A TOOLKIT FOR ROBUST THRESHOLDS

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Abstract. Consider a host (hyper)graph G which contains a spanning structure due to minimum degree considerations. We collect three results proving that when the edges of G are sampled at the appropriate rate then the spanning structure still appears with high probability in the sampled hypergraph. We prove such results for perfect matchings in dense hypergraphs above Dirac thresholds, for K_r-factors above the Hajnal–Szemerédi minimum degree condition, and for bounded-degree spanning trees. In each case our proof is based on constructing a spread measure and then applying recent results on the (fractional) Kahn–Kalai conjecture connecting the existence of such measures with an appropriate probabilistic threshold result. We note that our second result provides a shorter and more general version of a recent result of Allen, Böttcher, Corsten, Davies, Jenssen, Morris, Roberts, and Skokan which handles the case $r = 3$ with different techniques. In particular, we answer a question of theirs with regards to the number of K_r-factors in a graph above the Hajnal–Szemerédi minimum degree condition.

1. Introduction

A central pursuit in both random and extremal (hyper)graph theory is to determine thresholds for various properties. In the random setting the natural question is to determine for which $p = p(n)$ the random graph $\mathbb{G}(n, p)$ satisfies a particular property with high probability.[1] In the extremal setting it is natural to consider the *minimum-degree*, or *Dirac*, threshold, i.e., for which $d = d(n)$ every n-vertex graph G with $\delta(G) \geq d$ satisfies a particular property.

Prominent examples include thresholds for connectivity [11], for containing a fixed size subgraph [12], for containing a perfect matching [13, 14, 21, 24, 29, 35, 44], for containing a Hamilton cycle [10, 41], for Ramsey properties [17, 45], for containing a clique (or subgraph) factor [8, 20, 24], and for containing a given bounded-degree spanning tree [31, 39].

One interpretation of probabilistic thresholds, suggested by Krivelevich, Lee, and Sudakov [34], is as a measure of *robustness*. For example, K_n is extremely robust with respect to containing a perfect matching since the threshold for this property in $\mathbb{G}(n, p)$ is $\log n/n$. Below this density w.h.p. isolated vertices, which are very simple local obstructions, begin to appear.

In this work we collect several results that exemplify the philosophy that if the minimum degree of a hypergraph G is above the minimum-degree threshold for a particular property then it not only satisfies the property but satisfies it robustly. That is, up to multiplicative constants, the probabilistic threshold for the property does not depend on whether one considers subgraphs of G or subgraphs of the complete hypergraph. We refer the reader to a survey of Sudakov [49] where a number of previous results in this direction are collected.

In order to prove our results we take advantage of the recently established connection between so-called *spread measures* and thresholds [18]. In fact, a second purpose of this paper is to demonstrate

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[1] A sequence of events, indexed by n, holds *with high probability (w.h.p.)* if the probabilities of their occurrence tend to 1 as $n \to \infty$.

methods to construct such measures. The techniques we use are related to regularity, robust perfect matchings, random greedy algorithms, and iterative absorption.

We prove robustness results in three settings: (i) the Dirac threshold for containing a perfect matching in a k-uniform hypergraph, (ii) the Hajnal–Szemerédi minimum degree condition for containing a K_r-factor, and (iii) the minimum degree condition for containing bounded-degree spanning trees. It is worth mentioning that finding the probabilistic thresholds for these properties in random subgraphs of the complete (hyper)graph were major open problems in the field. The thresholds for (i) and (ii) were found by Johansson, Kahn, and Vu [24] and the threshold for (iii) was found by Montgomery [39]. The fact that we are able to consider the minimum-degree setting for all three in just one paper speaks to the strength of the spread-measure results in [18].

We remark that very recently Allen, Böttcher, Corsten, Davies, Jenssen, Morris, Roberts, and Skokan [1] proved (ii) for $r = 3$, but without spread techniques and using techniques related to Johansson–Kahn–Vu [24]. Results falling under (i) have been independently proved by Kang, Kelly, Kühn, Osthus, and Pfenninger [28] using a similar strategy but with differences in implementation. Additionally, for the Dirac threshold for $(k-1)$-degrees, they prove a sharper "stability" version; we refer to the discussion after Theorem 1.5 for more details.

The idea of using spread measures to bound thresholds comes from the *fractional expectation threshold vs. threshold* conjecture of Talagrand [51] which was only recently established in a breakthrough due to Frankston, Kahn, Narayanan, and Park [18]. We note that this result is a fractional version of the Kahn–Kalai *expectation threshold vs. threshold* conjecture [25] and that the full conjecture, which implies the fractional version, was resolved in very recent work of Park and the first author [40]. For our purposes, the crucial corollary is a connection between so-called *spread measures* and thresholds.

Definition 1.1. Consider a finite ground set Z and fix a nonempty collection of subsets $\mathcal{H} \subseteq 2^Z$. Let μ be a probability measure on $\mathcal{H}$. For $q > 0$ we say that μ is *q-spread* if for every set $S \subseteq Z$:

$$\mu\left(\{A \in \mathcal{H} \colon S \subseteq A\}\right) \le q^{|S|}.$$

For a finite set Z and $p \in [0,1]$ we denote by $Z(p)$ the binomial distribution on subsets of Z where each vertex is present with probability p, independently of the other vertices. For a hypergraph H we abuse notation and write $H(p)$ instead of $E(H)(p)$, i.e., the binomial distribution on the edge set of H.

Theorem 1.2 (From [18, Theorem 1.6])**.** *There exists a constant $C = C_{1.2} > 0$ such that the following holds. Consider a non-empty ground set Z and fix a nonempty collection of subsets $\mathcal{H} \subseteq 2^Z$. Suppose that there exists a q-spread probability measure on $\mathcal{H}$. Then $Z(\min(Cq\log|Z|, 1))$ contains an element of $\mathcal{H}$ as a subset with probability $1 - o_{|Z|\to\infty}(1)$.*

We note that given $\mathcal{H} \subseteq 2^Z$, determining the expectation-threshold or fractional expectation-threshold is a difficult task in general. In particular, the fractional expectation-threshold is the solution to a linear program with variables corresponding to the collection of subsets in $\mathcal{H}$, which for many applications is of exponential size. (The expectation-threshold is in general an *integer* linear program.) Nevertheless, the results of [18, 40] immediately imply several previously difficult results including the threshold for containing a perfect matching in a random hypergraph (due to Johansson, Kahn, and Vu [24]) or containing a given bounded-degree spanning tree in a random graph (due to Montgomery [39]). A crucial factor in these applications is that the uniform distribution has optimal spread. This follows, for example, from the fact that vertex permutations act transitively on the objects in question (see [18, Section 7]). In contrast, in the minimum-degree setting using vertex permutations is a non-starter, and it is not obvious that the uniform distribution has sufficiently small spread to find the thresholds. The focus of this work is on providing methods for constructing spread measures in situations where one cannot rely on the "spread from vertex permutations".

A similar difficulty is present in work on the threshold for containing a Steiner triple system [27, 47], where the connection between spread and thresholds was exploited alongside other tools for understanding Steiner systems and Latin squares.

We will now state each of our results (i-iii), outline their proofs, and discuss relations to existing literature.

1.1. **Dirac robustness in random hypergraphs.** For a k-uniform hypergraph H and $S \subseteq V(H)$, let $\deg_H(S)$ be the number of hyperedges containing S. For $1 \le \ell \le k$ let $\delta_\ell(H)$, the minimum ℓ-degree of H, be the smallest degree of an ℓ-set in $V(H)$.

Definition 1.3. For integers $1 \le \ell \le k$ and n, with $n \in k\mathbb{N}$, let $t(n,k,\ell)$ be the smallest d such that every n-vertex k-uniform hypergraph with $\delta_\ell(H) \ge d$ contains a perfect matching. The *ℓ-degree (Dirac) threshold* for perfect matchings in k-uniform hypergraphs is

$$\delta^+_{\ell,k} := \lim_{\substack{n\to\infty \\ k|n}} \frac{t(n,k,\ell)}{\binom{n}{k-\ell}}.$$

Remark 1. The existence of the limit in Definition 1.3 is not *a priori* clear but was proven as [16, Theorem 1.2]. The actual value of $\delta^+_{\ell,k}$ is known in only a few cases. For instance, it is an open problem to find $\delta^+_{1,k}$ for $k \ge 5$.

Furthermore, an immediate consequence of this definition is that for every $1 \le \ell < k$ and $\varepsilon > 0$ there exists some $n_{\ell,k,\varepsilon}$ such that every k-uniform hypergraph H on $n \ge n_{\ell,k,\varepsilon}$ vertices with $k \in k\mathbb{N}$ and $\delta_\ell(H) \ge (\delta^+_{\ell,k} + \varepsilon)\binom{n}{k-\ell}$ contains a perfect matching.

Recall that Johansson, Kahn, and Vu [24] proved that the threshold for the appearance of perfect matchings in $\mathbb{G}^{(k)}(n,p)$ is order $\log n/n^{k-1}$. We show that this holds more generally for binomial random subgraphs of any hypergraph satisfying the minimum-degree condition above.

Definition 1.4. For a hypergraph $\mathcal{H}$ and $p \in [0,1]$, let $\mathcal{H}(p)$ a random hypergraph where each hyperedge of $\mathcal{H}$ is retained with probability p, independently of all other choices.

Theorem 1.5. *Let $\varepsilon > 0$ and $k, \ell \in \mathbb{N}$ be fixed. There exists $C = C_{1.5}(\ell,k,\varepsilon)$ such that the following holds. Let $\mathcal{H}$ be an n-vertex hypergraph with $k|n$ and $\delta_\ell(\mathcal{H}) \ge (\delta^+_{\ell,k}+\varepsilon)\binom{n}{k-\ell}$. Then $\mathcal{H}(C\log n/n^{k-1})$ contains a perfect matching with high probability.*

Remark 2. Kang, Kelly, Kühn, Osthus, and Pfenninger [28] independently proved Theorem 1.5. Furthermore they prove that when $\ell = k-1$, the ε in the minimum degree, and thus also the ε-dependence in the threshold, can be removed thereby providing a tight result in this case and in particular providing a robust version of a result of Rödl, Ruciński, and Szemerédi [44].

Kang, Kelly, Kühn, Osthus, and Pfenninger also prove that Theorem 1.5 can be extended to "optimal matchings" in cases when $n/k \notin \mathbb{Z}$; we note that our techniques also (immediately) extend to such "optimal matchings".

In terms of robustness for the Dirac degree threshold, for graphs a result of Sudakov and Vu [50] shows that for $p = \omega(\log n/n)$ any subgraph of $\mathbb{G}(n,p)$ with minimum degree $(1/2+o(1))np$ has a perfect matching. This *local resilience* property is strictly stronger than the robustness of Theorem 1.5 in the case $(k,\ell) = (2,1)$ since such a "minimum-degree subgraph" can be specified by intersecting the minimum degree host with the random graph $\mathbb{G}(n,p)$. It remains a problem of substantial interest to derive any such resilient threshold (or closely related universality result) from spread related techniques.

For hypergraphs, analogues of results of Sudakov–Vu [50] were proven by Ferber and Kwan [16] for k-uniform hypergraphs where $p \gtrsim \max\{n^{-k/2+o(1)}, n^{-k+2}\log n\}$. In particular, for $G \sim \mathbb{G}^{(k)}(n,p)$ with p above the specified threshold, with high probability any $G' \subseteq G$ satisfying $\delta_\ell(G') \ge$

$(\delta^+_{\ell,k}+\varepsilon)\binom{n-\ell}{k-\ell}p$ contains a perfect matching. (The case when $\ell = k-1$ was handled in earlier work of Ferber and Hirschfeld [15].) Note that this result is nontrivial only when $p = \Omega(n^{\ell-k}\log n)$ since otherwise one has $\delta_\ell(G) = 0$ and there are no subgraphs satisfying the hypothesis of the statement. This indicates a difference between the notion of robustness considered in our work and the notion of local resilence considered in these previous works as the probabilistic ranges of interest in Theorem 1.5 and [16, Conjecture 1.3] only coincide when $\ell = 1$.

We now briefly outline the proof of Theorem 1.5; the short proof is given in Section 3. Fix integers $k > \ell \geq 1$, $\varepsilon > 0$, and a hypergraph $\mathcal{H}$ satisfying the conditions of Theorem 1.5. By Theorem 1.2, it suffices to construct an $O(1/n^{k-1})$-spread distribution on perfect matchings in $\mathcal{H}$.

We construct such a spread distribution on matchings via *iterative absorption*, which was first introduced by Kühn and Osthus [36] and Knox, Kühn, and Osthus [30] to prove results on Hamilton decompositions. This powerful method has played a prominent role in several recent breakthroughs. We mention only a proof that combinatorial designs exist [19] and the proof of the Erdős–Faber–Lovász conjecture [26]. The method is substantially simplified in the context of perfect matchings.

Fix $\eta \ll \varepsilon$ (independent of n) and uniformly at random choose a *vortex* $V(\mathcal{H}) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N = X$ with $|X| \leq n^{1/(k+1)}$ and $|V_{i+1}| \approx \eta|V_i|$ for every i.

The heart of the construction is a *cover-down* procedure: Suppose that for $0 \leq i < \ell$ we have constructed a matching $M_i \subseteq \mathcal{H}$ that covers $V_0 \setminus V_i$ and only a small number of vertices in V_i. We then construct an $O(1/|V_i|^{k-1})$-spread distribution on matchings $M_{i+1} \supseteq M_i$, contained in $\mathcal{H}$, that cover all vertices of $V_0 \setminus V_{i+1}$, and only a small number of vertices in V_{i+1}.

Inductively applying the cover-down procedure we obtain a matching M_ℓ that covers $V_0 \setminus X$, and only a small number of vertices in X. In fact, we will ensure that the number of vertices in $V(M_\ell) \cap X$ is small enough that $\delta_\ell(\mathcal{H}[X \setminus V(M)]) \geq (\delta^+_{\ell,k}+\varepsilon/2)\binom{|X|}{k-\ell}$, so by Remark 1 there exists a perfect matching $\widetilde{M} \subseteq \mathcal{H}[X \setminus V(M)]$. Therefore $M_\ell \cup \widetilde{M} \subseteq H$ is a perfect matching.

To see why this procedure has a spread of $O(1/n^{k-1})$, we heuristically analyze the probability that any specific hyperedge $T \in \mathcal{H}$ is used in M. (Since the goal is to apply Theorem 1.2 in the actual analysis one needs to make the analogous calculation for any *set* of hyperedges.) For any i, the probability that T is spanned by V_i is $(|V_i|/n)^k$. Since the matching M_{i+1} is $O(1/|V_i|^{k-1})$-spread, the probability that $T \in M_{i+1} \setminus M_i$ is $O((|V_i|/n)^k/|V_i|^{k-1}) = O(|V_i|/n^k) = O(1/n^{k-1})$. Finally, the probability that T is in $\widetilde{M}$ is at most the probability that T is spanned by X, which is $(|X|/n)^k \leq 1/n^{k-1}$.

1.2. **K_r-factors in edge-percolated graphs.** We next consider the robustness of the Hajnal–Szemerédi theorem [20] on the existence of K_r-factors in dense graphs.

Theorem 1.6. *Let $r \in \mathbb{N}$. There exists a constant $C = C_{1.6}(r)$ such that every graph G on $n \in r\mathbb{N}$ vertices with $\delta(G) \geq (1-1/r)n$ contains at least $(n/C)^{(r-1)n/r}$ K_r-factors. Additionally, if $p = C(\log n)^{2/(r(r-1))}/n^{2/r}$ then with high probability $G(p)$ contains a K_r-factor.*

We note that $p = ((r-1)!\log n)^{2/(r(r-1))}/n^{2/r}$ is the (sharp) threshold for the property that every vertex in $\mathbb{G}(n,p)$ is contained in an r-clique. Hence, Theorem 1.6 is optimal up to a constant factor. That this is indeed the threshold for a K_r-factor in $\mathbb{G}(n,p)$ was first proved by Johansson, Kahn, and Vu [24]. Furthermore, the threshold in the $r = 3$ case of Theorem 1.6 was proved very recently by Allen, Böttcher, Corsten, Davies, Jenssen, Morris, Roberts, and Skokan [1]. The counting statement in Theorem 1.6 proves [1, Conjecture 1] (which is new even for $r = 3$).

One difficulty in proving Theorem 1.6 is that there does not exist a q-spread probability measure on K_r-factors in G with $q < \binom{n-1}{r-1}^{-2/(r(r-1))} = \Omega(n^{-2/r})$ (indeed, each vertex is contained in at least one K_r that is chosen with probability at least $\binom{n-1}{r-1}^{-1}$). Hence, the best bound one can hope for from applying Theorem 1.2 directly is $O(\log n/n^{2/r})$. This exceeds the threshold in Theorem 1.6 by a fractional power of $\log n$. To circumvent this difficulty we prove Theorem 1.7, which asserts that an appropriately spread measure exists on perfect matchings in the *r-clique complex* of G. We then apply Theorem 1.2 to obtain an optimal bound on the threshold for perfect matchings in the clique complex. Finally, we apply results of Riordan [43] to couple percolations of the clique complex with percolations of G. This gives the correct logarithmic power in Theorem 1.6.

Theorem 1.7. *Let $r \in \mathbb{N}$. There exists a constant $C = C_r$ such that for every graph G on $n \in r\mathbb{N}$ vertices with $\delta(G) \geq (r-1)n/r$ if $\mathcal{H}$ is the set of r-cliques in G then there is a C/n^{r-1}-spread measure (with respect to ground set $\mathcal{H}$) supported on the K_r-factors of G.*

The proof of Theorem 1.7 essentially breaks into two steps. The first is proving a version of Theorem 1.7 in which the minimum degree condition is replaced by the assumption that one has an r-partite graph G where the graph between each pair of parts is super-regular; this is Theorem 1.9.

Definition 1.8. Given a pair of vertex sets X_1, X_2 in a graph G define $d_G(X_1, X_2) = \frac{e_G(X_1,X_2)}{|X_1||X_2|}$ (when the graph G is clear from context we may omit the subscript). A pair (A_1, A_2) is ε-regular if for all $X_i \subseteq A_i$ with $|X_i| \geq \varepsilon|A_i|$ we have that $|d(A_1, A_2) - d(X_1, X_2)| \leq \varepsilon$. We say a pair (A_1, A_2) is (d, ε)-regular if $d(A_1, A_2) = d$.

Furthermore we say (A_1, A_2) is (d, ε, δ)-super-regular if it is (d, ε)-regular and for all $v \in A_i$ we have $\deg(v, A_{3-i}) \geq \delta|A_{3-i}|$. We say a pair is (d, ε)-super-regular if it is $(d, \varepsilon, d - \varepsilon)$-super-regular. Finally we say a pair (A_1, A_2) is (d^+, ε)-super-regular if it is (d', ε)-super-regular for some $d' \geq d$.

Theorem 1.9. *Fix $r \geq 2$ and suppose $1/n \ll \varepsilon \ll d$. Let $G = (V, E)$ be a r-partite graph on $V = \bigcup_{i=1}^{r} A_i$ where $|A_i| = n$ for all $i \in [r]$. Suppose $G[A_i, A_j]$ is $(d_{i,j}, \varepsilon)$-super-regular with $d_{i,j} \geq d$ for all $i \neq j$. Let $\mathcal{H}$ be the r-uniform hypergraph where edges in $\mathcal{H}$ correspond to r-partite cliques of G. There exists a $O_{d,\varepsilon}(1/n^{r-1})$-spread distribution on the set of perfect matchings in $\mathcal{H}$.*

We note that the $r = 3$ version of Theorem 1.9 immediately implies (along with work of Riordan [43] which provides a coupling to the K_r-factor version) the crucial technical statement in [1] and one of the main contributions of this work is providing a pair of short proofs of Theorem 1.9.

The first proof of Theorem 1.9 proceeds via induction on r. The base case $r = 2$ is the key step; for the inductive step one chooses a spread matching between a pair of parts and then constructs an auxiliary $(r-1)$-partite graph where an edge chosen between the initial pair of parts is connected to a vertex in the remaining parts if and only if both endpoints connect to the vertex. A careful application of the union bound proves that the associated graph is super-regular and therefore the key step is the case $r = 2$. For this, the crucial idea is to consider a subgraph G' of the underlying bipartite graph G where one chooses to keep a uniformly random large constant number of neighbors of each vertex (an edge is kept if either of its vertices wish to keep it). This subgraph is trivially seen to be appropriately spread by construction and the desired result follows by using Hall's theorem to verify that G' has a perfect matching w.h.p. We note that this idea of constructing a "spread" matching via considering a random subgraph where each vertex has a constant number of out-neighbors also plays a role in forthcoming work of the last three authors on the planar assignment problem [46] and will also be used in the proof of Theorem 1.10. The proof here is given in the short Section 4.

The second proof follows closely along the lines of the proof of Theorem 1.5; in particular one considers the set of r-cliques as hyperedges in an r-uniform hypergraph and the heuristic derivation of spread in Section 1.1 is unchanged. The changes between the proof of Theorem 1.5 and Theorem 1.9 are due to the regularity boosting procedure in the cover-down step and finding the perfect matching in final vortex set. The second issue is handled immediately by the influential blow-up lemma of Komlós, Sárközy, and Szemerédi [33] while the first issue can be handled by a straightforward regularity boosting procedure based on the counting lemma. This proof is given in Section 5.

The basic strategy to deduce Theorem 1.7 from Theorem 1.9 is as follows: one applies Szemerédi's regularity lemma, finds a K_r-factor between the reduced graph of the regularity partition, and then uses Theorem 1.9 for each K_r in the factor on the reduced graph. However, this sketch is a gross oversimplification of the necessary stability analysis as one is forced to deal with various exceptional vertices which occur in the partition. Our stability analysis is combination of those given in [33] and [1]. In particular we handle the case where the underlying graph G is far from being r-partite using essentially an identical argument to that given in [1] (for $r = 2$ there is an additional special case where the graph is near the union of two complete graphs). The extremal case, however, where there is a subset of size $|V(G)|/r$ which is nearly empty, is handled via an argument closely related to that given in [33]; in particular we rely on the devices of special stars (which are somewhat simpler in the K_r-factor case) used throughout [33] to rebalance parts of a vertex partition so that one can convert various density constraints into minimum degree constraints. Furthermore the various necessary modifications can be made in a "spread" manner, which completes our analysis. The necessary stability analysis is carried out in Section 6.

Finally we note that Theorem 1.6 is not the first result establishing a robust version of the Hajnal–Szemerédi theorem. The counting statement in Theorem 1.6 generalizes a result of Sárközy, Selkow, and Szemerédi [48], where they proved that a Dirac-graph with has at least $(cn)^{n/2}$ perfect matchings for a small constant c. We note that the optimal constant c (which is achieved by the random graph of the appropriate density) was proven by Cuckler and Kahn [9]; the extension of such a result to K_r-factors would be of substantial interest. Second, a resilience version of the Corrádi–Hajnal theorem [8] (which is the $r = 3$ case of the Hajnal–Szemerédi theorem) was proven by Balogh, Lee, and Samotij [4]: with high probability $\mathbb{G}(n, p)$ with $p = \omega((\log n/n)^{1/2})$ is such that every subgraph with minimum degree $(2/3 + o(1))np$ contains a triangle factor covering all but $O(p^{-2})$ vertices. This theorem is optimal both with the range of p and the size of the exceptional set of vertices.

1.3. **Bounded-degree spanning trees.** Using a precursor of the influential blow-up lemma, Komlós, Sárközy, and Szemerédi [31] proved that for every $\Delta \in \mathbb{N}$ and $\varepsilon > 0$ every sufficiently large n-vertex graph with minimum degree at least $(1/2 + \varepsilon)n$ contains a copy of every spanning tree of maximum degree at most Δ. Since there exist disconnected n-vertex graphs with minimum degree $\lceil n/2\rceil - 1$, the "1/2" cannot be improved.

Regarding the corresponding threshold in random graphs, Montgomery [39] proved that w.h.p. for any given spanning tree with degrees bounded by Δ, the random graph $\mathbb{G}(n, O_\Delta(\log n/n))$ contains a copy. (In fact, Montgomery proved the stronger *universality* result that w.h.p. $\mathbb{G}(n, O_\Delta(\log n/n))$ contains a copy of *every* spanning tree with maximal degree at most Δ.) Note that a lower bound of $\log n/n$ for this threshold follows from considering the presence of isolated vertices. Therefore Montgomery's result is tight up to a multiplicative constant.

Our third result is that graphs satisfying the Kómlos–Sárközy–Szemerédi [31] minimum degree condition are robust with respect to containing a given bounded-degree spanning tree.

Theorem 1.10. *For every $\Delta \in \mathbb{N}$, $\delta > 0$, there exists $C = C_{1.10}(\Delta, \delta)$ such that the following holds. Suppose that G is an n-vertex graph satisfying $\delta(G) \geq (1/2 + \delta)n$ and T is an n-vertex tree with $\Delta(T) \leq \Delta$. Then w.h.p. $G(C \log n/n)$ contains a copy of T.*

Our proof of Theorem 1.10 closely follows that of [31] and in fact we quote the main preprocessing statement from their proof. Roughly, the proof in [31] proceeds by embedding most vertices of the tree via a greedy process into a regularity decomposition of the graph G. The remaining vertices, which either form paths of length 4 or stars of various sizes, are embedded via specialized lemmas which are each essentially special cases of the blow-up lemma. The initial embedding of the large portion of the tree can easily be made $O(1/n)$-spread by using a *random* greedy algorithm, and observing that there are always $\Omega(n)$ choices for where to place each vertex. The paths of length 4 lemma can be made spread by essentially quoting the $r = 3$ case of Theorem 1.9 and the embedding of stars can be made spread via using the robust bipartite matchings discussed in Section 4. The details of the proof are in Section 7.

We remark that Theorem 1.10 is not the first robust version of the results of Komlós, Sárközy, and Szemerédi; Balogh, Csaba, and Samotij [3] consider the resilience of $\mathbb{G}(n,p)$ with respect to containing *almost* spanning trees. Specifically, for every $\Delta \in \mathbb{N}$ and $\varepsilon, \eta > 0$, in $\mathbb{G}(n,p)$ with $p = \Omega_{\eta,\varepsilon,\Delta}(1/n)$, w.h.p. any subgraph G with at least $(1/2+\eta)$-fraction of the edges at each vertex contains any tree on $(1-\varepsilon)n$ vertices with degree bounded by Δ. Related work has also considered such universality with respect to containment of almost spanning structures of limited bandwidth in $\mathbb{G}(n,p)$ and resilient subgraphs; we refer the reader to [6] and references therein.

Organization. In Section 2 we collect a series of basic preliminaries which will be used throughout the paper. In Section 3 we give the proof of Theorem 1.5. In Section 4 we give the first proof of Theorem 1.9 and in Section 5 we give the second proof of Theorem 1.9. In Section 6 we prove the necessary stability analysis to deduce Theorem 1.7 and thus Theorem 1.6. Finally in Section 7 we prove Theorem 1.10.

Notation. We write $[n] = \{1, \ldots, n\}$. We write $f = O(g)$ to mean that $f \le Cg$ for some absolute constant C, and $g = \Omega(f)$ to mean the same. We put a subscript, say O_ε, to clarify if the constant may depend on some outside parameter. We write $f = o(g)$ if for all $c > 0$ we have $f \le cg$ once the implicit growing parameter (typically n) grows large enough, and $g = \omega(f)$ means the same. For parameters α, β, we write $\alpha \ll \beta$ to mean that α is less than some sufficiently chosen function of β. As discussed earlier, for a hypergraph H and subset $S \subseteq V(H)$ we write $\deg_H(S)$ for the number of hyperedges of H containing S. If $S \subseteq V(H)$ with $|S|$ at most the uniformity of H and $A \subseteq V(H)$ we also write $\deg_H(S, A)$ to mean the number of hyperedges of H-neighbors of S fully contained in A. Finally, we may implicitly round large real numbers to integers if they are counting objects or performing a similar role, and the exact number is not required to be precise.

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2. Preliminaries

We will repeatedly use the Chernoff bound for binomial and hypergeometric distributions (see for example [23, Theorems 2.1, 2.10]) without further comment.

Lemma 2.1 (Chernoff bound)**.** *Let X be either:*

- *a sum of independent random variables, each of which take values in $[0,1]$, or*
- *hypergeometrically distributed (with any parameters).*

Then for any $\delta > 0$ we have

$$\mathbb{P}[X \le (1-\delta)\mathbb{E}X] \le \exp(-\delta^2\mathbb{E}X/2), \qquad \mathbb{P}[X \ge (1+\delta)\mathbb{E}X] \le \exp(-\delta^2\mathbb{E}X/(2+\delta)).$$

We also record a lemma for comparing a sequence of random variables to an independent sum of Bernoulli random variables.

Lemma 2.2 ([42, Lemma 8])**.** *Let $X_1, \ldots, X_n$ be $\{0,1\}$-valued random variables such that for all $i \in [n]$, we have that $\mathbb{P}[X_i = 1 | X_1, \ldots, X_{i-1}] \le p$ then $\mathbb{P}[\sum_{i=1}^n X_i \ge t] \le \mathbb{P}[\mathrm{Bin}(n,p) \ge t]$ for all $t \ge 0$.*

Finally, we will use McDiarmid's inequality, which follows from [37, Lemma 1.2].

Lemma 2.3. *Let $X_1, \ldots, X_n$ be independent random variables, with each X_i taking values in a finite set Λ_i. Let $f : \prod_{i=1}^n \Lambda_i \to \mathbb{R}$ be a function satisfying: for some $L > 0$ if $\vec{x}, \vec{y} \in \prod_{i=1}^n \Lambda_i$ differ by at most one coordinate then $|f(\vec{x}) - f(\vec{y})| \leq L$. Then, for every $t > 0$ there holds*

$$\mathbb{P}\left[|f(X_1, \ldots, X_n) - \mathbb{E}[f(X_1, \ldots, X_n)]| \geq t\right] \leq 2 \exp\left(-2t^2/(2nL^2)\right).$$

Next we require a version of the Rödl nibble which gives a spread distribution over nearly complete hypergraph matchings in a dense situation. We sketch a short proof based on applying the Rödl nibble given in [2, Theorem 4.7.1] to a harshly subsampled random hypergraph.

Lemma 2.4. *Fix $\gamma, \eta > 0$ and suppose n is sufficiently large. Given a k-uniform hypergraph $\mathcal{H}$ on n vertices such that each vertex is in $\gamma n^{k-1} \pm n^{k-5/4}$ hyperedges of $\mathcal{H}$, there exists a distribution $\mathfrak{p}$ on nearly-complete matchings, i.e., those covering at least $(1-\eta)n$ vertices, such that $\mathfrak{p}$ is $O_{k,\gamma,\eta}(1/n^{k-1})$-spread.*

Proof. Consider sampling every hyper-edge of $\mathcal{H}$ with probability $(C/\gamma)n^{1-k}$ for a constant C to be specified later. Let the sampled hypergraph be $\mathcal{L}$. First note that $\max_{v_1,v_2} \operatorname{codeg}_{\mathcal{L}}(v_1, v_2)$ is stochastically dominated by $\operatorname{Bin}(n^{k-2}, (C/\gamma)n^{-(k-1)})$ or $\operatorname{Bin}(n^{k-2}, n^{-k+5/4})$ for n sufficiently large in terms of C. This implies that as there are n^2 pairs to union bound over, that with high probability all codegrees in $\mathcal{L}$ are bounded by 3.

We now consider the subset of edges in $\mathcal{L}$ where every vertex has $\mathcal{L}$-degree at most $C + C^{3/4}$; call this hypergraph $\mathcal{L}'$. Furthermore by Chernoff and Markov with probability at least 7/8 we have that $(1 - e^{-\Omega(C^{1/3})})n$ vertices have $\mathcal{L}$-degree in $[C \pm C^{3/4}]$. Now we wish to show that taking the induced (sampled) hypergraph $\mathcal{L}'$ does not dramatically reduce the degree of the vast majority of these vertices.

Reveal a vertex v and assume that $\max_{v' \neq v} \operatorname{codeg}(v, v') \leq 3$ and its degree is in $[C \pm C^{3/4}]$. Having revealed the outcome of edges coming out of v, we need to prove that it is highly likely that the remaining neighbors in all these edges have degree at most $C + C^{3/4}$, so that the $\mathcal{L}'$-degree of v is the same as the $\mathcal{L}$-degree and hence is in $[C \pm C^{3/4}]$. This occurs for v with probability $1 - e^{-\Omega(C^{1/3})}$ by Chernoff, having revealed every edge containing v (using that the original degrees in $\mathcal{H}$ are tightly controlled).

Therefore we find from Markov that with probability 3/4, say, $\mathcal{L}'$ has at least $(1 - e^{-\Omega(C^{1/3})})n$ vertices with degree within $C \pm C^{3/4}$, no vertices with degree above $C + C^{3/4}$, and all codegrees bounded by 3. The desired result then follows via applying [2, Theorem 4.7.1] to the non-isolated vertices of $\mathcal{L}'$, which gives a sparse vertex cover, and then deleting any hyperedge in the cover containing a vertex which is covered more than once. We obtain a partial hypergraph matching which is evidently spread due to the sampling of hyperedges defining $\mathcal{L}$. $\square$

Finally we will also require a number of basic definitions regarding regular and super-regular pairs. Recall that have already defined the notion of (d, ε, δ)-super-regular as well as (d^+, ε)-super-regular in Definition 1.8. We will require the following lemma which allows one to transfer between these notions at the cost of passing to a suitable subgraph; this appears as [1, Lemma 2.12].

Lemma 2.5. *For every $\varepsilon > 0$ and $n = n_{2.5}(\varepsilon)$ such that the following holds. Consider a bipartite graph on V_1, V_2 with parts of size n and G which is (ε^2, d^+)-super-regular for d such that $4\varepsilon \leq d \leq 1$ and $dn^2 \in \mathbb{N}$. Then there is a spanning subgraph G' of G so that (V_1, V_2) is $(4\varepsilon, d)$-super-regular in G'.*

We will next require the counting lemma which counts embeddings of a subgraph H into fixed parts of a collection of regular pairs. This follows immediately from the standard proof of the counting lemma.

Lemma 2.6 (Counting lemma)**.** *Fix $\varepsilon > 0$ and a graph H on vertices $v_1, \ldots, v_k$. There exists an absolute constant $C_H = C_{H,2.6}$ such that the following holds. Fix a graph k-partite $G = (V, E)$ where $V = \bigcup_{i=1}^k A_i$ and suppose that for each $(i, j) \in E(H)$ that $G[A_i, A_j]$ is $(d_{i,j}, \varepsilon)$-regular with $d_{i,j} \geq \varepsilon$. Fix any subsets $X_i \subseteq A_i$ with $|X_i| \geq \varepsilon|A_i|$. Then the number of homomorphisms from H to G with v_i mapping into X_i is*

$$\prod_{(i,j)\in E(H)} d_{i,j} \prod_{i=1}^{k} |X_i| \pm C_H \varepsilon \prod_{i=1}^{k} |A_i|.$$

We will also require that any (d, ε) super-regular bipartite graph has a subset of edges such that both it and its complement are super-regular.

Lemma 2.7. *Suppose that $1/n \ll \varepsilon \ll d \leq 2/3$. Then given a graph $G = (A_1 \cup A_2, E)$ such that $|A_i| = n$ and (A_1, A_2) is (d, ε)-super-regular we have a spanning subgraph $G' \subseteq G$ which is $(d \pm \varepsilon^{1/3}, \varepsilon^{1/3})$-super-regular and such that the bipartite complement $K_{A_1,A_2} \setminus G'$ is $(1-d \pm \varepsilon^{1/3}, \varepsilon^{1/3})$-super-regular.*

Proof. Let $A_1' = \{v \in A_1 \colon \deg_G(v, A_2) \geq (d + 2\varepsilon)n\}$ and define A_2' similarly. By (d, ε)-regularity applied to the sets A_1' and A_2 (and symmetric) we have that $|A_1'|, |A_2'| \leq \varepsilon n$. By removing all edges between A_1' and A_2', note that the degrees in $A_1 \setminus A_1'$ and $A_2 \setminus A_2'$ are unchanged and we still have a lower bound of $(d - \varepsilon)n$ for all such vertices. Now for each vertex $v \in A_i'$ choose a set of approximately dn edges to keep and remove the rest; this is possible to do on a vertex-by-vertex basis as all remaining edges have at most 1 endpoint in A_i'. Furthermore note that in this procedure all vertices outside $A_1' \cup A_2'$ have degrees adjusted by at most εn. Finally note that since at most $2\varepsilon n^2$ edges have been modified the desired result follows immediately by the definition of regularity. $\square$

A similar proof can in fact be used to show the following; this combined with Lemmas 2.5 and 2.7 show that any (d, ε, δ)-super-regular graph will have very super-regular spanning subgraphs.

Lemma 2.8. *Suppose that $1/n \ll \varepsilon \ll \delta \ll d \leq 2/3$. Then given a graph $G = (A_1 \cup A_2, E)$ such that $|A_i| = n$ and (A_1, A_2) is (d, ε, δ)-super-regular we have a spanning subgraph $G' \subseteq G$ which is $(\delta^+, \varepsilon^{1/3})$-super-regular.*

Proof sketch. Using (d, ε)-regularity, we can show that the vast majority of vertices have degree $(d \pm 2\varepsilon)n$, say. Call the exceptional vertices A_1', A_2' as before. We take random (δ/d)-samples of the edges $G[A_1 \setminus A_1', A_2 \setminus A_2']$, delete the edges $G[A_1', A_2']$, and then for each $v \in A_1' \cup A_2'$ independently choose δn edges to keep. $\square$

Furthermore we will need the influential blowup lemma of Komlós, Sárközy, and Szemerédi [32, Theorem 1].

Theorem 2.9 ([32, Theorem 1])**.** *Given a graph R of order r and parameters δ, Δ, there exists $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds. Suppose that one replaces the vertices of R with sets of size $n_1, \ldots, n_r$ and define a graph G_1 where each edge of R is blown up to a (δ, ε)-super-regular pair and a graph G_2 where each edge of R is blown up to a complete graph. Then if a graph H with maximum degree Δ embeds into G_2 it embeds into G_1.*

Finally we will also require a version of the regularity lemma which respects the minimum degree of the input graph. The version stated appears as [1, Lemma 2.6].

Definition 2.10. We say a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ is an ε-regular partition if $|V_0| \leq \varepsilon|V(G)|$, $|V_1| = \cdots = |V_t|$, and all but εt^2 pairs (V_i, V_j) are ε-regular. Given an ε-regular partition and $d \in [0, 1]$, we say that R is the (ε, d)-reduced graph with respect to the partition if $V(R) = [t]$ and $(i, j) \in E(R)$ if and only if (V_i, V_j) is a (d^+, ε)-regular pair.

Lemma 2.11 ([1, Lemma 2.6])**.** *For all $\varepsilon > 0$ and $m_0 \in \mathbb{N}$, there is $M_0 = M_{2.11}(m_0, \varepsilon)$ such that the following holds. For all $0 < d < \gamma < 1$, $n > M_0$, and graphs G with $\delta(G) \geq \gamma n$, there exists an ε-regular partition $V_0 \cup V_1 \cup \cdots \cup V_m$ with $m_0 \leq m \leq M_0$ such that the (ε, d)-reduced graph R has $\delta(R) \geq (\gamma - d - 2\varepsilon)m$.*

3. Robust conditions for perfect matchings in random hypergraphs

The first lemma will allow us to construct the vortex; a crucial feature of our analysis is that the randomness in the choice of the vortex is taken into account when calculating the spread. For this reason the lemma guarantees a *distribution* over vortices, rather than the existence of any particular one.

Lemma 3.1 (Vortex)**.** *For every $\alpha > 0$ and $\varepsilon \in (0, 1/10)$ there exists some $C = C_{3.1}(\varepsilon)$ such that if $\mathcal{H}$ is a k-uniform hypergraph on $n \geq C$ vertices satisfying $\delta_\ell(H) \geq (\alpha + \varepsilon)\binom{n}{k-\ell}$ then there exists a distribution on set sequences $V(\mathcal{H}) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N = X$ with the following properties:*

- ***V1*** *For every $0 \leq i < N$ there holds $|V_{i+1}| = (1 \pm \varepsilon^2/N^2)\varepsilon^2|V_i|$;*
- ***V2*** *$|X| \in [n^{1/(k+2)}, n^{1/(k+1)}]$;*
- ***V3*** *For every $S \in \binom{V(\mathcal{H})}{\ell}$ and every $0 \leq i \leq N$, there holds $\deg_{\mathcal{H}}(S, V_i) \geq (\alpha + \varepsilon/2)\binom{|V_i|}{k-\ell}$;*
- ***V4*** *For every vertex set $\{v_1, \ldots, v_m\} \subseteq V(\mathcal{H})$ and every vector $\vec{x} \in \{0, \ldots, N\}^m$ there holds*

$$\mathbb{P}\left[\bigwedge_{i=1}^{m}(v_i \in V_{x_i})\right] \leq \prod_{i=1}^{m} \frac{2|V_{x_i}|}{n}.$$

Proof. First, consider the distribution on set sequences $V(\mathcal{H}) = U_0 \supseteq \cdots \supseteq U_N$ obtained as follows: Set $U_0 = V(\mathcal{H})$. For as long as $|U_i| > n^{1/(k+1)}$, let U_{i+1} be a binomial random subset of U_i of density ε^2. Let $\mathcal{E}$ be the event that properties **V1** to **V3** hold. McDiarmid's inequality (Lemma 2.3) and a union bound imply that $\mathcal{E}$ holds w.h.p.

Next, let $\{v_1, \ldots, v_m\} \subseteq V(\mathcal{H})$ and $\vec{x} \in \{0, \ldots, N\}^m$. Clearly:

$$\mathbb{P}\left[\bigwedge_{i=1}^{m}(v_i \in U_{x_i})\right] = \prod_{i=1}^{m} \varepsilon^{2x_i}.$$

Let $V_0 \supseteq \cdots \supseteq V_N$ be the distribution obtained by conditioning $U_0 \supseteq \cdots \supseteq U_N$ on the occurrence of $\mathcal{E}$. By definition, $V_0 \supseteq \cdots \supseteq V_N$ satisfies properties **V1** to **V3**. Furthermore, for every nonempty $\{v_1, \ldots, v_m\} \subseteq V(\mathcal{H})$ and $\vec{x} \in \{0, \ldots, N\}^m$:

$$\mathbb{P}\left[\bigwedge_{i=1}^{m}(v_i \in V_{x_i})\right] = \mathbb{P}\left[\bigwedge_{i=1}^{m}(v_i \in U_{x_i})\middle|\mathcal{E}\right] \leq \frac{\mathbb{P}\left[\bigwedge_{i=1}^{m}(v_i \in U_{x_i})\right]}{\mathbb{P}[\mathcal{E}]} \leq (1 + o(1))\prod_{i=1}^{m} \varepsilon^{2x_i} \leq \prod_{i=1}^{m} \frac{2|V_{x_i}|}{n},$$

as desired. The last inequality comes from applying **V1** iteratively at most N times. □

Next, we show that if $\mathcal{H}$ satisfies the conditions in Theorem 1.5 then it contains a large regular subgraph. We note that Lemma 3.2 plays the role of a "regularity–boosting" lemma which has various previous applications in iterative absorption.

Lemma 3.2. *For every $\varepsilon > 0$ there exists some $n'_\varepsilon > 0$ such that if $\mathcal{H}$ is a k-uniform hypergraph on $n \geq n'_\varepsilon$ vertices with $k|n$ and $\delta_\ell(H) \geq (\delta^+_{\ell,k} + \varepsilon)\binom{n}{k-\ell}$ then for every $0 \leq d \leq \frac{1}{2}\varepsilon\binom{n}{k-\ell}$ there exists a d-regular subgraph $\mathcal{H}' \subseteq \mathcal{H}$.*

Proof. Using the notation of Remark 1, let $n'_\varepsilon = n_{\ell,k,\varepsilon/3}$. Supposing $\mathcal{H}$ satisfies the assumptions, then by Remark 1 it contains a perfect matching M. If we remove M from $\mathcal{H}$ we obtain a hypergraph with minimum degree $\delta(H) - 1$. Proceeding inductively, we can remove d disjoint perfect matchings from $\mathcal{H}$. We may then take $\mathcal{H}'$ as their union. □

We are ready to state the cover-down lemma.

Lemma 3.3 (Cover-down lemma). *For every $\varepsilon \in (0, 1/(10k!))$ there exists some $C = C_{3.3}(\varepsilon) > 0$ such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph with $|V(\mathcal{H})| \geq C_\varepsilon$. Let $U \subseteq V(\mathcal{H})$ satisfy $|U| = (1 \pm \varepsilon)\varepsilon^2 |V(\mathcal{H})|$. Suppose that $\mathcal{H}$ satisfies $\delta_\ell(\mathcal{H}) \geq (\delta^+_{\ell,k} + \varepsilon/3)\binom{|V(\mathcal{H})|}{k-\ell}$ and, for every $S \in \binom{V(\mathcal{H})}{\ell}$, it holds $\deg_{\mathcal{H}}(S, U) \geq (\delta^+_{\ell,k} + \varepsilon/3)\binom{|U|}{k-\ell}$. Then there exists a $C_\varepsilon/|V(\mathcal{H})|^{k-1}$-spread distribution on matchings $M \subseteq \mathcal{H}$ that satisfy:*

C1 *M covers every vertex in $V(\mathcal{H}) \setminus U$; and*

C2 *M covers at most $\varepsilon^2|U|$ vertices in U.*

Proof. For notational conciseness we set $V = V(\mathcal{H})$. We first construct the random matching M.

Step 1: Finding a regular subgraph. We will find a regular subgraph of $\mathcal{H}[V \setminus U]$ by applying Lemma 3.2. This lemma can only be applied if the vertex set is a multiple of k. To this end, let $W \subseteq V \setminus U$ be a set of at most $k-1$ vertices such that $V' \coloneqq V \setminus (U \cup W)$ is a multiple of k. Let $\mathcal{H}' \coloneqq \mathcal{H}[V']$. Observe that $\delta_\ell(\mathcal{H}') \geq \delta_\ell(\mathcal{H}) - (|W| + |U|)^{k-\ell} \geq (\delta^+_{\ell,k} + 0.3\varepsilon)\binom{|V'|}{k-\ell}$. Assuming that $|V|$ is sufficiently large, we can apply Lemma 3.2 to find a $\frac{1}{8}\varepsilon\binom{|V'|}{k-\ell}$-regular subgraph $\widetilde{\mathcal{H}} \subseteq \mathcal{H}'$.

Step 2: Finding a spread approximate matching. We apply Lemma 2.4 to $\widetilde{\mathcal{H}}$ to find an $O_\varepsilon(1/|V|^{k-1})$-spread random matching $\widetilde{M} \subseteq \widetilde{\mathcal{H}}$ covering all but at most $\varepsilon^6 |V(\widetilde{H})|$ vertices.

Step 3: Covering remaining vertices in $V \setminus U$. Conditioning on $\widetilde{M}$, let $v_1, \dots, v_m$ be an enumeration of the uncovered vertices $V \setminus (V(\widetilde{M}) \cup U)$, noting $m \leq \varepsilon^6|V(\widetilde{H})| + k - 1$. (Recall that this set includes W.) We extend $\widetilde{M}$ to a matching $M \subseteq \mathcal{H}$ using a random greedy algorithm: Iterating through $i = 1, \dots, m$, for each v_i choose, uniformly at random, a hyperedge $T_i \in H$ containing v_i and $k-1$ vertices in U that is vertex-disjoint from $\widetilde{M}$ and all T_j for $j < i$.

We note that this procedure is sure to be successful. Indeed, before choosing any hyperedge T_i, every vertex $v \in V$ satisfies $\deg_H(v, U) \geq \binom{|U|}{\ell-1}(\delta^+_{\ell,k} + \varepsilon/3)\binom{|U|}{k-\ell}\binom{k-1}{\ell-1}^{-1} \geq (\delta^+_{\ell,k} + \varepsilon/4)\binom{|U|}{k-1}$. Furthermore, since $\widetilde{M}$ is contained entirely in $V \setminus U$, none of these hyperedges intersect $\widetilde{M}$. Thus, there are at least $(\delta^+_{\ell,k} + \varepsilon/4)\binom{|U|}{k-1}$ choices for T_i. Additionally, every hyperedge T_j intersects at most $(k-1)|U|^{k-2}$ possible choices for T_i. Since $m(k-1)|U|^{k-2} \leq (\varepsilon^6|V(\widetilde{\mathcal{H}})| + k - 1)(k-1)|U|^{k-2} \leq \varepsilon^2|U|^{k-1}$, there are always at least $(\delta^+_{\ell,k} + \varepsilon/8)\binom{|U|}{k-1}$ choices available for T_i.

For the final matching we take $M \coloneqq \widetilde{M} \cup \{T_1, \dots, T_m\}$. Clearly, M covers all vertices in $V \setminus U$, proving **C1**. Moreover it covers $(k-1)m \leq \varepsilon^2|U|$ vertices in U, proving **C2**.

It remains to show that M is $O(1/|V|^{k-1})$-spread. Let $S \subseteq \mathcal{H}$ be a set of hyperedges. We need to show that $P_S \coloneqq \mathbb{P}[S \subseteq M] = (O(1/|V|))^{(k-1)|S|}$. First, we may assume that S is a matching. Furthermore, if $S \subseteq M$ then every hyperedge in S is either included in $\widetilde{M}$ (in which case it has all k vertices in $V \setminus U$) or it is one of the hyperedges $T_1, \dots, T_m$ (in which case it has exactly one vertex in $V \setminus U$). So we may assume that every hyperedge in S has either one or k vertices in $V \setminus U$. Let S_k be those hyperedges in S with all vertices in $V \setminus U$, and let $S_1 = S \setminus S_k$ be those hyperedges in S with only one vertex in $V \setminus U$. We now have:

$$P_S = \mathbb{P}\big[S_k \subseteq \widetilde{M}\big]\mathbb{P}\big[S_1 \subseteq M \setminus \widetilde{M} | S_k \subseteq \widetilde{M}\big].$$

By construction, $\widetilde{M}$ is $O(1/|V|^{k-1})$-spread, so $\mathbb{P}[S_k \subseteq \widetilde{M}] = (O(1/|V|^{k-1}))^{|S_k|}$. Next, we observe that after conditioning on any outcome of $\widetilde{M}$, it holds that $S_1 \subseteq M \setminus \widetilde{M}$ only if for every hyperedge $T \in S_1$, the hyperedge chosen to match the (unique) vertex in $T \setminus U$ was T. Since every such choice is made uniformly from at least $(\delta^+_{\ell,k} + \varepsilon/8)\binom{|U|}{k-1} = \Omega(|V|^{k-1})$ possibilities, it follows that $\mathbb{P}[S_1 \subseteq M \setminus \widetilde{M} | S_k \subseteq \widetilde{M}] = (O(1/|V|^{k-1}))^{|S_1|}$. Thus $P_S = (O(1/|V|))^{(k-1)|S|}$, as desired. ☐

We are now in position to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. We assume, without loss of generality, that $\varepsilon < 1/(100k!)$.

Using Lemma 3.1, let $V(\mathcal{H}) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N = X$ be a random sequence of sets satisfying properties **V1** to **V4** in Lemma 3.1.

We will inductively construct (random) matchings $\emptyset = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N$, satisfying the following properties for every $0 \le i \le N$. For notational convenience we set $V_{N+1} = \emptyset$.

(1) M_i is $O(1/|V_i|^{k-1})$-spread;
(2) M_i covers all vertices in $V(H) \setminus V_i$;
(3) $|V(M_i) \cap V_i| \le 2\varepsilon^2 |V_i|$; and
(4) $V(M_i) \cap V_{i+1} = \emptyset$.

We begin by taking $M_0 = \emptyset$. Now, suppose that for $0 \le i < N$ we have constructed M_i with the properties above. Let $V_i' = V_i \setminus (V(M_i) \cup V_{i+2})$ and let $\mathcal{H}_i = \mathcal{H}[V_i']$. Note $V_{i+1} \subseteq V_i'$. Observe that $\delta_\ell(\mathcal{H}_i) \ge (\delta^+_{\ell,k} + \varepsilon/3)\binom{|V_i'|}{k-\ell}$. Indeed, for every $S \in \binom{V'}{\ell}$, by **V3** it holds $\deg_{\mathcal{H}}(S, V_i) \ge (\delta^+_{\ell,k} + \varepsilon/2)\binom{|V_i|}{k-\ell}$. At most $(|V(M_i) \cap V_i| + |V_{i+2}|)\binom{|V_i|}{k-\ell-1} \le 3\varepsilon^2 |V_i| \binom{|V_i|}{k-\ell-1} \le \frac{\varepsilon}{10}\binom{|V_i'|}{k-\ell}$ of these hyperedges are not contained in V'. Therefore $\deg_{\mathcal{H}_i}(S) \ge (\delta_{\ell,k} + \varepsilon/3)\binom{|V_i'|}{k-\ell}$, as desired. By applying Lemma 3.3 to $\mathcal{H}_i$ with $U = V_i' \cap V_{i+1}$ we obtain an $O_\varepsilon(1/|V_i|^{k-1})$-spread matching M_i' covering all vertices in $V_i' \setminus V_{i+1}$, at most $2\varepsilon^2 |V_{i+1}|$ vertices in V_{i+1}, and no vertices in V_{i+2}. (We can apply this after checking a similar minimum degree condition with respect to $\deg_{\mathcal{H}_i}(S, U)$.) By taking $M_{i+1} = M_i \cup M_i'$ we complete the inductive step.

Finally, to obtain a perfect matching, note that if M_N satisfies the properties above then $\delta_\ell(\mathcal{H}[V(\mathcal{H}) \setminus V(M_N)]) \ge (\delta^+_{\ell,k} + \varepsilon/3)\binom{|X|}{k-\ell}$. Furthermore, $|V(\mathcal{H}) \setminus V(M_N)|$ is a multiple of k since it was obtained from $V(\mathcal{H})$ by removing a matching. Therefore, by Remark 1, there exists a perfect matching $\widetilde{M} \subseteq \mathcal{H}[V(\mathcal{H}) \setminus V(M_N)]$. Take $M = M_N \cup \widetilde{M}$.

It remains to prove that M is $O_\varepsilon(1/n^{k-1})$-spread. Let $S \subseteq \mathcal{H}$ be a set of hyperedges, which we may assume are vertex-disjoint. We need to show that $P_S := \mathbb{P}[S \subseteq M] = (O_\varepsilon(1/n^{k-1}))^{|S|}$. Let $T_1, \ldots, T_m$ be an enumeration of the hyperedges in S. For a vector $\vec{x} \in [N+1]^m$, let $P(\vec{x})$ be the probability that for every $j \in [m]$, the hyperedge T_j is in $M_{x_j} \setminus M_{x_j - 1}$ if $x_j \le N$, and $T_j \in \widetilde{M}$ if $x_j = N+1$. We will show that

$$P(\vec{x}) = \left(\prod_{i=1}^{N} \left(O_\varepsilon \left(\frac{|V_{i-1}|}{n^k} \right) \right)^{|\{j : x_j = i\}|} \right) \left(\frac{|V_N|}{n} \right)^{k|\{j : x_j = N+1\}|}. \tag{3.1}$$

This will suffice, since then

$$\begin{aligned} P_S = \sum_{\vec{x} \in [N+1]^m} P(\vec{x}) &= \sum_{\vec{x} \in [N+1]^m} \left(\prod_{i=1}^{N} \left(O_\varepsilon \left(\frac{|V_{i-1}|}{n^k} \right) \right)^{|\{j : x_j = i\}|} \right) \left(\frac{|V_N|}{n} \right)^{k|\{j : x_j = N+1\}|} \\ &= \left(\frac{O_\varepsilon(1)}{n^{k-1}} \right)^m \sum_{\vec{x} \in [N+1]^m} \left(\prod_{i=1}^{N} (\varepsilon^{2i})^{|\{j : x_j = i\}|} \right) \left(\frac{|V_N|^k}{n} \right)^{|\{j : x_j = N+1\}|} \\ &= \left(O_\varepsilon \left(\frac{1}{n^{k-1}} \right) \right)^m \prod_{j=1}^{m} \left(\sum_{i=1}^{N} \varepsilon^{2i} + \frac{n^{k/(k+1)}}{n} \right) = \left(O_\varepsilon \left(\frac{1}{n^{k-1}} \right) \right)^m. \end{aligned}$$

We now prove (3.1). For $1 \le i \le N+1$, let C_i be the event that $\{T_j : x_j = i\} \subseteq \mathcal{H}[V_{i-1}]$ and let D_i be the event that $\{T_j : x_j = i\} \subseteq M_i \setminus M_{i-1}$ if $i \le N$, and $\{T_j : x_j = i\} \subseteq \widetilde{M}$ if $i = N+1$. We then have

$$P(\vec{x}) \le \mathbb{P}\left[\bigcap_{i=1}^{N+1} C_i \right] \prod_{i=1}^{N+1} \mathbb{P}\left[D_i \,\middle|\, \bigcap_{i=1}^{N+1} C_i, D_1 \cap \cdots \cap D_{i-1} \right].$$

By the randomness guarantee in the vortex construction (**V4** in Lemma 3.1), we have:

$$\mathbb{P}\left[\bigcap_{i=1}^{N+1} C_i\right] = \prod_{i=1}^{N+1}\left(O\left(\frac{|V_{i-1}|}{n}\right)\right)^{k|\{j:\, x_j=i\}|}.$$

Next, we note that conditioned on any outcome of M_{i-1} and the vortex, the matching $M_i \setminus M_{i-1}$ is $O_\varepsilon(1/|V_i|^{k-1})$-spread. Thus, for every $i \leq N$:

$$\mathbb{P}\left[D_i \middle| \bigcap_{i=1}^{N+1} C_i, D_1 \cap \cdots \cap D_{i-1}\right] = \left(O_\varepsilon\left(\frac{1}{|V_i|^{k-1}}\right)\right)^{|\{j:\, x_j=i\}|}.$$

Finally, we use the trivial bound $\mathbb{P}\left[D_{N+1} \middle| \bigcap_{i=1}^{N+1} C_i, D_1 \cap \cdots \cap D_\ell\right] \leq 1$ to obtain:

$$P(\vec{x}) \leq \left(\prod_{i=1}^{N+1}\left(O\left(\frac{|V_{i-1}|}{n}\right)\right)^{k|\{j:\, x_j=i\}|}\right)\left(\prod_{i=1}^{N}\left(O_\varepsilon\left(\frac{1}{|V_i|^{k-1}}\right)\right)^{|\{j:\, x_j=i\}|}\right),$$

which implies (3.1) upon using $|V_{i-1}| = O_\varepsilon(|V_i|)$. □

4. K_r-factors in r-partite super-regular systems via robust-perfect matchings

We now give the first of two proofs of Theorem 1.9. A key ingredient is the case where $r = 2$, and in particular proving that in a bipartite graph $G = (A, B, E)$ with $|A| = |B| = n$ one can find a spread perfect matching. This tool will again be crucial in the proof of Theorem 1.10. To do so, we consider the following subgraph of G. For each vertex v of G, choose a uniform and independent random set of C neighbors of v (with repetitions). Let H be the graph containing all of the edges chosen by either vertex.

Lemma 4.1. *Let G be (d,δ)-super-regular for $\delta \ll d$. Then, with probability at least $3/4$, for C sufficiently large depending only on d, the subgraph H contains a perfect matching.*

Proof. We will prove that H satisfies Hall's condition with high probability. In particular, we want to show that there is no subset T of B of size k for which there is a subset S of A of size $k+1$ and the neighborhood of any vertex in S is contained in T; and similarly for T a subset of A of size k and S a subset of B of size $k+1$. Note that if all vertices in S have their neighborhood contained in T, then all vertices in $B \setminus T$ have their neighborhood contained in $A \setminus S$. Thus, by symmetry, we only need to consider the case $k \leq n/2$, since for $k > n/2$ we have $|A \setminus S| < n/2$. In the following, let T be a subset of B of size k, and S a subset of A of size $k+1$. We bound the probability that $N_H(v) \subseteq T$ for all $v \in S$. Let $\eta = 4\delta^{1/3}$.

Case 1: $k \in (\eta n, n/2]$. In this case, by the assumption that G is (d,δ)-regular, for $\varepsilon = \delta^{1/3}/d$, the number of vertices v with $|N_G(v) \cap T| > (k/n + \varepsilon)dn$ is at most $\delta^{1/3}n$. Thus there are at least $k - \delta^{1/3}n$ vertices v in S with $|N_G(v) \cap T| \leq (k/n+\varepsilon)dn$. For each such v, the chance that $N_H(v) \subseteq T$ is at most $(k/n + \varepsilon^{1/2})^C$. Hence, the probability that $N_H(S) \subseteq T$ is at most $(k/n + \varepsilon^{1/2})^{C(k-\delta n)}$. By the union bound, the probability that there exists S and T with $N_H(S) \subseteq T$ is at most

$$\begin{aligned}\binom{n}{k}\binom{n}{k+1}(k/n + \varepsilon^{1/2})^{C(k-\delta^{1/3}n)} &\leq \left(\frac{e^2n^2}{k^2}\right)^{k+1}\left(\frac{k}{n} + \varepsilon^{1/2}\right)^{C(k-\delta^{1/3}n)} \\ &\leq \left(\frac{e^2n^2}{k^2}\right)^{2k}\left(\frac{k}{n} + \varepsilon^{1/2}\right)^{Ck/2} = \left(\frac{e^4n^4}{k^4}\cdot\left(\frac{k}{n} + \varepsilon^{1/2}\right)^{C/2}\right)^k.\end{aligned}$$

Note that $\frac{k}{n} + \varepsilon^{1/2} \le \min(2/3, 2(k/n)/d)$, and hence

$$\left(\frac{e^4n^4}{k^4} \cdot \left(\frac{k}{n} + \varepsilon^{1/2}\right)^{C/2}\right)^k \le \left(\frac{e^4n^4}{k^4}\left(\frac{k}{n} + \varepsilon^{1/2}\right)^4 \cdot (2/3)^{C/2-2}\right)^k \le \left(\frac{(2e)^4}{d^4} \cdot (2/3)^{C/2-2}\right)^k < 2^{-k},$$

assuming that C is sufficiently large in d.

Case 2: $k \le \eta n$. In this case, for each $v \in S$, $|N_G(v) \cap T| \le k \le \eta n$. Hence, the chance that $N_H(v) \subseteq T$ is at most $(2k/(dn))^C$. Hence, the probability that $N_H(S) \subseteq T$ is at most $(2k/(dn))^{Ck}$. By the union bound, the probability that there exists S and T with $N_H(S) \subseteq T$ is at most

$$\binom{n}{k}\binom{n}{k+1}(2k/(dn))^{Ck} \le \left(\frac{e(2k)^{C-4}}{d^C n^{C-4}}\right)^k < \left(\frac{e(2k/n)^{C-4}}{d^C}\right)^k.$$

Combining the cases, by the union bound, the probability that H does not satisfy Hall's condition is at most

$$2\sum_{k\le\eta n}\left(\frac{e(2k/n)^{C-2}}{d^C}\right)^k + 2^{-\eta n+1} = o(1). \qquad \square$$

Using Lemma 4.1, we can give a general procedure for finding spread matchings in super-regular bipartite graphs.

Theorem 4.2. *Let $d > 0$ and $\delta \ll d$. Let G be a (d,δ)-super-regular bipartite graph with parts of size n. There exists a distribution μ on perfect matchings in G which is $O_d(1/n)$-spread.*

Proof. From G pick a subgraph H as in Lemma 4.1, which has a perfect matching with probability at least $3/4$. Condition on this event and pick and output an arbitrary perfect matching W of H. This induces a distribution μ on perfect matchings of G. We show that μ is $O_d(1/n)$-spread. Indeed, given any subset S of edges of G, if S is not a matching, then $\mu(W \supseteq S) = 0$. If S is a matching, W can only contain S if for each edge $e = \{x,y\} \in S$, either x or y picks the other vertex as one of the C neighbors, which happens with probability at most $2C/n$. Furthermore, the above events are independent across different edges of the matching S. Hence, $\mu(W \supseteq S) \le (4/3)(2C/n)^{|S|}$. Thus, μ is $(4C/n)$-spread. $\square$

One can now prove Theorem 1.9 via an inductive argument on r.

Lemma 4.3. *Let $G = (V_1, V_2, V_3, E)$ be a (d,δ)-super-regular tripartite graph with $|V_1| = |V_2| = |V_3|$. Assume that for each edge $\{v_2, v_3\} \in E(G)$, there are at least $(d^2 - \delta^{1/2})n$ vertices $v_1 \in V_1$ which are adjacent to both v_2 and v_3. Let μ_1 be the distribution on perfect matchings M_1 between V_2 and V_3 given by Theorem 4.2, which is p-spread for some $p \le C/n$. Construct a graph Γ_{M_1} where the vertices are the edges in M_1 and vertices in V_1, and an edge $e = \{v_2, v_3\}$ of M_1 is connected to a vertex v_1 of V_1 if and only if v_1 is adjacent to both v_2 and v_3. There exists d', c' depending only on d and C such that for δ sufficiently small in terms of d, C, with probability at least $1 - \exp(-c'n)$, we have that Γ_{M_1} contains a $(d', 16C/\log(1/\delta)^{1/4})$-super-regular subgraph.*

Proof. Let $\eta = (\log \delta^{-1})^{-1/2}$. We first prove that Γ_{M_1} is $4\eta^{1/2}$-regular with high probability. Indeed, for each subset S_1 of V_1 of size ρn and $\rho \ge 4\eta^{1/2}$, we say that $e \in E(G_1)$ is bad if the number of common neighbors of the endpoints of e in S_1 is not in $(d^2 \pm \eta)\rho n$, and say that $v_i \in V_i$ is bad for $i \in \{2,3\}$ if the number of neighbors of v_i in S_1 is not in $(d \pm \eta)\rho n$. The number of bad v_i is at most δn. For each v_i which is not bad, the number of bad edges e adjacent to v_i is at most δn. We say that a perfect matching M_1 of $G[V_2, V_3]$ is bad if it contains at least ηn bad edges which are adjacent only to good vertices. The number of bad perfect matchings is at most

$$\binom{n}{\eta n}(\delta n)^{\eta n} \cdot n^{n-\eta n}.$$

For each such perfect matching, the probability (under μ_1) that it is realized is at most $(C/n)^n$. Hence, the probability that there are at least ηn bad edges adjacent to good vertices selected in M_1 is at most

$$\binom{n}{\eta n}(\delta n)^{\eta n} \cdot (n)^{n-\eta n} \cdot (C/n)^n \leq \exp(-(\log \delta^{-1})\eta n) C^n \exp(\eta n \log(\delta^{-1})/2)$$
$$\leq \exp(-(\log \delta^{-1})^{1/2} n/4).$$

Note that if there are at most ηn bad edges adjacent to good vertices in M_1, then the number of vertices in $E(M_1)$ whose number of edges to S_1 is not $(d^2 \pm \eta)\rho n$ is at most $(2\delta + \eta)n$. In that case, for any subset T of $E(M_1)$ of size at least ρn, the number of edges between T and S_1 is $(d^2 \pm 2(\eta + 2\delta)\rho^{-1})\rho n|T|$. Hence, by the union bound over S_1, we obtain that Γ_{M_1} is $4\eta^{1/2}$-regular with probability at least

$$2^n \exp(-(\log \delta^{-1})^{1/2} n/4) < 2^{-n}.$$

By assumption, the minimum degree of each $e \in M_1$ in Γ_{M_1} is at least $(d^2 - \delta^{1/2})n$. For each vertex $v_1 \in V_1$, let $E(v_1)$ be the set of edges in $E(G_1)$ whose endpoints are both adjacent to v_1. Then each vertex in $V_2 \cup V_3$ is adjacent to at least $(d^2 - \delta^{1/2})n$ edges in $E(v_1)$. By the remark following Theorem 4.2, the probability that the degree of v_1 in Γ_{M_1} is at most $(d^2/(2e^2))^C n$ is at most $\exp(-(d^2/(4e^2))^C n)$. Hence, by the union bound, with probability at least $1-n\exp(-(d^2/(4e^2))^C n)$, the minimum degree of Γ_{M_1} is at least $(d^2/(2e^2))^C n$. The conclusion of the lemma then follows from Lemmas 2.5 and 2.8. □

Proof of Theorem 1.9. We prove the result by induction on r. The case $r = 2$ is shown in Theorem 4.2.

Let $G_{r-1,r}$ be the graph induced on vertex sets V_{r-1} and V_r. By Theorem 4.2, there is a distribution μ_1 on perfect matchings M_1 of $G_{r-1,r}$ which is (C_d/n)-spread for C_d depending only on d. By Lemma 4.3, if for $i < r-1$, we construct the graph Γ_{i,M_1} which has as vertices edges $e = \{v_{r-1}, v_r\}$ in M_1 and vertices v_i in V_i for which v_i is adjacent to both v_{r-1} and v_r, then with probability at least $1 - \exp(-c'n)$, Γ_{i,M_1} has a subgraph which is $(d', 16C/(\log \delta^{-1})^{1/4})$-super-regular. By the union bound, with high probability, this property holds for all $i < r-1$. Now we have an $(r-1)$-partite graph G' where $G'[V_i, V_j] = G[V_i, V_j]$ for $i, j < r-1$ and $G'[V_i, V_{r-1}] = \Gamma_{i,M_1}$ for $i < r-1$, for which each pair of parts is $(d'^{+}, 16C/(\log \delta^{-1})^{1/4})$-super-regular. By the inductive hypothesis, we have a distribution μ on perfect matchings $\widetilde{M}$ of G' which is $O_d(1/n^{r-2})$-spread. The perfect matching $\widetilde{M}$ the corresponds to a perfect matching M of G. We now verify that M is $O_d(1/n^{r-1})$-spread.

Fix a subset S of hyperedges. As before, we can assume that S is a matching. Let S_1 be the matching of $G_{r-1,r}$ induced by S. If $M \supseteq S$, then $M_1 \supseteq S_1$, which holds with probability at most $(C/n)^{|S|}$ for some C depending only on d. Furthermore, conditioned on a consistent realization of M_1, we need the matching $\widetilde{M}$ to contain a corresponding set of edges of $E(G')$ of size $|S|$, which holds with probability at most $(O_d(1/n^{r-2}))^{|S|}$. Hence, the probability that M contains S is at most $(C'/n^{r-1})^{|S|}$. Thus, the distribution μ is $O_d(1/n^{r-1})$-spread. □

5. K_r-FACTORS IN r-PARTITE SUPER-REGULAR SYSTEMS VIA ITERATIVE ABSORPTION

In order to prove Theorem 1.9, we will require a regularity boosting lemma for the clique complex above a set of super-regular pairs. Regularity boosting plays a crucial role in the applications of iterative absorption; see e.g. [5, Lemma 4.2]. While our regularity boost generally follows a similar strategy of using local gadgets to adjust the initial uniform weighting on the hypergraph to give a good fractional weighting, generally partite instances requires a great deal more care (see e.g. work of Montgomery [38]); however in our case a substantially simpler proof suffices.

Lemma 5.1 (Fractional matching)**.** *Fix $r \geq 2$ and suppose $1/n \ll \varepsilon \ll d, 1/r \leq 2/3$. Let $G = (V, E)$ be an r-partite graph on $V = \bigcup_{i=1}^{r} A_i$ where $|A_i| = n$ for all $i \in [r]$. Suppose $G[A_i, A_j]$ is $(d_{i,j}, \varepsilon)$-super-regular with $d_{i,j} \geq d$ for all $i \neq j$. Let $\mathcal{H}$ be the r-uniform hypergraph where edges in $\mathcal{H}$ correspond to r-partite cliques of G. Then there exists a weighting $\omega\colon \mathcal{H} \to [0,1]$ such that for all $v \in \bigcup_{i=1}^{r} A_i$ we have that*

$$\sum_{\substack{H \ni v \\ H \in \mathcal{H}}} \omega(H) = \frac{1}{2} n^{r-1} \prod_{1 \leq i < j \leq r} d_{i,j}.$$

By sampling cliques according to ω and applying the Chernoff bound we have the following immediate corollary.

Corollary 5.2. *Fix $r \geq 2$ and suppose $1/n \ll \varepsilon \ll d, 1/r \leq 2/3$. Let $G = (V, E)$ be a r-partite graph on $V = \bigcup_{i=1}^{r} A_i$ where $|A_i| = n$ for all $i \in [r]$. Suppose $G[A_i, A_j]$ is $(d_{i,j}, \varepsilon)$-super-regular with $d_{i,j} \geq d$ for all $i \neq j$. Let $\mathcal{H}$ be the r-uniform hypergraph where edges in $\mathcal{H}$ correspond to r-partite cliques of G. Then there exists a weighting $\omega\colon \mathcal{H} \to \{0,1\}$ such that for all $v \in \bigcup_{i=1}^{r} A_i$ we have that*

$$\sum_{\substack{H \ni v \\ H \in \mathcal{H}}} \omega(H) = \frac{1}{2} n^{r-1} \prod_{1 \leq i < j \leq r} d_{i,j} \pm n^{r-4/3}.$$

We now prove Lemma 5.1.

Proof of Lemma 5.1. By applying Lemma 2.7, we may assume that each pair (A_i, A_j) is $((d_{i,j} - 4\varepsilon)^+, \varepsilon^{1/3})$-super-regular and that the complement is $((1 - d_{i,j} + 4\varepsilon)^+, \varepsilon^{1/3})$-super-regular. The weight function ω will be a perturbation of the function which is uniformly $1/2$ on $\mathcal{H}$.

Fix a vertex $v \in A_i$. By the degree lower bounds for v to A_j with $j \neq i$ and by Lemma 2.6 applied to $|N(v) \cap A_j|$ for $j \neq i$ to count copies of K_{r-1}, we have that

$$\deg_{\mathcal{H}}(v) = \sum_{\substack{H \in \mathcal{H} \\ H \ni v}} 1 = n^{r-1} \prod_{1 \leq i < j \leq r} d_{i,j} \pm \varepsilon^{1/4} n^{r-1}.$$

Note that this implies that $|E(\mathcal{H})| = n^r \prod_{1 \leq i < j \leq r} d_{i,j} \pm \varepsilon^{1/4} n^r$. We define the *defect* of a vertex v as

$$D_v = \deg_{\mathcal{H}}(v) - \frac{|E(\mathcal{H})|}{n}.$$

Note that $|D_v| \leq 2\varepsilon^{1/4} n^{r-1}$.

We will now define weight-shifting gadgets. For a pair of vertices $v_1, v_2 \in A_r$, let $\mathcal{R}_{v_1,v_2}$ be the set of $(2r-1)$-tuples of distinct vertices $(v_1', \ldots, v_{2r-1}')$ where $v_{2r-1}' \in A_r$, $v_i', v_{i+r-1}' \in A_i$ for $i \in [r-1]$, and $(v_1, v_1', \ldots, v_{r-1}')$, $(v_{2r-1}', v_1', \ldots, v_{r-1}')$, $(v_2, v_r', \ldots, v_{2r-2}')$, and $(v_{2r-1}', v_r', \ldots, v_{2r-2}')$ are all in $\mathcal{H}$. By applying the counting lemma (Lemma 2.6) we have that

$$|\mathcal{R}_{v_1,v_2}| = n^{2r-1} \prod_{1 \leq i < j \leq r-1} d_{i,j}^2 \prod_{1 \leq i \leq r-1} d_{i,r}^4 \pm \varepsilon^{1/4} n^{2r-1}.$$

For each $R \in \mathcal{R}_{v_1,v_2}$, define $f_{v_1,v_2,R}^r\colon \mathcal{H} \to \mathbb{R}$ by assigning 0 to everything outside the four distinguished r-cliques of R, assigning $(D_{v_1} - D_{v_2})/(2n|\mathcal{R}_{v_1,v_2}|)$ to $(v_{2r-1}', v_1', \ldots, v_{r-1}')$ and $(v_2, v_r', \ldots, v_{2r-2}')$, and assigning $-(D_{v_1} - D_{v_2})/(2n|\mathcal{R}_{v_1,v_2}|)$ to $(v_1, v_1', \ldots, v_{r-1}')$ and $(v_{2r-1}', v_r', \ldots, v_{2r-2}')$. Define $f_{v_1,v_2}^r\colon \mathcal{H} \to \mathbb{R}$ as the sum of all $f_{v_1,v_2,R}^r$ over $R \in \mathcal{R}_{v_1,v_2}$. One can define the analogous construction for each pair of vertices in the same part for different $j \in [r]$, not just A_r. The modified function ω will simply be

$$\omega(H)=\frac{2^{-1}n^{r-1}\prod_{1\le i<j\le r}d_{i,j}}{|E(\mathcal{H})|n^{-1}}\Bigg(1+\sum_{\substack{1\le i\le r\\ v_1,v_2\in A_i}}f^i_{v_1,v_2}(H)\Bigg).$$

Notice that the functions $f^i_{v_1,v_2}$ are mean zero upon averaging over all $H\in\mathcal{H}$ by construction and therefore to prove the desired result it suffices to prove that for each pair of vertices $v_1,v_2\in A_i$ there holds

$$\sum_{\substack{H\in\mathcal{H}\\ H\ni v_1}}\omega(H)=\sum_{\substack{H\in\mathcal{H}\\ H\ni v_2}}\omega(H)$$

and that $\omega(H)\in[0,1]$ for every $H\in\mathcal{H}$. For the first claim notice that if $v\in A_\ell$ then by construction

$$\begin{aligned}\bigg(\frac{2^{-1}n^{r-1}\prod_{1\le i<j\le r}d_{i,j}}{|E(\mathcal{H})|n^{-1}}\bigg)^{-1}\sum_{\substack{H\in\mathcal{H}\\ H\ni v}}\omega(H)&=\sum_{H\ni v}\Bigg(1+\sum_{\substack{1\le i\le r\\ v_1,v_2\in A_i}}f^i_{v_1,v_2}(H)\Bigg)\\&=\sum_{H\ni v}\Bigg(1+2\sum_{v'\in A_\ell}f^\ell_{v,v'}(H)\Bigg)=\sum_{H\ni v}1+\sum_{v'\in A_\ell}\frac{(D_{v'}-D_v)}{n}\\&=\deg_{\mathcal{H}}(v)-D_v+\frac{1}{n}\sum_{v'\in A_\ell}D'_v=\frac{|E(\mathcal{H})|}{n}.\end{aligned}$$

Thus it remains to prove that $\omega(H)\in[0,1]$ for every $H\in\mathcal{H}$. We saw above that for every $v_1,v_2\in A_i$ there holds $|\mathcal{R}^i_{v_1,v_2}|\ge d^{O(1)}n^{2r-1}$. Furthermore each hyperedge is given non-zero weight by at most $4rn^{r+1}$ gadgets $f^i_{v_1,v_2}$. Finally, there holds

$$|f^i_{v_1,v_2}(H)|\le(|D_{v_1}|+|D_{v_2}|)/(2n|\mathcal{R}^i_{v_1,v_2}|)\le 4\varepsilon^{1/4}n^{r-1}/(2n^{2r}d^{O(1)})$$

for every $H\in\mathcal{H}$, and every $v_1,v_2\in A_i$. This implies that

$$\Bigg|\sum_{\substack{1\le i\le r\\ v_1,v_2\in A_i}}f^i_{v_1,v_2}(H)\Bigg|\le 4rn^{r+1}\times 4\varepsilon^{1/4}n^{r-1}/(2n^{2r}d^{O(1)})=\varepsilon^{1/4}d^{-O(1)}.$$

As $\varepsilon\ll d$ and $\frac{2^{-1}\prod_{1\le i<j\le r}d_{i,j}n^{r-1}}{|E(\mathcal{H})|n^{-1}}\in[1/3,2/3]$ by the counting lemma (Lemma 2.6), the desired result follows immediately. □

We are now in position to prove Theorem 1.9. Given Corollary 5.2 and Theorem 2.9, the proof (via iterative absorption) is analogous to that of Theorem 1.5. First we prove a vortex lemma similar to Lemma 3.1, but starting with the setup in Theorem 1.9.

Lemma 5.3 (Vortex). *Fix $r\ge 2$ and suppose $1/n\ll\eta\ll\varepsilon\ll d$. Let $G=(V,E)$ be an r-partite graph on partition $V=\bigcup_{j=1}^r A_j$ where $|A_j|=n$ for all $j\in[r]$. Suppose $G[A_j,A_k]$ is $(d_{j,k},\varepsilon)$-super-regular with $d_{j,k}\ge d$ for all $j\ne k$. Then there exists a distribution on set sequences $V(G)=V_0\supseteq V_1\supseteq\cdots\supseteq V_N=X$ with the following properties:*

- ***V1*** *For every $0\le i\le N$ there holds $|V_i\cap A_j|=|V_i|/r$ for all $j\in[r]$;*
- ***V2*** *For every $0\le i<N$ there holds $|V_{i+1}|=(1\pm\eta/N^2)\eta|V_i|$;*
- ***V3*** *$|X|\in[n^{1/(r+2)},n^{1/(r+1)}]$;*
- ***V4*** *For every $0\le i\le N$ and $j\ne k$ there holds $G[V_i\cap A_j,V_i\cap A_k]$ is $(d^{(i)}_{j,k},\varepsilon^{1/20})$-super-regular for some $d^{(i)}_{j,k}=d_{j,k}\pm 2\varepsilon$;*
- ***V5*** *For every $0\le i<N$, $j\ne k$, and $v\in V_i\cap A_j$ there holds $\deg_G(v,V_{i+1}\cap A_k)\ge(d_{i,j}-2\varepsilon)|V_{i+1}|/r$;*

V6 *For every vertex set* $\{v_1, \dots, v_m\} \subseteq V(H)$ *and every vector* $\vec{x} \in \{0, \dots, N\}^m$ *there holds*

$$\mathbb{P}\Big[\bigwedge_{i=1}^{m}(v_i \in V_{x_i})\Big] \le \prod_{i=1}^{m} \frac{2|V_{x_i}|}{n}.$$

Proof. First, consider the distribution of set sequences $V(G) = U_0 \supseteq \cdots \supseteq U_N$ obtained as follows: Set $U_0 = V(G)$. For as long as $|U_i| > n^{1/(r+1)}$, for each $j \in [r]$ let $U_{i+1} \cap V_j$ be a uniformly random subset of $U_i \cap V_r$ of size exactly $\lceil \eta |U_{i+1}|/r \rceil$. Observe that **V1**, **V3**, and **V3** hold by definition. Chernoff's inequality for hypergeometric distributions (Lemma 2.1) and a union bound imply that **V5** holds w.h.p.

Showing that **V4** holds w.h.p. requires a little more work. Let $i \in [N]$ and $j, k \in [r]$ be distinct. We use regularity and Chernoff's inequality to show that with all but exponentially small probability there holds $d(U_i \cap A_j, U_i \cap A_k) = d_{j,k} \pm 2\varepsilon$. In order to show that $G[U_i \cap A_j, U_i \cap A_k]$ is $\varepsilon^{1/20}$-regular we use the well-known equivalence between the sums of codegrees and regularity [7]. That is, we first use McDiarmid's inequality (Lemma 2.3) to prove that with all but exponentially small probability there holds

$$\sum_{u,v \in U_i \cap A_j} |\{w \in U_i \cap A_k : uw, vw \in E(G)\}| = (1 \pm o(1)) \left(\frac{|U_i|}{|V|}\right)^3 \sum_{u,v \in A_j} |\{w \in A_k : uw, vw \in E(G)\}|.$$

Since (A_j, A_k) is $(d_{j,k}, \varepsilon)$-regular the sum on the right is equal to $n^3 d_{j,k}^2 \pm \varepsilon^{1/4} n^3$. Hence the sum on the left is equal to $|U_i \cap A_j|^3 d_{j,k}^2 \pm \varepsilon^{1/5} |U_i|^3$. But this, in turn, implies that $G[U_i \cap A_j, U_i \cap A_k]$ is $\varepsilon^{1/20}$-regular. In order to show super-regularity it remains to verify the minimum degree condition; this follows immediately from **V5**.

Next, let $\{v_1, \dots, v_m\} \subseteq V(G)$ and $\vec{x} \in \{0, \dots, N\}^m$. Clearly

$$\mathbb{P}\Big[\bigwedge_{i=1}^{m}(v_i \in U_{x_i})\Big] \le \prod_{i=1}^{m} \frac{3|U_{x_i}|}{2n},$$

since each next set U_{i+1} of the binomial set process can be coupled inside a $\eta(1 + 1/N^2)$-binomial random subset, say, of the current set U_i (and multiplying over at most N steps).

Let $\mathcal{E}$ be the event that properties **V1** to **V5** hold for $U_0, \dots, U_N$. Let $V_0 \supseteq \cdots \supseteq V_N$ be the distribution obtained by conditioning $U_0 \supseteq \cdots \supseteq U_N$ on the occurrence of $\mathcal{E}$. By definition, $V_0 \supseteq \cdots \supseteq V_N$ satisfies properties **V1** to **V5**. Furthermore, for every nonempty $\{v_1, \dots, v_m\} \subseteq V(G)$ and $\vec{x} \in \{0, \dots, N\}^m$:

$$\mathbb{P}\Big[\bigwedge_{i=1}^{m}(v_i \in V_{x_i})\Big] = \mathbb{P}\Big[\bigwedge_{i=1}^{m}(v_i \in U_{x_i}) \Big| \mathcal{E}\Big] \le \frac{\mathbb{P}\big[\bigwedge_{i=1}^{m}(v_i \in U_{x_i})\big]}{\mathbb{P}[\mathcal{E}]} \le \prod_{i=1}^{m} \frac{2|V_{x_i}|}{n},$$

as desired. The last inequality comes from applying **V2** iteratively at most N times. □

Next we prove a cover-down lemma similar to Lemma 3.3.

Lemma 5.4 (Cover-down lemma)**.** *Fix* $r \ge 2$ *and suppose* $1/m \ll \eta \ll \varepsilon \ll d$. *Let* $G = (V, E)$ *be a* r*-partite graph on partition* $V = \bigcup_{j=1}^{r} A_j$ *where* $|A_j| = m$ *for all* $j \in [r]$. *Suppose* $G[A_j, A_k]$ *is* $(d_{j,k}, \varepsilon)$*-super-regular with* $1/2 \ge d_{j,k} \ge d$ *for all* $j \ne k$. *Let* $U \subseteq V(G)$ *satisfy* $|U \cap A_j| = |U|/r$ *for all* $j \in [r]$ *and* $|U| = (1 \pm \eta)\eta(rm)$. *Suppose that for all* $j \ne k$ *and* $v \in A_j$ *we have* $\deg_G(v, U \cap A_k) \ge (d_{i,j} - \varepsilon)|U|/r$, *and for* $j \ne k$ *we have that* $G[U \cap A_j, U \cap A_k]$ *is* $(d'_{j,k}, \varepsilon)$*-super-regular for some* $1/2 \ge d'_{j,k} \ge d$. *Then there exists a* C_η/m^{r-1}*-spread distribution on partial* K_r*-factors* M *of* G *that satisfies:*

C1 M *covers every vertex in* $V(G) \setminus U$;
C2 M *covers at most* $\eta|U|$ *vertices in* U.

Proof. We first construct the random partial K_r-factor M.

Step 1: Finding a regular clique system. We will find a regular collection of cliques of $G[V \setminus U]$ by applying Corollary 5.2. Let $V' := V \setminus U$ and $G' := G[V']$. Observe that $G'[V' \cap A_j, V' \cap A_k]$ is $(d'_{j,k}, 2\varepsilon)$-super-regular for some $d'_{j,k} \in [d_{j,k} \pm 2\varepsilon]$ since we removed a small (with respect to ε) fraction. Since $|V' \cap A_j| \geq m/2$ is sufficiently large with respect to ε and the part sizes are equal by the given conditions, we can apply Corollary 5.2 to find a set $\widetilde{\mathcal{H}}$ of r-cliques of G' so that every $v \in V'$ is contained in

$$\frac{1}{2}n^{r-1} \prod_{1 \leq i < j \leq r} d'_{i,j} \pm n^{r-4/3}$$

many cliques.

Step 2: Finding a spread approximate matching. We apply Lemma 2.4 to $\widetilde{\mathcal{H}}$ to find an $O_{\eta,r}(1/m^{k-1})$-spread matching $\widetilde{M} \subseteq \widetilde{\mathcal{H}}$ covering all but at most $\eta^6|V'|$ vertices of V'.

Step 3: Covering remaining vertices in $V \setminus U$. Conditioning on $\widetilde{M}$, let $v_1, \dots, v_t$ be an enumeration of the uncovered vertices $V \setminus (V(\widetilde{M}) \cup U)$, noting $t \leq \eta^6|V'|$. Note that there are an equal amount in each part A_j for $j \in [r]$. We extend $\widetilde{M}$ to a partial K_r-factor M covering all of $V \setminus U$ (and some of U) using a random greedy algorithm: Iterating through $i = 1, \dots, t$, for each v_i choose, uniformly at random, an r-clique T_i of G containing v_i and $k-1$ vertices in U that is vertex-disjoint from $\widetilde{M}$ and all T_j for $j < i$.

We note that this procedure is sure to be successful. Indeed, before choosing any hyperedge T_i, every vertex $v \in A_j$ satisfies $\deg_G(v, U \cap A_k) \geq (d_{j,k} - \varepsilon)|U|/r$. Furthermore, since $\widetilde{M}$ is contained entirely in $V \setminus U$, none of these hyperedges intersect $\widetilde{M}$. Thus, for some T_i with $v_i \in A_j$, there are at least

$$\prod_{k \neq j}(|U|/r)\Bigg(\prod_{k \neq j}(d_{j,k} - \varepsilon) \prod_{\substack{1 \leq k_1 < k_2 \leq r \\ k_1, k_2 \neq j}} d_{k_1,k_2} - C\varepsilon\Bigg)$$

choices for T_i by Lemma 2.6 and the super-regularity of the pairs $(U \cap A_{k_1}, U \cap A_{k_2})$. Additionally, every other hyperedge T_j intersects at most $(r-1)|U|^{r-2}$ possible choices for T_i. Since $t(r-1)|U|^{r-2} \leq (\eta^6 m)(r-1)|U|^{r-2} \leq \eta|U|^{r-1}$, there are always at least say

$$(|U|/(2r))^{r-1} \prod_{1 \leq j < k \leq r} d_{j,k}$$

choices available for T_i regardless of the prior choices.

For the final matching we take $M := \widetilde{M} \cup \{T_1, \dots, T_t\}$. Clearly, M covers all vertices in $V \setminus U$, proving **C1**. Moreover it covers $(r-1)t \leq \eta|U|$ vertices in U, proving **C2**.

It remains to show that M is $O_\eta(1/m^{r-1})$-spread. Let S be a set of r-cliques of G. We need to show that $P_S := \mathbb{P}[S \subseteq M] = (O(1/m))^{(r-1)|S|}$. First, we may assume that S is a partial K_r-factor. Furthermore, if $S \subseteq M$ then every r-clique in S is either included in $\widetilde{M}$ (in which case it has all k vertices in $V \setminus U$) or it is one of the hyperedges $T_1, \dots, T_t$ (in which case it has exactly one vertex in $V \setminus U$). So we may assume that every hyperedge in S has either one or k vertices in $V \setminus U$. Let S_k be those hyperedges in S with all vertices in $V \setminus U$, and let $S_1 = S \setminus S_k$ be those hyperedges in S with only one vertex in $V \setminus U$. We now have:

$$P_S = \mathbb{P}\big[S_k \subseteq \widetilde{M}\big]\mathbb{P}\big[S_1 \subseteq M \setminus \widetilde{M} | S_k \subseteq \widetilde{M}\big].$$

By construction, $\widetilde{M}$ is $O_\eta(1/m^{k-1})$-spread, so $\mathbb{P}[S_k \subseteq \widetilde{M}] = (O(1/m^{k-1}))^{|S_k|}$. Next, we observe that after conditioning on any outcome of $\widetilde{M}$, it holds that $S_1 \subseteq M \setminus \widetilde{M}$ only if for every hyperedge $T \in S_1$, the hyperedge chosen to match the (unique) vertex in $T \setminus U$ was T. Since every such choice is made uniformly from at least $(|U|/(2r))^{r-1}\prod_{1\le j<k\le r} d_{j,k} = \Omega_\eta(m^{k-1})$ possibilities, it follows that $\mathbb{P}[S_1 \subseteq M \setminus \widetilde{M} | S_k \subseteq \widetilde{M}] = (O(1/m^{k-1}))^{|S_1|}$. Thus $P_S = (O(1/m))^{(k-1)|S|}$, as desired. $\square$

Now we prove Theorem 1.9.

Proof of Theorem 1.9. First, note that we may assume $d_{j,k} \le 1/3$ for all $j \ne k$ by considering a random $1/3$-sample of the edges of the graph G and adjusting parameters appropriately. Additionally, we then choose parameters so that

$$1/n \ll \eta \ll \varepsilon \ll d \le 1/3,$$

treating our choice of η ultimately as a function of d, ε. By renaming parameters we may assume that all the pairs in G are ε^{20}-super-regular.

Using Lemma 5.3, let $V(G) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_N = X$ be a random sequence of sets satisfying properties **V1** to **V6** in Lemma 5.3.

We will inductively construct (random) partial K_r-factors $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N$, satisfying the following properties for every $0 \le i \le N$. For notational convenience we set $V_{N+1} = \emptyset$.

(1) M_i is $O(1/|V_i|^{k-1})$-spread;
(2) M_i covers all vertices in $V(G) \setminus V_i$;
(3) $|V(M_i) \cap V_i| \le 2\eta |V_i|$; and
(4) $V(M_i) \cap V_{i+1} = \emptyset$.

We begin by taking $M_0 = \emptyset$. Now, suppose that for $i < N$ we have constructed M_i with the properties above. Let $V_i' = V_i \setminus (V(M_i) \cup V_{i+2})$ and let $G_i = G[V_i']$. Note $V_{i+1} \subseteq V_i'$. Observe that G_i and $U = V_{i+1} \setminus V_{i+2} \subseteq V(G_i)$ satisfy the hypotheses of Lemma 5.4 for $m = |V_i'|$ and slightly modified parameters $d_{j,k}, d'_{j,k}, \varepsilon$, using **V1** and **V2** for the set sizes, **V5** for the degree condition, and **V4** for $i, i+1$ for the super-regularity conditions (plus $V_{i+1} \subseteq V_i'$ as well as the fact that removing V_{i+2} is removing a negligible η-fraction of V_{i+1} and thus does not affect these conditions severely). By applying Lemma 5.4 to $G_i, U = V_{i+1} \setminus V_{i+2}$ in this situation we obtain an $O_\eta(1/|V_i|^{k-1})$-spread partial K_r-factor M_i' covering all vertices in $V_i' \setminus V_{i+1}$, at most $2\eta|V_{i+1}|$ vertices in V_{i+1}, and no vertices in V_{i+2}. By taking $M_{i+1} = M_i \cup M_i'$ we complete the inductive step.

Finally, to obtain a perfect matching, note that if M_N satisfies the properties above then **V4** for $i = N$ and the fact that we only delete a small fraction of V_N means that we have a super-regular remainder, to which Theorem 2.9 applies (with H being a disjoint collection of r-cliques and $R = K_r$ underlying the r-partite structure of $G_1 = G[V(G) \setminus V(M_N)]$ induced by $A_1, \ldots, A_r$). That is, we have a K_r-factor $\widetilde{M}$ of $G[V(G) \setminus V(M_N)]$. Take $M = M_N \cup \widetilde{M}$.

It remains to prove that M is $O(1/n^{r-1})$-spread. Let S be a set of r-cliques. We need to show that $P_S := \mathbb{P}[S \subseteq M] = (O_\eta(1/n^{r-1}))^{|S|}$. Let $T_1, \ldots, T_m$ be an enumeration of the r-cliques in S. For a vector $\vec{x} \in [N+1]^m$, let $P(\vec{x})$ be the probability that for every $j \in [m]$, the clique T_j is in $M_{x_j} \setminus M_{x_j - 1}$ if $x_j \le N$, and $T_j \in \widetilde{M}$ if $x_j = N+1$. We can, essentially identically to the proof of (3.1) within the proof of Theorem 1.5, show that

$$P(\vec{x}) = \left(\prod_{i=1}^{N} \left(O_\eta\left(\frac{|V_{i-1}|}{n^r} \right) \right)^{|\{j\colon x_j = i\}|} \right) \left(\frac{|V_N|}{n} \right)^{r|\{j\colon x_j = N+1\}|}. \tag{5.1}$$

We truncate the details: we use the randomness guarantee **V6** in Lemma 5.3 and the $O_\eta(1/|V_i|^{k-1})$-spreadness of the $M_i' = M_{i+1} \setminus M_i$. Then, similarly, we deduce

$$P_S = \sum_{\vec{x} \in [N+1]^m} P(\vec{x}) = \sum_{\vec{x} \in [N+1]^m} \left(\prod_{i=1}^{N} \left(O_\eta\left(\frac{|V_{i-1}|}{n^k} \right) \right)^{|\{j\colon x_j = i\}|} \right) \left(\frac{|V_N|}{n} \right)^{r|\{j\colon x_j = N+1\}|}$$

$$= \left(\frac{O_\eta(1)}{n^{r-1}}\right)^m \sum_{\vec{x}\in[N+1]^m} \left(\prod_{i=1}^N (\eta^i)^{|\{j\colon x_j=i\}|}\right) \left(\frac{|V_N|^r}{n}\right)^{|\{j\colon x_j=N+1\}|}$$

$$= \left(O_\eta\left(\frac{1}{n^{r-1}}\right)\right)^m \prod_{j=1}^m \left(\sum_{i=1}^N \eta^i + \frac{n^{r/(r+1)}}{n}\right) = \left(O_\eta\left(\frac{1}{n^{r-1}}\right)\right)^m. \qquad \square$$

6. Robust Hajnal–Szemerédi Theorem and counting K_r-factors

We are now in position to use our analysis in order to prove that a version of the robust Hajnal–Szemerédi theorem, Theorem 1.6, holds. We first show that Theorem 1.6 follows from Theorem 1.7, which will be the main focus for the rest of the section.

Proof of Theorem 1.6 from Theorem 1.7. We rely on theorems of Riordan ([43, Theorem 1] for $r \geq 4$ and [43, Theorem 16] for $r = 3$) that couple the random binomial hypergraph with the clique complex of the random binomial graph. The result we need is that there exists a constant $a > 0$ such that for every (fixed) $r \in \mathbb{N}$ and $p \leq \log^2(n)/n^{r-1}$, denoting $q = ap^{1/\binom{r}{2}}$, there exists a coupling of $\mathbb{G}^{(r)}(n,p)$ and $\mathbb{G}(n,q)$ such that w.h.p. $H' \sim \mathbb{G}^{(r)}(n,p)$ is contained in the r-clique complex of $G' \sim \mathbb{G}(n,q)$. (We note that Riordan proves a substantially more precise result when $r \geq 4$ obtaining the optimal constant a for the complete hypergraph; an analogous result for $r = 3$ was proven by [22].)

Denote the r-clique complex of G by $\mathcal{H}$. Theorem 1.7 and Theorem 1.2 imply that for some $C_1 > 0$, denoting $p = C_1 \log n/n^{r-1}$, w.h.p. $\mathcal{H}(p)$ (i.e., the random binomial subgraph of $\mathcal{H}$ with rate p) contains a perfect matching. Riordan's theorems imply that there exists a coupling of $G(ap^{1/\binom{r}{2}})$ and $\mathcal{H}(p)$ such that for $G' \sim G(ap^{1/\binom{r}{2}})$ and $H' \sim \mathcal{H}(p)$ w.h.p. H' is contained in the clique complex of G'. In particular, w.h.p. both H' both contains a perfect matching and is contained in the clique complex of G', which together imply that G' contains a K_r-factor.

For the counting result, note that we have an $O(1/n^{r-1})$-spread measure on the set of K_r-factors, each of which is composed of n/r many r-cliques. Therefore, for some $C > 0$, each factor occurs with probability at most $(C/n^{r-1})^{n/r}$ by the spread condition, so there are at least $(n^{r-1}/C^{r-1})^{n/r} = (n/C)^{(r-1)n/r}$ total factors. $\square$

We define the key property of being somewhat near an extremal structure (complete balanced r-partite graph) in a specific sense.

Definition 6.1. We say that graph G is (r,α)-sparse if there is $A \subseteq G$ with $|A| = \lfloor |V(G)|/r \rfloor$ such that $d_G(A) \leq \alpha$. We say it is α-disconnected if there is a partition $V(G) = A \cup B$ with $|A| = \lfloor |V(G)|/2 \rfloor$ and $d_G(A,B) \leq \alpha$.

6.1. Non-sparse setting.

We next show the result when G is not (r,α)-sparse for appropriate α.

Lemma 6.2. *Let $r|n$ and $\alpha < \alpha_{6.2}(r)$ and $\theta = \theta_{6.2}(r,\alpha,\alpha')$. Let G be an n-vertex graph. If $\delta(G) \geq (r-1)n/r - \theta n$ and G is not (r,α)-sparse, and furthermore if $r = 2$ then G is not α-disconnected, then there is a $C_{6.2}(r,\alpha)/n^{r-1}$-spread distribution on the set of K_r-factors of G.*

The proof is a slight simplification of the proof for triangles presented in [1, Lemma 9.1]; as the details are not as delicate in the non-extremal case we will be brief. The main task of the algorithm is noting that given a regularity partition, one can find a K_r-factor of the reduced graph covering almost all vertices, and the small remainder (and vertices within the regularity partition of exceptional degree) can be handled in a spread manner.

We will first require a version of the Hajnal–Szemerédi Theorem itself [20].

Theorem 6.3. *Let $n, k \geq 2$ be integers and let $0 \leq x < 1$. Suppose that G is an n-vertex graph with $\delta(G) \geq \big(\frac{k-1}{k} - x\big)n$. Then G contains a K_k-matching of size at least $(1 - k(k-1)x)\lfloor \frac{n}{k} \rfloor$.*

The crucial input from [1] is a robust fractional version of the Hajnal–Szemerédi Theorem.

Theorem 6.4 ([1, Theorem 7.4])**.** *Fix $k \geq 2$ and $\eta > 0$. There exists $\gamma = \gamma(k, \eta) > 0$ such that the following holds for all m. Let G be a connected graph on m vertices with $\delta(G) \geq ((k-1)/k - \gamma)m$ and $\alpha(G) < (1/k - \eta)m$. Let $\lambda\colon V(G) \to \mathbb{N}$ be a weight function such that $\lambda(u) = (1 \pm \frac{\gamma}{2})(\frac{1}{m}\sum_{v \in V(G)} \lambda(v))$ and $\lambda(u) \geq m^{2k}$ for all $u \in V(G)$, and k divides $\sum_{v \in V(G)} \lambda(v)$. Then there exists a weight function $\omega\colon K_k(G) \to \mathbb{N} \cup \{0\}$ such that $\sum_{\substack{K \in K_k(G) \\ K \ni u}} \omega(K) = \lambda(u)$ for all $u \in V(G)$.*

We now prove Lemma 6.2.

Proof of Lemma 6.2. Fix a sequence of constants $0 < \frac{1}{m_0} \ll \theta \ll \varepsilon \ll d \ll \alpha \ll 1$ satisfying various constraints throughout the proof. Given a graph G, apply Lemma 2.11 and consider the (ε, d)-reduced graph R which is returned and note that R has $m \in [m_0, M_0]$ vertices. Let the underlying ε-regular partition of $V(G)$ be $V_0 \cup V_1 \cup \cdots \cup V_m$ and note by Lemma 2.11 that $\delta(R) \geq ((r-1)/r - 4d)m$.

We first claim that the independence number of G is suitably large. This is the analogue of [1, Claim 9.4].

Claim 6.5. *We have that $\alpha(R) < (\frac{1}{r} - \alpha^2)m$.*

Proof. Suppose that R has an independent set S of size $(1/r - \alpha^2)m$. Let $S' = \bigcup_{j \in S} V_j \subseteq V(G)$. By the definition of the (ε, d)-reduced graph, we have that $e_G(S') \leq (2\varepsilon + d)n^2$ and $|S'| \geq (1-\varepsilon)(1/r - \alpha^2)n \geq (1/r - 2\alpha^2)n$. Adding an arbitrary set of $n/r - |S'|$ many additional vertices to S', gives a set of size exactly n/r with at most $4\alpha^2 n^2$ edges in G. This contradicts the fact that G is not (r, α)-sparse. $\square$

We next require that R is a connected graph. This is necessary to verify the connectedness assumption which appears within Theorem 6.4; this is the unique place where the assumption that G is not α-disconnected is required.

Claim 6.6. *The graph R is connected.*

Proof. For $r \geq 3$, the claim is immediate as $\delta(R) \geq 3m/5$. For $r = 2$, note that we have that $\delta(R) \geq (1/2 - 4d)m$. Therefore, if R is not connected, then there are at most 2 connected components each of size at least $(1/2 - 4d)m$. Let the connected components of R be S_1 and S_2 and define $S_i' = \bigcup_{j \in S_i} V_j$ for $i \in [2]$. By the definition of an (ε, d)-reduced graph, we have that $e_G(S_1', S_2') \leq (2\varepsilon + d)n^2$ and that $|S_1'|, |S_2'| \geq (1-\varepsilon)(1/2 - 4d)n \geq (1/2 - 5d)n$. This immediately implies, using $|V_0| \leq \varepsilon n$, that $e_G(V_0 \cup S_1', S_2') \leq 5dn^2$ and that $||(V_0 \cup S_1')| - |S_2'|| \leq 11dn$. Rebalancing $(V_0 \cup S_1')$ and S_2' to give an equipartition of $V(G)$, we obtain a contradiction to the fact that G is not α-disconnected. $\square$

We now consider the reduced graph R and the induced partition on the vertex set $V_0 \cup V_1 \cup \cdots \cup V_m$. By applying Claim 6.5 and Theorem 6.3, there exists a partial K_r-factor $\mathcal{T}_r$ of the reduced graph R which cover all but $O_r(dm)$ vertices in R. Let $\mathcal{T}^* = V(\mathcal{T}_r)$ and for each clique $\{i_1, \ldots, i_r\}$ in $\mathcal{T}_r$, we can pass to a subset $V_{i_1}^*, \ldots, V_{i_r}^*$ with $|V_{i_r}^*| = (1 - r\varepsilon)|V_1|$ and $(V_{i_j}^*, V_{i_\ell}^*)$ being $(2\varepsilon, (d-\varepsilon)^+, d - k\varepsilon)$-super-regular (see [1, Lemma 2.9]).

We define $X = V_0 \cup \bigcup_{j \notin \mathcal{T}^*} V_j \cup \bigcup_{i \in [m]} (V_i \setminus V_i^*)$. Note that $|X| \lesssim_r dn$. We now proceed with the following algorithm.

- Order the vertices in X as $\{v_1, \ldots, v_{|X|}\}$ arbitrarily. Define $G_0 = V(G)$; G_i will correspond to the vertex set after the vertex v_i has been matched.
- For each vertex v_ℓ in X, choose a uniformly random clique extending v_ℓ within $G_{\ell-1}$ which does not contain an additional vertex of X. Update G_ℓ to be the vertices in $G_{\ell-1}$ minus the set of vertices in the chosen clique.

We now prove a number of basic properties of the algorithm

Claim 6.7. *The algorithm satisfies the following properties:*

- *The algorithm always runs to completion;*
- *The random set of cliques created by the algorithm is $O_r(1/n^{r-1})$-spread;*
- *For any subset of vertices $S \subseteq V(G) \setminus X$ we have*

$$\mathbb{P}[|S \cap (G_0 \setminus G_{|X|})| \geq \sqrt{d}|S|] \leq \exp(-\Omega_r(\sqrt{d}|S|)).$$

Proof. For the first part, we consider the degree of v_ℓ in the remaining graph. Notice that the degree of v_ℓ is always at least $((r-1)/r)n - r|X| \geq ((r-1)/r - C_r d)n$ for an appropriate constant C_r. Note that any subset of size T of size $(r-1)n/r$ in G has at least

$$\begin{aligned} e(G[T]) &= \frac{1}{2}\sum_{v\in T}\deg_G(v) - \frac{1}{2}|G[T,T^c]| \geq |T|/2\cdot((r-1)n/r - \theta n) - \frac{1}{2}|T|(n/r)\\ &\geq |T|/2\cdot((r-2)n/r - d^2 n) \geq |T|^2/2\cdot((r-2)/(r-1) - d)\end{aligned}$$

edges. This implies that the density of the edge set of the neighbors of X is at least $(r-2)/(r-1) - C_r' d$ and therefore by supersaturation for Turan's theorem there are at least $\Omega_r(n^{r-1})$ possible cliques at each stage. (Notice that the Turan threshold for finding a K_{r-1} is $(r-3)/(r-2)$ which is strictly below the density specified.) This essentially immediately implies the first two statements in the claim.

For the third claim, notice that there are at most $|S|n^{r-2}$ cliques of size r containing an element of S and thus the result follows notice that $|X| \leq \sqrt{d}n/r$ and the binomial domination lemma (Lemma 2.2) noting that at each stage we can remove at most r vertices in S. ☐

By applying Claim 6.7 we have with probability at least $1/2$ that

- $|V_i^* \cap G_{|X|}| \geq (1-\sqrt{d})|V_i^*|$
- For each edge in the clique factor $\mathcal{T}_r$ we have that the corresponding pair of parts formed by $V_i^* \cap G_{|X|}$ and $V_j^* \cap G_{|X|}$ are still $(4\varepsilon, (d/2)^+, d/4)$-super-regular by considering the number of vertices deleted in each part, and the number of neighbors of a given vertex which are deleted, controlled via the third bullet of Claim 6.7.
- For a pair (V_i^*, V_j^*) where each (i,j) appears in R, we have that $(V_i^* \cap G_{|X|}, V_j^* \cap G_{|X|})$ is $(4\varepsilon, (d/2)^+)$-regular. This is immediate by considering the number of vertices deleted from V_i to obtain $V_i^* \cap G_{|X|}$.

For the sake of clarity define $V_i' = V_i^* \cap G_{|X|}$ for $i \in V(\mathcal{T}_r)$. The key issue at this point however is the various parts corresponding to each clique of the K_r-factor in $\mathcal{T}_r$, while relatively close in size, are not balanced appropriately. The next crucial trick is to use Theorem 6.4 in order to remove a certain number of triangles and "rebalance" the part sizes in V_i'. For the sake of simplicity we let R' denote the restriction of the graph R onto the vertices of $V(\mathcal{T}_r)$ in the obvious manner.

In order to apply Theorem 6.4, we define $\lambda(i) = |V_i'| - \lceil n/m(1-d^{1/3})\rceil$. Notice that $\sum_{i\in V(\mathcal{T}_r)}\lambda(i)$ is divisible by r and $\sum_{i\in V(\mathcal{T}_r)}|V_i'|$ is the number of remaining vertices and there are $r|\mathcal{T}_r|$ remaining parts. Furthermore $\lambda(i) \in (d^{1/3} \pm d^{1/4})(n/(10m))$, $\delta(R') \geq ((r-1)/r - C_r d)m$, $\alpha(R') < (1/r - \alpha^2)m$, and $|r|\mathcal{T}_r|| = m(1 \pm O_r(d))$. As $d \ll \alpha$, there exists a weight function on $\omega\colon K_r(R') \to \mathbb{N}$ by Theorem 6.4 such that for all $i \in V(\mathcal{T}_r)$ we have that

$$\sum_{\substack{K\in K_r(R')\\ K\ni i}} \omega(K) = \lambda(i).$$

(Note that strictly speaking one needs to check that the graph on R' is connected, but the proof given in Claim 6.6 is obviously robust to perturbations of the vertex set of R of size $O(dm)$.)

Notice that if we can remove $\omega(K)$ cliques in a disjoint manner from the corresponding r-partite set of regular pairs, then we are left $|V_i'| - \lambda(i) = \lceil n/m(1-d^{1/3})\rceil$ vertices in each part. This immediately gives the desired result, provided the parts left are sufficiently well behaved. The crucial difficulty is guaranteeing the necessary super-regularity at the end of the algorithm; to do so we split $V_i' = V_i'' \cup V_i'''$ where each vertex of V_i' is placed with probability $1/2$ independently at random in V_i'' or V_i'''. Consider the clique K which contains i in $\mathcal{T}_r$; then for all remaining $j \in K$, we have for each vertex $v \in V_j'$ that $\deg(v, V_i'') \geq (d/8)|V_i''|$ by super-regularity. Furthermore we have that $|V_i''| = (1/2 \pm 1/3)|V_i'|$ by the union bound with high probability.

We now proceed with the following algorithm.

- Order the cliques in $K_r(R')$ in an arbitrary manner.
- Given a clique $K \in K_r(R')$ consider the parts V_i''' where $i \in K$. For $\omega(K)$ steps, iteratively remove a random clique with each vertex in V_i''' which does not intersect previously chosen cliques.

We now note that the above algorithm trivially runs to completion and is appropriately spread. To see that the algorithm runs to completion note that we remove at most $\lambda(i)$ vertices from a part V_i''' and hence we have that least (say) $|V_i'''|/2$ choices for the vertex in each part and the counting-lemma guarantees there are $\Omega(n^r)$ choices in each stage. Furthermore the algorithm is trivially sufficiently spread as there are only $O(n)$ rounds and the choice of each clique is uniformly random among a set of size $\Omega(n^r)$ at each step (conditional on the previous choices). Finally pairs for each K are still suitably super-regular as the minimum degree condition is preserved by looking at edges in V_i'' and regularity is preserved as we are left with a constant fraction of each vertex set V_i'. We then take any pair of parts appearing in a clique in $\mathcal{T}_r$, apply Lemma 2.5, and then apply Theorem 1.9 in order to give an $O_{d,\varepsilon}(1/n^{r-1})$-spread factor covering the remaining vertices. This completes the proof. □

6.2. **Reduction to non-sparse setting.** We now prove Theorem 1.7 by reducing to applications of Lemma 6.2 and Theorem 1.9.

Proof of Theorem 1.7. We are given G on $r|n$ vertices $V = V(G)$ with $\delta(G) \geq (r-1)n/r$ and wish to create. The argument is similar to [33, Section 6]. Given r, consider parameters

$$1/n \ll \alpha_1 \ll \alpha_2 \ll \cdots \ll \alpha_r \ll 1/r$$

with appropriate space between each pair.

We iterate over $i \in \{1, \ldots, r\}$ and at each step, if possible, find $A_i \subseteq V \setminus (A_1 \cup \cdots \cup A_{i-1})$ such that $|A_i| = n/r$ and $d_G(A_i) \leq \alpha_i$. Let $m \in \{0, \ldots, r\}$ be the number of steps that successfully go through. Let $A_{\geq 1} = A_1 \cup \cdots \cup A_m$ and $B = V(G) \setminus A$. We write $A_0 = B$ for convenience, and let the resulting partition be denoted $\mathcal{A}$. Note that $m = 0$ corresponds directly to Lemma 6.2, except in the case $r = 2$ where there is a slight difference; in general we will apply Lemma 6.2 with r replaced by $r - m$ to create a K_{r-m}-factor on a set roughly similar to B, then find a K_{m+1}-factor of a corresponding reduced near-partite graph using Theorem 1.9.

Let $\alpha = \alpha_m$ and $\beta = \alpha_{m+1}$, note $\alpha \ll \beta$, and note that $G[A_1], \ldots, G[A_m]$ have densities bounded by α while $G[B]$ is such that every set of size n/r has density at least β. Consider η satisfying $\beta \ll \eta \ll 1/r$ with appropriate space between the pairs. We define a notion of vertices that do not look like they respect the partition, in the sense that their degrees are not what they should be if $G[A_1], \ldots, G[A_m]$ were empty.

Definition 6.8 (Bad and exceptional vertices)**.** Given a partition $\mathcal{P} = P_1 \cup \cdots \cup P_m \cup P_0$ of V with $P_{\geq 1} = P_1 \cup \cdots \cup P_m$ and $|P_i| = n/r$ for $i \in [m]$, we define *bad* and *exceptional* vertices as follows. For $i \in [m]$, let $v \in V$ be called (i, η^*)*-bad wrt* $\mathcal{P}$ if $\deg_G(v, P_i) \geq \eta^*|P_i|$. We say that v is $(0, \eta^*)$*-bad wrt* $\mathcal{P}$ if $\deg_G(v, P_{\geq 1}) \leq (1-\eta^*)|P_{\geq 1}|$. For $j \in [m]$ we say that a vertex $v \in V$ is

j-exceptional wrt $\mathcal{P}$ if $\deg_G(v, P_j) \le \eta^*|P_j|/2$. For $i \in [m]$ we say that $v \in V$ is *$(0, \eta^*)$-exceptional wrt $\mathcal{P}$* if $\deg_G(v, P_0) \le (r - m - 1 + \eta^*/2)n/r$. We will often drop η^* and specification of $\mathcal{P}$ from the notation where it is clear.

For the purpose of the argument we have defined these notions over all of $V(G)$, but we will be most interested in i-bad vertices contained in P_i as well as j-exceptional vertices that are not contained in P_j.

Notice that if v is $(j, 1/8)$-exceptional with respect to $\mathcal{P}$ then almost all of its "degree deficit" (which is at most $n/r - 1$) is used up by the edges between v and P_j, and hence v is nearly complete to P_i for all $i \ne j$ hence is $(i, 1/8)$-bad for all $i \notin \{0, j\}$. Furthermore, if $i = 0$ and $j \ne i$ then by inspection we see v is still $(i, 1/8)$-bad. Thus, if v is $(j, 1/8)$-exceptional then it is $(i, 1/8)$-bad for all $i \ne j$.

Finally, given a partition $\mathcal{P}$ of some vertex set $P_1 \cup \cdots \cup P_m \cup P_0$, we say it is *balanced* if $|P_1| = \cdots = |P_m|$ and $|P_0| = (r - m)|P_1|$. We say a clique K_r is balanced with respect to $\mathcal{P}$ if it has 1 vertex in each P_i for $i \in [m]$, and $r - m$ in P_0.

6.2.1. *Connected case.* We first assume that either $m \ne r - 2$ or if $m = r - 2$ then that $G[B]$ is not α_{m+1}-disconnected. Using $d_G(A_i) \le \alpha$ and say $\alpha \le \eta^8$, we see that there are at most $\alpha^{2/3}n$ many (i, η^2)-bad vertices wrt $\mathcal{A}$ for each $i \in [m]$ by Markov. A similar Markov argument shows there are also at most $\alpha^{2/3}n$ many $(0, \eta^2)$-bad vertices wrt $\mathcal{A}$. Indeed, by the minimum degree condition and $d_G(A_i) \le \alpha$ we see that the number of edges $e_G(A_i, B)$ is at least $(r - m)n^2/r^2 - 2\alpha n^2$ hence

$$e_G(A, B) \ge m(r - m)n^2/r^2 - 2m\alpha n^2.$$

But $|B| = (r - m)n/r$ and $\deg_G(v, A) \le mn/r$, so Markov applied to the quantities $mn/r - \deg_G(v, A)$ over $v \in B$ yields the desired result.

Step 1: Cleaning the partition. We modify $\mathcal{A}$ into a partition $\mathcal{A}'$ slightly in the following manner:

- Initialize with $\mathcal{A}' = \mathcal{A}$.
- If there are distinct $i, j \in \{0, \dots, m\}$ and $v \in A'_i$ which is (i, η^2)-bad wrt $\mathcal{A}$ and $w \in A'_j$ which is $(i, \eta^{1/2})$-exceptional wrt $\mathcal{A}$, we swap the positions of v, w in A'_i, A'_j.
- Continue this operation until there are no possible choices, then terminate.

We claim this actually terminates, and in fact terminates in at most $\alpha^{1/2}n$ steps total. Indeed, note that every step the following quantity strictly decreases, and it starts at size at most $\alpha^{3/5}n$ by the above analysis: the number of $(i, v) \in \{0, \dots, m\} \times V(G)$ such that $v \in A'_i$ and v is (i, η^2)-bad wrt $\mathcal{A}$. This is because at the start of a valid step both v, w contribute to this condition but at the end of it, only v can contribute (and everything else is left unchanged since we are measuring wrt the original partition $\mathcal{A}$). We are using that $(i, 1/8)$-exceptional vertices are $(j, 1/8)$-bad with respect to all other indices $j \ne i$.

Now since $\mathcal{A}'$ differs from $\mathcal{A}$ by few vertices, we actually find that for each $i \in \{0, \dots, m\}$, either there are no $v \in A'_i$ which is (i, η)-bad wrt $\mathcal{A}'$ or there are no (i, η)-exceptional vertices wrt $\mathcal{A}'$ not in A'_i (or both). Furthermore, $d_G(A'_i) \le \alpha^{1/2}$ for all $i \in [m]$, say, and the part sizes of $\mathcal{A}'$ are balanced in the same way as before. The goal at this point is to cover vertices with "balanced" cliques, i.e., ones where there are $r - m$ vertices in A'_0 and 1 in each A'_i for $i \ne 0$. However, the exceptional vertices will require a different treatment.

Step 2: Covering exceptional vertices. Our first goal is to cover all exceptional vertices by r-cliques in a spread manner, leaving a balanced partition behind. The idea is that an (i, η)-exceptional vertex, even though it is in some A'_j, is better utilized if we "swap" it to A'_i. To counterbalance this, something must be "swapped" back. Equivalently, we will cover i-exceptional vertices in A'_j by r-cliques which contain one "extra" vertex in A'_j and one fewer vertex in A_i (compared to a balanced clique), and this will be accompanied by a clique with one extra vertex in A_i' and one fewer vertex in A_j (compared to balanced).

Suppose there are $x_i \neq 0$ many (i,η)-exceptional vertices wrt $\mathcal{A}'$ that are not in A_i'. Since there exist such vertices, the earlier cleaning step shows that there are no $v \in A_i'$ that are (i,η)-bad wrt $\mathcal{A}'$. Now we double-count $e_G(A_i', V \setminus A_i')$. If $i \neq 0$ every exceptional vertex contributes at most $\eta n/2$ edges, and the rest contribute at most n/r edges, so that

$$e_G(A_i', V \setminus A_i') \leq (r-1)n^2/r^2 - x_i(n/r - \eta n/2).$$

On the other hand, each of n/r vertices in A_i' has degree at least $(r-1)n/r$, so the above demonstrates that $e_G(A_i') \geq x_i n/(3r)$, say. A similar argument works for $i = 0$, except that the bound is replaced by $e_G(A_0', V \setminus A_0') \leq m(r-m)n^2/r^2 - x_i(n/r - \eta n/2)$ and thus in fact

$$e_G(A_0', A_0') \geq (r-m)(r-m-1)n^2/r^2 + x_i n/(3r) \geq x_i n/(3r).$$

So either way $G[A_i']$ has at least $x_i n/(3r)$ edges, and since there are no (i,η)-bad vertices in A_i', it has maximum degree at most ηn. It also has n/r vertices for $i \neq 0$ and $(r-m)n/r$ vertices for $i = 0$. Now perform the following random process: sample each edge with probability $6r/n$ and delete edges whose endpoints are included in more than 1 sampled edge, then condition on having at least x_i remaining edges with the property that both endpoints are not (i,η)-bad. Then uniformly at random choose x_i such edges.

The maximum degree condition along with the fact that there are very few i-bad vertices in A_i' demonstrates that the desired event we condition on occurs with probability at least $1/100$, and then we see that the choice of edges is $O(1/n)$-spread. Having done this, for all $i \in \{0,\ldots,m\}$ we now cover all (i,η)-exceptional vertices not in A_i' and all of these chosen edges by r-cliques in a sufficiently spread manner. Notice that the total amount of these is small, say at most $\alpha^{2/5}n$ in total. Furthermore, every (i,η)-exceptional vertex not in A_i' has used up most of its "degree deficit" on A_i', which means that it is nearly complete to $V \setminus A_i'$, and every constructed edge in A_i' has both endpoints not (i,η)-bad hence they similarly are nearly complete to $V \setminus A_i'$. We therefore easily find that the (i,η)-exceptional vertices in A_j' (with $j \neq i$) are contained in $\Omega(n^{r-1})$ many r-cliques with one "extra" vertex in A_j' and one fewer in A_i'; similarly, the constructed edges within A_i' whose endpoints are not (i,η)-bad are contained in $\Omega(n^{r-2})$ many r-cliques with one "extra" vertex within A_i' and one fewer in some prescribed A_j'. (Here we are using the robust counting version of Turán's theorem or similar argumentation.)

Choosing such r-cliques iteratively uniformly at random such that they do not overlap the previously chosen cliques, we can easily argue that the resulting collection of r-cliques is $O(1/n^{r-1})$-spread. Furthermore, since we chose x_i edges within each A_i', the removal of all these r-cliques leaves a resulting (random) partition $\mathcal{A}''$, $V' = A_1'' \cup \cdots \cup A_m'' \cup A_0''$ which is still balanced. Note that we only removed at most say $\alpha^{1/3}n$ vertices total so far.

Step 3: Covering 0-bad vertices. Our second goal is to cover all $(0,\eta)$-bad vertices (wrt $\mathcal{A}'$) that remain in A_0'' (in a spread manner and leaving a balanced partition behind to which we can begin to apply Theorem 1.9). Again, there are very few of them. At this point, since we removed all (i,η)-exceptional vertices (wrt $\mathcal{A}'$) outside A_i' for each i, every such $v \in A_0''$ has degree at least $\eta n/(2r)$ to each $A_1', \ldots, A_m'$ hence degree at least $\eta n/(3r)$ to each $A_1'', \ldots, A_m''$.

We uniformly at random choose vertices in each such neighborhood (that have not yet been chosen), conditional on forming an $(m+1)$-clique with v and conditional on each such vertex in say A_j'' being (j,η)-good wrt $\mathcal{A}'$ (there are at most say $\alpha^{1/3}n$ such bad vertices, but at least $\eta n/(3r)$ choices, and $\alpha \ll \eta$, so this conditioning does not distort the randomness too much). Then the common neighborhood of these m extra vertices within A_0'' is nearly all of A_0'', and we thus easily argue that there are $\Omega_\eta(n^{r-1})$ total choices of balanced clique K_r containing v. (Again, we are using the robust counting version of Turán's theorem.) We can iteratively at random disjointly remove such cliques over all the $(0,\eta)$-bad vertices in A_0'', of which there are at most say $\alpha^{1/3}n$. Similar to before, this can be shown to be $O_\eta(1/n^{r-1})$-spread (conditional on the prior randomness).

Again, we have removed a total of at most $O(\alpha^{1/3}n)$ vertices total, say, and we have a remaining partition $\mathcal{A}'''$, $V'' = A_1''' \cup \cdots \cup A_m''' \cup A_0'''$ which is still balanced. Now we furthermore have no $(0,\eta)$-bad vertices wrt $\mathcal{A}'$ in A_0''' alongside having no (i,η)-exceptional vertices wrt $\mathcal{A}'$ in $V'' \setminus A_i'''$.

Step 4: Covering the remainder with Theorem 1.9. Note that $G[A_0''']$ and $G[A_0]$ are close up to deleting order $O_\alpha(n)$ vertices. Thus, since $\alpha \ll \beta$ this graph satisfies the hypotheses of Lemma 6.2 with r replaced by $r-m$ and α replaced by say $\beta/2$, using also that $\mathcal{A}'''$ is balanced hence $|A_0'''|/(r-m) \in \mathbb{N}$.

Now we perform the following process. By Lemma 6.2 applied to $G[A_0''']$ in the manner above, there is a $O_\beta(1/n^{r-m-1})$-spread distribution on the set of K_{r-m}-factors of $G[A_0''']$. Sample from this distribution. Now, since every vertex here is not $(0,\eta)$-bad wrt $\mathcal{A}'$, every vertex in these $(r-m)$-cliques have degree at least $(1-\eta)|A_{\ge 1}'|$ to $A_{\ge 1}'$. Hence each of these $(r-m)$-cliques have common degree at least $(1-\eta^{1/2})|A_{\ge 1}'''|$ to $A_{\ge 1}'''$. Create an auxiliary graph G' which is $G[A_{\ge 1}''']$ along with a collection of vertices C corresponding to these $(r-m)$-cliques, each clique connected to its common neighbors in $A_{\ge 1}'''$ within G. We see that each pair of parts among the A_i''' have the same size and have density at least $1-\alpha^{1/4}$ say. Furthermore, A_i''' and C have the same size and the density of edges between them is also at least $1-\alpha^{1/4}$ (the average vertex v in A_i''' is missing say $\alpha^{1/3}$-fraction of the crossing edges to A_0''' in G, and therefore the fraction of $(r-m)$-cliques that do not fully connect to v is on average at most say $r\alpha^{1/3}$). Therefore, these are automatically regular pairs with error parameter depending on α. On the other hand, the covering that we have done so far has ensured that between every pair, the minimum degree is at least say $\eta n/(4r)$, and $\alpha \ll \eta$.

Thus we can apply Lemmas 2.5 and 2.8 to obtain super-regular pairs and then use Theorem 1.9 to obtain a $O_\alpha(1/n^m)$-spread distribution on K_{m+1}-factors in G'. This corresponds to a K_r-factor of $G[\bigcup \mathcal{A}''']$, so we have constructed a full K_r-factor. Furthermore, we can easily argue due to the spread of the K_{r-m} factor and then spread of the K_{m+1}-factor in G' that the factor produced at this stage, conditional on the randomness in the previous steps, is $O_\alpha(1/n^{r-1})$-spread. We are done with this case, putting together the various spread steps.

6.2.2. *Disconnected case.* We now consider the only remaining case, that $m = r-2$ and $G[B]$ is β-disconnected (recall $\beta = \alpha_{m+1}$). The argumentation is very similar to the connected case in Section 6.2.1, with two differences. First, since $G[B]$ is β-disconnected, we can essentially break $G[B]$ into two nearly-complete parts which are mostly disconnected. Therefore almost all the balanced cliques we use will have the additional property that they have 2 vertices in one of these parts. However, in the case that these nearly-complete parts both have odd size, we will need to use 1 clique with the property that it has 1 vertex in each of the parts. Beyond this, the pruning of exceptional and 0-bad vertices is essentially the same. We will therefore not repeat said argumentation in detail, merely stating the essential points.

Step 1: Cleaning the partition. We modify $\mathcal{A}$ into $\mathcal{A}'$ similar to the connected case (Section 6.2.1). $\mathcal{A}'$ is still balanced, differs from $\mathcal{A}$ by few vertices (say $\alpha^{1/3}n$), and for each $i \in \{0,\ldots,m\}$ either there are no $v \in A_i'$ which is (i,η)-bad wrt $\mathcal{A}'$ or there are no (i,η)-exceptional vertices wrt $\mathcal{A}'$ not in A_i' (or both). Since $\mathcal{A}$ is close to $\mathcal{A}'$ and $\alpha \ll \beta$, we know that $B' = A_{\ge 1}'$ has the property that $G[B']$ is 2β-disconnected, say.

Step 2: Partitioning $B' = A_0'$. Consider a partition $B' = B_1' \cup B_2'$ constructed as follows: start with the equipartition $B' = B_1^{(0)\prime} \cup B_2^{(0)\prime}$ guaranteed by the 2β-disconnectedness of $G[B']$ (Definition 6.1). Then, for each time $t \ge 1$, we define $B_1^{(t)\prime} \cup B_2^{(t)\prime}$ by swapping a vertex $v \in B_i^{(t-1)\prime}$ for some $i \in \{1,2\}$ such that $\deg_G(v, B_i^{(t-1)\prime}) \le n/(4r)$. If this is no longer possible we terminate and set $(B_1', B_2') = (B_1^{(t-1)\prime}, B_2^{(t-1)\prime})$. Each non-terminating step clearly decreases the cut size by at least $n/(2r)$ each time, and the initial cut size is $e(G[B_1^{(0)\prime}, B_2^{(0)\prime}]) \le \beta(2n/r)^2/2$. Therefore there are at most βn total steps, meaning that the process terminates and furthermore the part sizes are βn-close to the original. In particular, we find $e(G[B_1', B_2']) \le \beta n^2$, say. Additionally, we see that $\delta(G[B_i']) \ge n/(4r)$ for $i \in \{1, 2\}$. We deduce that $e(G[B_i']) \ge (1 - O(\beta))(n/r)^2/2$. So the induced graph in each part B_i' is almost-complete, and we have a reasonable minimum degree condition within each part.

Step 3: Fixing parities. Notice $|B_1'| + |B_2'| = 2(n/r)$ is even. If B_1', B_2' have even size then there is no need to do anything additional and we can move on to the next step. However, if they are both odd size then we wish to make them even in some way. To do this, we find attempt to find (in a spread way) a single K_r which is balanced and has 1 vertex in each B_i'. Let us assume $|B_1'| \ge |B_2'|$.

Since $|B_1'| \ge |B_2'|$ we have $|B_2'| \le n/r$ so $\deg_G(v, B_1') \ge 1$ for each $v \in B_2'$ by the minimum degree condition on G. Choose a uniformly random edge in $G[B_1', B_2']$ such that its vertex in B_2' is not (i, η)-exceptional for any $i \in [r-2]$ and is not $(0, \eta/(4r))$-bad. Notice that by Markov, most of the vertices of B_2' satisfy this property and combining with $\deg_G(v, B_1') \ge 1$ for all $v \in B_2'$ shows that the are at least say $n/(4r)$ choices for this edge, which is thus $O(1/n)$-spread. Call the edge $e_0 = (u_0, v_0)$ where $u_0 \in B_1'$ and $v_0 \in B_2'$. In the case $r > 2$, if u_0 is not (i, η)-exceptional for any $i \ne 0$, then the common neighbors of u_0, v_0 among A_i number at least $\eta n/4$. We easily find many balanced cliques containing this edge in this case, using that almost all vertices (up to error depending only on α) are non-exceptional and good in the relevant senses. We then choose a uniformly random such clique which is $O(1/n^{r-1})$-spread. Now after removing this one K_r, the remaining partition $\mathcal{A}^*$ is balanced and the new B_1^*, B_2^* are even in size. If $r = 2$ then just choosing the spread edge is enough as we do not need to extend it further, and no condition on 0-badness is needed or even meaningful (and we now move to the next step).

When $r > 2$ and the endpoint u_0 is (i, η)-exceptional for some $i \ne 0$, then $\deg_G(v, A_{\ge 1}') \le (r - 3 + \eta/2)n/r$ so $\deg_G(v, B') \ge (2 - \eta)n/r$. We can then move v to B_{3-i}' to create $B' = B_1^* \cup B_2^*$. Now both sizes are even in size again, and the minimum degree in both parts is still at least $n/(4r)$. Now we move on to the next step.

Step 4: Exceptional and 0-bad vertices. At this stage, we cover the exceptional and 0-bad vertices with basically the same arguments as in Section 6.2.1. When covering (i, η)-exceptional vertices not in A_i' when $i \ne 0$, we can run the same procedure and easily ensure that the two (slightly unbalanced) r-cliques used are such that all vertices within B' appear in the same part B_1' or B_2'.

We can do the same for $(0, \eta)$-exceptional vertices, but we must be a bit careful: when we choose the r-cliques with one fewer vertex in B', we cannot necessarily choose where the 1 vertex in B' is; and thus we first choose all of those cliques and then choose the edges within A_0' (which are then extended to cliques with 1 extra vertex in B'). We therefore need to be able to choose up to x_0 (spread) edges not just within $B' = A_0'$ but specifically within B_1' and B_2' as necessary to balance the parity. (They must also have non-bad endpoints.) Here we use that $G[B_1'], G[B_2']$ are almost-complete graphs instead of the argumentation used in **Step 2** of Section 6.2.1, and then choose extension to cliques with 1 extra vertex in A_0' where the third vertex in A_0' is in the same part B_i'. It is not hard to see that this is possible in a spread manner.

When covering 0-bad vertices (now that exceptional vertices are removed), we create an $(r-1)$-clique formed with one vertex from each A_i' for $i \in [r-2]$ and our 0-bad vertex v with various goodness properties on the new vertices, and then extend to a balanced r-clique; we can easily guarantee that the final added vertex (which is in B') is in the same part of B' as v using the $n/(4r)$ minimum degree condition within the parts.

Step 5: Covering the remainder with Theorem 1.9. After doing all this covering, which only deletes around $O_\alpha(n)$ vertices, we can again use Theorem 1.9 (here Lemma 6.2 is not really needed). This time, however, we apply it two times, once to a partition where we take appropriately regular subsets of $A''_1, \dots, A''_{r-2}$ of size $|B''_1|/2$ as well as an appropriately regular bipartition of B''_1, and a second time where we apply to the complements of these subsets and an appropriately regular bipartition of B''_2. All these partitions and subsets can be found randomly, as the necessary host sets already satisfy appropriately super-regularity. The minimum degree conditions follow since $\beta \ll \eta$ and B''_1, B''_2 differ from n/r by an amount depending only on β. □

7. A spread distribution on bounded-degree trees

Fix $\Delta \in \mathbb{N}$ and $\varepsilon > 0$. Let G be an n-vertex graph satisfying $\delta(G) \geq (1/2 + \varepsilon)n$ and let T be an n-vertex tree with maximal degree at most Δ. To prove Theorem 1.10 it suffices to exhibit an $O(1/n)$-spread distribution on the copies of T in G. To do so we closely follow the original proof of Komlós, Sárközy, and Szemerédi [31] that G contains at least one copy of T. We show that if this algorithm is appropriately randomized then the resulting distribution on copies of T is $O_{\Delta,\varepsilon}(1/n)$-spread.

Most of the embedding algorithm consists of embedding vertices of T into G "random greedily" (i.e., the image of each successive vertex is chosen uniformly at random from a set of suitable choices). For this type of algorithm it is natural to analyze "vertex spread", which we now define.

Definition 7.1. Let X and Y be finite sets and let μ be a distribution over injections $\varphi\colon X \to Y$. For $q \in [0,1]$, we say that μ is *q-vertex-spread* if for every two sequences of distinct vertices $x_1, \dots, x_k \in X$ and $y_1, \dots, y_k \in Y$:

$$\mathbb{P}\left[\bigwedge_{i=1}^{k} \varphi(x_i) = y_i\right] \leq q^k.$$

We will prove that a randomized version of the Komlós, Sárközy, and Szemerédi tree embedding algorithm is $O(1/n)$-vertex-spread. The next lemma implies that this is sufficient.

Lemma 7.2. *Let G be an n-vertex graph and let T be an n-vertex tree with maximal degree at most $\Delta \in \mathbb{N}$. Suppose that there exists an $O_{\Delta,\varepsilon}(1/n)$-vertex-spread distribution on graph embeddings φ of T into G. Then φ is an $O_{\Delta,\varepsilon}(1/n)$-spread distribution on copies of T in G.*

Proof. Let $C \geq 1$ be a constant such that φ is (C/n)-vertex-spread. Let $\varphi(E(T))$ denote the (random) set of edges in the embedding of T into G. We will show that for every edge set $S \subseteq E(G)$:

$$\mathbb{P}\left[S \subseteq \varphi(E(T))\right] \leq \left(\frac{\Delta C^2}{n}\right)^{|S|}.$$

This will imply the lemma.

Let $S \subseteq E(G)$. We may assume that S is a non-empty forest. Denote the number of connected components in S by ℓ. Let $V(S)$ be the set of vertices incident to S. We observe that $|V(S)| = |S| + \ell$. We claim that there are at most $n^\ell \Delta^{|S|}$ embeddings of of S into T. Indeed, let $v_1, v_2, \dots, v_{|S|+\ell}$ be an ordering of $V(S)$, where $v_1, \dots, v_\ell$ are in distinct connected components and each of the remaining vertices is incident to a vertex that appeared previously (for instance, one may take a breadth-first ordering of $V(S)$). We will count the number of ways to embed $V(S)$ into T one vertex at a time. There are fewer than n^ℓ ways to embed $v_1, \dots, v_\ell$ into T. Then, each v_i is incident to a previously embedded vertex. Since $\Delta(T) \leq \Delta$ there are at most Δ choices to embed each v_i. Hence, the number of embeddings is at most $n^\ell \Delta^{|S|}$, as claimed.

Now, for a given embedding $\psi\colon V(S) \to V(T)$, by the vertex-spread assumption for φ, we have

$$\mathbb{P}\left[\bigwedge_{i=1}^{|S|+\ell} \varphi(\psi(v_i)) = v_i\right] \leq \left(\frac{C}{n}\right)^{|S|+\ell}.$$

Applying a union bound over the at most $n^\ell \Delta^{|S|}$ choices of ψ we conclude that

$$\mathbb{P}\left[S \subseteq \varphi(E(T))\right] \leq n^\ell \Delta^{|S|} \left(\frac{C}{n}\right)^{|S|+\ell} \leq \left(\frac{\Delta C^2}{n}\right)^{|S|},$$

where the second inequality follows from the fact that $\ell \leq |S|$ and the assumption that $C \geq 1$. This completes the proof. $\square$

The remainder of this section is devoted to proving the next lemma. Together with the previous claim and Theorem 1.2 it implies Theorem 1.10.

Lemma 7.3. *For every $\Delta \in \mathbb{N}$ and $\delta > 0$ there exists some $n_{7.3} = n_{7.3}(\Delta, \delta) > 0$ and $C_{7.3} = C_{7.3}(\Delta, \delta) > 0$ such that for every graph G on $n \geq n_{7.3}$ vertices with $\delta(G) \geq (1/2 + \delta)n$ and every tree T on n vertices with $\Delta(T) \leq \Delta$ there exists a $(C_{7.3}/n)$-vertex-spread distribution on graph embeddings of T into G.*

7.1. **Preliminaries.** In this section we will construct distributions over extensions of graph embeddings in super-regular pairs. We will generally be in the setting where we are attempting to extend a partial embedding of T into G to a larger partial embedding. It is thus helpful to introduce notation for rooted embeddings. Suppose that H and G are graphs, that $R \subseteq V(H)$ is a vertex set, and that $\varphi\colon R \hookrightarrow V(G)$ is an injective partial embedding of H. We denote the set of graph embeddings $\widetilde{\varphi}\colon H \hookrightarrow G$ that extend φ by $X(H, G, R, \varphi)$. We say that a distribution over elements $\widetilde{\varphi} \in X(H, G, R, \varphi)$ is *q-vertex-spread* if the induced distribution over $\widetilde{\varphi}|_{V(H)\setminus R}$ is q-vertex-spread.

A key fact proved in [31] is that one can embed bounded-degree stars in super-regular pairs. The next lemma shows that this can be done in a spread manner. We need the following definition.

Definition 7.4. Given a bipartite graph $G = (A, B, E)$ and a vector $\vec{d} = (d_a\colon a \in A) \in \mathbb{N}^A$ let $\mathcal{S}_{\vec{d}}$ be the graph consisting of the disjoint union of the stars $(S_a\colon a \in A)$, where $S_a = K_{1,d_a}$ for each $a \in A$. Let R_A be the set of the roots of these stars and let $\varphi : R_A \hookrightarrow A$ map the root of S_a to a. A *$\vec{d}$-matching* in G from A to B is an element of $X(\mathcal{S}_{\vec{d}}, G, R_A, \varphi)$.

The next lemma is a randomized version of [31, Lemma 2.1]; the proof is similar to that of Lemma 4.1.

Lemma 7.5. *Let $\delta > 0$ and $\Delta \in \mathbb{N}$ be fixed. Suppose that $G = (A, B, E)$ is a bipartite graph that is $(\delta, \varepsilon, \delta/2)$-super-regular, with $\varepsilon \leq \delta/(10\Delta)$. Suppose that $\vec{d} \in \mathbb{N}^A$ is a vector satisfying $\sum_{a\in A} d_a \leq |B|$ and $\max_{a\in A} d_a \leq \Delta$. There exists some $C = C(\delta, \Delta)$ and a $(C/|B|)$-vertex-spread distribution over $X(\mathcal{S}_{\vec{d}}, G, R_A, \varphi)$.*

Proof. It suffices to show that there exists an $O(1/|B|)$-spread distribution on subgraphs $H \subseteq G$ that w.h.p. contain $\vec{d}$-matchings from A to B (i.e., $X(\mathcal{S}_{\vec{d}}, H, R_A, \varphi) \neq \emptyset$). Indeed, suppose that μ is such a distribution. Let ν be the distribution on $X(\mathcal{S}_{\vec{d}}, G, R_A, \varphi)$ obtained by first sampling $H \sim \mu$, conditioned on H containing a $\vec{d}$-matching from A to B, and then choosing an element of $X(\mathcal{S}_{\vec{d}}, H, R_A, \varphi)$ arbitrarily. Let $v_1, \ldots, v_k \in V(\mathcal{S}_{\vec{d}}) \setminus R_A$ be distinct and $u_1, \ldots, u_k \in B$. We wish to show that

$$\mathbb{P}_{\widetilde{\varphi}\sim\nu}\left[\bigwedge_{i=1}^{k} \widetilde{\varphi}(v_i) = u_i\right] \leq (O(1/|B|))^k.$$

Observe that every v_i has a unique neighbor a_i in $\mathcal{S}_{\vec{d}}$, and that $\widetilde{\varphi}(v_i) = u_i$ only if $\varphi(a_i)u_i \in E(H)$. Hence

$$\mathbb{P}_{\widetilde{\varphi}\sim\nu}\left[\bigwedge_{i=1}^{k} \widetilde{\varphi}(v_i) = u_i\right] \leq \mathbb{P}_{H\sim\mu}\left[\{\varphi(a_1)u_1, \varphi(a_2)u_2, \ldots, \varphi(a_k)u_k\} \subseteq E(H) | X(\mathcal{S}_{\vec{d}}, H, R_A, \varphi) \neq \emptyset\right]$$

$$\leq (1+o(1))\mathbb{P}_{H\sim\mu}\left[\{\varphi(a_1)u_1, \varphi(a_2)u_2, \dots, \varphi(a_k), u_k\} \subseteq E(H)\right].$$

Since μ is $O(1/|B|)$-spread the last quantity is bounded from above by $(O(1/|B|))^k$, as desired.

We now construct the desired distribution on subgraphs $H \subseteq G$. Let D be a large constant (depending only on δ and Δ). Let $H' = G(D/|B|)$ (i.e., the binomial subgraph of G with density $D/|B|$). Let $H'' \subseteq G$ be a random graph constructed as follows: For each $v \in A \cup B$ choose, uniformly at random and independently of all other choices, a set of D edges in G incident to v and add them to H''. Set $H = H' \cup H''$. Clearly, the distribution on H is $O(D/|B|)$-spread. We will show that w.h.p. H contains a $\vec{d}$-matching from A to B.

For $X \subseteq A$, let $d_X \coloneqq \sum_{a\in X} d_a$. It suffices to show that w.h.p. H satisfies the König–Hall criterion

$$\forall X \subseteq A, |N_H(X)| \geq d_X.$$

For notational conciseness set $\varepsilon \coloneqq \delta/(10\Delta)$. We first consider $0 < |X| \leq \varepsilon|B|$. We will bound the probability that for some $Y \subseteq B$ with $|Y| = d_X$ we have $N_{H''}(X) \subseteq Y$. Indeed, given such X and Y, we have

$$\mathbb{P}\left[N_{H''}(X) \subseteq Y\right] \leq \left(\frac{\binom{|Y|}{D}}{\binom{\delta|B|/2}{D}}\right)^{|X|} \leq \left(\frac{2e|Y|}{\delta|B|}\right)^{D|X|} \leq \left(\frac{2e\Delta|X|}{\delta|B|}\right)^{D|X|}.$$

We now apply a union bound over choices of such X and Y to obtain:

$$\sum_{k=1}^{\varepsilon|B|}\binom{|A|}{k}\binom{|B|}{\Delta k}\left(\frac{2e\Delta k}{\delta|B|}\right)^{Dk} \leq \sum_{k=1}^{\varepsilon|B|}\left(\frac{2^D e^{1+\Delta+D}\Delta^{D-\Delta}}{\delta^D} \times \frac{k^{D-\Delta-1}}{|B|^{D-\Delta-1}}\right)^k = o(1).$$

Next, we consider $|X|$ such that $\varepsilon|B| \leq |X|$ and $d_X \leq (1-\varepsilon)|B|$. Let $Y \in \binom{B}{d_X}$ (so, in particular, $|Y| \geq |X| \geq \varepsilon|B|$). Since G is (δ,ε)-regular we have $e_G(X, B\setminus Y) > \delta|X|(|B|-|Y|)/2 \geq \delta\varepsilon^2|B|^2/2$. Therefore

$$\mathbb{P}\left[e_{H'}(X,Y) = 0\right] \leq \left(1 - \frac{D}{|B|}\right)^{\delta\varepsilon^2|B|^2/2} \leq \exp\left(-\frac{D\delta\varepsilon^2}{2}|B|\right).$$

Since there are fewer than $2^{|B|}$ choices for X and Y, applying a union bound, the probability that there exist such X and Y with $e_{H'}(X, B\setminus Y) = 0$ is at most $2^{2|B|}\exp\left(-D\delta\varepsilon^2|B|/2\right)$, which tends to zero provided D is sufficiently large.

Observe that the two cases above complete the proof when $|B| > 2d_A$. Henceforth, we assume that $|B| \leq 2d_A \leq 2\Delta|A|$.

It remains to consider X such that $d_X > (1-\varepsilon)|B|$. This implies $|A\setminus X| \leq \varepsilon|B|$ (since $|B| \geq d_A = d_X + d_{A\setminus X} \geq (1-\varepsilon)|B| + |A\setminus X|$). Let $Y \subseteq B$ satisfy $|Y| = d_X - 1$. Thus $|B\setminus Y| \geq |A\setminus X|$ (since $|B\setminus Y| \geq d_A - |Y| > d_A - d_X = d_{A\setminus X} \geq |A\setminus X|$). Now, if $N_H(X) \subseteq Y$ then $N_H(B\setminus Y) \subseteq A\setminus X$. Hence, there exists a subset of B of size $|B| - d_X + 1 \leq 2\varepsilon|B|$, all of whose neighbors are contained in a smaller set. We use a union bound to show that w.h.p. there is no such set:

$$\sum_{k=1}^{2\varepsilon|B|}\binom{|B|}{k}\binom{|A|}{k}\left(\frac{\binom{k}{D}}{\binom{\delta|A|/2}{D}}\right)^k \leq \sum_{k=1}^{2\varepsilon|B|}\left(\frac{e^2 2^{D+1}\Delta}{\delta^D} \times \frac{k^{D-2}}{|A|^{D-2}}\right)^k = o(1).$$

Thus, w.h.p. H satisfies the König–Hall condition and contains a $\vec{d}$-matching from A to B. $\square$

A second key claim that is proved in [31] allows the embedding of forests of length-3 paths into super-regular pairs. We use the following definitions.

Definition 7.6. A *four-layer (d,ε)-super-regular graph* is a graph $G = (V, E)$ with a vertex partition $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where all parts are the same size, and for every $i = 1, 2, 3$ the induced bipartite graph $G[V_i, V_{i+1}]$ is (d^+, ε)-super-regular.

For $m \in \mathbb{N}$ let $\mathcal{P}_m$ be the graph consisting of m vertex disjoint copies of length-3 paths. Denote by $V_{\text{out}}(\mathcal{P}_m)$ the set of $2m$ leaves in $\mathcal{P}_m$ and denote by $V_{\text{in}}(\mathcal{P}_m)$ the set of $2m$ inner vertices in $\mathcal{P}_m$.

If $\pi\colon V_1 \to V_4$ is a bijection then $\varphi\colon V_{\text{out}}(\mathcal{P}_{|V_1|}) \to V_1 \cup V_4$ is π-*respecting* if for every $v \in V_1$ both $\varphi^{-1}(v)$ and $\varphi^{-1}(\pi(v))$ belong to the same path in $\mathcal{P}_{|V_1|}$.

The next lemma is a randomized version of [31, Theorem 2.1].

Lemma 7.7. *For every $d > 0$ there exist $C = C_{7.7}(d), \varepsilon = \varepsilon_{7.7}(d), n_{7.7}(d) > 0$ such that the following holds for all $n \geq n_{7.7}$. If G is a four-layer (d, ε)-super-regular graph on $4n$ vertices then for any bijection $\pi : V_1 \to V_4$ and any π-respecting bijection $\varphi\colon V_{\text{out}}(\mathcal{P}_n) \to V_1 \cup V_4$ there exists a (C/n)-vertex-spread distribution on $X(\mathcal{P}_n, G, V_{\text{out}}(\mathcal{P}_n), \varphi)$.*

Proof. We prove the lemma by applying Theorem 1.9 to the auxiliary graph G' which is obtained by identifying V_1 and V_4 according to π. In other words, the vertex set of G' is $V_1 \cup V_2 \cup V_3$ and the edge set consists of $G[V_1, V_2] \cup G[V_2, V_3]$ and $\{xy \in V_1 \times V_3\colon \pi(x)y \in G[V_4, V_3]\}$. As long as ε is sufficiently small and n is sufficiently large then G' satisfies the assumptions of Theorem 1.9. Hence, for some $C > 1$ there exists a (Cn^{-2})-spread distribution μ on perfect matchings in the 3-clique complex $\mathcal{H}$ of G'.

Observe that there is a natural correspondance between elements $X(\mathcal{P}_n, G, V_{\text{out}}(\mathcal{P}_n), \varphi)$ and perfect matchings in $\mathcal{H}$. Explicitly, the embedding $\widetilde{\varphi} \in X(\mathcal{P}_n, G, V_{\text{out}}(\mathcal{P}_n), \varphi)$ corresponds to the perfect matching $M \subseteq \mathcal{H}$ consisting of all triples $v_1v_2v_3 \in \mathcal{H}$ such that $v_1v_2v_3\pi(v_1)$ is the image under $\widetilde{\varphi}$ of a path in $\mathcal{P}_n$. Hence, μ induces a distribution ν on $X(\mathcal{P}_n, G, V_{\text{out}}(\mathcal{P}_n), \varphi)$. We will show that ν is (C/n)-vertex-spread.

Let $v_1, \dots, v_k \in V_{\text{in}}(\mathcal{P}_n)$ be a sequence of distinct vertices and let $u_1, \dots, u_k \in V_2 \cup V_3$. Let $M \sim \mu$ and let $\widetilde{\varphi} \in X(\mathcal{P}_n, G, V_{\text{out}}(\mathcal{P}_n), \varphi)$ be the corresponding embedding. We wish to show that

$$\mathbb{P}\left[\bigwedge_{i=1}^{k} \widetilde{\varphi}(v_i) = u_i\right] \leq (C/n)^k.$$

Observe that every vertex $v \in V_{\text{in}}(\mathcal{P}_n)$ has a unique neighbor $v' \in V_{\text{in}}(\mathcal{P}_n)$ and these vertices are connected to a unique pair in $V_{\text{out}}(\mathcal{P}_n)$. Hence, specifying the image of v and v' is equivalent to prescribing that a specific triangle appear in M. Let a be the number of vertices in $v \in \{v_1, \dots, v_k\}$ such that $v' \in \{v_1, \dots, v_k\}$. For each of the remaining $k - a$ vertices v there are at most n ways to embed v'. Using the spread of μ, for each such choice, the probability that all the corresponding triangles will be in M is at most $(Cn^{-2})^{a/2+k-a}$. Applying a union bound we conclude that

$$\mathbb{P}\left[\bigwedge_{i=1}^{k} \widetilde{\varphi}(v_i) = u_i\right] \leq n^{k-a}\left(\frac{C}{n^2}\right)^{a/2+k-a} \leq \left(\frac{C}{n}\right)^k. \qquad \square$$

Before stating the next lemma we introduce some notation. Suppose that φ is a partial embedding of a graph H into a graph G, defined on $D \subseteq V(H)$. We call the vertices in $V(G) \setminus \varphi(D)$ *unoccupied by* φ. For $v \in V(H) \setminus D$ we write $A(\varphi, v, H, G)$ for the set of vertices $u \in V(G)$ that are unoccupied by φ and that are adjacent to the images of all embedded neighbors of v (formally, for every $w \in D$ such that $wv \in E(H)$ it holds $u\varphi(w) \in E(G)$). This is the set of *available* locations for v. If H and G are bipartite (as will always be the case for us) we additionally restrict vertices so that they respect the bipartition.

The next lemma, which is closely related to the blow-up lemma, allows one to extend an embedding of a graph within a regular pair in a spread manner. It also allows the designation of "buffers" which are either target locations for certain vertices or sets that should be left mostly unoccupied.

Lemma 7.8. *Let $\Delta \in \mathbb{N}$ and $\alpha, d \in (0, 1)$. There exists some $\varepsilon = \varepsilon(\Delta, \alpha, d) > 0$ such that for every $k \in \mathbb{N}$ the following holds for every $n \geq n_0 = n_0(k, \Delta, \alpha, d)$. Suppose that $G = (A, B, E)$ is a (d^+, ε)-regular pair with $2n \geq |A|, |B| \geq n$ and that $H = (C, D, E_H)$ is a bipartite graph satisfying*

- *every connected component of H has size at most $\varepsilon^2 n$,*
- $|C| \le (1-\alpha)|A|$ *and* $|D| \le (1-\alpha)|B|$, *and*
- $\Delta(H) \le \Delta$.

Suppose further that $F \subseteq C \cup D$ and $B_1, B_2 \subseteq A \cup B$ satisfy

- $|F \cap C| = |B_1 \cap A|$ *and* $|F \cap D| = |B_1 \cap B|$.
- $|B_2 \cap A| = |A| - |C|$ *and* $|B_2 \cap B| = |B| - |D|$.
- $B_1 \cap B_2 = \emptyset$.

Finally, suppose that for a set $S \subseteq V(H)$ of size at most k there is a graph embedding $\varphi_0 \colon H[S] \hookrightarrow G$ such that for every $v \in V(H) \backslash S$ it holds $|A(\varphi_0, v, H, G)| \ge (d/2)^\Delta n$. There exists an $O(1/n)$-vertex-spread distribution on graph embeddings $\varphi \in X(H, G, S, \varphi_0)$ such that

- *At most $\varepsilon^{1/3} n$ vertices of B_2 are occupied by φ, and*
- *At most $\varepsilon^{1/3} n$ vertices of F are not mapped by φ to B_1.*

Proof. We describe a randomized embedding procedure that embeds the vertices of $V(H) \setminus S$ one by one, where the image of each vertex is chosen uniformly at random from a set of size $\Omega(n)$. This guarantees the vertex-spread. Additionally, we design the algorithm in such a way that vertices from B_2 are chosen at most $\varepsilon^{1/3} n$ times, and vertices from F will fail to be mapped to B_1 at most $\varepsilon^{1/3} n$ times. The lemma follows by considering the corresponding distribution over embeddings.

Naively, one might try a simple random greedy algorithm, where each successive vertex is mapped to a uniformly random available vertex, all while avoiding B_2. However, special care is needed to handle the buffers. A particular concern is that since (for example) $|A \setminus B_2| = |C|$, if we have embedded nearly all vertices in C then the remaining set of unoccupied vertices in $A \setminus B_2$ might be so small that regularity fails, in which case the embedding algorithm might get stuck.

To circumvent this we set aside additional buffer zones within B_2 which we allow ourselves to use in the embedding procedure. These are large enough to guarantee that regularity never fails but still small enough that almost all vertices in B_2 remain unoccupied. We define two buffer zones, as follows. Let Z_1 and Z_2 be disjoint subsets of B_2, each consisting of $\sqrt{\varepsilon} n$ vertices in each of $B_2 \cap A$ and $B_2 \cap B$ (so that $|Z_1 \cap A| = |Z_1 \cap B| = |Z_2 \cap A| = |Z_2 \cap B| = \sqrt{\varepsilon} n$).

For each vertex we will now define a target set where it will be embedded. Let $N(S)$ denote the set of neighbors of S. For $v \in N(S)$ let $X(v) \subseteq A(\varphi_0, v, H, G) \setminus (Z_1 \cup Z_2)$ be a set of size $\sqrt{\varepsilon}(d/2)^\Delta n/(2\Delta k)$ such that:

- For distinct $u, v \in N(S)$ the sets $X(u)$ and $X(v)$ are mutually disjoint;
- For $u, v \in N(S)$ that lie in different sides of the partition of H the pair $(X(u), X(v))$ is a $((4d/5)^+, \sqrt{\varepsilon})$-regular pair in G; and
- for every $v \in N(S)$, if (W_1, W_2) is the ordering of A, B such that $X(v) \subseteq W_1$, then $(X(v), W_2)$ is a $((d/2)^+, \varepsilon^{3/4})$-regular pair.

We note that such a choice of set $\{X(v)\}_{v \in N(S)}$ is possible since the regularity properties are satisfied w.h.p. by choosing uniformly random disjoint sets of the appropriate size. We write $Z_3 \coloneqq \bigcup_{v \in N(S)} X(v)$. Observe that $|Z_3| = |N(S)| \sqrt{\varepsilon}(d/2)^\Delta n/(2\Delta k) \le \sqrt{\varepsilon} n/2$.

Now, for $v \in V(H)$ let $W(v) = A$ if $v \in C$ and let $W(v) = B$ if $v \in D$. For $v \in F \setminus (S \cup N(S))$ set its target set as $X(v) \coloneqq W(v) \cap ((B_1 \setminus Z_3) \cup Z_1)$. Finally, for $v \in V(H) \setminus (S \cup N(S) \cup F)$ set $X(v) \coloneqq W(v) \cap (V(G) \setminus (B_1 \cup B_2 \cup Z_3) \cup Z_2)$.

The upshot of choosing the sets $X(v)$ in this way that is that if (as will indeed be the case) we succeed in embedding every v into $X(v)$ then vertices in F will fail to be embedded into B_1 at most $\varepsilon^{1/3} n$ times. This is because such a vertex is either in $N(S)$ or else, if it is not embedded into B_1, it must have been embedded into Z_1. Since $|Z_1| + |N(S)| \le 2\sqrt{\varepsilon} n + k\Delta \le \varepsilon^{1/3} n$ this is an upper bound on the number of "wrongly embedded" vertices in B_1. Similarly, vertices in B_2 will be used at most $|Z_1| + |Z_2| + |Z_3| \le 5\sqrt{\varepsilon} n \le \varepsilon^{1/3} n$ times. Additionally, for as long as we embed vertices only into their target sets, for $v \in V(H) \setminus (S \cup N(S))$ there will always be at least $\sqrt{\varepsilon}n$ unoccupied vertices in its target set. This is ensured by the excess vertices added by the buffers: each vertex in $X(v)$ appears in at most $|X(v)| - \sqrt{\varepsilon}n$ other target sets.

We observe that for every $v \in V(G) \setminus S$ it holds $|X(v)| \geq \sqrt{\varepsilon}(d/2)^{\Delta}n/2 = \Omega_{d,k,\Delta}(n)$ and, furthermore, if $v \notin N(S)$ then $|X(v)| \geq \min\{\alpha, 1-\alpha\}n$. Additionally, if $u, v \in V(H)$ lie in different sides of the partition of H then the pair $(X(u), X(v))$ is $((3d/5)^+, \varepsilon^{3/5})$-regular.

The embedding algorithm is as follows.

- Set $\varphi = \varphi_0$.
- Let $v_1, v_2, \ldots, v_{|V(H)|-|S|}$ be an ordering of $V(H) \setminus S$ where the vertices lying in each connected component of H appear as a contiguous interval of vertices.
- For each $i = 1, \ldots, |V(H)| - |S|$:
  - Let $N_i \subseteq V(H)$ be the set of neighbors of v_i that have not already been embedded.
  - Let X_i be the set of vertices $v \in A(\varphi, v_i, H, G) \cap X(v_i)$ that, for each $u \in N_i$, satisfy $\deg_G(v, A(\varphi, u, H, G) \cap X(u)) \geq \frac{d}{2}|A(\varphi, u, H, G) \cap X(u)|$.
  - Choose some $v \in X_i$ uniformly at random and update $\varphi(v_i) = v$.

We claim that the algorithm is guaranteed to succeed and that the set X_i always has size at least $\varepsilon^2 n = \Omega_{\Delta,d,\alpha}(n)$. Indeed, we note that at step i of the algorithm, if each of the sets $A(\varphi, v_i, H, G) \cap X(v_i)$ and $\{A(\varphi, u, H, G) \cap X(u)\}_{u \in N_i}$ contains at least a $(\Delta+1)\varepsilon^{3/5}$-fraction of the vertices in their respective target sets then by regularity, for every $u \in N_i$, the inequality $\deg_G(v, A(\varphi, u, H, G) \cap X(u)) \geq \frac{d}{2}|A(\varphi, u, H, G) \cap X(u)|$ holds for all but at most $\varepsilon^{3/5}|X(v_i)|$ vertices in $A(\varphi, v_i, H, G) \cap X(v_i)$. Hence, if this is the case, then $|X_i| \geq (\Delta+1)\varepsilon^{3/5}|X(v_i)| - \Delta\varepsilon^{3/5}|X(v_i)| = \varepsilon^{3/5}|X(v_i)| \geq \varepsilon^2 n$.

We now show that as long as vertex v has not been embedded the set $A(\varphi, v, H, G) \cap X(v)$ indeed contains at least a $(\Delta+1)\varepsilon^{3/5}$-fraction of $X(v)$. We consider two cases. First, if $v \in N(S)$ then $X(v)$ is disjoint from all other target sets. Thus, it decreases only when a neighbor of v is embedded, in which case (by the algorithm's design) it decreases by a factor f with $f \geq d/2$. Hence, since v has at most Δ neighbors the number of available locations is always at least $(d/2)^{\Delta}|X(v)| \geq (\Delta+1)\varepsilon^{3/5}|X(v)|$, where the inequality holds provided ε is sufficiently small.

In the second case $v \in V(H) \setminus (S \cup N(S))$. In this case the number of locations can decrease in two ways: in the first, a neighbor of v is embedded, in which case the number of available locations can decrease multiplicatively by a factor of f with $f \geq d/2$. As before, this can happen at most Δ times. In the second, the set of available locations can decrease if some vertex in $X(v)$ becomes occupied, in which case it decreases by 1. However, as long as no neighbor of v is embedded the number of available locations is precisely the number of unoccupied vertices in $X(v)$ which is of size at least $\sqrt{\varepsilon}n$ (this is the excess space guaranteed by the buffers). Recall that the embedding is done connected component by connected component and that each connected component has size at most $\varepsilon^2 n$. Hence, after the first neighbor of v is embedded at most $\varepsilon^2 n$ additional vertices in $X(v)$ become occupied before v itself is embedded. Thus the number of available locations for v is at least $(\sqrt{\varepsilon} - \varepsilon^2)n(d/2)^{\Delta} \geq (\Delta+1)\varepsilon^{3/5}|X(v)|$, where the last inequality holds provided ε is sufficiently small in terms of d and α. □

7.2. **A randomized tree embedding algorithm.** We now describe our adaptation of the tree embedding algorithm in [31, Section 5.5]. That algorithm has eight steps. Steps 1–6 are "preprocessing" of the tree T and the graph G, and we make no changes to these steps. The output of these steps is a regularization of G together with an assignment of $V(T)$ to regularized clusters that determine into which cluster each tree vertex will be embedded. We do not give details of this construction; instead, we simply list its salient properties in Claim 7.9, below. For additional details we refer the reader to [31].

The actual embedding is carried out in steps 7 and 8. One approach to prove Lemma 7.3 would be to randomize the embedding strategy and note that it is $O(1/n)$-vertex-spread. Indeed, Step 7 consists of a random greedy algorithm, where vertices of T are embedded greedily into G one at a time, where for each vertex there are $\Omega_{\Delta,\varepsilon}(n)$ suitable choices. By choosing the embedding random-greedily it becomes appropriately spread. In Step 8 vertices are embedded by employing derandomized versions of either Lemma 7.5 or Lemma 7.7; if these are replaced by the appropriate randomized counterpart the embedding strategy as a whole regains the necessary spread.

We do not follow the embedding strategy in [31] exactly. This is mostly for organizational purposes, as well as to avoid duplicating large parts of [31] verbatim. Instead, we first embed a small number of vertices that serve as "bridges" between super-regular pairs in the regularized graph. We then embed the remainder of the tree into the super-regular pairs.

We begin by describing the outcome of the preprocessing steps in [31]. Following [31], we will make T rooted by fixing an arbitrary root. Any subgraph of T can then be viewed as a rooted forest. We define a *secondary leaf* in a forest as a non-leaf vertex all of whose children are leaves.

Claim 7.9. *In the setting of Lemma 7.3, for every $\varepsilon > 0$ there exists some $M = M(\varepsilon) > 0$, $\alpha = \alpha(\Delta) > 0$, such that if we fix any vertex $r \in V(T)$ as a root then there exists a decomposition of G into clusters $\mathcal{C}$ with the following properties.*

(1) $|\mathcal{C}| \leq M$.
(2) For every $C \in \mathcal{C}$ it holds $2n/M \geq |C| \geq n/(2M)$.
(3) There exists a perfect matching $\mathcal{M}$ of the clusters in $\mathcal{C}$ such that every pair in $\mathcal{M}$ is $((\delta/2)^+, \varepsilon)$-super-regular. For $C \in \mathcal{C}$ we denote its match by C'.

There also exists an assignment $a\colon V(T) \to \mathcal{C}$, a set $S \subseteq V(T)$, and a constant $K = K(\Delta, \delta, \varepsilon)$ with the following properties:

(1) $|S| \leq K$.
(2) For every $C \in \mathcal{C}$ it holds $|a^{-1}(C)| = |C|$.
(3) For every edge $uv \in E(T)$, the pair $(a(u), a(v))$ is ε-regular with density at least $\delta/2$.
(4) For every edge $uv \in E(T)$, if $(a(u), a(v)) \notin \mathcal{M}$ then $u, v \in S$.

Finally, for every cluster pair $(C, C') \in \mathcal{M}$, let $F_{C,C'} = T[a^{-1}(C), a^{-1}(C')]$ be the subforest of T that is spanned by the vertices assigned to C and C'. Then each connected component of $F_{C,C'}$ has size at most $\varepsilon^2 n$. Additionally, there exist sets $F^1_{C,C'}, F^2_{C,C'} \subseteq V(F_{C,C'}) \setminus S$ such that one of the following holds:

(1) $F^2_{C,C'}$ consists of $\alpha|C|$ leaves of $F_{C,C'}$, equally divided between $a^{-1}(C)$ and $a^{-1}(C')$, and $F^1_{C,C'}$ consists of the parents (within $F_{C,C'}$) of $F^2_{C,C'}$.
(2) $F^2_{C,C'}$ consists of $\alpha|C|$ secondary leaves in C and their children, and $F^1_{C,C'}$ consists of the parents of the secondary leaves (all within $F_{C,C'}$).
(3) $F^2_{C,C'}$ consists of $\alpha|C|$ secondary leaves in C' and their children, and $F^1_{C,C'}$ consists of the parents of the secondary leaves (all within $F_{C,C'}$).
(4) For a set of $\alpha|C|$ vertex-disjoint length-3 paths in $F_{C,C'}$ in which the internal vertices all have degree 2, $F^2_{C,C'}$ consists of the $2\alpha|C|$ internal vertices and $F^1_{C,C'}$ consists of the $2\alpha|C|$ endpoints.

We observe that in the setup above, for every $v \in V(T) \setminus S$, if $a(v) = C$ then all neighbors of v are assigned to C'.

We are ready to describe the randomized embedding procedure.

Proof of Lemma 7.3. Take the output of Claim 7.9 with respect to ε sufficiently small so that Lemma 7.8 can be applied to the super-regular pairs (provided n is sufficiently large).

We initialize a partial embedding φ whose domain is the empty set. We call a vertex of G *occupied* if a vertex of T has been assigned to it (otherwise it is *unoccupied*).

We begin by embedding the vertices of S. For this we will use a random greedy algorithm. The remaining vertices of T will be embedded using Lemmas 7.5, 7.7, and 7.8.

Let $s_1, \dots, s_k$ be an ordering of the vertices of S where each s_i is incident (in T) to at most one vertex that precedes it in the ordering. (Such an ordering is possible since T is a tree.) Iterating through $i = 1, \dots, k$, let $A_i \subseteq a(s_i)$ be the set of vertices that are adjacent to all $\varphi(u)$ for all neighbors of s_i that precede it in the order (of which there is at most one). Then, let $B_i \subseteq A_i$ be the set of vertices $v \in A_i$ such that $\deg_G(v, a(u)) \geq \delta|a(u)|/3$ for every $u \in V(T)$ that is adjacent to s_i. Choose some $v \in B_i$ uniformly at random, and set $\varphi(s_i) = v$.

Using the fact that in this stage we embed only $O(1)$ vertices and the regularity properties of the decomposition, there are always at least $\delta n/(8M)$ choices for each embedding. Hence $\varphi|_S$ is $8M/(\delta n) = O(1/n)$-vertex-spread.

We now embed the remaining vertices of T. We do this separately for each cluster pair $(C, C') \in \mathcal{M}$. Let $(C, C') \in \mathcal{M}$. We will use Lemma 7.8 to embed the vertices in $F_{C,C'}$ besides those in $F^2_{C,C'}$, and then use either Lemma 7.5 or Lemma 7.7 to embed $F^2_{C,C'}$. Before applying Lemma 7.8 we set aside buffer zones in (C, C'). For $i = 1, 2$, let $B_i \subseteq C \setminus S$ have size $|F^i_{C,C'} \cap a^{-1}(C)|$. Similarly, let $B'_i \subseteq C' \setminus S$ have size $|F^i_{C,C'} \cap a^{-1}(C')|$. We choose these sets in such a way that for all $i, j = 1, 2$, the pairs (B_i, B'_j) are $((\delta/3)^+, 2\varepsilon)$-super-regular. We also choose the buffer zones so that they are mutually disjoint. (To see that this is possible note that if appropriately-sized disjoint sets are chosen uniformly at random then w.h.p. they satisfy the super-regularity.)

We now apply Lemma 7.8 to extend φ to a partial embedding that embeds $F_{C,C'} \setminus F^2_{C,C'}$ such that:

(1) All but $10\varepsilon|C|$ vertices of $F^1_{C,C'}$ are embedded to $B_1 \cup B'_1$.
(2) The set of unoccupied vertices differs from $B_2 \cup B'_2$ by at most $10\varepsilon|C|$.

Let $D_1 = \varphi(F^1_{C,C'}) \cap C$ and $D'_1 = \varphi(F^1_{C,C'}) \cap C'$. Let D_2 and D'_2 be the set of unoccupied vertices in C and C' respectively. Observe that for every $i, j \in \{1, 2\}$ the pair (D_i, D'_j) is $((\delta/10)^+, 20\varepsilon)$-super-regular. It remains to extend φ so that it embeds $F^2_{C,C'}$ into $D_2 \cup D'_2$. We consider three cases, depending on how $F^2_{C,C'}$ was constructed.

In the first case $F^2_{C,C'}$ is a set of leaves, evenly divided between "even" leaves in $a^{-1}(C)$ and "odd" leaves in $a^{-1}(C')$. We apply Lemma 7.5 twice, first to match the even leaves to their parents (which are already embedded) and then to match the odd leaves to their parents.

In the second and third cases $F^2_{C,C'}$ consists of $\alpha|C|$ secondary leaves (all in either $a^{-1}(C)$ or $a^{-1}(C')$) and their children, and $F^1_{C,C'}$ consists of the secondary leaves' parents. We again apply Lemma 7.5 twice: first to embed the secondary leaves and then to embed their children.

Finally, in the fourth case, $F^2_{C,C'}$ consists of the internal vertices of $\alpha|C|$ vertex-disjoint paths (in which all vertices in $F^2_{C,C'}$ have degree 2). We apply Lemma 7.7 to embed the desired length-3 paths.

In all cases, Lemmas 7.5 and 7.7 ensure that the embedding is completed in an $O(1/n)$-vertex spread manner, as desired. □

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