Abstract. We prove that for all fixed $p > 2$, the translative packing density of unit $\ell_p$-balls in $\mathbb{R}^n$ is at most $2^{(\gamma_p + o(1))n}$ with $\gamma_p < -1/p$. This is the first exponential improvement in high dimensions since van der Corput and Schaake (1936).

1. Introduction

The sphere packing problem asks for the densest packing of non-overlapping unit balls in $\mathbb{R}^n$. This is an old and difficult problem whose exact solution is only known in dimensions 1, 2, 3, 8, and 24. The problem is already non-trivial in two dimensions (see [8] for a short proof). The three-dimensional sphere packing problem is known as Kepler’s conjecture, and it was solved by Hales [9] via a monumental computer-assisted proof. The problem in eight dimensions was recently resolved by Viazovska [23] in a stunning breakthrough, and the method was then quickly extended to solve the problem in twenty-four dimensions [3]. Dimensions 8 and 24 are special due to the existence of highly dense and symmetric lattices known as the $E_8$ lattice (dimension 8) and the Leech lattice (dimension 24). See the survey [2] and its references for background and recent developments.

In this paper, we study translative packings of $\ell_p$-balls in high dimensions. Denote the $\ell_p$-balls with radius $R$ in $\mathbb{R}^n$ by $B_n^p(R) := \{ x \in \mathbb{R}^n : \|x\|_p \leq R \}$ and the unit $\ell_p$-ball by $B_n^p := B_n^p(1)$. Here $\|x_1, \ldots, x_n\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ is the $\ell_p$-norm. The name superball refers to $\ell_p$-balls with $p > 2$ [19]. Superballs are more cube-like compared to the familiar $\ell_2$-balls. See [10, 11, 5] for studies of $\ell_p$-ball packings in $\mathbb{R}^3$. Although $\ell_p$-balls do not possess rotational symmetry, in this paper we only consider translations of identical $\ell_p$-balls, not allowing rotations. The best known lower bounds on high dimensional superball packing densities do not use rotations [6] (see Section 3.2).

Let $\Delta_p(n)$ denote the maximum translative packing density of copies of $B_n^p$ in $\mathbb{R}^n$. Here density is the fraction of space occupied by these balls. For fixed $p \in [1, \infty)$, let

$$\gamma_p := \limsup_{n \to \infty} \frac{1}{n} \log_2 \Delta_p(n)$$

be the exponential rate of optimal packing densities in high dimensions. The precise value of $\gamma_p$ is unknown for any $p \in [1, \infty)$, and the current best upper and lower bounds are quite far apart. For Euclidean balls, $p = 2$, the best high dimensional upper bound (apart from constant factors) is due to Kabatiansky and Levenshtein [12]:

$$\Delta_2(n) \leq 2^{(\kappa_{KL} + o(1))n}, \quad \text{where} \quad \kappa_{KL} := -0.5990 \ldots.$$

See Cohn and Zhao [4] and Sardari and Zargar [20] for constant factor improvements over Kabatiansky and Levenshtein [12]. For lower bounds, we have $\Delta_p(n) \geq 2^{-n}$ for all $n$ and $p \geq 1$ since every maximal packing has density at least $2^{-n}$. For $p = 2$, there have only been subexponential improvements, with the current best lower bound due to Venkatesh [22]. In summary, the best bounds on $\gamma_2$ are $-1 \leq \gamma_2 \leq \kappa_{KL} = -0.5990 \ldots$.

YZ was supported by NSF Awards DMS-1362326 and DMS-1764176, and the MIT Solomon Buchsbaum Fund.
For $p > 2$, the current best upper bound on the exponential rate of superball packing densities was first proved by van der Corput and Schaake [21] via Blichfeldt’s method [1] (e.g., see [24, Section 6.3]), giving

$$\gamma_p \leq -1/p \quad \text{for } p > 2.$$ 

There have been subsequent subexponential upper bound improvements on $\Delta_p(n)$ for $p > 2$, e.g., Rankin [16, 17]. We defer to Section 3 for a discussion of known bounds on $\gamma_p$ in other regimes.

In this paper, we prove a new upper bound on $\gamma_p$ for all $p > 2$, giving the first exponential improvement since 1936 on the upper bound of superball packing densities in high dimensions.

**Theorem 1.1.** For all $p \geq 2$,

$$\gamma_p \leq \inf_{0 < \theta < \pi/2} \left( \frac{1 + \sin \theta}{2 \sin \theta} \log_2 \frac{1 + \sin \theta}{2 \sin \theta} - \frac{1 - \sin \theta}{2 \sin \theta} \log_2 \frac{1 - \sin \theta}{2 \sin \theta} + \frac{2}{p} \log_2 \sin \frac{\theta}{2} \right).$$

In particular, $\gamma_p < -1/p$ for all $p \geq 2$.

See Figure 1 for a plot of the bounds.

**Remark.** Theorem 1.1 with $p = 2$ recovers $\gamma_p \leq \kappa_{\text{KL}}$. Our upper bound on $\gamma_p$ is continuous with $p$, whereas the previous best bounds were not continuous\(^1\) at $p = 2$. The fact that our bound at $p = 2$ recovers the Kabatiansky–Levenshtein bound is not a coincidence, as our proof relies on the Kabatiansky–Levenshtein bound for spherical codes.

### 2. Proof of main theorem

**2.1. Kabatiansky–Levenshtein spherical code bound.** Denote the $\ell_p$-sphere in $\mathbb{R}^n$ of radius $R$ by $S^{n-1}_p(R) := \{ x \in \mathbb{R}^n : \|x\|_p = R \}$ and the unit $\ell_p$-sphere by $S^{n-1}_p := S^{n-1}_p(1)$. Let $A_p(n, d)$ to be the maximum number of points on $S^{n-1}_p$ with pairwise $\ell_p$-distance at least $2d$, i.e, an $\ell_p$-spherical code. Note that $A_p(n, d) = 1$ unless $d \in [0, 1]$. Note that $A_2(n, \sin(\theta/2))$ is the maximum size of a

\(^1\)It is unknown whether $p \mapsto \gamma_p$ is continuous. Lemma 3.1 implies that $\gamma_p$ is continuous at all but at most countably many points.
spherical code in $\mathbb{R}^n$ with pairwise angle at least $\theta$. Kabatiansky and Levenshtein [12] proved that for all $0 < \theta < \pi/2$,
\[
\limsup_{n \to \infty} \frac{1}{n} \log_2 A_2(n, \sin(\theta/2)) \leq a(\theta)
\]  
(2.1)
where
\[
a(\theta) := \frac{1 + \sin \theta}{2 \sin \theta} \log_2 \frac{1 + \sin \theta}{2} - \frac{1 - \sin \theta}{2 \sin \theta} \log_2 \frac{1 - \sin \theta}{2 \sin \theta}.
\]

A projection argument (see [4, Section 2]) shows that
\[
\Delta_2(n) \leq \sin^n(\theta/2) A_2(n + 1, \sin(\theta/2)),
\]
so (2.1) gives
\[
\lim_{n \to \infty} \frac{1}{n} \log_2 \Delta_2(n) \leq a(\theta) + \log_2 \frac{\theta}{2}.
\]
The bound $\gamma_2 \leq 0.5990 \ldots$ is obtained by choosing $\theta = \theta_{KL} = 1.0995 \ldots$ to minimize the upper bound above.

2.2. $\ell_p$-twist. Fix $p \geq 2$. Define
\[
x^* := \text{sgn}(x)|x|^{p/2}, \quad x \in \mathbb{R}.
\]
For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, write $x^* := (x_1^*, \ldots, x_n^*)$, and for $X \subseteq \mathbb{R}^n$, write $X^* := \{x^* : x \in X\}$. Observe that for all $x, y \in \mathbb{R}$,
\[
|x^* - y^*| \geq 2^{1-p/2}|x - y|^{p/2}. \tag{2.2}
\]
Indeed, without loss of generality it suffices to consider two cases: $x \geq 0 \geq y$ and $x \geq y \geq 0$. The former case is an immediate consequence of Hölder’s inequality (or the convexity of $x \mapsto x^{p/2}$). In the latter case, we have
\[
x^{p/2} - y^{p/2} \geq (x - y)^{p/2} \geq 2^{1-p/2}(x - y)^{p/2}.
\]
Here we use $(w + z)^{p/2} \geq w^{p/2} + z^{p/2}$ for $w, z > 0$, which can be proved by first normalizing to $w + z = 1$ and noting that $w^{p/2} + z^{p/2} \leq w + z = 1$.

Lemma 2.1. For all $p \geq 2$ and $d \in (0, 1]$, we have $A_p(n, d) \leq A_2(n, dp^{p/2})$.

Proof. Let $X \subseteq S^{n-1}_p$ with $|X| = A_p(n, d)$ and $\|x - y\|_p \geq 2d$ for all distinct $x, y \in X$. We have $\|x^*\|_2 = \|x\|_p = 1$ for all $x \in X$, so $X^* \subseteq S^{n-1}_{2^{1-p}}$. For distinct $x, y \in X$, we have
\[
\|x^* - y^*\|_2^2 = \sum_{i=1}^n |x_i^* - y_i^*|^2 \geq 2^{2-p} \sum_{i=1}^n |x_i - y_i|^p \geq 2^2 dp^p,
\]
by (2.2). Thus $X^*$ is a subset of $S^{n-1}_{2^{1-p}}$ whose points have pairwise $\ell_2$-distance at least $2dp^p$. Hence $|X| = |X^*| \leq A_2(n, dp^{p/2})$.

Remark. The same argument shows that $A_p(n, d) \leq A_q(n, dp^{p/q})$ for all $1 \leq q \leq p$ and $d \in (0, 1]$.

Lemma 2.2. For every $p \geq 1, d \in (0, 1]$, and $n \in \mathbb{N}$, we have $\Delta_p(n) \leq d^n A_p(n + 1, d)$.

Proof. Let $\rho < \Delta_p(n)$ be arbitrary. Consider a translative packing $\{x + B^n_p(d) : x \in X\}$ in $\mathbb{R}^n$ with density greater than $\rho$, where $X \subseteq \mathbb{R}^n$ is the set of centers of the $\ell_p$-balls. By an averaging argument\(^3\), there exists some translate of a unit $\ell_p$-ball that contains at least $d^{-n}\rho$ points of $X$. Translating $X$ if necessary, we may assume that $|X \cap B^n_p| \geq d^{-n}\rho$. Add an $(n + 1)$-st coordinate to each point in $X \cap B^n_p$ to obtain a set $X'$ of points on the unit $\ell_p$-sphere in $\mathbb{R}^{n+1}$. In other words, $X'$

---

\(^2\)A simple geometric argument (see [13, (17)]) shows that the upper bound (2.1) can be improved for $\theta < \theta_{KL} := 1.0995 \ldots$ to $a(\theta_{KL}) + \log_2 \sin(\theta_{KL}/2) - \log_2 \sin(\theta/2)$, but this improvement does not benefit our bounds.

\(^3\)A uniform random translation of a unit $\ell_p$-ball inside $[-R, R]^n$ contains more than $d^{-n}\rho + o_{R \to \infty}(1)$ points of $X$. 

---
is obtained by projecting the points of $X$ contained in the unit ball “upward” to the hemisphere one dimension higher. Since the points in $X$ are pairwise at least $2d$ apart in $\ell_p$-distance, the same holds for $X'$. So $X'$ is an $\ell_p$-spherical code whose points are pairwise separated by $\ell_p$-distance at least $2d$, and hence $d^{-n} \rho \leq |X \cap B^n_p| = |X'| \leq A_p(n + 1, d)$. Since $\rho$ can be arbitrarily close to $\Delta_p(n)$, we obtain the claimed inequality. \qed

Remark. As in \cite{4}, the above argument can be modified so that we do not need to add a new dimension when $d \notin [1/2, 1]$, resulting in a slightly better bound $\Delta_p(n) \leq d^{-n} A_p(n, d)$. We omit the details of this modification since this improvement does not affect the exponential asymptotics.

\textbf{Proof of Theorem 1.1.} Applying Lemmas 2.1 and 2.2, we have, for every $0 < \theta < \pi/2$,
\[ \Delta_p(n) \leq \sin(\theta/2)^{2n/p} A_p(n + 1, \sin(\theta/2)^{2/p}) \leq \sin(\theta/2)^{2n/p} A_2(n + 1, \sin(\theta/2)). \]
Applying (2.1), we obtain
\[ \gamma_p = \limsup_{n \to \infty} \frac{1}{n} \log_2 \Delta_p(n) \leq a(\theta) + \frac{2}{p} \log_2 \sin(\theta/2). \]
The main result follows by taking the infimum of the bound over $\theta \in (0, \pi/2)$.

Setting $\theta = \pi/2 - \eta$, we have, with $p \geq 2$ fixed and $\eta \to 0^+$,
\[ \gamma_p \leq -\frac{1}{p} - \frac{\eta}{p \ln 2} + o(\eta). \]
So choosing $\eta > 0$ sufficiently small gives $\gamma_p < -1/p$ for all $p \geq 2$. \qed

3. Remarks

3.1. Asymptotics. Setting $\theta = \pi/2 - (p \ln p)^{-1}$, we obtain
\[ \gamma_p \leq -\frac{1}{p} - \frac{1}{\ln 4} \cdot \frac{1}{p^2 \ln p} + O\left(\frac{1}{p^2 \ln^2 p}\right), \quad \text{as } p \to \infty. \]
Taking $\theta = \theta_{KL}$ gives
\[ \gamma_p \leq \kappa_{KL} + \frac{2 - p}{p} \log_2 \sin \frac{\theta_{KL}}{2}, \quad \text{for all } p \geq 2. \]
Thus, as $\epsilon \to 0^+$,
\[ \gamma_{2+\epsilon} \leq \kappa_{KL} - \left(\frac{1}{2} \log_2 \sin \frac{\theta_{KL}}{2}\right) \epsilon + O(\epsilon^2) = (-0.5990\ldots) + (0.4650\ldots) \epsilon + O(\epsilon^2). \]

3.2. Review of other bounds on $\gamma_p$. Here we survey other existing bounds on $\gamma_p$.

For $p = 2$, the best known bounds are $-1 \leq \gamma_2 \leq \kappa_{KL} = -0.5990\ldots$ as discussed earlier.

For $p > 2$, the best known upper bounds are the ones given in this paper. For lower bounds, extending on methods developed by Rush \cite{18} and Rush–Sloane \cite{19}, Elkies, Odlyzko, and Rush \cite{6} proved $\gamma_p > -1$ for all $p > 2$, thereby exponentially beating the Minkowski–Hlawka lower bound. See \cite{6} for the precise bound. Their bounds have the following asymptotics:
\[ \gamma_p \geq -\left(1 + o(1)\right) \frac{\ln \ln p}{p \ln 2}, \quad \text{as } p \to \infty, \]
and
\[ \gamma_{2+\epsilon} \geq -1 + \left(\frac{\sqrt{\pi \zeta(3)}}{2 \ln 2} + o(1)\right) \frac{\epsilon}{\ln^{3/2}(1/\epsilon)}, \quad \text{as } \epsilon \to 0^+. \]
Here $\zeta$ denotes the Riemann zeta function. See \cite{14} for some later improvements using algebraic-geometric codes for some specific integers $p$. 

For $1 \leq p < 2$, no improvement over the Minkowski–Hlawka lower bound $\gamma_p \geq -1$ is known. The best upper bound on $\gamma_p$ is due to Rankin [15], based on Blichfeldt’s method [1]:

$$\gamma_p \leq \inf_{\frac{1}{2} \leq \ell \leq \frac{1}{2} + \frac{1}{p}} \left( b(p) - b(q) - 1 + 1/p + (1/q - 1/p) \log_2 \left( \frac{2 - 1/q}{1 - 1/q} \right) \right)$$

(3.1)

where

$$b(p) := \lim_{n \to \infty} \frac{1}{n} \log_2 \text{vol} B_p^n(n^{1/p}) = 1 + \log_2 \Gamma \left( 1 + \frac{1}{p} \right) + \frac{1}{p} \log_2 (pe).$$

(3.2)

Recall that $\text{vol} B_p^n = 2^n \Gamma(1 + 1/p)^n / \Gamma(1 + n/p)$.

For packings of congruent cross-polytopes (i.e., unit $\ell_1$-balls) allowing rotations, Fejes Tóth, Fodor, and Vígh [7] proved an exponentially decaying upper bound in high dimensions. For translatively packing of unit $\ell_1$-balls, the upper bound (3.1) remains best known in high dimensions.

We note that the above bound (3.1) can be improved on the region $p \in [1.494 \ldots, 2)$ using the Kabatiansky–Levenshtein bound via the following folklore observation.

**Lemma 3.1.** For $1 \leq p \leq q \leq \infty$, $\gamma_p - b(p) \leq \gamma_q - b(q)$.

**Proof.** By monotonicity of norms, we have $n^{-1/p} \|x\|_p \leq n^{-1/q} \|x\|_q$, so $B_p^n(n^{1/p}) \supseteq B_q^n(n^{1/q})$. Any packing of $B_p^n(n^{1/p})$ can be shrunk into a packing of $B_q^n(n^{1/q})$. Hence

$$\frac{\Delta_p(n)}{\text{vol} B_p^n(n^{1/p})} \leq \frac{\Delta_q(n)}{\text{vol} B_q^n(n^{1/q})}.$$

Taking log, dividing by $n$, and letting $n \to \infty$ yields the lemma. \hfill \Box

Using $\gamma_2 \leq \kappa_{KL}$, we find that

$$\gamma_p \leq \kappa_{KL} - b(2) + b(p) = (-0.5990 \ldots) - b(2) + b(p) \quad \text{for} \quad 1 \leq p < 2.$$  

(3.3)

Thus

$$\gamma_p \leq \min\{\text{RHS of (3.1)}, \text{RHS of (3.3)}\}$$

See Figure 1 for an illustration of the above bounds.

**Acknowledgments**

This work began during Y.Z.’s internship at Microsoft Research New England, and he would like to thank Henry Cohn for discussions and mentorship and Microsoft Research for its hospitality.

**References**


