IMPROVING THE $\frac{1}{3}-\frac{2}{3}$ CONJECTURE FOR WIDTH TWO POSETS

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Abstract. Extending results of Linial (1984) and Aigner (1985), we prove a uniform lower bound on the balance constant of a poset P of width 2. This constant is defined as $\delta(P) = \max_{(x,y)\in P^2}\min\{\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)\}$, where $\mathbb{P}(x \prec y)$ is the probability x is less than y in a uniformly random linear extension of P. In particular, we show that if P is a width 2 poset that cannot be formed from the singleton poset and the three element poset with one relation using the operation of direct sum, then

$$\delta(P) \geq \frac{-3+5\sqrt{17}}{52} \approx 0.33876\ldots.$$

This partially answers a question of Brightwell (1999); a full resolution would require a proof of the $\frac{1}{3}-\frac{2}{3}$ Conjecture that if P is not totally ordered then $\delta(P) \geq \frac{1}{3}$.

Furthermore, we construct a sequence of posets T_n of width 2 with $\delta(T_n) \to \beta \approx 0.348843\ldots$, giving an improvement over a construction of Chen (2017) and over the finite posets found by Peczarski (2017). Numerical work on small posets by Peczarski suggests the constant β may be optimal.

1. Introduction

Definition 1.1. Given a fixed, underlying poset $(P, \leq)$, $\mathbb{P}(x \prec y)$ is the probability that x precedes y in a uniformly random linear extension of P. We define $\mathbb{P}(x \prec x) = 0$.

A conjecture dating back to 1968 states that in any finite partial order not a chain, there is a pair (x, y) such that $\mathbb{P}(x \prec y) \in \left[\frac{1}{3}, \frac{2}{3}\right]$. Kislitsyn [10], Fredman [7], and Linial [11] independently made this so-called $\frac{1}{3}-\frac{2}{3}$ Conjecture. Each had in mind an application to sorting theory. In particular, this conjecture implies that the number of comparisons needed to fully sort elements that are already known to be in the partial order P is at most $(1+o(1))\log_{\frac{3}{2}} e(P)$, within a constant factor of the trivial information-theoretic lower bound $\log_2 e(P)$. Here $e(P)$ is the number of linear extensions of P.

Definition 1.2. The *balance constant* of poset P is

$$\delta(P) = \max_{(x,y)\in P^2}\min\{\mathbb{P}(x \prec y), \mathbb{P}(y \prec x)\}.$$

We can thus restate the $\frac{1}{3}-\frac{2}{3}$ Conjecture as follows.

Conjecture 1.3. *If P is a finite poset that is not totally ordered, then $\delta(P) \geq \frac{1}{3}$.*

Brightwell [3] deemed it "one of the major open problems in the combinatorial theory of partial orders."

This conjecture is known to be true for certain classes of posets: width 2 by Linial [11], height 2 by Trotter, Gehrlein, and Fishburn [15], 6-thin by Peczarski [13], semiorders by Brightwell [4], and posets whose Hasse diagram is a tree by Zaguia [16].

We can ask how the balance constant interacts with the fundamental operations of disjoint union and direct sum. (Direct sum will be important in characterizing equality cases.) It is clear that if $\oplus$

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denotes direct sum of posets and $\sqcup$ disjoint union of posets then

$$\delta(P \oplus Q) = \max\{\delta(P), \delta(Q)\},$$
$$\delta(P \sqcup Q) \geq \max\{\delta(P), \delta(Q)\}.$$

We more formally define $P \oplus Q$ on the union set such that $x \leq y$ precisely when $x \leq_P y$ or $x \leq_Q y$ or both $x \in P$ and $y \in Q$. Also, $P \sqcup Q$ is defined on the union such that $x \leq y$ precisely when $x \leq_P y$ or $x \leq_Q y$. Thus the $\frac{1}{3} - \frac{2}{3}$ Conjecture is true for direct sums and disjoint unions of posets which satisfy it. It follows easily that the conjecture holds, for instance, for series-parallel posets, i.e. N-free posets. Brightwell, Felsner, and Trotter [2] showed that if P is not totally ordered, than $\delta(P) \geq \frac{5-\sqrt{5}}{10}$, improving on methods of Kahn and Saks [9]. See the survey of Brightwell [3] for more information on general progress.

Aigner [1] showed that the only width 2 posets that achieve equality ($\delta(P) = \frac{1}{3}$) are those formed from $\mathbf{1}$ and $\mathcal{E}$ using the operation of direct sum: $\mathbf{1}$ is the poset with one element and $\mathcal{E}$ is the poset with three elements and exactly one relation $x \leq y$. Alternatively, $\mathcal{E} = (\mathbf{1} \oplus \mathbf{1}) \sqcup \mathbf{1}$.

Brightwell [3] posed the question of understanding in general the structure of the set $\mathcal{B} = \{\delta(P) : P \text{ a finite partial order}\}$, asking whether there is a gap after $\frac{1}{3}$. Of course, a result of this form would be much stronger than the $\frac{1}{3} - \frac{2}{3}$ Conjecture.

We answer this question in the affirmative in the width 2 setting, thus extending the results of Aigner [1] and Linial [11]. In particular, we prove

Theorem 1.4. *If P is a finite, width 2 poset that cannot be formed from $\mathbf{1}$ and $\mathcal{E}$ using the operation of direct sum, then*

$$\delta(P) \geq \lambda = \frac{-3 + 5\sqrt{17}}{52} = 0.33876\ldots.$$

The proof relies on a path-counting interpretation of linear extensions of width 2 posets, and as noted in Section 5, computer results seem to indicate that we can improve the constant, potentially to 0.348842 or so.

On the other side of the issue, Chen [5] exhibited a sequence of width 2 posets whose balance constants approach $\frac{93-\sqrt{6697}}{32} \approx 0.3488999\ldots$. Using our path-counting interpretation, we can easily compute balance constants of similar families, and in particular show

Theorem 1.5. *There is a sequence T_n of width 2 posets with*

$$\delta(T_n) \to \beta = \frac{5864893 + 27\sqrt{57}}{16812976} \approx 0.348843\ldots.$$

In Section 2 we outline the key path-counting interpretation of linear extensions of width 2 posets, and prove essential properties of the correspondence. In Section 3 we prove Theorem 1.4 using those properties. In Section 4 we prove Theorem 1.5 by outlining the computations, also based on the path-counting interpretation of linear extensions, that determine $\delta(T_n)$. Auxiliary calculations for this section are contained in Appendix A. Finally, we discuss the optimal constants and other outstanding questions in Section 5.

2. Path-Counting Interpretation of Linear Extensions

2.1. **The Interpretation.** Let P be a finite, width 2 poset. We can reinterpret linear extensions of P in a natural way. Write $P = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$, where $a_1 \leq \cdots \leq a_m$ and $b_1 \leq \cdots \leq b_n$ (there may also be relations of the form $a_i \leq b_j$ or $b_j \leq a_i$).

Definition 2.1. The *grid diagram* of P is formed as follows. Draw an $m \times n$ grid. Label the segments along the left axis by $a_1, \ldots, a_m$ from the top, and label the segments along the top axis by $b_1, \ldots, b_n$ from the left. Let $C_{i,j}$ the cell in the ith row from the top and jth column from the left. We also label all $(m+1)(n+1)$ grid points so that the top left is $(0, 0)$ and the bottom right is

(m,n). Thus (i,j) is the bottom right corner of $C_{i,j}$. Now, if $a_i \le b_j$ then color the cell $C_{i,j}$ red, while if $a_i \ge b_j$ then color the cell $C_{i,j}$ blue. Let R be the set of red cells, and B the set of blue cells. Finally, let S be the set of $C_{i,j}$, such that $1 \le i \le m$, $1 \le j \le n$, and $\mathbb{P}(a_i \prec b_j) \le \frac{1}{2}$.

It will later be convenient to sometimes color cells $C_{0,i}$ for $1 \le i \le n$ red and cells $C_{j,0}$ for $1 \le j \le m$ blue, but we do not include such cells in the definition of the sets R and B.

An example of a width 2 poset P and its corresponding grid diagram is shown in Figure 1. Notice also that the grid diagram of a poset may depend on its presentation $P = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$, since there may be multiple ways to decompose it into chains. This is a technicality which will not be too important.

Proposition 2.2. *In the grid diagram of finite width 2 poset P, the sets R and B form Young diagrams. R is right- and top-justified, and B is left- and bottom-justified. Furthermore, $R \cap B = \varnothing$.*

Proof. We see that if $C_{i,j}$ is filled in red, then $a_i \le b_j$. Thus if additionally $i' \le i$ and $j \le j'$ then $a_{i'} \le a_i \le b_j \le b_{j'}$. In particular, $i' \le i$, $j \le j'$, and $(i,j) \in R$ together imply that $(i',j') \in R$. Therefore R indeed corresponds to a Young diagram which is right- and top-justified. Similarly, B corresponds to a Young diagram which is left- and bottom-justified. Clearly these two Young diagrams do not intersect. $\square$

Any linear extension $\prec$ of P can be written as a rearrangement of the symbols $a_1, \ldots, a_m, b_1, \ldots, b_n$, interpreted in increasing order. It must contain $a_1, \ldots, a_m$ in that order, and $b_1, \ldots, b_n$ in that order. Hence it corresponds to a path from the top left corner $(0,0)$ to the bottom right corner (m,n) of the grid diagram of P that only goes down and right: the order of the segments used gives precisely the linear extension as written above.

Furthermore, it is easy to see that the paths corresponding to linear extensions of P are precisely those that stay between B and R.

Definition 2.3. A down-right path from $(0,0)$ to (m,n) in the grid diagram of P is *valid* if it stays between B and R.

Additionally, notice that $a_i \prec b_j$ if and only if the corresponding path goes below the cell $C_{i,j}$: the extension must make its jth horizontal move after making its ith vertical move.

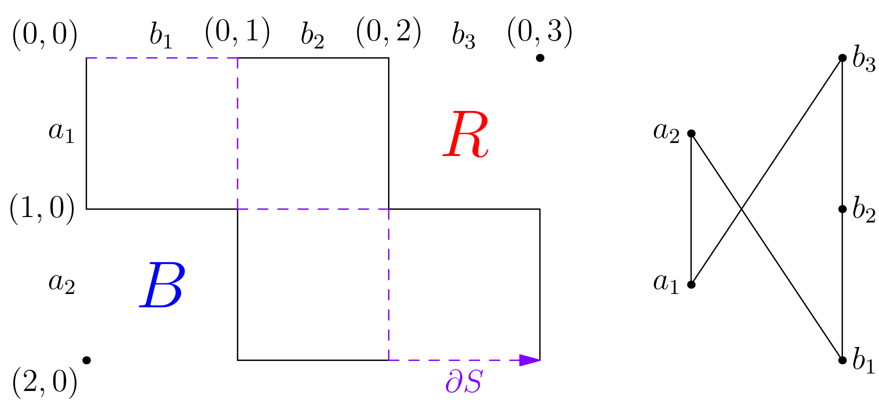

FIGURE 1. Example of poset and grid diagram.

Now, we study the structure of S.

Proposition 2.4. *In the grid diagram of finite width 2 poset P, the set S forms a left- and bottom-justified Young diagram satisfying $R \subseteq S$ and $S \cap B = \varnothing$.*

Proof. Notice that if $a_{i'} \prec b_{j'}$, $i \leq i'$, and $j' \leq j$ then $a_i \prec b_j$. Thus

$$\mathbb{P}(a_{i'} \prec b_{j'}) \leq \mathbb{P}(a_i \prec b_j)$$

if $a_{i'} \prec b_{j'}$, $i \leq i'$, and $j' \leq j$. In particular, $i \leq i'$, $j' \leq j$, and $(i, j) \in S$ imply $(i', j') \in S$. Thus S is a left- and bottom-justified Young diagram. Also, if $a_i \geq b_j$ then clearly $\mathbb{P}(a_i \prec b_j) = 0$ so that $(i, j) \in S$. Thus $R \subseteq S$. Similarly, we see that if $a_i \leq b_j$ then $\mathbb{P}(a_i \prec b_j) = 1$, hence $S \cap B = \varnothing$. $\square$

We will often consider the path ∂S along the border of S, from the top left corner to bottom right corner of our grid diagram. From this path S can be reconstructed. It is a path which stays between R and B, by Proposition 2.4. An example is shown in Figure 1. ∂S will be denoted by a dotted arrow in all figures.

Incidentally, ∂S thus corresponds to a linear extension $\prec_0$ of P given by $x \prec_0 y$ if and only if $\mathbb{P}(x \prec y) \geq \frac{1}{2}$. (Fishburn [6] showed that this relation is not necessarily transitive for general posets P.)

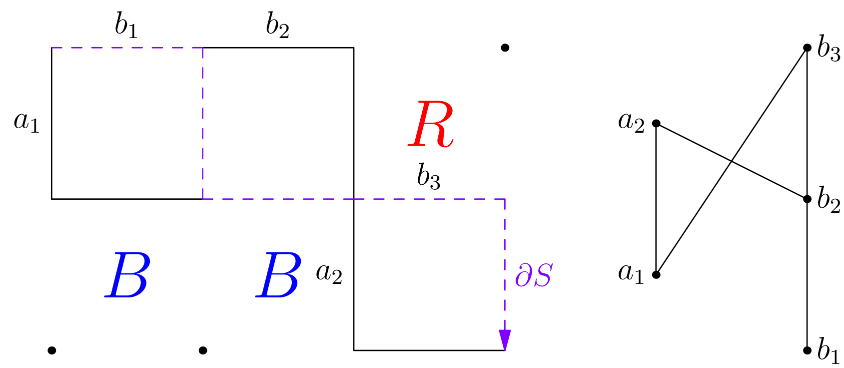

FIGURE 2. Example of poset sum.

Finally, it will be useful to understand the structure of a grid diagram of a poset direct sum.

Proposition 2.5. *If in the grid diagram of finite width* 2 *poset P the sets B and R have cells that share a vertex, then P decomposes as a direct sum. Otherwise, if either the top left or bottom right cell of the grid diagram is in $B \cup R$, then P decomposes as a direct sum.*

Proof. First suppose that B and R have cells that share a vertex (i, j). Consider the induced subposets P_1 and P_2 of P obtained by restricting to $\{a_1, \ldots, a_i, b_1, \ldots, b_j\}$ and $\{a_{i+1}, \ldots, a_m, b_{j+1}, \ldots, b_n\}$, respectively.

Since (i, j) is the vertex of some cell of B, we see that the cells $C_{i',j'}$ for $i' \geq i + 1$ and $j' \leq j$ are in B. Similarly, the cells $C_{i',j'}$ for $i' \leq i$ and $j' \geq j + 1$ are in R. Unwinding the definition of B and R, this demonstrates that every element of P_1 is less than every element of P_2 when considered as part of the entire poset P. Thus $P = P_1 \oplus P_2$, as desired.

Now suppose that the top left cell of the grid diagram is in R. The other three cases are symmetric. Then $C_{1,1} \in R$, which means $C_{1,j} \in R$ for all $1 \leq j \leq n$. Thus $a_1 \leq b_j$ for all $1 \leq j \leq n$, while $a_1 \leq a_i$ for all $2 \leq i \leq m$ by definition. Therefore every element of $\{a_1\}$ is less than every element of $\{a_2, \ldots, a_m, b_1, \ldots, b_m\}$, and a similar argument to above shows that P decomposes as a direct sum. $\square$

An example of direct sum decomposition is shown in Figure 2.

2.2. **Path-Counting Inequalities.** Fix an underlying poset $P = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ with $a_1 \leq \cdots \leq a_m$ and $b_1 \leq \cdots \leq b_n$ as earlier. We construct the grid diagram of P, defining B, R, and S as above, and we additionally assume that P does not decompose as a direct sum.

Definition 2.6. Let $t_{i,j}$ be the number of down-right paths from $(0,0)$ to (i,j) that stay between B and R. Let $r_{i,j}$ be the number of up-left paths from (m,n) to (i,j) that stay between B and R.

An example of this definition is shown in Figure 3. These numbers satisfy recursive relations $t_{i,j} = t_{i-1,j} + t_{i,j-1}$ and $r_{i,j} = r_{i+1,j} + r_{i,j+1}$ provided (i,j) is connected to $(0,0)$ via a down-right path that stays between B and R. We define $t_{i,j} = r_{i,j} = 0$ unless $0 \le i \le m$ and $0 \le j \le n$. (Notice that the recursions do not hold for $(i,j) \in \{(2,0),(0,3)\}$ in Figure 3, and instead $t_{i,j} = r_{i,j} = 0$.)

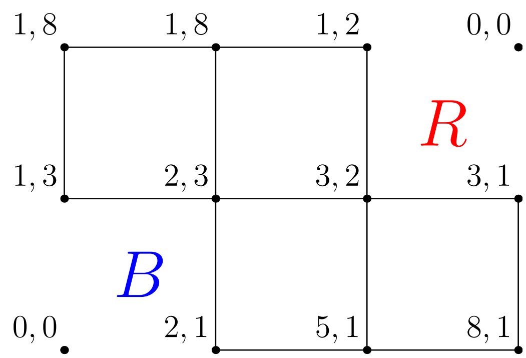

FIGURE 3. Example of $t_{i,j}$ and $b_{i,j}$, respectively.

We will show that these sequences naturally give rise to log-concave sequences. Recall that a sequence $(c_i)_{i=1}^k$ is *log-concave* if $c_i > 0$ for $1 \le i \le k$ and $c_i^2 \ge c_{i-1}c_{i+1}$ for $2 \le i \le k-1$. We say that $(c_i)_{i=1}^k$ is *log-concave with surrounding zeros* if the zeros only form a block at the beginning and end of the sequence, and the remainder is log-concave.

First, we need a lemma about general log-concave sequences.

Lemma 2.7. *If $(c_i)_{i=1}^k$ is log-concave then so is $(d_i)_{i=1}^k$ where*

$$d_i = \sum_{j=1}^{i} c_i.$$

This is straightforward to prove with explicit computations and induction, but it is also a special case of a result of Hoggar [8] which states that the convolution of two log-concave sequences is log-concave. Convolution with a sequence of 1s demonstrates the desired result.

Lemma 2.8. *For every $0 \le j \le n$, the sequences $(t_{i,j})_{i=0}^m$ and $(r_{i,j})_{i=0}^m$ are log-concave with surrounding zeros. For every $0 \le i \le m$, the sequences $(t_{i,j})_{j=0}^n, (r_{i,j})_{j=0}^n$ are log-concave with surrounding zeros.*

Proof. Notice that no paths staying between B and R end at a point that is strictly within B or R, so there are potentially many 0s in these sequences, but we can see that they surround the positive part in the middle.

By symmetry it suffices to prove that $(t_{i,j})_{i=0}^m$ is log-concave with surrounding zeros. We do this by induction on j.

The base case $j = 0$ is trivial: $t_{i,0} \in \{0,1\}$ for $0 \le i \le m$, and the sequence starts with $t_{0,0} = 1$ before permanently transitioning into 0s after some point.

Now assume that $j = \ell + 1$, and assume the truth of the required assertion for $j = \ell$. Suppose that $(t_{i,\ell})_{i=0}^m$ is of the form $0, 0, \ldots, 0, c_1, c_2, \ldots, c_t, 0, \ldots, 0$, where $c_i > 0$ for $1 \le i \le t$. Then we know that $(c_i)_{i=1}^t$ is log-concave.

Let k be the first index such that the lattice point to the right of c_k is not in the strict interior of R. Figure 4 depicts this situation, with the region between the solid lines denoting the cells not in

$B \cup R$. We see that the next column of values, $(t_{i,\ell+1})_{i=0}^{m}$, is

$$0, \dots, 0, c_k, c_k + c_{k+1}, \dots, c_k + \dots + c_{t-1}, c_k + \dots + c_t, c_k + \dots + c_t, \dots, c_k + \dots + c_t, 0, \dots, 0,$$

using the recurrence $t_{i,j} = t_{i-1,j} + t_{i,j-1}$ in the obvious way. Now apply Lemma 2.7 and notice that $(c_k + \dots + c_t)^2 \geq (c_k + \dots + c_{t-1})(c_k + \dots + c_t)$ to establish the log-concavity of the nonzero portion, which finishes. □

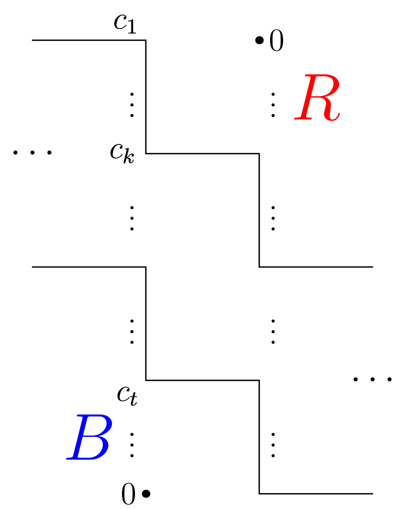

FIGURE 4. Log-concavity proof figure.

Finally, it is useful to note that the number of valid paths through (i, j) is $r_{i,j}t_{i,j}$.

3. Bounding $\delta(P)$

In this section we prove Theorem 1.4, which we restate here.

Theorem 1.4. *If P is a finite, width 2 poset that cannot be formed from $\mathbf{1}$ and $\mathcal{E}$ using the operation of direct sum, then*

$$\delta(P) \geq \lambda = \frac{-3 + 5\sqrt{17}}{52} = 0.33876\dots.$$

Proof. Since $\delta(P \oplus Q) = \max\{\delta(P), \delta(Q)\}$, as noted in Section 1, we may assume that P cannot be decomposed as a direct sum. By hypothesis, we know P is not $\mathbf{1}$ or $\mathcal{E}$.

Now if the grid diagram of P is a $1 \times n$ or $m \times 1$ rectangle, then $B = R = \varnothing$ (else P decomposes as a direct sum).

We then easily see that there is a pair $x, y \in P$ with $\mathbb{P}(x \prec y) = \frac{1}{2}$ if n is odd, and if $n = 2k$ is even then there is a pair with $\mathbb{P}(x \prec y) = \frac{k}{2k+1} \geq \frac{2}{5}$ when $k \geq 2$. Notice that $n = 2$ is impossible since P is not $\mathcal{E}$.

Therefore, we can assume that P is not decomposable as a direct sum, and has both dimensions at least 2 in its grid diagram.

Now fix our underlying poset $P = \{a_1, \dots, a_m, b_1, \dots, b_n\}$ with $a_1 \leq \dots \leq a_m$ and $b_1 \leq \dots \leq b_n$ as in Section 2. We have $m, n \geq 2$ and know that P does not decompose as a direct sum. We define B, R, and S as before.

We will be doing casework on the configuration of B, R, and S in the top left corner of the grid diagram of P. As noted earlier, we will often consider the path ∂S along the border of S. In figures ∂S is denoted by a dotted line where a solid grid line would normally be. The special property ∂S has is that if a cell $C_{i,j}$ is below and to the left of ∂S, i.e. if $C_{i,j} \in S$, then at most $\frac{1}{2}$ of all valid paths pass below $C_{i,j}$. Thus, since $\delta(P)$ is the balance constant, we can deduce that actually at most a $\delta(P)$ fraction of valid paths pass below $C_{i,j}$. This property, which we call the *balance property*, as well as its mirror for cells above and to the right of ∂S will be exploited several times in the following argument.

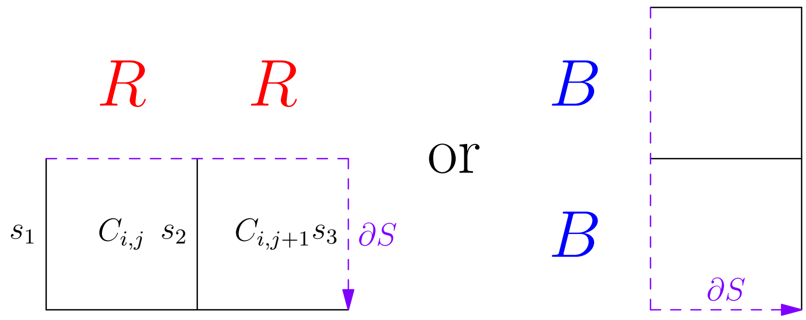

FIGURE 5. Lemma 3.1 configurations.

For the statement of our first lemma, recall that the cells $C_{0,j}$ for $1 \leq j \leq n$ are implicitly colored red and $C_{i,0}$ for $1 \leq i \leq m$ are implicitly colored blue; thus, the configurations identified the lemma might be flush against the top or left of the grid diagram of P.

Lemma 3.1 ($\frac{2}{5}$-Lemma)**.** *If one of the two images in Figure 5 appears within the grid diagram of P with blue and red cells in the corresponding places and with ∂S including the three segments shown, then*

$$\delta(P) \geq \frac{2}{5}.$$

Proof. Let $\delta = \delta(P)$.

Without loss of generality we work with the situation on the left. Let a, b, and c be the number of valid paths of the grid diagram of P that pass through segments s_1, s_2, and s_3 respectively. Let $p = e(P)$ be the total number of valid paths, or equivalently linear extensions of P. Notice that the paths may also go through segments parallel to s_1, s_2, and s_3 that are outside of the small portion depicted in Figure 5.

Now, since cell $C_{i,j+1}$ is to the left of ∂S, the balance property implies that the number of valid paths that pass to its left is at most δp. Indeed, the fraction of valid paths passing on segments to the left is precisely the probability $\mathbb{P}(a_i \prec b_{j+1})$, and since the fraction is at most $\frac{1}{2}$, it must actually be at most δ. Similarly, the number of valid paths passing this vertical section strictly to the right of segment s_3 is at most δp. For this we use the balance property applied to cell $C_{i,j+2}$; if we are at the edge of the grid then this number of paths to the right is in fact 0. Now, the total number of valid paths is p, hence the number of valid paths passing through segment s_3 must be at least $(1-2\delta)p$. That is, $c \geq (1-2\delta)p$.

However, it is also clear that $a \geq b \geq c$. Indeed, we can exhibit an injection from valid paths passing through s_3 to valid paths passing through s_2 (and similar for s_2 and s_1): any such path has a portion from $(i-1, j)$ to $(i, j+1)$ that we reflect over the cell $C_{i,j+1}$.

Thus $a, b \geq (1-2\delta)p$. However, we showed earlier that the number of paths passing to the left of cell $C_{i,j+1}$ was at most δp. This yields

$$2(1-2\delta)p \leq a + b \leq \delta p,$$

which gives the desired result as $p > 0$. $\square$

Now suppose that P also satisfies $\delta(P) < \lambda = \frac{-3+5\sqrt{17}}{52}$. We shall derive a contradiction using Lemma 3.1 and some cases. Without loss of generality, ∂S, which starts at $(0,0)$, goes through $(1,0)$.

Lemma 3.2 (Structure Lemma)**.** *The top left of the grid diagram of P is one of the 9 diagrams depicted in Figure 6.*

Proof. Recall that no blue cells are above ∂S and no red cells are below it. Notice that since P does not decompose as a direct sum, $C_{1,1}$ is neither colored red nor blue.

We know $\delta(P) < \frac{2}{5}$, which means that by Lemma 3.1 the path ∂S must go through $(1, 1)$. Notice that ∂S must continue past this point since we know that the grid diagram has both dimensions at least 2.

If ∂S continues through $(1, 2)$, then consider $C_{1,2}$. Either it is empty, yielding Case 1, or it is red. If it is red, we can again apply Lemma 3.1 to deduce that ∂S must then immediately go downwards, passing through $(2, 2)$. Indeed, if it ∂S farther right and then later goes downwards, then a figure as in Lemma 3.1 will appear. Recalling that P is not a direct sum, the cells $C_{2,1}$ and $C_{2,2}$ are not blue and thus we are in Case 2.

Otherwise ∂S goes through $(2, 1)$. If $C_{2,1}$ is blue we get Case 3. Otherwise it has no color.

Now we look at where ∂S continues. If it goes through $(3, 1)$ next then there are two possibilities. If $C_{3,1}$ is not blue, then we obtain Case 4. Otherwise, we see by Lemma 3.1 that ∂S must immediately go right. Now we see that $C_{3,2}$ and $C_{2,2}$ should not be red since otherwise this would violate the condition that P does not decompose as a direct sum. Thus they have no color: as they are above ∂S, they cannot be blue. Thus there are only two cases, depending on whether or not $C_{1,2}$ is red. Not red gives Case 5 and red gives Case 6.

The other possibility is that ∂S continues through $(2, 2)$ instead. Now we look at $C_{1,2}$. If it is empty, we obtain Case 7. Otherwise, it is red. Now, look at $C_{2,2}$. If it is empty, we obtain Case 9.

Finally, we have the case in which both $C_{1,2}$ and $C_{2,2}$ are red. Then, by similar arguments using Lemma 3.1, ∂S must immediately move downwards. Then we see by virtue of P not decomposing as a direct sum that $C_{3,1}$ and $C_{3,2}$ are empty. Thus we are left with Case 8. □

Now we dispatch the 9 cases in order. Many of the proofs only need to use linear inequalities, but the last case needs the log-concavity inequalities of Lemma 2.8.

Again, let $p = e(P) = r_{0,0} = t_{m,n}$ be the number of valid paths.

All the cases are similar to the first (although Cases 4 and 9 have some modifications), so they are condensed.

Case 1.

Proof. Let $(a, b, c, d) = \frac{1}{p}(r_{1,0}, r_{1,1}, r_{1,2}, r_{0,2})$, as indicated in Figure 6. We then easily see that the fraction of valid paths that pass through $(1, 0)$ is precisely a. Similarly, the fraction of paths going through $(1, 1)$ is $2b$, the fraction going through $(1, 2)$ is $3c$, and the fraction going through $(0, 2)$ is d. (We are using the fact from Section 2 that the number of valid paths through (i, j) is $t_{i,j}r_{i,j}$.)

Since cell $C_{1,1}$ is above ∂S, the balance property tells us that the fraction of valid paths that go above $C_{1,1}$ is at most δ. We can see that this fraction is precisely $\frac{1 \cdot r_{0,1}}{p} = b + d$, using the recurrence for the $r_{i,j}$ sequence. Thus $b + d \leq \delta$.

The cell $C_{2,2}$ is below δS. Thus the fraction of paths above it is at least $1 - \delta$ (if this cell does not exist in the grid diagram, then the fraction is exactly 1, which also satisfies this inequality). These paths go through the segment from $(0, 1)$ to $(0, 2)$ or the segment from $(1, 1)$ to $(1, 2)$. The total number of such paths is $2(cp) + 1(dp) = (2c + d)p$. Thus the fraction is $2c + d$, yielding $2c + d \geq 1 - \delta$.

Additionally, it is clear from the recurrence relation that $a \geq b \geq c \geq 0$ and $d \geq c \geq 0$. Finally, $p = r_{0,0} = r_{0,1} + r_{1,0} = r_{0,1} + r_{1,1} + r_{0,2} = (a + b + d)p$, hence $a + b + d = 1$.

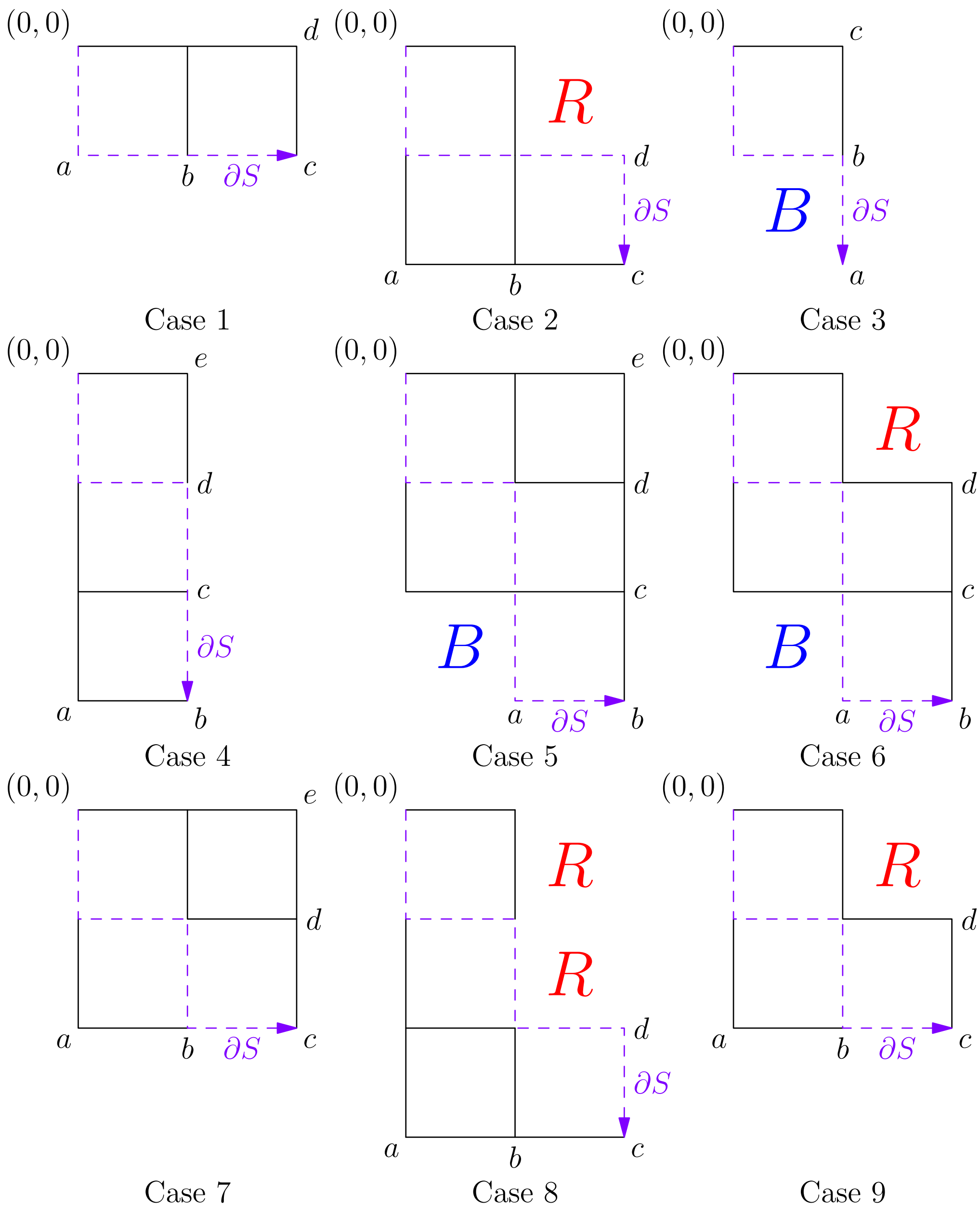

FIGURE 6. Cases of Lemma 3.2.

Thus, overall, we have

$$\begin{aligned}
\delta &\geq b+d,\\
2c+d &\geq 1-\delta,\\
a &\geq b,\\
b &\geq c,\\
d &\geq c,\\
c &\geq 0,\\
a+b+d &= 1.
\end{aligned}$$

In this linear programming relaxation, we can show that $\delta \geq 25$. Indeed, multiply the above relations respectively by $\frac{3}{5}, \frac{2}{5}, 0, \frac{3}{5}, \frac{1}{5}, 0, 0$ and add.

This contradicts our assumption that $\delta(P) < \lambda = \frac{-3+5\sqrt{17}}{52}$. □

Case 2.

Proof. Let $(a, b, c, d) = \frac{1}{p}(r_{2,0}, r_{2,1}, r_{2,2}, r_{1,2})$. Similar to Case 1, we see $a \geq b \geq c \geq 0$ and $d \geq c$.

Using that $C_{1,1}$ is above ∂S, we find that the fraction of valid paths going above it is at most δ. Thus $b + d \leq \delta$, since we easily see $r_{0,1} = r_{1,1} = r_{2,1} + r_{1,2} = (b + d)p$.

Using that $C_{2,2}$ is below ∂S, the fraction of valid paths going above it is at least $1 - \delta$. Hence we see $2d \geq 1 - \delta$.

Finally, we use $C_{2,3}$. The fraction of valid paths that go to the left of it is at least $1 - \delta$. (Again, if $C_{2,3}$ is not in the grid diagram since the diagram only has two columns, then the fraction is in fact 1.) The fraction of such paths is $a + 2b + 2c \geq 1 - \delta$.

Hence

$$\begin{aligned}
\delta &\geq b + d, \\
2d &\geq 1 - \delta, \\
a + 2b + 2c &\geq 1 - \delta, \\
a &\geq b, \\
b &\geq c, \\
d &\geq c, \\
c &\geq 0, \\
a + 2b + 2d &= 1.
\end{aligned}$$

Multiplying these relations respectively by $\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{2}{5}, 0, 0, -\frac{1}{5}$ and adding yields $\delta \geq \frac{2}{5}$. (Note that the last relation is an equality, hence the negative coefficient is justified.)

This again contradicts our assumption. □

Case 3.

Proof. We see $0 \leq a \leq b \leq c$, and using $C_{1,1}$ gives $b \geq 1 - \delta$. We also have $b + c = 1$. Thus $1 = b + c \geq 2b \geq 2(1 - \delta)$, hence $\delta \geq \frac{1}{2}$, contradicting our assumption. □

Case 4.

Proof. Again $a \geq b \geq 0, e \geq d \geq c \geq b$, and $a + c + d + e = \frac{r_{0,0}}{p} = 1$. Using that $C_{1,1}$ is above ∂S yields $e \leq \delta$; using $C_{2,1}$ gives $a + c \leq \delta$; using $C_{3,2}$ yields $a + 3b \geq 1 - \delta$.

Finally, we (for the first time) need Lemma 2.8 on the log-concavity of the t and r sequences. In particular, it implies that $r_{2,1}^2 \geq r_{1,1}r_{3,1}$, which after dividing by p yields $c^2 \geq bd$. Thus

$$\begin{aligned}
\delta &\geq e, \\
\delta &\geq a + c, \\
a + 3b + \delta &\geq 1, \\
c^2 &\geq bd, \\
a &\geq b \geq 0, \\
e &\geq d \geq c \geq b, \\
a + c + d + e &\geq 1.
\end{aligned}$$

(We have weakened the equality to an inequality.) We claim that any real numbers satisfying these inequalities will also satisfy $\delta > \frac{9}{25} = 0.36$, which will finish this case. Assume for the sake of contradiction that there is some solution with $\delta \leq 0.36$. Notice that multiplying all of (δ, a, b, c, d, e) by $\mu > 1$ preserves all of these inequalities, hence we may assume that $\delta = 0.36$. Hence we can assume without loss of generality that $\delta = 0.36$.

Now $a + c \leq 0.36$ and $a + c + d + e \geq 1$ so $d + e \geq 0.64$. Also, $0.36 \geq e$, so $d \geq 0.28$. Then $c^2 \geq bd \geq 0.28b$.

We have $a \geq b$ and $a \geq 0.64 - 3b$ while $a \leq 0.36 - c$, giving $0.36 - c \geq b$ and $0.36 - c \geq 0.64 - 3b$. Thus $b + c \leq 0.36$ and $3b - c \geq 0.28$. Recall that $c \geq b$.

Thus $c \leq 0.36 - b$ and $c \leq 3b - 0.28$, yielding $b \leq 3b - 0.28$ and thus $b \geq 0.14$ as well as $b \leq 0.36 - b$ and thus $b \leq 0.18$. Therefore $b \in [0.14, 0.18]$, which contradicts $0.28b \leq c^2 \leq (3b - 0.28)^2$. Thus we are done with this case.

(As it happens, the optimal constant for the system above is $\delta = \frac{-1+2\sqrt{13}}{17}$.) □

Case 5.

Proof. Again $a \geq b \geq 0, e \geq d \geq c \geq b$, and $3a+3c+2d+e = 1$. Using $C_{1,1}$ yields $2a+2c+d \geq 1-\delta$. Using $C_{2,1}$ yields $a+c \leq \delta$. Using $C_{3,2}$ yields $3a \geq 1-\delta$. Finally, using $C_{4,2}$ gives $3b+3c+2d+e \geq 1-\delta$. It is easily checked that this linear program yields $\delta \geq \frac{2}{5}$, giving the desired contradiction. □

Case 6.

Proof. Again $a \geq b \geq 0$, and $d \geq c \geq b$, and $3a + 3c + 2d = 1$. Using $C_{1,1}$ yields $2a + 2c + d \geq 1 - \delta$. Using $C_{2,1}$ yields $a+c \leq \delta$. Using $C_{3,2}$ yields $3a \geq 1-\delta$. Finally, using $C_{4,2}$ yields $3b+3c+2d \geq 1-\delta$. (These are essentially the same as last case, except $e = 0$ and we do not have $e \geq d$.)

It is easily checked that this linear program gives $\delta \geq \frac{5}{13}$, giving the desired contradiction. □

Case 7.

Proof. Again $a \geq b \geq c \geq 0$, and $e \geq d \geq c$, and $a + 2b + 2d + e = 1$. Using $C_{1,1}$ yields $b + d + e \leq \delta$; using $C_{2,1}$ yields $a \leq \delta$; using $C_{2,2}$ yields $a + 2b \geq 1 - \delta$; using $C_{3,2}$ yields $3c + 2d + e \geq 1 - \delta$. It is easily checked that this linear program yields $\delta \geq \frac{7}{19}$, giving the desired contradiction. □

Case 8.

Proof. Again $a \geq b \geq c \geq 0$, and $d \geq c$, and $a + 3b + 3d = 1$. Using $C_{1,1}$ yields $b + d \leq \delta$; using $C_{2,1}$ yields $a + b + d \leq \delta$; using $C_{3,2}$ yields $a + 3b \leq \delta$; using $C_{3,3}$ yields $a + 3b + 3c \geq 1 - \delta$. It is easily checked that this linear program yields $\delta \geq \frac{9}{23}$, giving the desired contradiction. □

Case 9.

Proof. Again $a \geq b \geq c \geq 0$, and $d \geq c$, and $a + 2b + 2d = 1$. Using $C_{1,1}$ yields $b + d \leq \delta$; using $C_{2,1}$ yields $a \leq \delta$; using $C_{2,2}$ yields $2d \leq \delta$; using $C_{3,2}$ yields $3c + 2d \geq 1 - \delta$. Finally, the log-concavity result of Lemma 2.8 gives $b^2 \geq ac$, similar to before. Thus, in particular we have

$$\begin{aligned}
\delta &\geq b + d, \\
\delta &\geq a, \\
\delta &\geq 2d, \\
3c + 2d + \delta &\geq 1, \\
b^2 &\geq ac, \\
a \geq b \geq c &\geq 0, \\
d &\geq c, \\
a + 2b + 2d &\geq 1.
\end{aligned}$$

As in Case 4, we weakened the equality to an inequality. Now assume for the sake of contradiction that there was a solution (δ, a, b, c, d) to these inequalities with $\delta < \lambda = \frac{-3+5\sqrt{17}}{52}$. Then multiply each

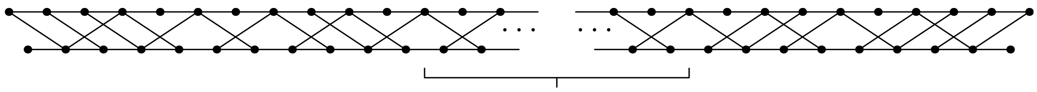

FIGURE 7. Hasse diagram of T_n, elements increasing from left to right.

of (δ, a, b, c, d) by $\mu > 1$ to make $\delta = \lambda$. All the inequalities are preserved, except that $3c+2d+\delta > 1$ and $a+2b+2d > 1$ are now strict.

Now $b+d \leq \lambda$ and $a+2b+2d > 1$ give $a > 1-2\lambda$, hence $b^2 > (1-2\lambda)c$ (or $b = c = 0$, which immediately implies $2\lambda \geq \lambda + 2d > 1$, a contradiction). Additionally, the inequalities $a \leq \lambda$ and $a+2b+2d > 1$ give $b+d > \frac{1-\lambda}{2}$. Thus we have the following list of inequalities:

$$b+d \leq \lambda, 2d \leq \lambda, c \leq d,$$
$$3c+2d > 1-\lambda, b^2 > (1-2\lambda)c, 2b+2d > 1-\lambda.$$

In particular, we have $3c > 1-\lambda-2d$ along with $c \leq d$ and $c < \frac{b^2}{1-2\lambda}$. Thus $5d > 1-\lambda$ and $3b^2 > (1-2\lambda)(1-\lambda-2d)$. This gives us

$$b+d \leq \lambda, 2d \leq \lambda,$$
$$5d > 1-\lambda, 2b+2d > 1-\lambda, 3b^2 > (1-2\lambda)(1-2\lambda-2d).$$

Now let $x = b+d$ and $y = d$. Then we know $x \in \left(\frac{1-\lambda}{2}, \lambda\right]$ and $y \in \left(\frac{1-\lambda}{5}, \frac{\lambda}{2}\right]$, as well as

$$3(x-y)^2 - (1-2\lambda)(1-\lambda-2y) > 0.$$

Since $\lambda < \frac{1}{2}$ we know that $x > y$, so that $\lambda - y \geq x - y > 0$. Thus $3(\lambda-y)^2 - (1-2\lambda)(1-\lambda-2y) > 0$. Since $\lambda = \frac{-3+5\sqrt{17}}{52}$, we can check that the roots of this quadratic are $\frac{1-\lambda}{5} < \frac{53\lambda-13}{15}$. Thus we have $y < \frac{1-\lambda}{5}$ or $y > \frac{53\lambda-13}{15} > \frac{\lambda}{2}$. But this contradicts our assertion that $y \in \left(\frac{1-\lambda}{5}, \frac{\lambda}{2}\right]$. We have our desired contradiction. ☐

Thus all cases are exhausted, and the theorem is proved. ☐

4. A Sequence of Posets with Small Balance Constant

We now construct the posets T_n and give the major calculations demonstrating that

$$\delta(T_n) \to \beta = \frac{5864893 + 27\sqrt{57}}{16812976},$$

thus proving Theorem 1.5. Many of the minor calculations have been relegated to Appendix A.

The poset T_n depends on the positive integer parameter n, and has Hasse diagram as in Figure 7. There are n copies of the trapezoidal figure in the middle section. Notice that the top strand has $2n+21$ elements, and the bottom has $2n+20$.

The grid diagram is shown in Figure 8 with $t_{i,j}$ numbers. It is made to be taller than it is wide by 1. Notice that the n trapezoidal objects from the Hasse diagram correspond to the n different 3×2 rectangles, which are denoted by $R_1, \ldots, R_n$. We let $a_m = t_{2m+8,2m+8}$ and $b_m = t_{2m+9,2m+8}$ for $1 \leq m \leq n+1$. Thus if $1 \leq m \leq n$ then a_m and b_m are the values of $t_{i,j}$ in the top left of R_m, as shown in Figure 8. Similarly, if $1 \leq m \leq n$ then a_{m+1} and b_{m+1} are the values of $t_{i,j}$ in the bottom right of R_m.

This allows us to (using the $t_{i,j}$ recurrences) determine the values at every point in the grid. Notice that $a_{m+1} = 3a_m + 3b_n$ and $b_{m+1} = 4a_m + 6b_m$ since the pair (a_{m+1}, b_{m+1}) corresponds to the bottom right portion of R_m.

We can also explicitly compute $t_{i,j}$ for $0 \leq i \leq 11$ and $0 \leq j \leq 10$, although the results are not pictured here. Such computation yields $(a_1, b_1) = (19212, 35784)$. This allows us to solve the linear

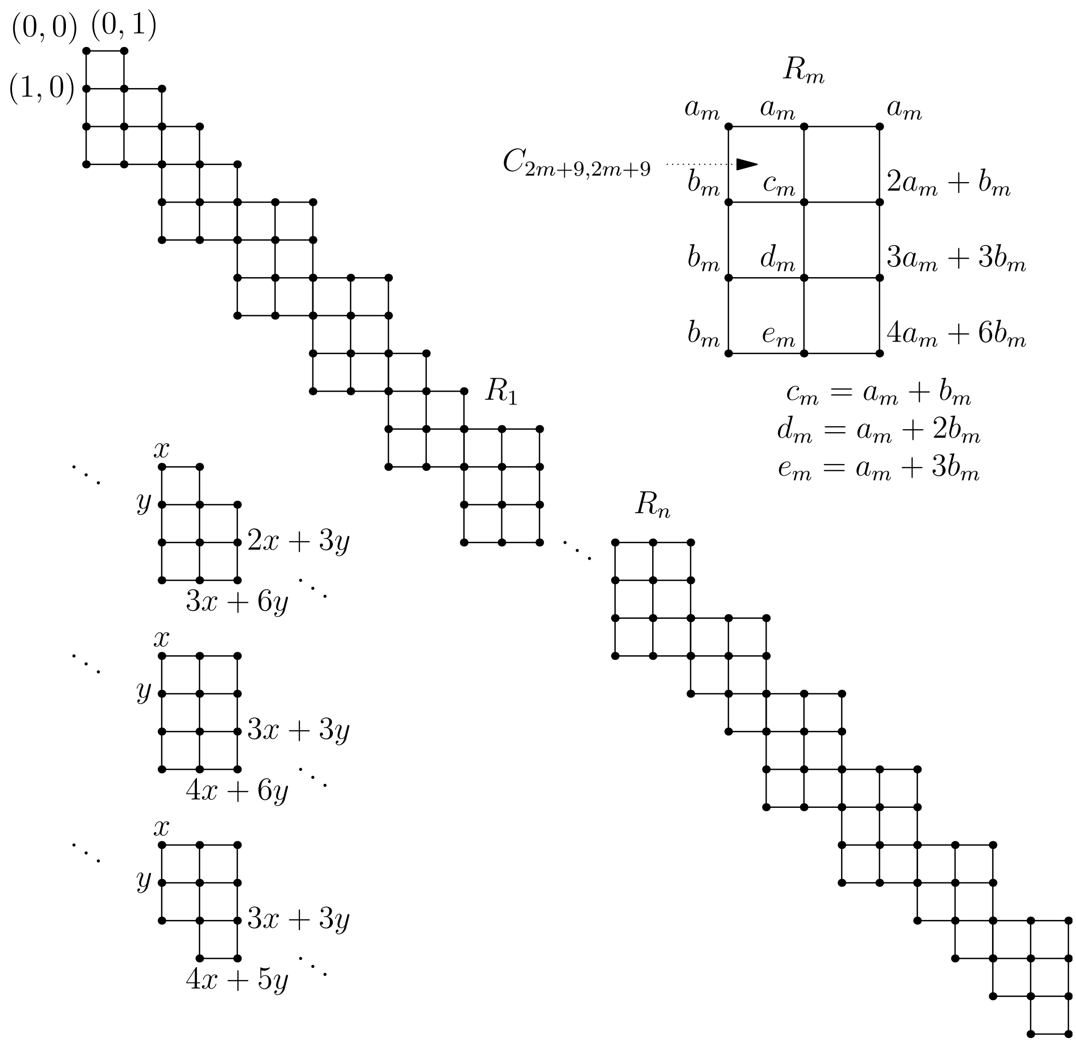

FIGURE 8. Grid diagram of T_n, with $t_{i,j}$ relations.

recurrences for (a_m, b_m), hence determining $t_{i,j}$ values within $R_1, \ldots, R_n$. Additionally, with the pair (a_{n+1}, b_{n+1}) we can determine the $t_{i,j}$ values for the bottom portion of the grid diagram of T_n. In particular, we can check that $t_{2n+21,2n+20} = p = 16572a_{n+1} + 19212b_{n+1}$.

Notice that the grid diagram is symmetric under 180 degree rotation, which means that $r_{i,j} = t_{2n+21-i,2n+20-j}$. Now, the total number of paths through (i,j) is $r_{i,j}t_{i,j} = t_{2n+21-i,2n+20-j}t_{i,j}$. For a given cell $C_{i,j}$, we want to calculate the fraction of paths that go below or above it. Notice that for $C_{1,1}$, the number of paths above it equals the number of paths through $(0,1)$, which is $r_{0,1} = t_{2n+21,2n+19} = 5781a_{n+1} + 6702b_{n+1}$. Hence the fraction is

$$\mathbb{P}(y \prec x) = \frac{5781a_{n+1} + 6702b_{n+1}}{16572a_{n+1} + 19212b_{n+1}},$$

where y is the smallest of the chain of length $2n+20$ while x is the smallest of the chain of length $2n+21$.

Solving the recurrence shows that a_n and b_n are combinations of $(9 \pm \sqrt{57})^n$, and that the above fraction limits to $\beta = \frac{5864893+27\sqrt{57}}{16812976}$ as $n \to \infty$.

We claim that the above probability is the closest to $\frac{1}{2}$, i.e. it equals the balance constant $\delta(T_n)$. This is enough to finish.

We have complete control over the number of paths through any square: for example, the number of paths through the top left corner of R_m is $a_m \cdot (4a_{n+1-m} + 6b_{n+1-m})$. Furthermore, most cells $C_{i,j}$ are already blue or red (which corresponds to $\mathbb{P}(x \prec y)$ values of $0, 1$). The ones that are not fall into finitely many classes: the ones at the two ends, and ones within R_m. Each of these have fraction explicitly computable as a fraction of these recurrent sequences. For instance, $\frac{a_m(4a_{n+1-m}+6b_{n+1-m})}{16572a_{n+1}+19212b_{n+1}}$ is one such ratio. Thus the matter at hand reduces to finitely many inequalities of linearly recurrent sequences (for which we have explicit formulas). Checking the details is not so interesting. Refer to Appendix A for these details.

This justifies Theorem 1.5.

5. Further remarks

5.1. **Optimal Constants.** Using computers to extend the casework in the method we use a little further seems to indicate that, in fact, for width 2 posets not obtainable from $\mathbf{1}$ and $\mathcal{E}$ using direct sums, we have $\delta(P) \geq 0.348842$ or so. However, due to numerical precision issues it is not stated as a result here. Nevertheless, we expect it to not be hard to use quantifier elimination programs to verify the constant to this level of precision. There seem to be fundamental obstructions to further exploring the tree of cases, however: at some point it seems to be impossible to effectively prune the branches of the tree while approaching the optimal constant.

5.2. **Conjectures and Further Questions.** Regardless of the difficulties mentioned above, we ask whether the family exhibited above is optimal.

Conjecture 5.1. *If P is a finite, width 2 poset that cannot be formed from $\mathbf{1}$ and $\mathcal{E}$ using the operation of direct sum, then*

$$\delta(P) \geq \beta.$$

Numerical results of Peczarski [14] on small posets suggest that the optimal constant is near 0.348843. Combining with the speculations above, this suggests that β is in roughly the right range to be the optimal constant. We can also ask whether our results extend to *all* posets, not just width 2.

Conjecture 5.2. *There exists an absolute constant $\varepsilon > 0$ such that if P is a finite poset not obtainable from $\mathbf{1}$ and $\mathcal{E}$ using direct sums, then*

$$\delta(P) \geq \frac{1}{3} + \varepsilon.$$

(Can we take $\varepsilon = \beta - \frac{1}{3}$?)

Olson and Sagan [12] asked if, given any poset P of width at least 3, there exists a poset Q with smaller width such that $\delta(Q) > \delta(P)$. We ask a more general and precise question.

Conjecture 5.3. *Let $\mathcal{B}_w = \{\delta(P) : P$ is a finite poset of width $w\}$ and let $\delta_w = \inf \mathcal{B}_w$. Then*

$$0 = \delta_1 < \frac{1}{3} = \delta_2 < \delta_3 < \cdots.$$

Kahn and Saks [9] pose the following conjecture.

Conjecture 5.4.

$$\lim_{w\to\infty} \delta_w = \frac{1}{2}.$$

Other questions about balance constants of partial orders can be found in the survey of Brightwell [3]. We conclude by asking the following.

Question 5.5. What, in general, can be said about the topologies of $\mathcal{B}$ and $\mathcal{B}_w$? What can be said of the structure of the fibers $\delta^{-1}(r)$ for $r \in \mathbb{Q} \cap \left[0, \frac{1}{2}\right]$?

Acknowledgements

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Appendix A. Construction Computations

We use the notations and facts introduced in Section 4. First we will study and compute the values of $t_{i,j}$ for all relevant points (i,j). Then, using the fact that the number of valid paths passing through (i,j) is $t_{i,j}r_{i,j} = t_{2n+21-i,2n+20-j}t_{i,j}$ as noted in Section 4, we compute the number of valid paths going above and below each cell $C_{i,j}$. We then relate these fractions to the balance constant and prove the required inequalities. In the end there will be 27 inequalities in the positive integer n and 3 inequalities in the positive integers $1 \le m \le n$.

For convenience, we will denote $p_{i,j} = t_{i,j}r_{i,j}$ and $p = p_{0,0}$. Note that p is the number of linear extensions of T_n.

A.1. Computing $t_{i,j}$.

Using $t_{0,0} = 1$ and the recurrence for $t_{i,j}$ from Section 2 allows us to explicitly compute $t_{i,j}$ for $0 \le i \le 11$ and $0 \le j \le 10$. They are compiled in Figure 9.

Notice that, as mentioned in Section 4, this means $(a_1, b_1) = (19212, 35784)$. Now, we also want to compile the values of $t_{2n+21-i,2n+20-j}$ for $0 \le i \le 11$ and $0 \le j \le 10$. We can use the recurrence of Section 2 along with the initial values $t_{2n+10,2n+10} = a_{n+1}$ and $t_{2n+11,2n+10} = b_{n+1}$ to compute these as linear combinations of a_{n+1} and b_{n+1}.

Thus we can write $t_{2n+21-i,2n+20-j} = c_{i,j}a_{n+1} + d_{i,j}b_{n+1}$ for positive integers $c_{i,j}$ and $d_{i,j}$ when $0 \le i \le 11$ and $0 \le j \le 10$. These are tabulated in Figure 10 and Figure 11.

Putting it all together, we see that the number of valid paths $p_{i,j}$ through (i,j) when $0 \le i \le 11$ and $0 \le j \le 10$ is precisely $t_{i,j}t_{2n+21-i,2n+20-j} = (t_{i,j}c_{i,j})a_{n+1} + (t_{i,j}d_{i,j})b_{n+1}$. The total number of valid paths is this expression evaluated at $(i,j) = (0,0)$, which is $p = 16572a_{n+1} + 19212b_{n+1}$.

Recall that the fraction of valid paths going above $C_{1,1}$ was

$$\frac{p_{0,1}}{p} = \frac{5781a_{n+1} + 6702b_{n+1}}{16572a_{n+1} + 19212b_{n+1}} < \frac{1}{2},$$

which we claimed to be the balance constant $\delta(T_n)$. Thus we must show that every other fraction is either greater than $\frac{1}{2}$ or at most this quantity.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	0	0	0	0	0	0	0	0	0
1	1	2	2	0	0	0	0	0	0	0	0
2	1	3	5	5	0	0	0	0	0	0	0
3	1	4	9	14	14	0	0	0	0	0	0
4	0	0	9	23	37	37	37	0	0	0	0
5	0	0	9	32	69	106	143	0	0	0	0
6	0	0	0	0	69	175	318	318	318	0	0
7	0	0	0	0	69	244	562	880	1198	0	0
8	0	0	0	0	0	0	562	1442	2640	2640	0
9	0	0	0	0	0	0	562	2004	4644	7284	7284
10	0	0	0	0	0	0	0	0	4644	11928	19212
11	0	0	0	0	0	0	0	0	4644	16572	35784

Figure 9. Values of $t_{i,j}$ for $0 \le i \le 11$ and $0 \le j \le 10$.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	16572	5781	0	0	0	0	0	0	0	0	0
1	10791	5781	2184	0	0	0	0	0	0	0	0
2	5010	3597	2184	771	0	0	0	0	0	0	0
3	1413	1413	1413	771	300	0	0	0	0	0	0
4	0	0	642	471	300	129	36	0	0	0	0
5	0	0	171	171	171	93	36	0	0	0	0
6	0	0	0	0	78	57	36	15	4	0	0
7	0	0	0	0	21	21	21	11	4	0	0
8	0	0	0	0	0	0	10	7	4	1	0
9	0	0	0	0	0	0	3	3	3	1	0
10	0	0	0	0	0	0	0	0	2	1	0
11	0	0	0	0	0	0	0	0	1	1	1

FIGURE 10. Values of $c_{i,j}$ for $0 \leq i \leq 11$ and $0 \leq j \leq 10$.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10
0	19212	6702	0	0	0	0	0	0	0	0	0
1	12510	6702	2532	0	0	0	0	0	0	0	0
2	5808	4170	2532	894	0	0	0	0	0	0	0
3	1638	1638	1638	894	348	0	0	0	0	0	0
4	0	0	744	546	348	150	42	0	0	0	0
5	0	0	198	198	198	108	42	0	0	0	0
6	0	0	0	0	90	66	42	18	5	0	0
7	0	0	0	0	24	24	24	13	5	0	0
8	0	0	0	0	0	0	11	8	5	2	0
9	0	0	0	0	0	0	3	3	3	2	1
10	0	0	0	0	0	0	0	0	1	1	1
11	0	0	0	0	0	0	0	0	0	0	0

FIGURE 11. Values of $d_{i,j}$ for $0 \leq i \leq 11$ and $0 \leq j \leq 10$.

We just need to compute the fraction of valid paths that go above and below each $C_{i,j}$, where $1 \leq i \leq 2n+21$ and $1 \leq j \leq 2n+20$. As noted earlier, these correspond to the probabilities $\mathbb{P}(x \prec y)$ associated to the poset T_n. If $C_{i,j}$ is a colored cell, then the desired fractions are trivially 0 and 1 in some order and we need not consider $C_{i,j}$. Therefore we can suppose that $C_{i,j}$ is uncolored.

Recall that the rectangle R_m in the grid diagram of T_n was composed of the 6 cells $C_{i,j}$ for $2m+9 \leq i \leq 2m+11$ and $2m+9 \leq j \leq 2m+10$.

Now, there are two cases to consider: $C_{i,j}$ is uncolored with $1 \leq i \leq 11$ and $1 \leq j \leq 10$, or $C_{i,j}$ is in some rectangle R_m for $1 \leq m \leq n$ where $(i,j) \in \{(2m+9, 2m+9), (2m+9, 2m+10), (2m+10, 2m+10)\}$. The first case, we can see, consists of 27 different cells to explicitly consider.

We only need to consider half of the uncolored cells because of the rotational symmetry of the grid diagram of T_n.

A.2. **The First 27 Cells.** This is the case $C_{i,j}$ for $1 \leq i \leq 11$ and $1 \leq j \leq 10$. We compute either the number of valid paths going above or below $C_{i,j}$ as indicated in the case.

As an example, we compute the number of valid paths going under $C_{6,5}$. Notice that every valid path goes through exactly one of the points $(6,4)$, $(5,5)$, or $(4,6)$. Furthermore, the valid paths going below $C_{6,5}$ are precisely those going through $(6,4)$. Thus our desired count is $p_{6,4} = 69 \cdot 78a_{n+1} + 69 \cdot 90b_{n+1}$.

Figure 12 lists the number of valid paths going either below or above $C_{i,j}$, for $1 \le i \le 11$ and $1 \le j \le 10$ such that $C_{i,j}$ is uncolored.

cell	above or below	valid path count
$C_{1,1}$	above	$5781a_{n+1} + 6702b_{n+1}$
$C_{2,1}$	below	$5010a_{n+1} + 5808b_{n+1}$
$C_{2,2}$	above	$4368a_{n+1} + 5064b_{n+1}$
$C_{3,1}$	below	$1413a_{n+1} + 1638b_{n+1}$
$C_{3,2}$	below	$5652a_{n+1} + 6552b_{n+1}$
$C_{3,3}$	above	$3855a_{n+1} + 4470b_{n+1}$
$C_{4,3}$	below	$5778a_{n+1} + 6696b_{n+1}$
$C_{4,4}$	above	$4200a_{n+1} + 4872b_{n+1}$
$C_{5,3}$	below	$1539a_{n+1} + 1782b_{n+1}$
$C_{5,4}$	below	$5472a_{n+1} + 6336b_{n+1}$
$C_{5,5}$	above	$4773a_{n+1} + 5550b_{n+1}$
$C_{5,6}$	above	$1332a_{n+1} + 1554b_{n+1}$
$C_{6,5}$	below	$5382a_{n+1} + 6210b_{n+1}$
$C_{6,6}$	above	$5148a_{n+1} + 6006b_{n+1}$
$C_{7,5}$	below	$1449a_{n+1} + 1656b_{n+1}$
$C_{7,6}$	below	$5124a_{n+1} + 5856b_{n+1}$
$C_{7,7}$	above	$4770a_{n+1} + 5724b_{n+1}$
$C_{7,8}$	above	$1272a_{n+1} + 1590b_{n+1}$
$C_{8,7}$	below	$5620a_{n+1} + 6182b_{n+1}$
$C_{8,8}$	above	$4792a_{n+1} + 5990b_{n+1}$
$C_{9,7}$	below	$1686a_{n+1} + 1686b_{n+1}$
$C_{9,8}$	below	$6012a_{n+1} + 6012b_{n+1}$
$C_{9,9}$	above	$2640a_{n+1} + 5280b_{n+1}$
$C_{10,9}$	below	$9288a_{n+1} + 4644b_{n+1}$
$C_{10,10}$	above	$0000a_{n+1} + 7284b_{n+1}$
$C_{11,9}$	below	$4644a_{n+1} + 0000b_{n+1}$
$C_{11,10}$	below	$16572a_{n+1} + 0000b_{n+1}$

FIGURE 12. Number of valid paths going either above or below $C_{i,j}$.

To finish the case of $1 \le i \le 11$ and $1 \le j \le 10$, we just need to show that each of these counts is less than the count in the first row. We already know that the first row is $p_{0,1} < \frac{p}{2}$, and since $\frac{p_{0,1}}{p}$ is our claimed balance constant, proving this will finish.

By inspection, all the rows except the row corresponding to $C_{9,8}$, $C_{10,9}$, $C_{10,10}$, and $C_{11,10}$ give counts clearly less than the top row. We need to show that $6012a_{n+1} + 6012b_{n+1}$, and $9288a_{n+1} + 4644b_{n+1}$, and $7284b_{n+1}$, and $16572a_{n+1}$ are each less than $5781a_{n+1} + 6702b_{n+1}$. If we write $f = \frac{a_{n+1}}{b_{n+1}}$, then these four inequalities are respectively equivalent to $f < \frac{230}{77}$, and $f < \frac{98}{167}$, and $f > \frac{194}{1927}$, and $f < \frac{2234}{3597}$.

Thus we need to show $\frac{a_{n+1}}{b_{n+1}} \in \left(\frac{194}{1927}, \frac{98}{167}\right)$. Recall that $(a_1, b_1) = (19212, 35784)$, and $a_{m+1} = 3a_m + 3b_m$, and $b_{m+1} = 4a_m + 6b_m$. Thus, letting $f_i = \frac{a_i}{b_i}$, we find $f_1 = \frac{19212}{35784}$ and $f_{m+1} = \frac{3f_m+3}{4f_m+6}$. It is not hard to show that f_i is a strictly increasing positive sequence with limit $\frac{-3+\sqrt{57}}{8}$. Furthermore, clearly f_1 and the limit are within the required establishes the desired result.

A.3. **The Cell $C_{i,j}$ is in R_m.** This is the final case to check. As remarked earlier, we only need to consider $(i,j) \in \{(2m+9, 2m+9), (2m+9, 2m+10), (2m+10, 2m+10)\}$, which is half of the cells in all the R_m rectangles, because we can capitalize on the rotational symmetry of the grid diagram of T_n.

Case 1. $(i,j) = (2m+9, 2m+9)$

Then $C_{i,j}$ is the top left cell of R_m. We will compute the number of paths going above $C_{i,j}$, which we see is precisely $p_{i-1,j} = p_{2m+8,2m+9}$, that is, the number of paths through the top right corner of $C_{i,j}$.

We see, referencing Figure 8, that $t_{2m+8,2m+9} = a_m$. Also, $r_{2m+8,2m+9} = t_{2n-2m+13,2n-2m+11} = a_{n+1-m} + 3b_{n+1-m}$. (This value is depicted as e_{n+1-m} in Figure 8.)

Recall that it suffices to show $p_{i-1,j} < p_{0,1}$, or

$$a_m(a_{n+1-m} + 3b_{n+1-m}) < 5781a_{n+1} + 6702b_{n+1},$$

subject to the condition $1 \leq m \leq n$.

It is sufficient to show that $a_m(a_{n+1-m} + 3b_{n+1-m}) \leq (2a_m + b_m)b_{n-m+1}$, which then reduces the desired inequality to one that we later prove in Case 3.

This new inequality is equivalent to

$$a_m(a_{n+1-m} + b_{n+1-m}) \leq b_m b_{n-m+1},$$
$$\frac{a_m}{b_m} \cdot \left(\frac{a_{n-m+1}}{b_{n-m+1}} + 1\right) \leq 1.$$

Recall from earlier that $f_i = \frac{a_i}{b_i}$ for $i \geq 1$ is an increasing positive sequence with limit $\frac{-3+\sqrt{57}}{8}$. Since

$$\left(\frac{-3+\sqrt{57}}{8}\right) \cdot \left(\frac{-3+\sqrt{57}}{8} + 1\right) < 1,$$

the desired inequality is true.

Case 2. $(i,j) = (2m+9, 2m+10)$

Then $C_{i,j}$ is the top right cell of R_m. We will compute the number of paths going above $C_{i,j}$, which we see is precisely $p_{i-1,j} = p_{2m+8,2m+10}$.

We see, referencing Figure 8, that $t_{2m+8,2m+10} = a_m$. Also, $r_{2m+8,2m+10} = t_{2n-2m+13,2n-2m+10} = b_{n-m+1}$.

It suffices to show $p_{i-1,j} < p_{0,1}$, or

$$a_m b_{n-m+1} < 5781a_{n+1} + 6702b_{n+1},$$

subject to the condition $1 \leq m \leq n$. This inequality is clearly weaker than that of Case 3, so we again defer to that case.

Case 3. $(i,j) = (2m+10, 2m+10)$

Then $C_{i,j}$ is the middle right cell of R_m. We will compute the number of paths going above $C_{i,j}$, which we see is precisely $p_{i-1,j} = p_{2m+9,2m+10}$.

We see, referencing Figure 8, that $t_{2m+9,2m+10} = a_m + 2b_m$. Additionally, $r_{2m+9,2m+10} = t_{2n-2m+12,2n-2m+10} = b_{n-m+1}$.

It suffices to show $p_{i-1,j} < p_{0,1}$, or

$$(2a_m + b_m)b_{n-m+1} < 5781a_{n+1} + 6702b_{n+1},$$

subject to the condition $1 \leq m \leq n$.

Recall that $f_i = \frac{a_i}{b_i}$ for $i \geq 1$ is an increasing positive sequence with limit $\frac{-3+\sqrt{57}}{8}$. Hence

$$\begin{aligned}(2a_m + b_m)b_{n-m+1} &< \left(2 \cdot \frac{-3+\sqrt{57}}{8} + 1\right) b_m b_{n-m+1} \\ &\leq \frac{1+\sqrt{57}}{4} b_1 b_n \\ &\leq 44151a_n + 57555b_n \\ &= 5781a_{n+1} + 6702b_{n+1}.\end{aligned}$$

The first inequality follows from $f_m < \frac{-3+\sqrt{57}}{8}$, the second inequality follows from the log-convexity of $(b_i)_{i=1}^n$, the third inequality follows from $b_1 = 35784$ and

$$\frac{-16203 + 2982\sqrt{57}}{14717} < \frac{19212}{35784} \leq f_n,$$

and the last relation is an equality, using the recurrence $a_{n+1} = 3a_n + 3b_n$ and $b_{n+1} = 4a_n + 6b_n$.

The reason $(b_i)_{i=1}^n$ is log-convex is that for positive integers i we have

$$b_i = \frac{(12906 + 2542\sqrt{57})\left(\frac{9+\sqrt{57}}{2}\right)^i + (-12906 + 2542\sqrt{57})\left(\frac{9-\sqrt{57}}{2}\right)^i}{\sqrt{57}},$$

where $-12906 + 2542\sqrt{57} > 0$. We can then actually check that the function $b : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$b(x) = \frac{(12906 + 2542\sqrt{57})\left(\frac{9+\sqrt{57}}{2}\right)^x + (-12906 + 2542\sqrt{57})\left(\frac{9-\sqrt{57}}{2}\right)^x}{\sqrt{57}}$$

when $x > 0$ satisfies $b''(x) > 0$ for all $x > 0$, which implies the desired log-convexity at values $b_i = b(i)$, where $1 \leq i \leq n$. (In general, a function $\log(\alpha_1 \cdot \alpha_2^x + \alpha_3 \cdot \alpha_4^x)$ has second derivative $\frac{\alpha_1\alpha_3(\alpha_2\alpha_4)^x(\log\alpha_2 - \log\alpha_4)^2}{(\alpha_1\alpha_2^x + \alpha_3\alpha_4^x)^2} > 0$.)

A.4. **Conclusion.** All the cases are complete, finishing our computations. Thus, indeed,

$$\delta(T_n) = \frac{5781a_{n+1} + 6702b_{n+1}}{16572a_{n+1} + 19212b_{n+1}} \to \frac{5864893 + 27\sqrt{57}}{16812976},$$

as claimed.

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