

PATHS OF GIVEN LENGTH IN TOURNAMENTS

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ABSTRACT. We prove that every n -vertex tournament has at most $n \left(\frac{n-1}{2}\right)^k$ walks of length k .

We determine the maximum density of directed k -edge paths in an n -vertex tournament. Our focus is on the case of fixed k and large n . The expected number of directed k -edge paths in a uniform random n -vertex tournament is $n(n-1)\cdots(n-k)/2^k = (1+o(1))n(n/2)^k$. In this short note we show that one cannot do better, thereby confirming an unpublished conjecture of Jacob Fox, Hao Huang, and Choongbum Lee. The *length* of a path or walk refers to its number of edges.

Theorem 1. *Every n -vertex tournament has at most $n \left(\frac{n-1}{2}\right)^k$ walks of length k .*

Every regular tournament (with odd n) has exactly $n \left(\frac{n-1}{2}\right)^k$ walks of length k , thereby attaining the upper bound in the theorem. On the other hand, the transitive tournament minimizes the number of k -edge paths (or walks) among n -vertex tournaments. Indeed, a folklore result (with an easy induction proof) says that every tournament contains a directed Hamilton path. So every $(k+1)$ -vertex subset contains a path of length k . Hence every n -vertex tournament contains at least $\binom{n}{k+1}$ paths of length k , with equality for a transitive tournament.

Here is a “proof by picture” of Theorem 1. A more detailed proof is given later. A different proof, using entropy, by Dingding Dong and Tomasz Ślusarczyk, is given in the appendix.

$$\begin{aligned}
 (\bullet \rightarrow \bullet)^{-1} \left(\overbrace{\bullet \rightarrow \cdots \rightarrow \bullet}^{k \text{ edges}} \right)^2 &\leq \overbrace{\begin{array}{c} \bullet \rightarrow \cdots \rightarrow \bullet \\ \bullet \rightarrow \cdots \rightarrow \bullet \end{array}}^{k-1 \text{ edges}} \quad \text{Cauchy-Schwarz} \\
 &\leq \overbrace{\begin{array}{c} \bullet \rightarrow \cdots \rightarrow \bullet \\ \bullet \rightarrow \cdots \rightarrow \bullet \end{array}}^{k-2 \text{ edges}} \quad 2ab \leq a^2 + b^2 \\
 &\leq \overbrace{\begin{array}{c} \bullet \rightarrow \cdots \rightarrow \bullet \\ \bullet \rightarrow \cdots \rightarrow \bullet \end{array}}^{k-2 \text{ edges}} \cdot \left(\frac{n-1}{2}\right)^2 \quad d^+(x)d^-(x) \leq \left(\frac{n-1}{2}\right)^2 \\
 &\dots \text{Iterating} \dots \\
 &\leq \bullet \rightarrow \bullet \cdot \left(\frac{n-1}{2}\right)^{2k-2}.
 \end{aligned}$$

Let us mention some related problems and results. The most famous open problem with this theme is Sidorenko’s conjecture [7, 10], which says that for a fixed bipartite graph H , among graphs of a given density, quasirandom graphs minimize H -density. For recent progress on Sidorenko’s conjecture see [5, 6].

Zhao and Zhou [13] determined all directed graphs that have constant density in all tournaments; they are all disjoint unions of trees that are each constructed in a recursive manner, as conjectured by Fox, Huang, and Lee. As discussed at the end of [13], it would be interesting to characterize directed graphs H where is the H -density in tournaments maximized by the quasirandom tournament (such H is called *negative*), and likewise when “maximized” is replaced by “minimized” (such H is called *positive*). Our main result here implies that all directed paths are negative. It would be interesting

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to see what happens for other edge-orientations of a path. Starting with a negative (resp. positive) digraph, one can apply the same recursive construction as in [13] to produce additional negative (resp. positive) digraphs, namely by taking two disjoint copies of the digraph and adding a single edge joining a pair of twin vertices.

The problem of maximizing the number of directed k -cycles in a tournament is also interesting and not completely understood. Recently, Grzesik, Král', Lovász, and Volec [9] showed that quasirandom tournaments maximize the number of directed k -cycles whenever k is not divisible by 4. On the other hand, when k is divisible by 4, quasirandom tournaments do not maximize the density of directed k -cycles. The maximum directed k -cycle density is known for $k = 4$ [3, 4] and $k = 8$ [9] but open for all larger multiples of 4. See [9] for discussion.

A related problem is determining the maximum number $P(n)$ of Hamilton paths in a tournament (the problem for Hamilton cycles is related). By considering the expected number of Hamilton paths in a random tournament, one has $P(n) \geq n!/2^{n-1}$. This result, due to Szele [11], is considered the first application of the probabilistic method. This lower bound has been improved by a constant factor [1, 12]. Alon [2] proved a matching upper bound of the form $P(n) \leq n^{O(1)}n!/2^{n-1}$ (also see [8] for a later improvement).

Proof of Theorem 1. We may assume that $k \geq 1$. Let $f(x, y) = 1$ if (x, y) is a directed edge in the tournament, and 0 otherwise. Let $g_t(x)$ denote the number walks of length t ending at x . Let $d^+(x)$ and $d^-(x)$ denote the out-degree and the in-degree of x , respectively. We have, for each $t \geq 1$,

$$g_t(y) = \sum_x g_{t-1}(x)f(x, y).$$

Define

$$A_t := \sum_y g_t(y)^2 d^+(y) = \sum_{y,z} g_t(y)^2 f(y, z).$$

We have, for each $t \geq 1$,

$$\begin{aligned} A_t &= \sum_{x,x',y} g_{k-1}(x)f(x, y)g_{k-1}(x')f(x', y)d^+(y) \\ &\leq \sum_{x,x',y} \left(\frac{g_{k-1}(x)^2 + g_{k-1}(x')^2}{2} \right) f(x, y)f(x', y)d^+(y) \\ &= \sum_{x,x',y} g_{k-1}(x)^2 f(x, y)f(x', y)d^+(y) \\ &= \sum_{x,y} g_{k-1}(x)^2 f(x, y)d^-(y)d^+(y) \\ &\leq \left(\frac{n-1}{2} \right)^2 \sum_{x,y} g_{k-1}(x)^2 f(x, y) \quad [\text{since } d^-(y)d^+(y) \leq \left(\frac{n-1}{2} \right)^2] \\ &= \left(\frac{n-1}{2} \right)^2 A_{t-1}. \end{aligned}$$

So, for all $t \geq 0$,

$$A_t \leq A_0 \left(\frac{n-1}{2} \right)^{2t} \leq n \left(\frac{n-1}{2} \right)^{2t+1}.$$

Let W_k be the number of walks of length k . Applying the Cauchy–Schwarz inequality,

$$W_k = \sum_y g_{k-1}(y)d^+(y) \leq \sqrt{\sum_y g_{k-1}(x)^2 d^+(y)} \sqrt{\sum_y d^+(y)} \leq \sqrt{A_{k-1}A_0} \leq n \left(\frac{n-1}{2} \right)^k. \quad \square$$

The above proof also gives the following stability result.

Theorem 2 (Stability). *For $k \geq 2$, an n -vertex tournament satisfying*

$$\sum_x \left| d^+(x) - \frac{n-1}{2} \right| \geq \varepsilon \binom{n}{2}$$

has at most $(1 - \frac{\varepsilon^2}{2})n \left(\frac{n-1}{2}\right)^k$ walks of length k .

Note that by symmetry, we can replace d^+ by d^- in the hypothesis of Theorem 2.

Proof. We use the notation from the earlier proof. We have

$$\begin{aligned} W_2 &= \sum_x d^+(x)d^-(x) \leq \sum_x d^+(x)(n-1-d^+(x)) \\ &= \sum_x \left(\left(\frac{n-1}{2}\right)^2 - \left(\frac{n-1}{2} - d^+(x)\right)^2 \right) \\ &\leq n \left(\frac{n-1}{2}\right)^2 - \frac{1}{n} \left(\sum_x \left| \frac{n-1}{2} - d^+(x) \right| \right)^2 \\ &\leq (1 - \varepsilon^2)n \left(\frac{n-1}{2}\right)^2. \end{aligned}$$

From the proof of Theorem 1, we have

$$W_k^2 \leq A_{k-1}A_0 \leq \left(\frac{n-1}{2}\right)^{2(k-2)} A_1A_0.$$

Using $A_0 \leq n(n-1)/2$ and $A_1 = \sum_x d^-(x)^2d^+(x)$, we obtain

$$W_k^2 \leq n \left(\frac{n-1}{2}\right)^{2k-3} \sum_x d^-(x)^2d^+(x).$$

In the proof of Theorem 1, we defined $g_k(x)$ to be the number of k -edge walks ending at x . By running the same proof for the number of k -edge walks starting at x , we deduce

$$W_k^2 \leq n \left(\frac{n-1}{2}\right)^{2k-3} \sum_x d^-(x)d^+(x)^2.$$

Taking the average of the two bounds, we obtain

$$\begin{aligned} W_k^2 &\leq n \left(\frac{n-1}{2}\right)^{2k-3} \sum_x d^-(x)d^+(x) \left(\frac{d^-(x) + d^+(x)}{2}\right) \\ &\leq n \left(\frac{n-1}{2}\right)^{2k-2} \sum_x d^-(x)d^+(x) \\ &\leq (1 - \varepsilon^2)n^2 \left(\frac{n-1}{2}\right)^{2k} \leq \left(\left(1 - \frac{\varepsilon^2}{2}\right) n \left(\frac{n-1}{2}\right)^k \right)^2. \end{aligned} \quad \square$$

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APPENDIX A. AN ENTROPY PROOF
BY DINGDING DONG AND TOMASZ ŚLUSARCZYK

Here is another proof of Theorem 1 using entropy. Given a discrete random variable X taking values in Ω , its entropy is defined as

$$H(X) = - \sum_{x \in \Omega} \mathbb{P}(X = x) \log \mathbb{P}(X = x).$$

We have the uniform bound

$$H(X) \leq \log |\Omega|.$$

The chain rule says that if X and Y are jointly distributed random variables, then

$$H(X, Y) = H(X) + H(Y|X),$$

where the conditional entropy $H(Y|X)$ is defined as

$$H(Y|X) = \sum_{x \in \Omega} \mathbb{P}(X = x) H(Y|X = x).$$

Here $H(Y|X = x)$ is the entropy of the conditional distribution of Y given $X = x$.

Entropy proof of Theorem 1. Consider a random walk X_1, \dots, X_{k+1} chosen uniformly from the set of all W_k walks of length k in the given tournament. This random walk is Markovian in the sense that the distribution of (X_i, \dots, X_{k+1}) conditional on (X_1, \dots, X_i) is the same as the distribution of (X_i, \dots, X_{k+1}) conditional on X_i . Indeed, this conditional distribution is uniform over all walks (X_i, \dots, X_{k+1}) with a given starting vertex X_i . In particular, $H(X_j|X_{j-1}, \dots, X_1) = H(X_j|X_{j-1})$.

Applying the chain rule, we have

$$\begin{aligned} \log W_k = H(X_1, \dots, X_{k+1}) &= H(X_1, X_2) + \sum_{j=2}^k H(X_{j+1}|X_1, \dots, X_j) \\ &= H(X_1, X_2) + \sum_{j=2}^k H(X_{j+1}|X_j). \end{aligned}$$

Likewise,

$$H(X_1, \dots, X_{k+1}) = H(X_{k+1}, X_k) + \sum_{j=2}^k H(X_{j-1}|X_j).$$

Taking the average of the two bounds, we obtain

$$H(X_1, \dots, X_{k+1}) = \frac{H(X_1, X_2) + H(X_{k+1}, X_k)}{2} + \frac{1}{2} \sum_{j=2}^k (H(X_{j-1}|X_j) + H(X_{j+1}|X_j)).$$

For each $2 \leq j \leq k$ and vertex x , by the uniform bound, $H(X_{j-1}|X_j = x) \leq \log d^-(x)$ and $H(X_{j+1}|X_j = x) \leq \log d^+(x)$. Also, $d^-(x)d^+(x) \leq (n-1)^2/4$. Thus

$$H(X_{j-1}|X_j = x) + H(X_{j+1}|X_j = x) \leq \log d^-(x) + \log d^+(x) \leq 2 \log \left(\frac{n-1}{2} \right).$$

Thus

$$H(X_{j-1}|X_j) + H(X_{j+1}|X_j) \leq 2 \log \left(\frac{n-1}{2} \right).$$

Also $H(X_j, X_{j+1}) \leq \log \binom{n}{2}$ by the uniform bound. Therefore

$$\log W_k = H(X_1, \dots, X_{k+1}) \leq \log \binom{n}{2} + (k-1) \log \left(\frac{n-1}{2} \right) = \log \left(n \left(\frac{n-1}{2} \right)^k \right). \quad \square$$

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