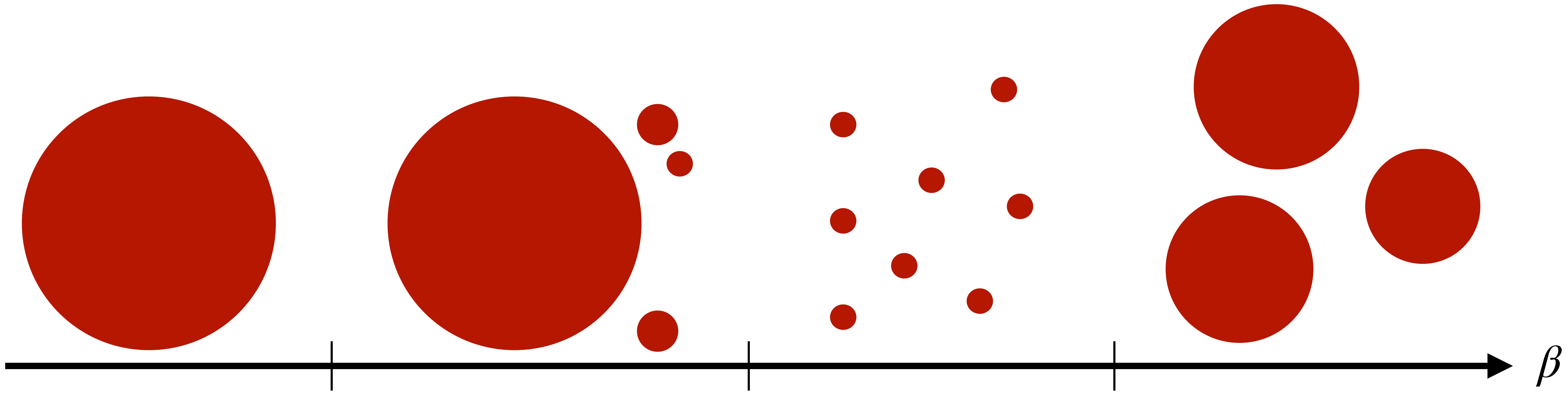

Talagrand, 2010

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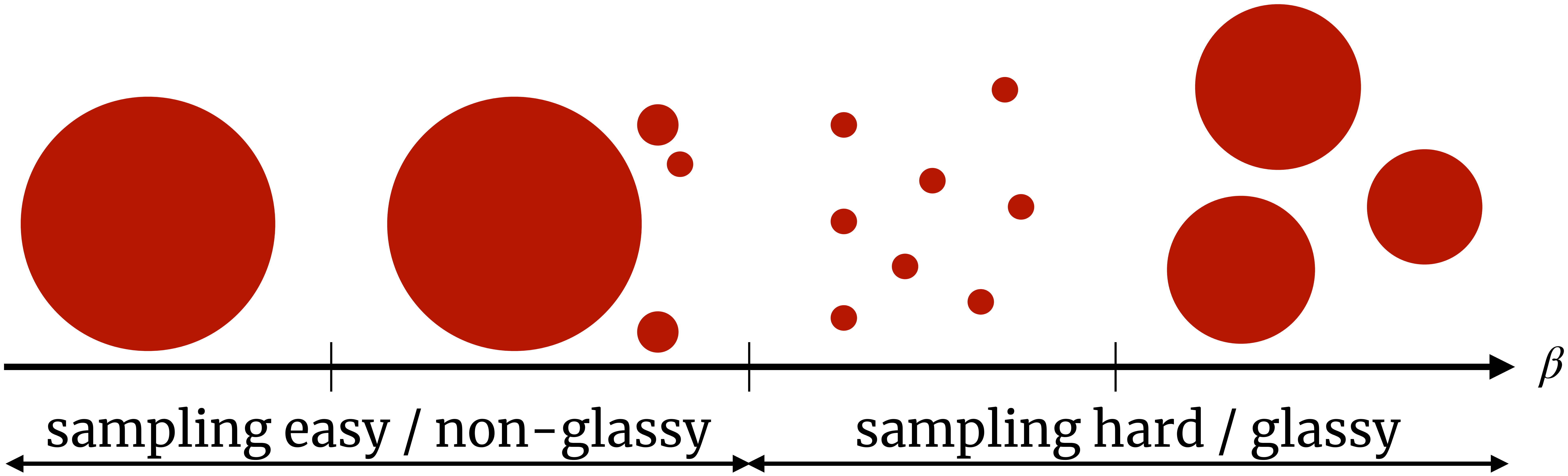

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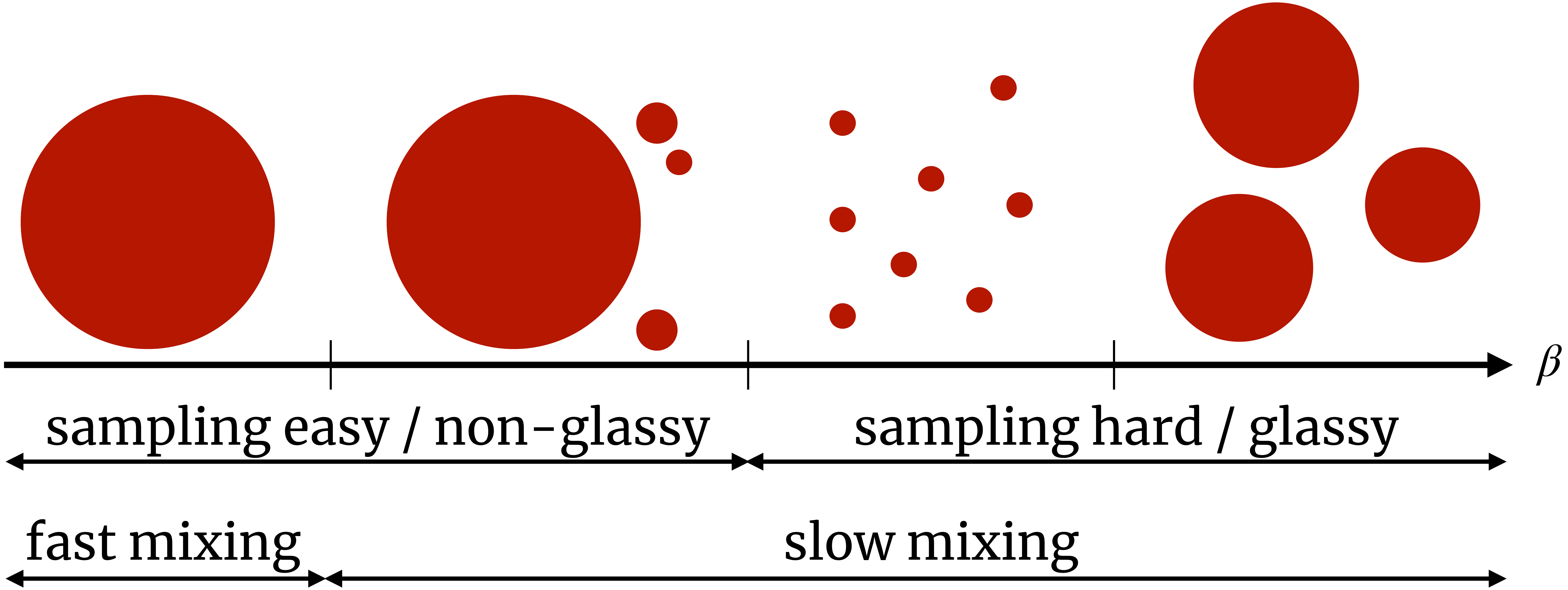

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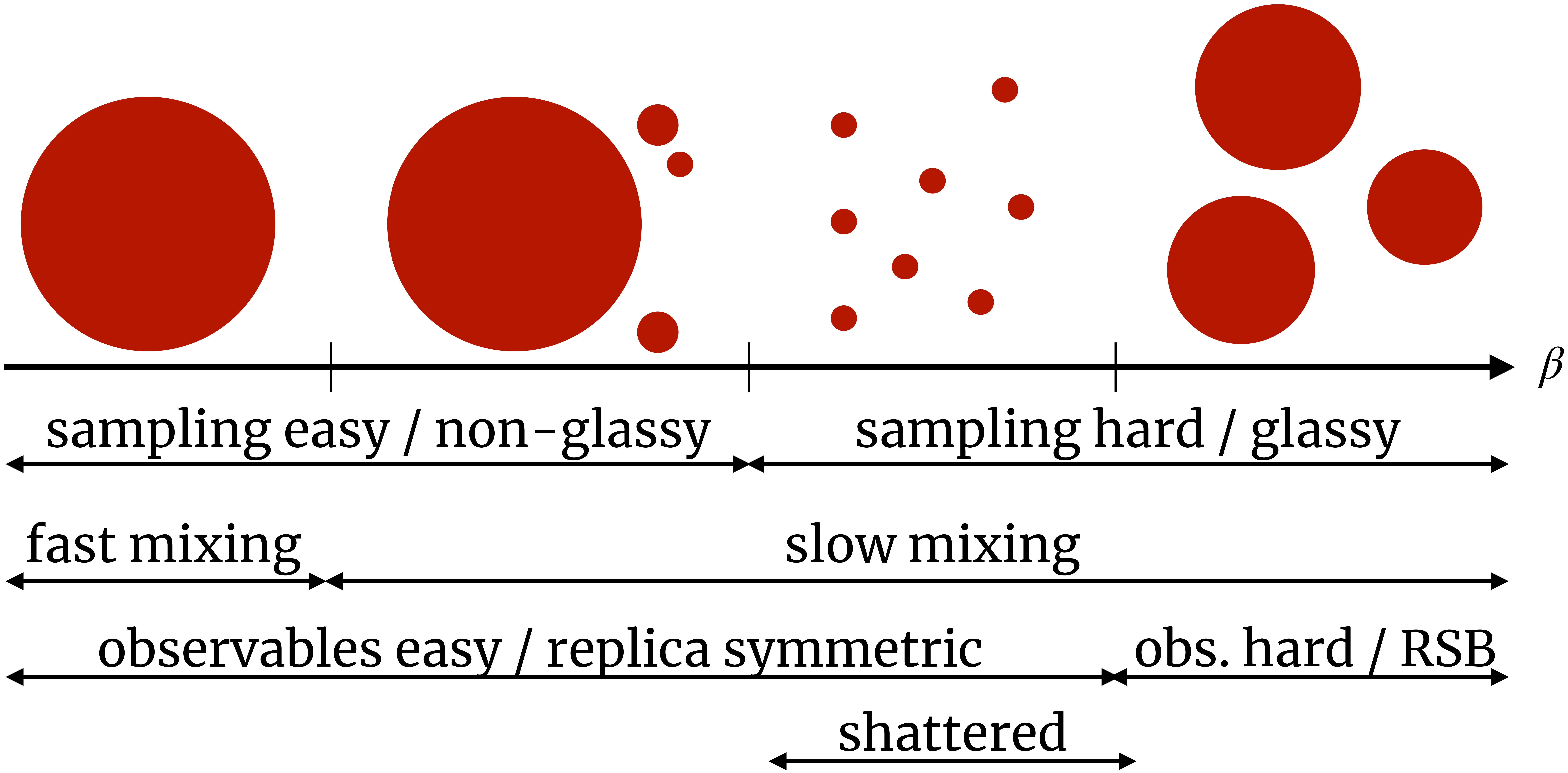
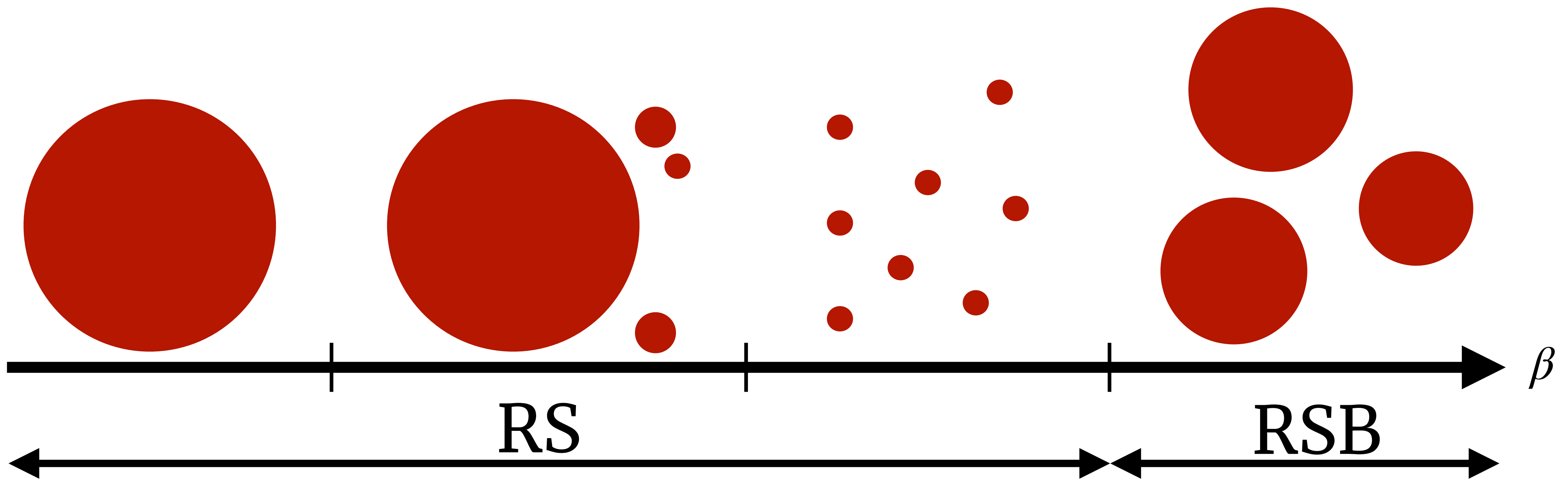
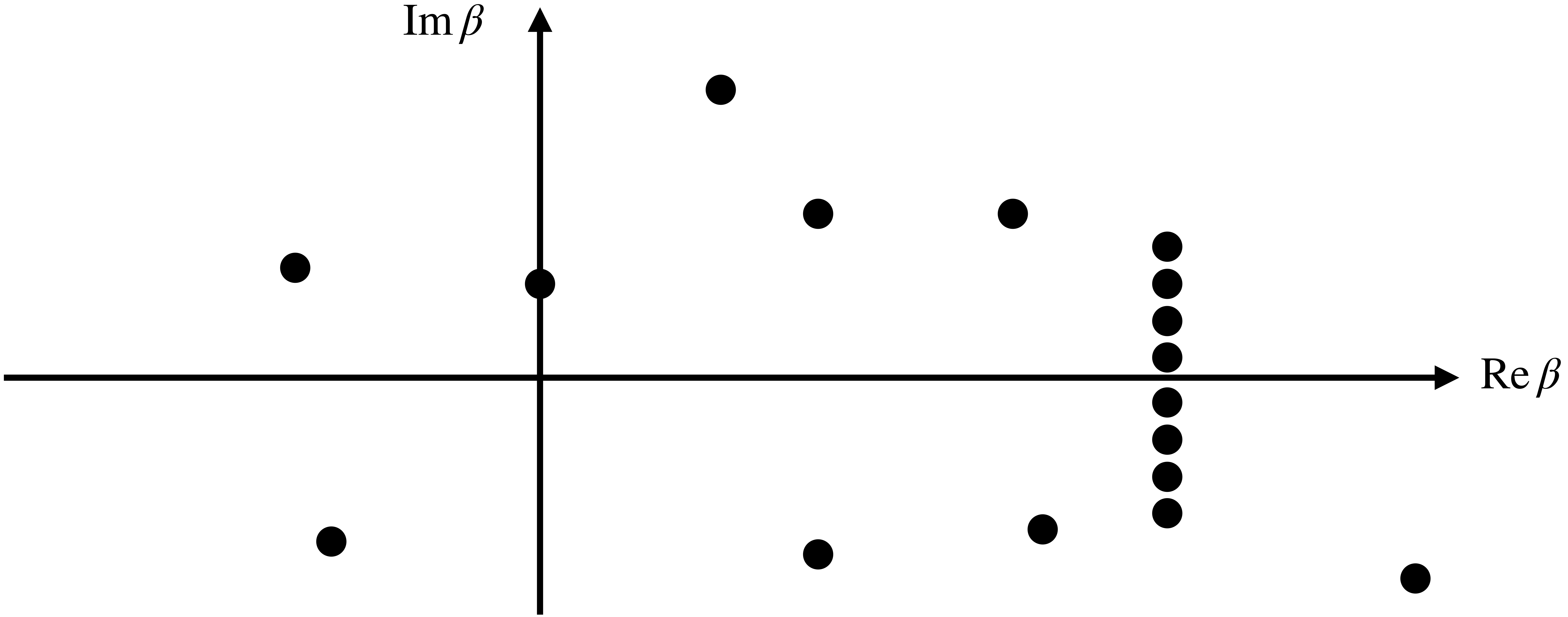


Talagrand, 2010

Fisher zeros

$$Z(\beta) = 0$$





T.-D. Lee and C.-N. Yang, 1952.
M. E. Fisher, 1965.

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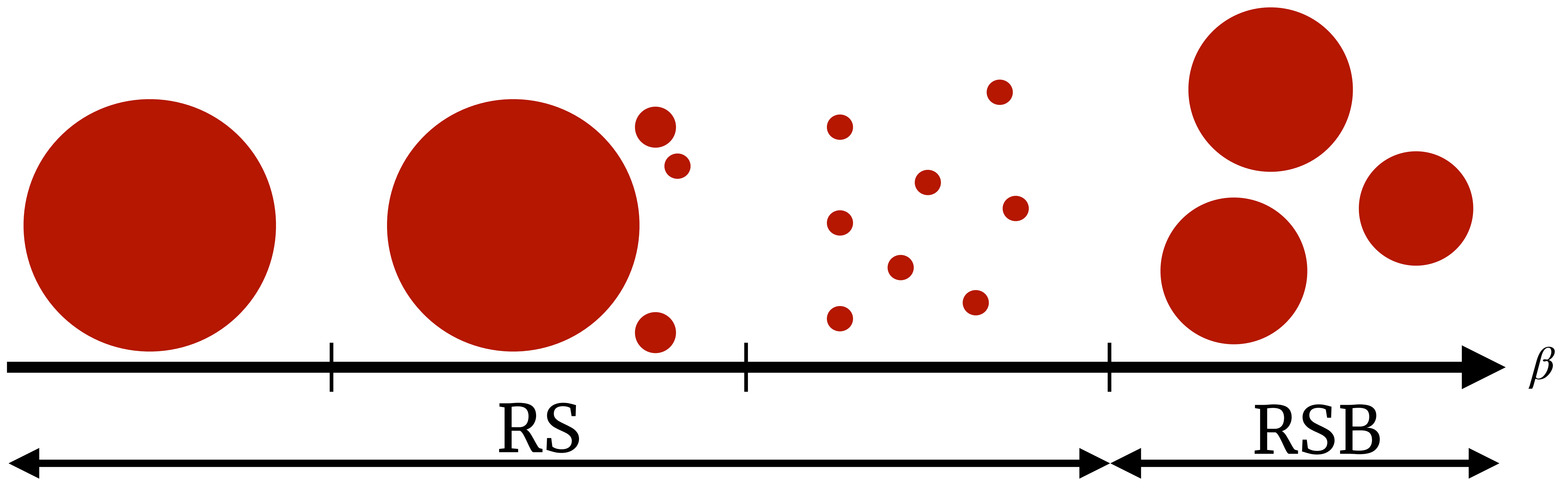

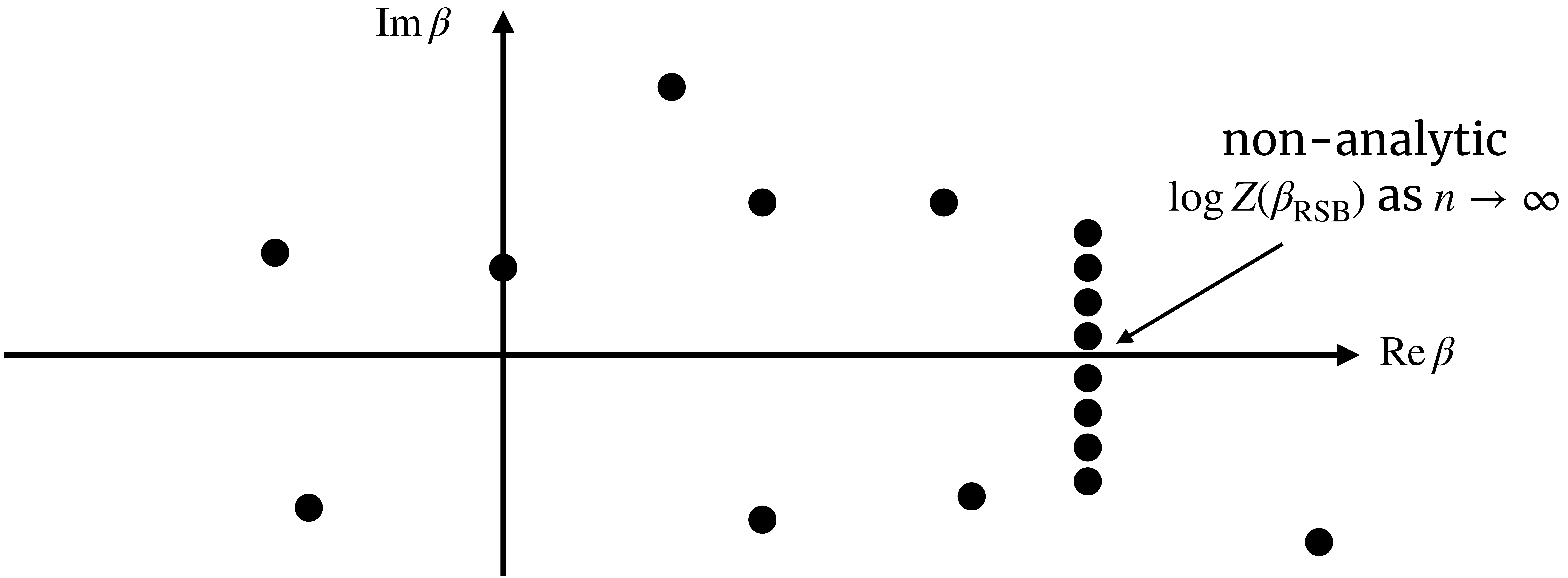

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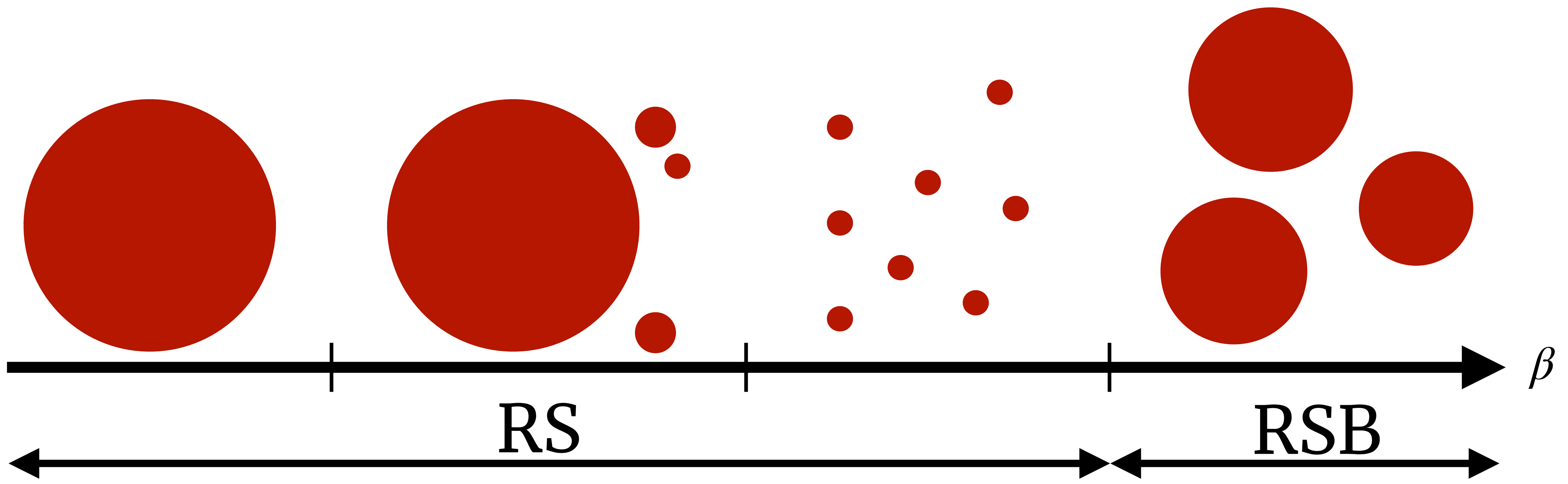
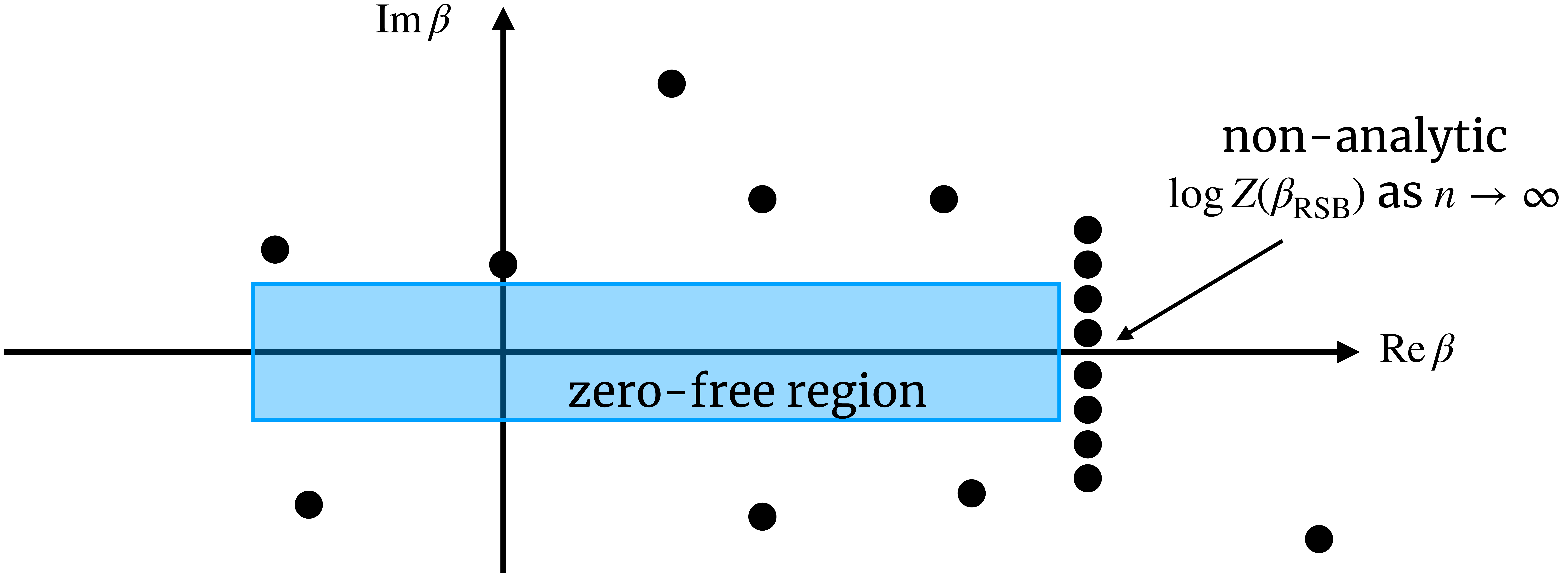




T.-D. Lee and C.-N. Yang, 1952.
M. E. Fisher, 1965.

Quantum glasses

Quantum replica method

$$e^{m \log Z} \approx 1 + m \log Z \implies \mathbb{E} \log Z = \lim_{m \to 0} \frac{\mathbb{E}[Z^m] - 1}{m}$$

1. Integrate over paths $s(\tau)$ parameterized by imaginary time $\tau \in [0, \beta]$ to compute $\mathrm{Tr}(e^{-\beta H})$.

2. Replica ansatz integrates overlap $G(\tau, \tau') \sim \langle s(\tau), s(\tau') \rangle$.

3. Saddle point method on integral gives free energy.

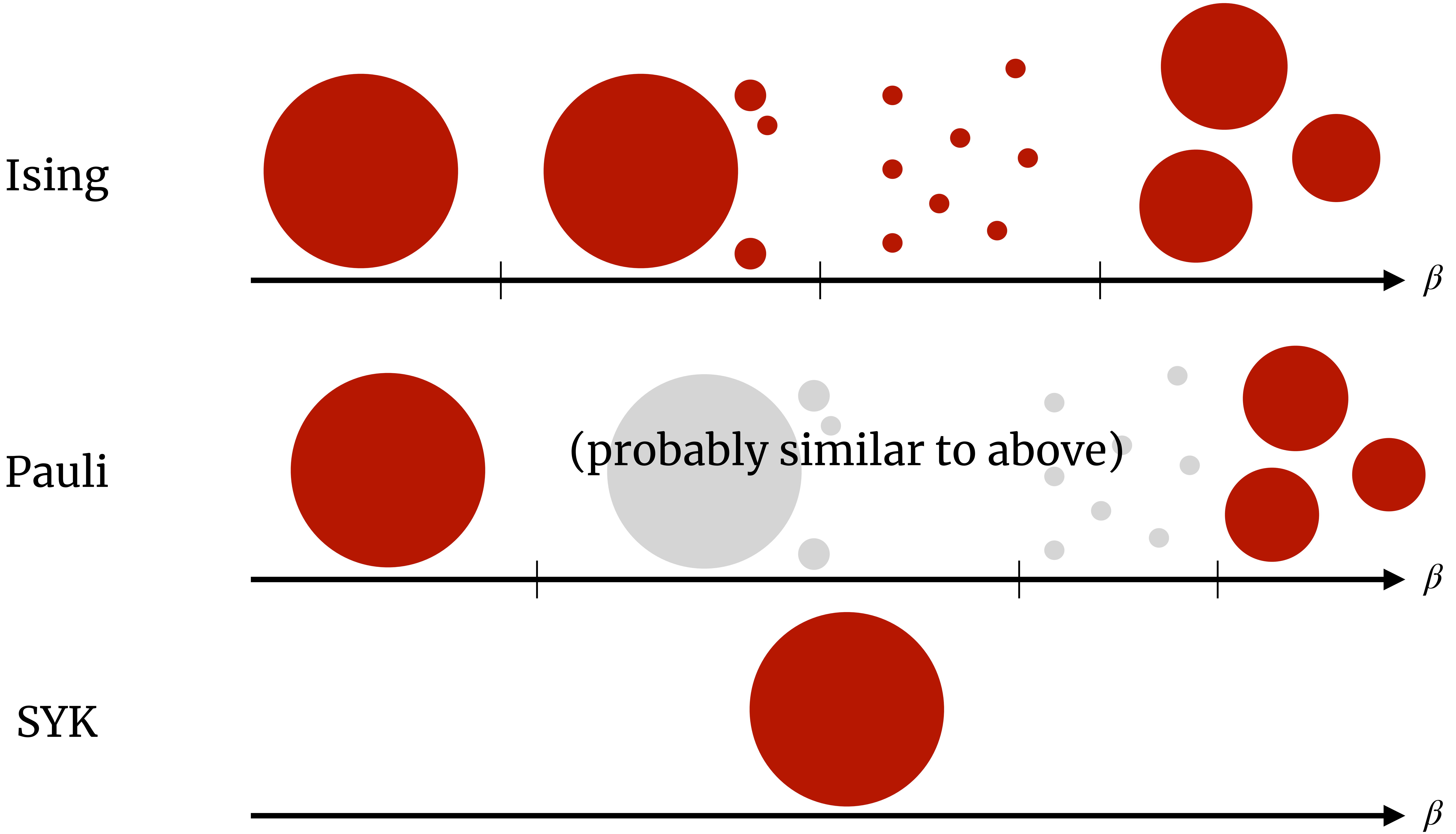
Quantum (Pauli) glasses

$$\langle \rho, \rho' \rangle_{\text{Pauli}} = \sum_{i=1}^{n} \sum_{\mu \in \{X,Y,Z\}} \text{Tr}(\rho \sigma_i^{\mu}) \text{Tr}(\rho' \sigma_i^{\mu})$$

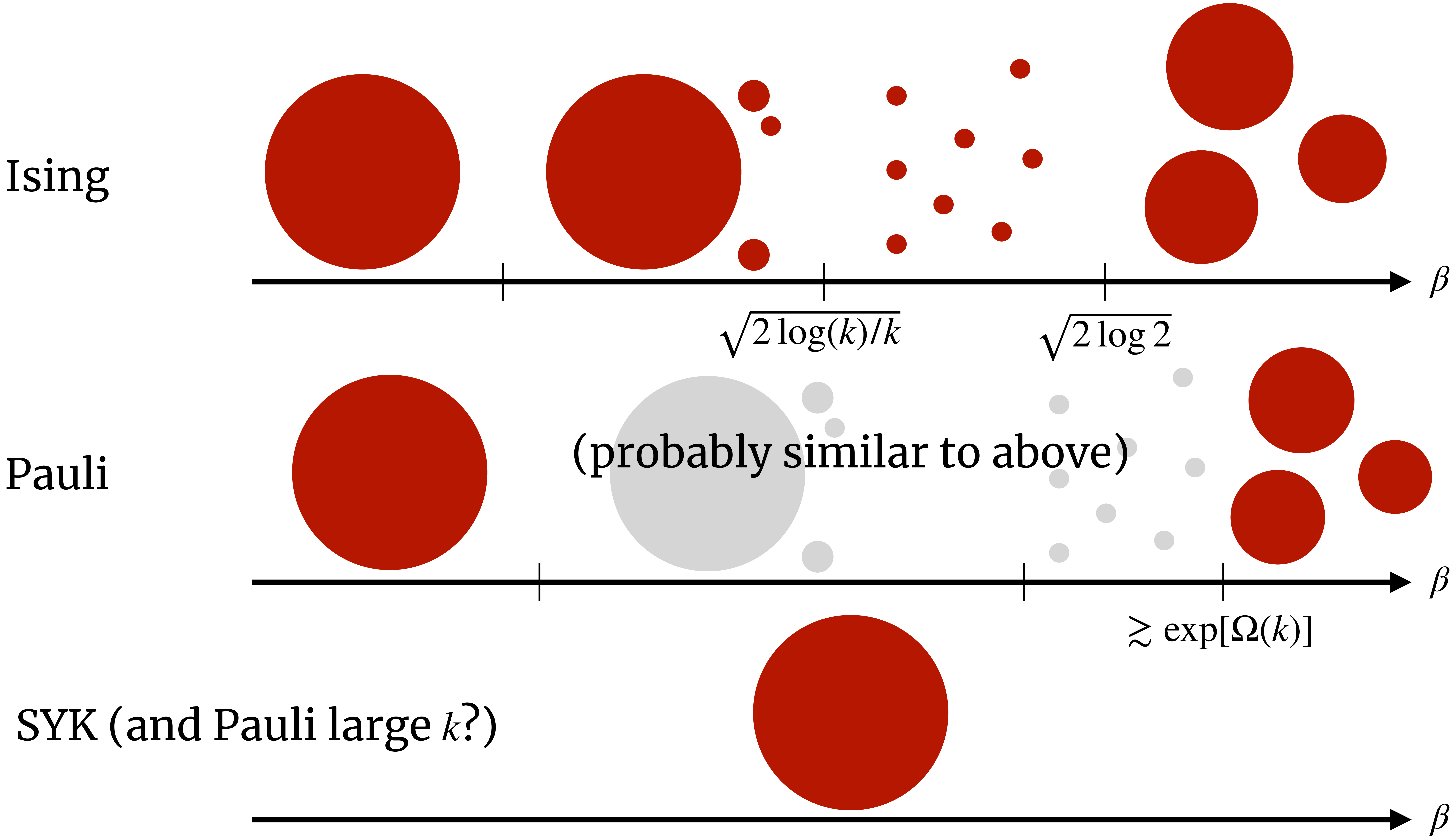
A glassy state ρ_β satisfies: there exists decomposition $\rho_\beta = \sum_{i=1}^{m} c_i \rho_i$ s.t.

1. Distinct clusters: $\|\rho_i - \rho_j\|_{\text{Pauli}}^2 = \Omega(n)$

2. Equal clusters: $c_i = 1/\text{poly}(m)$ for $m \geq 2$



S. Sachdev and J. Ye, 1993.
A. Kitaev, 2015.



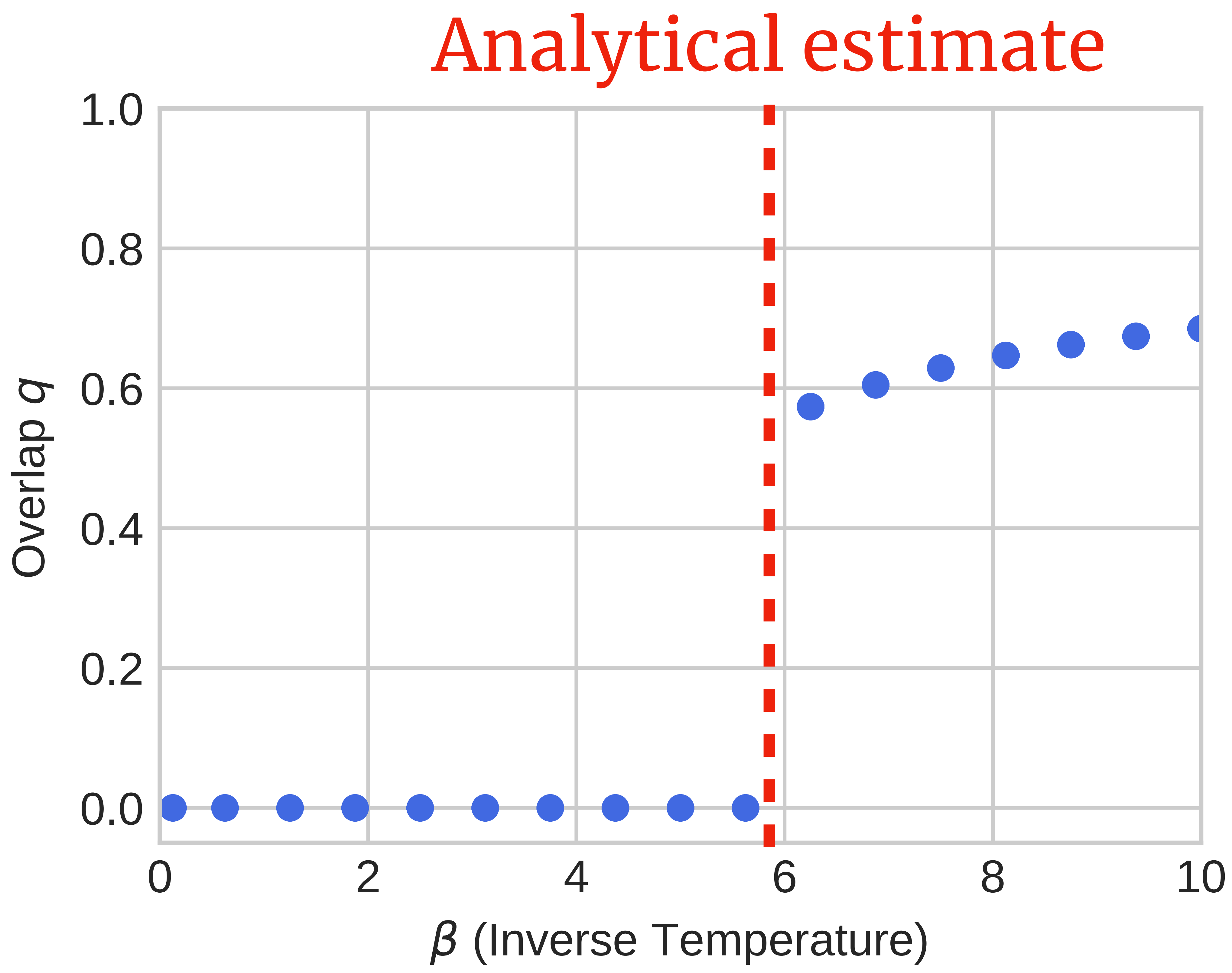
B. Swingle and M. Winer, 2023.

3-local Pauli Hamiltonians

Solve saddle point equations with replica symmetric ansatz.

$$\begin{pmatrix} G(\tau) & q & \cdots & q & q \\ q & G(\tau) & \cdots & q & q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q & q & \cdots & G(\tau) & q \\ q & q & \cdots & q & G(\tau) \end{pmatrix}$$

Variational bound with $G(\tau) = G$.
(See paper for full RSB eqs.)



A. Bray and M. Moore, 1980.
N. Kavokine, M. Müller, A. Georges, and O. Parcollet, 2024.

Algorithmic phase transitions: quantum hardness

Quantum optimal transport

Classical: minimum Hamming distance to change distribution μ to ν

Quantum: $\|\rho - \rho'\|_{W_1} = \min_{\substack{X_i^\dagger = X_i \\ \operatorname{Tr} X_i = 0}} \left\{ \sum_i \|X_i\|_1 \; : \; \sum_i X_i = \rho - \rho', \; \operatorname{Tr}_i X_i = 0 \right\}$

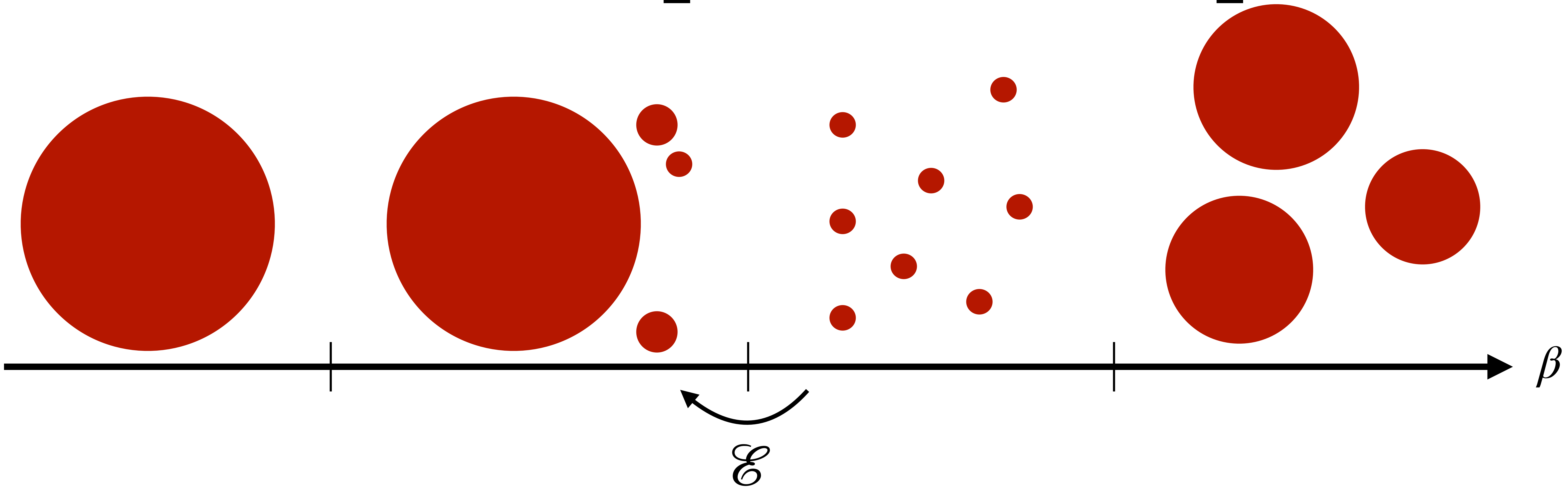
→ ℓ-local unitary changes W_1 by $O(\ell)$

Wasserstein complexity: $\mathrm{WC}(\mathscr{E}) = \max_\rho \|\rho - \mathscr{E}(\rho)\|_{W_1}$

→ lower bound on (Nielsen) circuit complexity

G. De Palma, M. Marvian, L. Trevisan, and S. Lloyd, 2020.

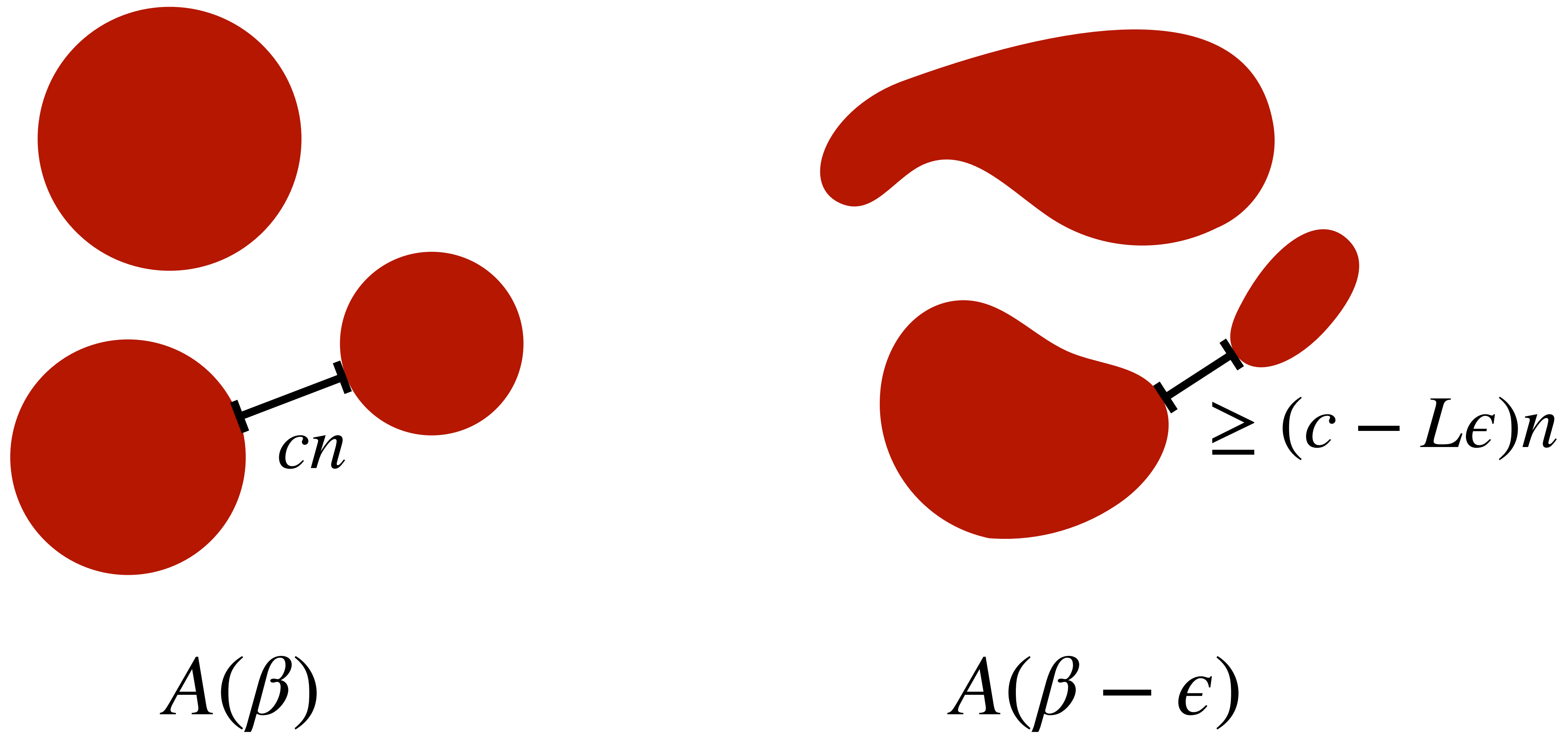
Quantum optimal transport



Theorem: for glassy ρ_β and non-glassy $\rho_{\beta'}$, all $\mathscr{E}$ such that $\mathscr{E}(\rho_\beta) = \rho_{\beta'}$ must also satisfy for constant $c > 0$

$$\max_{\rho} \|\rho - \mathscr{E}(\rho)\|_{W_1} \geq cn \,.$$

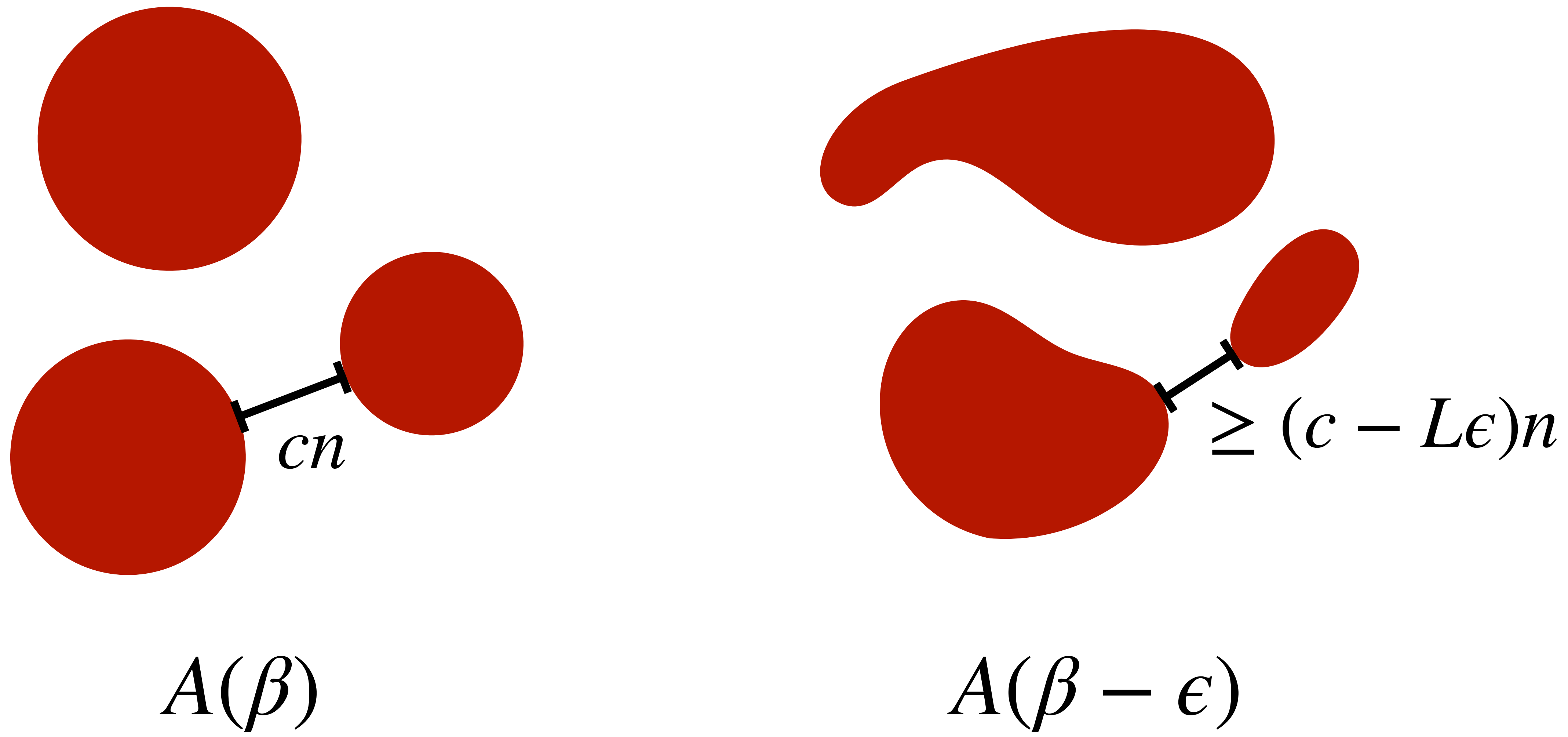
Stable quantum algorithms



Definition: an algorithm $A(\beta)$ that outputs a density matrix is stable if there exists channel $\mathscr{E}$ such that $\mathscr{E}(A(\beta)) = A(\beta')$ and for constant $L > 0$,

$$\max_{\rho} \|\rho - \mathscr{E}(\rho)\|_{W_1} \leq Ln\,|\beta - \beta'|\,.$$

Stable quantum algorithms

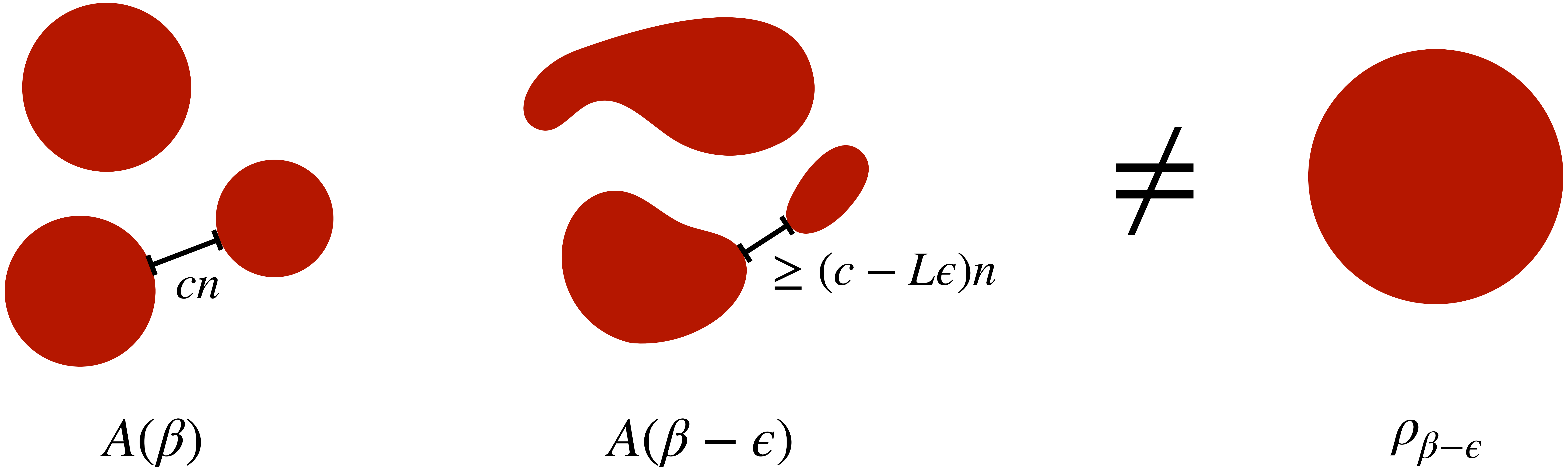


Example: local Lindbladian dynamics with a Lipschitz filter function for constant time. (Analogous to Langevin.)

$$\mathscr{L}(\rho) = -i[A,\rho] - \sum_j \gamma_j(\beta)\left(L_j \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho\}\right), \quad |\gamma_i(\beta) - \gamma_i(\beta')| \leq L|\beta - \beta'|$$

Compare: A. El Alaoui, A. Montanari, and M. Sellke, 2025.

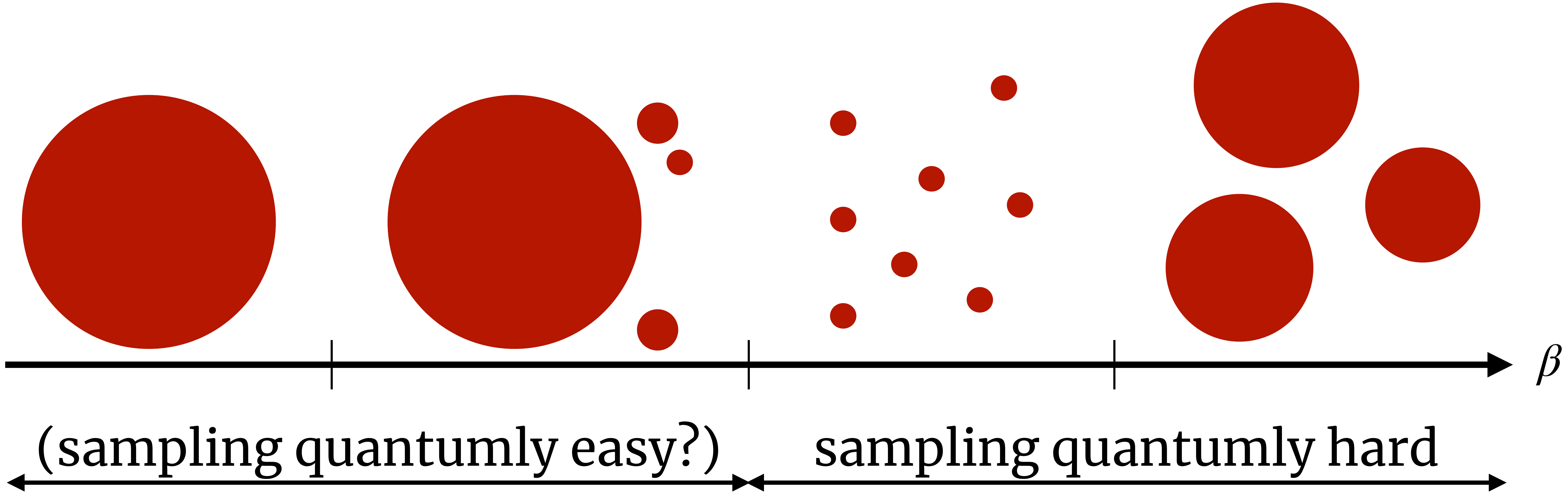
Glassiness implies hardness



Theorem: a stable quantum algorithm cannot prepare ρ_β if there is a glass transition.

Proof: choose β, β' around the transition so $\max_\rho \|\rho - \mathcal{E}(\rho)\|_{W_1} \leq Ln\,|\beta - \beta'| < cn$.

Average-case complexity



Algorithmic phase transitions: classical algorithms

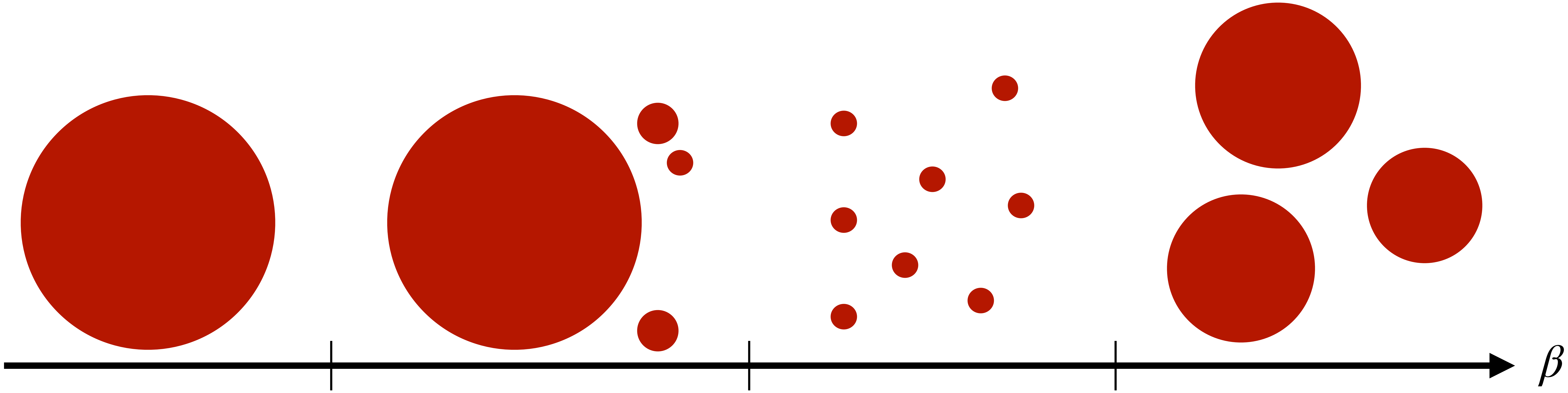
Quantum aspects of SYK

Quantum aspects of SYK:

- Sign problem (no basis makes off-diagonal elements have same sign)
- Polynomial circuit lower bound: $\Omega(n^{1+k/2})$ gates at any constant β
- Quantumly fast-mixing (rigorously $\mathbb{E} \log Z \approx \log \mathbb{E} Z$)

E. R. Anschuetz, C.-F. Chen, B. T. Kiani, and R. King, 2025.

Average-case complexity



(sampling quantumly easy?) | sampling quantumly hard

poly q. circuit lower bound

$\implies$ observables classically hard?

Barvinok's algorithm

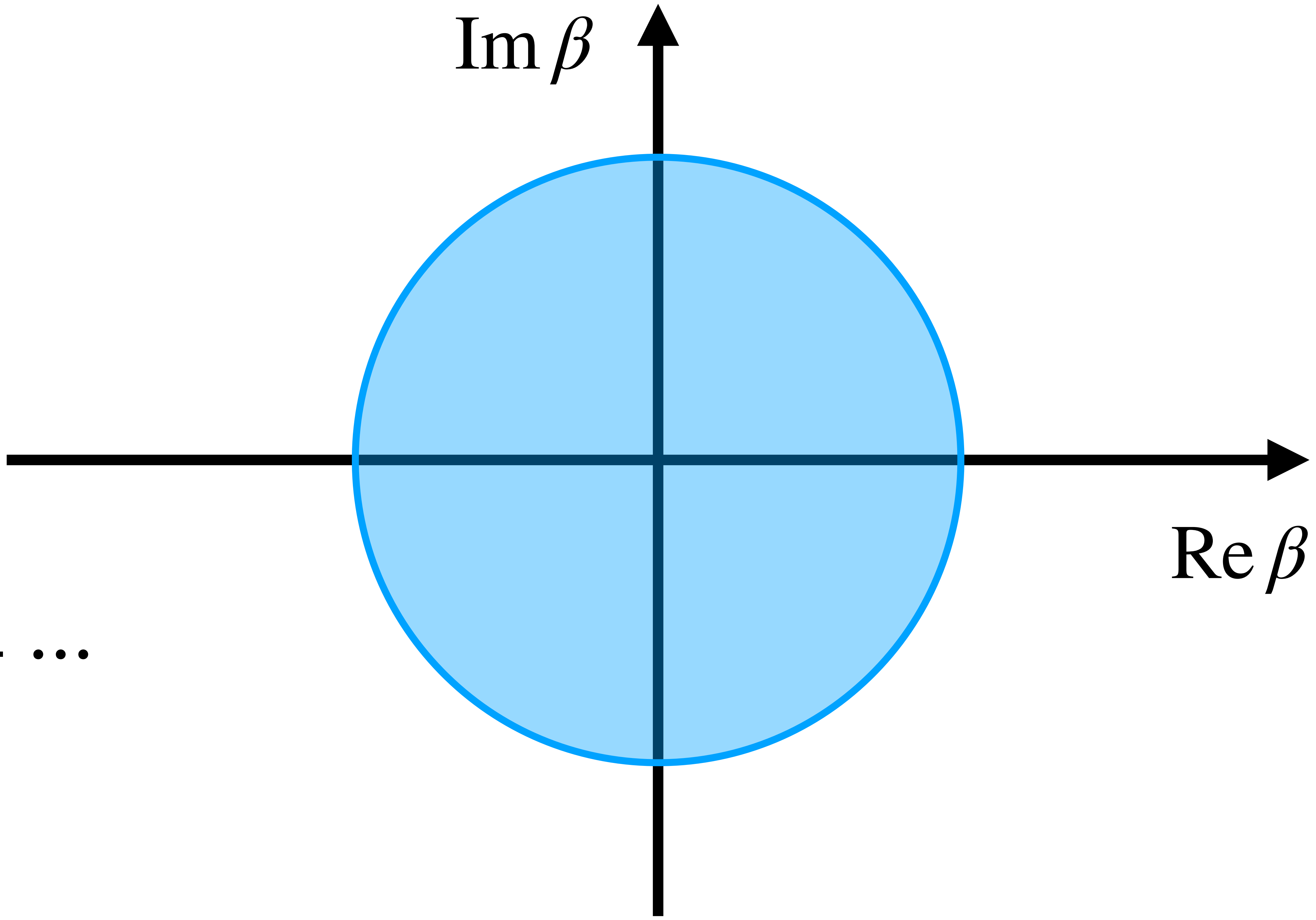
For $Z(\beta, \lambda) = \operatorname{Tr} e^{-\beta(H+\lambda O)}$, estimate

$$\operatorname{Tr}(O\rho_\beta) = -\frac{1}{\beta}\partial_\lambda \log Z(\beta, \lambda)\Big|_{\lambda=0}$$

by computing the partition function

$$Z(\beta, \lambda) = 1 - \beta \operatorname{Tr}(H + \lambda O) + \frac{1}{2}\beta^2 \operatorname{Tr}((H + \lambda O)^2) - \cdots$$

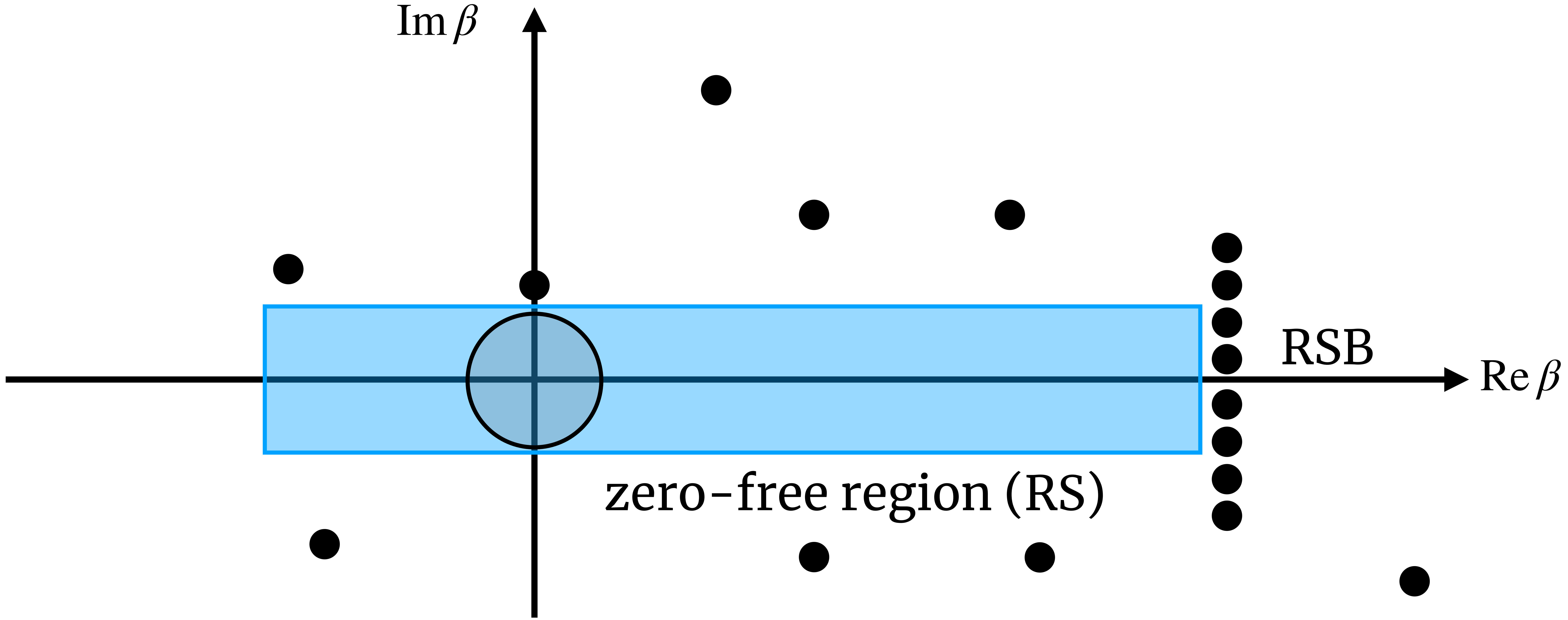
which converges if Z is analytic in a disk.



A. Barvinok , 2014.

Fisher zeros

$$Z(\beta) = 0$$



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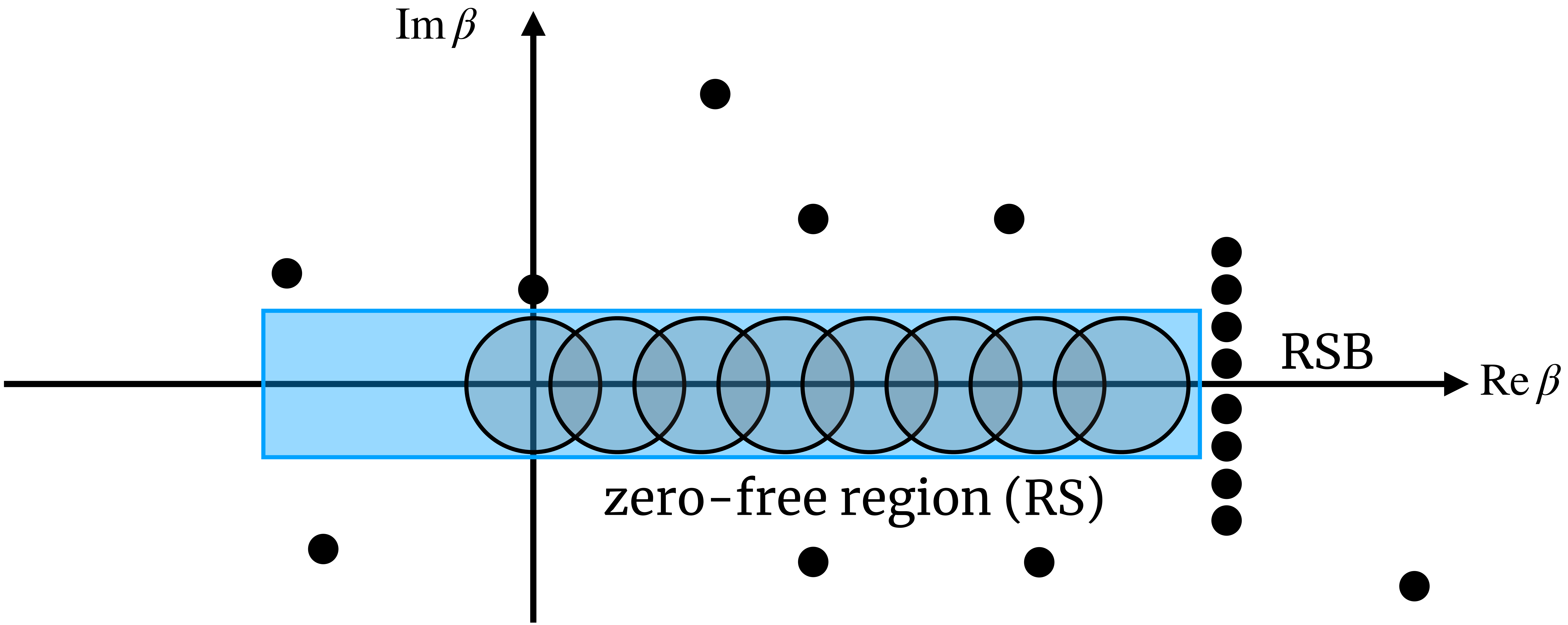
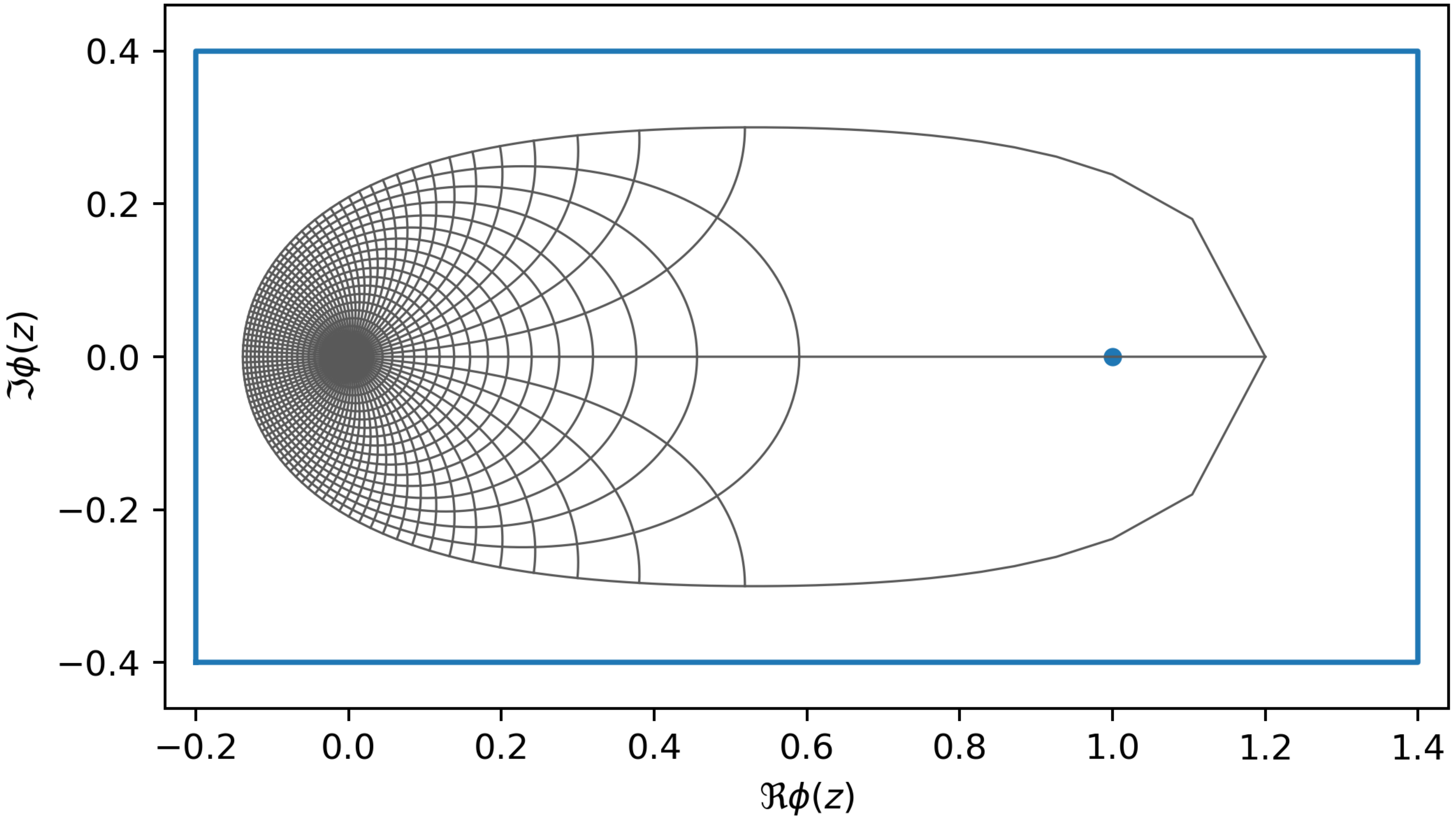


Cost of Barvinok's algorithm



No phase transition $\Longrightarrow$

zero-free rectangle of $O(1)$ height $\Longrightarrow$

$n^{e^{\beta}\log(n/\epsilon)}$ to ϵ-approximate observables

Zero-free disk

Jensen's formula: $$\sum_{\beta \in \mathbb{D}(0,R):Z(\beta)=0} \log \frac{R}{|\beta|} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \log \left| \frac{Z(Re^{i\theta})}{Z(0)} \right|$$

In a disk of radius $r < R$, the number of zeros is at most

$$\#\{\beta \in \mathbb{D}(0,r) : Z(\beta) = 0\} \leq \left(\log \frac{R}{r}\right)^{-1} \frac{1}{2\pi}\int_0^{2\pi} d\theta \log \left| \frac{Z(Re^{i\theta})}{Z(0)} \right|$$

Instead of quenched free energy, use Jensen's inequality (+ Markov):

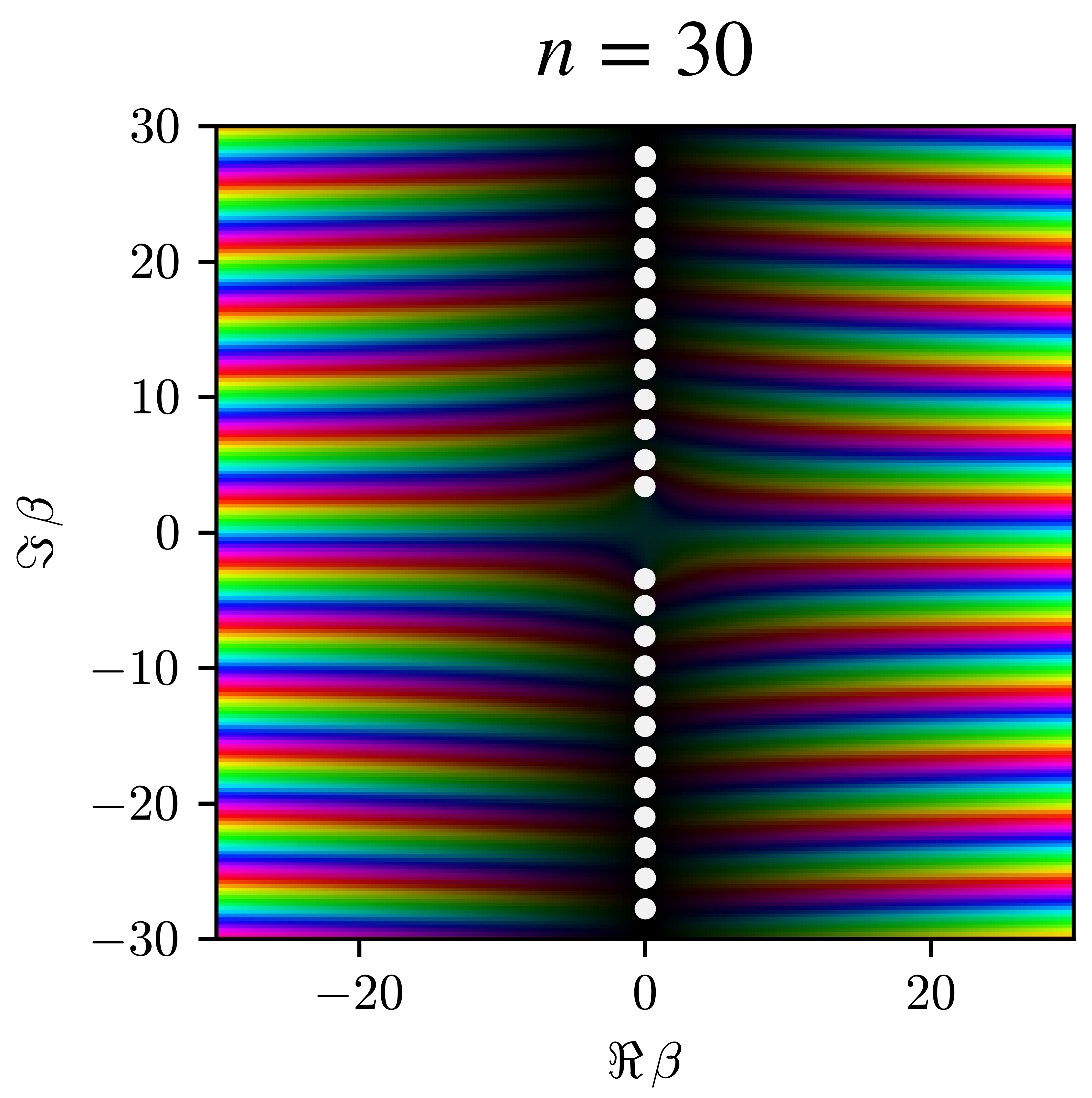
$$\mathbb{E} \log \left| \frac{Z(Re^{i\theta})}{Z(0)} \right| \leq \frac{1}{2} \log \mathbb{E} \left| \frac{Z(Re^{i\theta})}{Z(0)} \right|^2$$

Zero-free region

We can compute $\mathbb{E}|Z|^2$ to control complex zeros.

[Lengthy RS computation...]

Result: only zeros for $|\mathrm{Im}\,\beta| \gtrsim 1.3$ on imaginary axis.

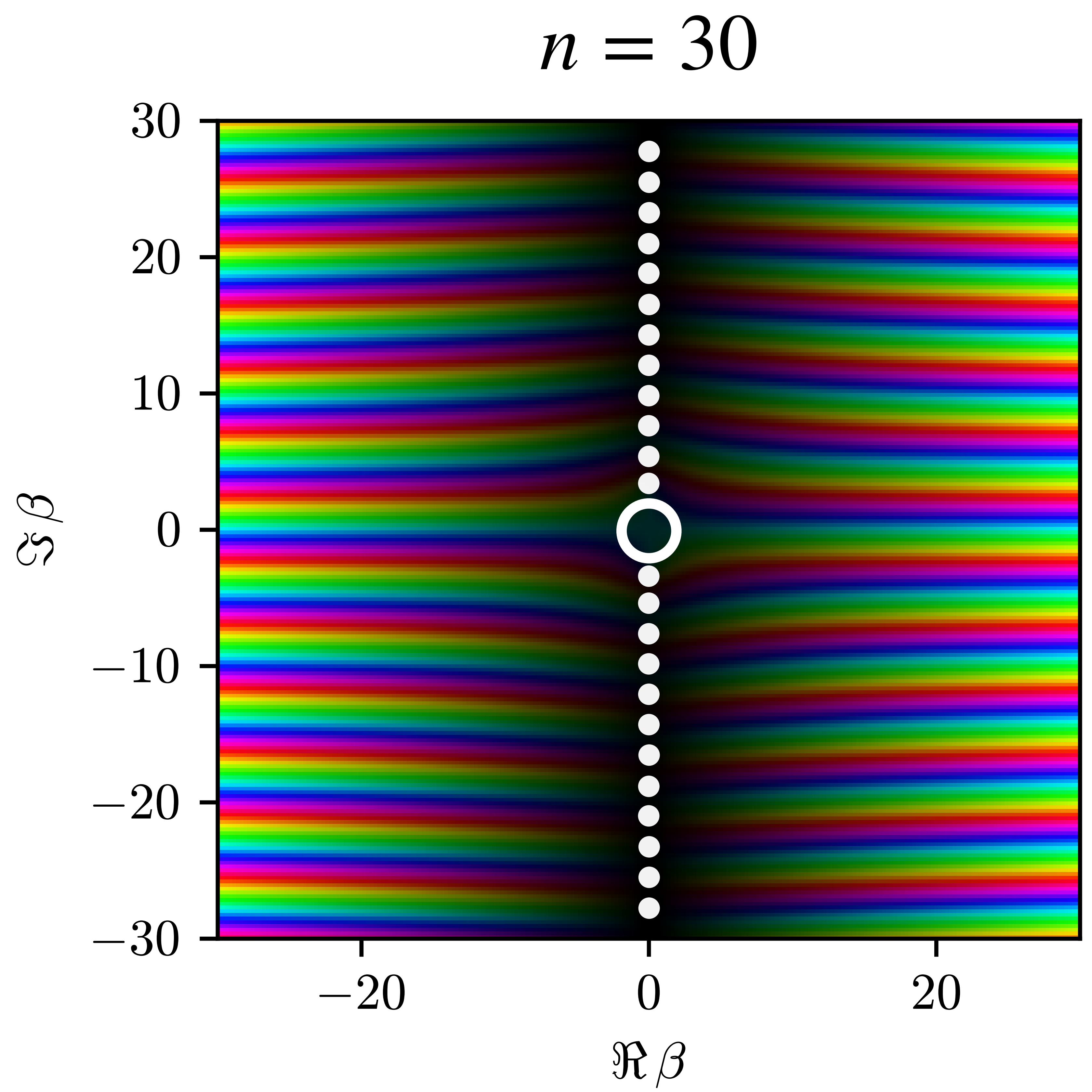


Zero-free region

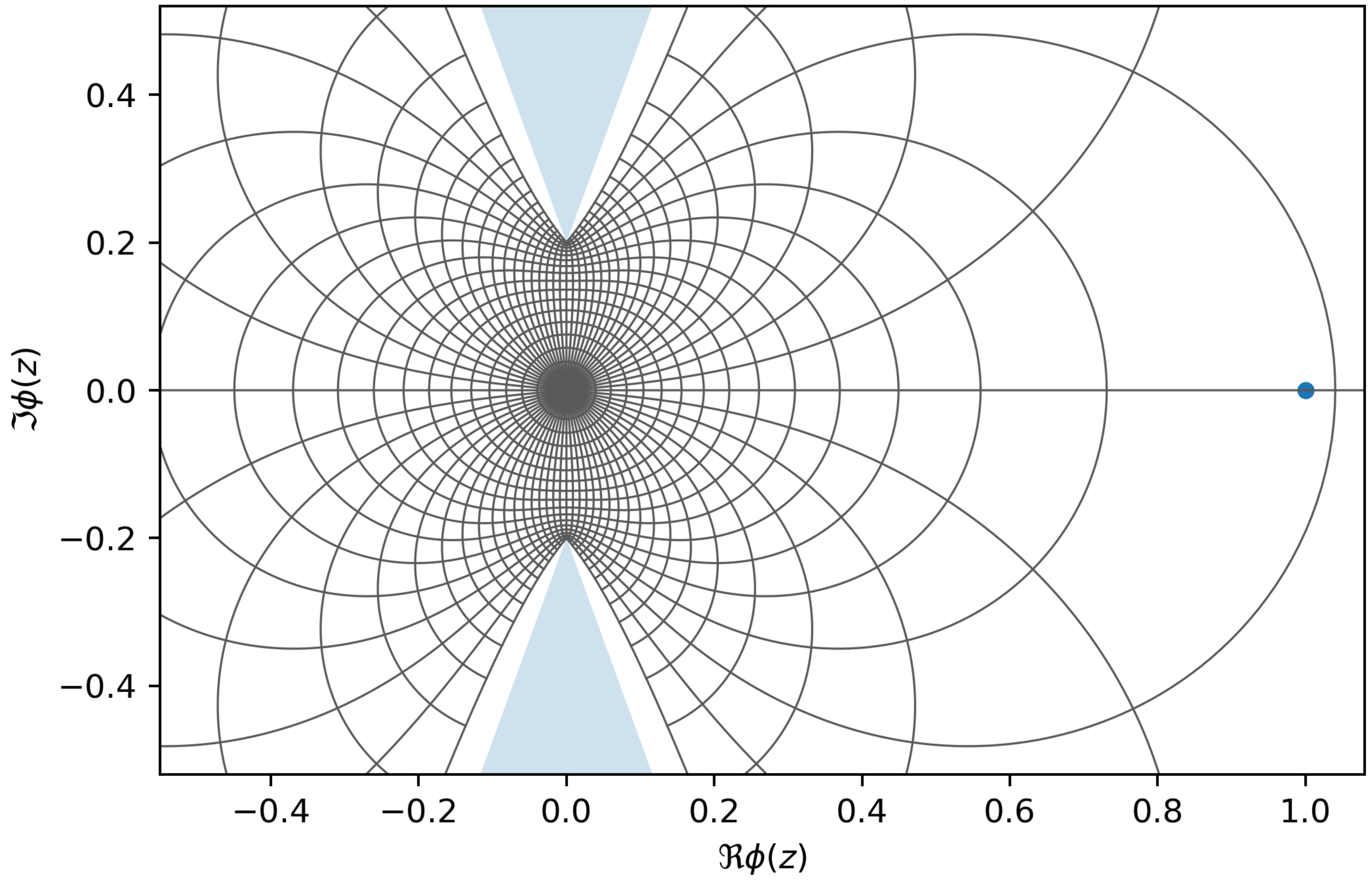
Rigorous: zero-free disk $|\beta| \lesssim 0.2$.

Computing the annealed free energy is open rigorously!
→ Unlike in classical models, SYK annealed partition function is *not* zero-free everywhere.

Cluster expansion (KP) with Wick contractions as polymers.



Cost of Barvinok's algorithm (SYK)

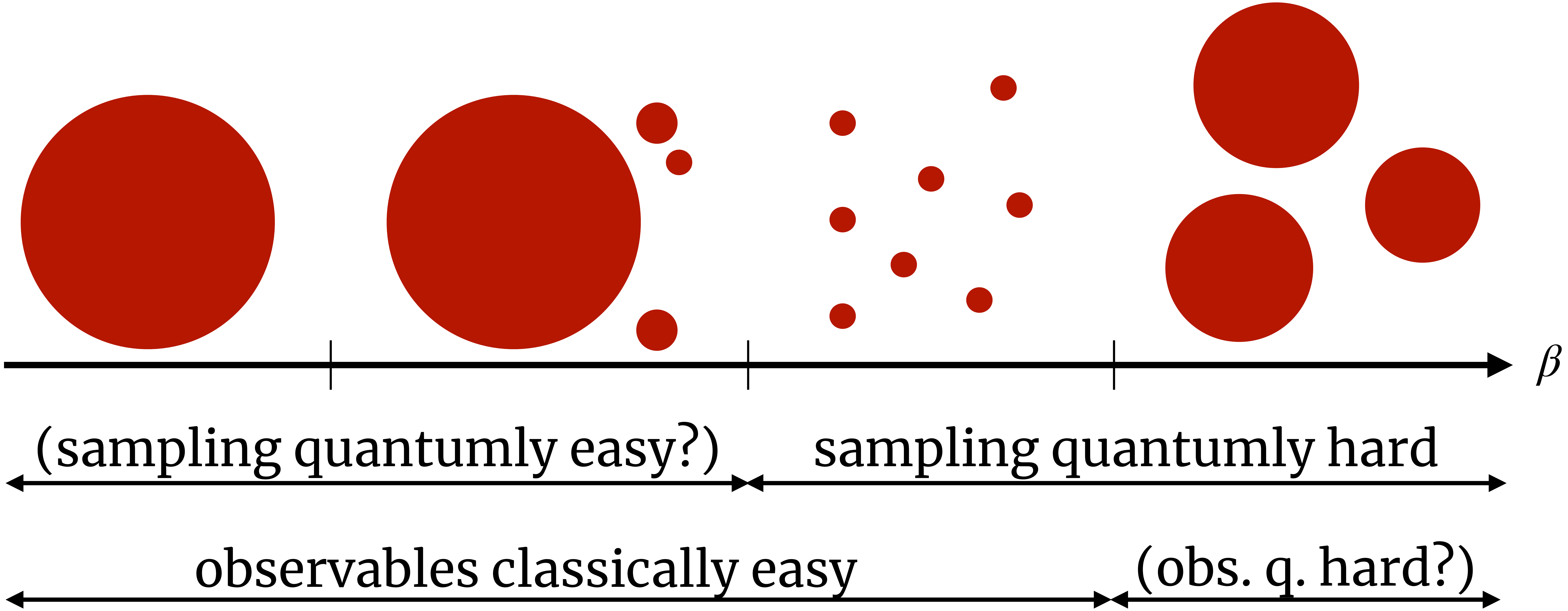


Extend to perturbed H by local $O \Longrightarrow$

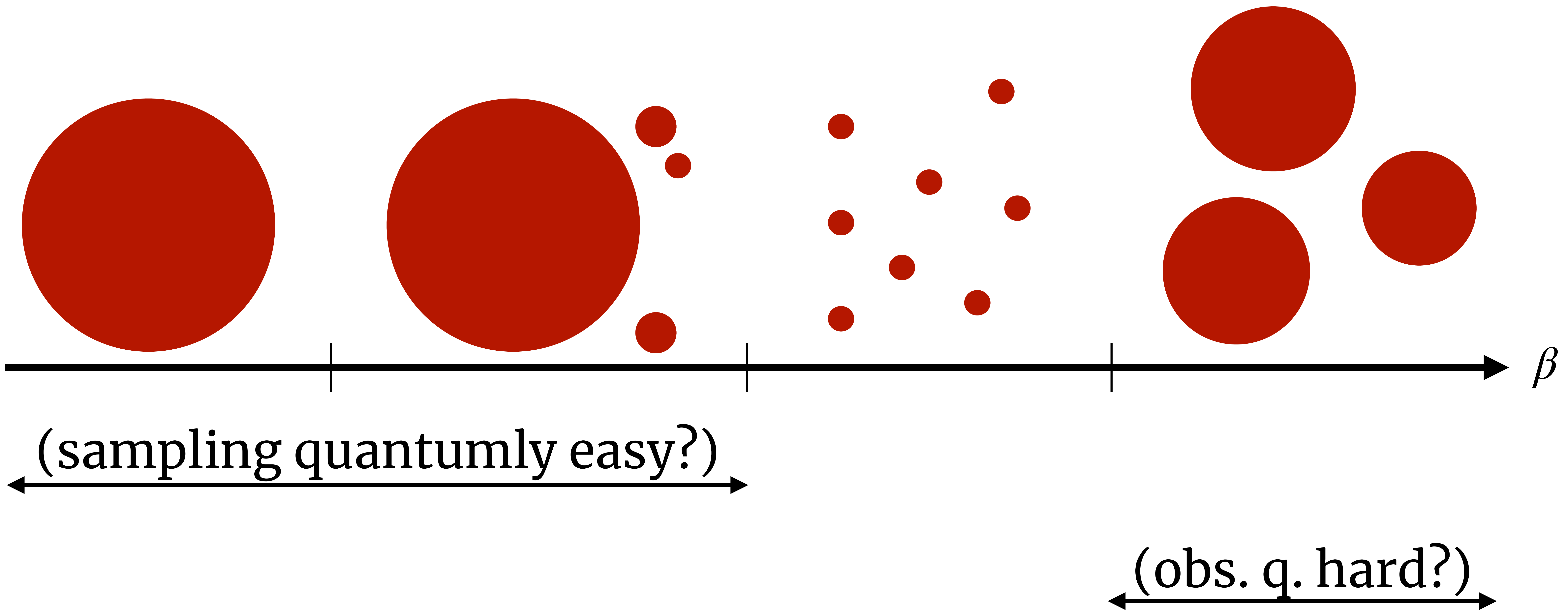
$n^{\beta \log(n\beta/\epsilon)}$ to ϵ-approximate observables for any constant β, local O

(This can resolve instance-to-instance fluctuations!)

Average-case complexity



Open questions



Open questions

Tighter relationship between physics and algorithms?

→ fast mixing implies zero-freeness? vice versa?

classical: Dobrushin $\implies$ zero-free, self-reducible + zero-free $\implies$ SSM

Stronger notions of average-case quantum hardness?

→ exponential-time lower bounds? non-local quantum algorithms?

Distinctly quantum aspects of quantum spin glasses?

→ SYK physics for Pauli glasses at constant k?

Thank you!