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Lecture 1

Introduction

Geometry

Geometry is the study of study "shapes" or "spaces" (e.g. subsets of $\mathbb{R}^n$; metric spaces) with distances, angles,… There is a focus on manifolds (i.e. spaces that look locally like $\mathbb{R}^n$ for some n.

Example. The surface of the earth is a 2 dimensional manifold embedded in $\mathbb{R}^3$

Example. The shape of a Y in $\mathbb{R}^2$ does not look like $\mathbb{R}^n$ at the intersection of the three lines although it looks like $\mathbb{R}$ on each line individually

Goals:

1. When is a subset of $\mathbb{R}^n$ a manifold?
2. Whitney's Theorem: Every n-dimensional manifold can be embedded in $\mathbb{R}^{2n+1}$

Topology

Topology is so called "rubber band geometry", it is the study of topological properties of spaces. Topological properties do not change under deformations like bending or stretching (no breaking).

Example. The square, triangle, and circle all have different geometry but are the same topologically.

Example. Two circles and one circle are not the same topologically

Example. 2-dimensional manifolds

cube = sphere $\neq$ plane $\neq$ donut (torus) $\neq$ Mobius band $\neq$ Klein bottle (embedded in $\mathbb{R}^4$)

There are many applications of these concepts. To list a few:

- **Physics**
    - Cosmology: shape of the universe (3D)
    - String theory: universe is 10 or 11 dimensional
- **CS**: data analysis
- **Biology**: shape of DNA, RNA
- **Economics**: models for the economy
- **Math**: analysis, differential geometry, algebraic geometry, …

Metric Geometry

Definition. *A* <u>*metric space*</u> *is a set X together with a function $d: X \times X \to \mathbb{R}$ satisfying:*

$$\begin{array}{lll} (1) & d(x,y) \geq 0 & \forall x,y \\ & d(x,y) = 0 & \Leftrightarrow x = y \\ (2) & d(x,y) = d(y,x) & \forall x,y \\ (3) & d(x,y) \leq d(x,z) + d(z,y) & \forall x,y,z \end{array}$$

Example.

$$\begin{array}{rl} \text{Discrete Metric} & d_0(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \\ \text{Manhattan Metric} & d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \\ \text{Euclidean Metric} & d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \\ \text{Infinity Metric} & d_\infty(x,y) = \max_{i=1,\ldots n} |x_i - y_i| \end{array}$$

Definition. *Given (X,d) is a metric space and $a \in X$, $\epsilon > 0$, an* <u>*open ball*</u> $B_\epsilon(a) = \{x \in X | d(x, a) < \epsilon\}$

Example. In $\mathbb{R}^2$ we have

$$\begin{array}{rcl} \text{Discrete Metric} & \Rightarrow & \text{point} \\ d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| & \Rightarrow & \text{diamond shaped} \\ d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} & \Rightarrow & \text{circle} \\ d_\infty(x,y) = \max_{i=1,\ldots n} |x_i - y_i| & \Rightarrow & \text{square} \end{array}$$

Definition. *(X,d) is a metric space. A subset $U \subset X$ is called* <u>*open*</u> *with respect to d if for every $a \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset U$.*

Example. $(0,1) \subset \mathbb{R}$ is open; $(0,1]$ is not open.

Example. In the discrete metric, every subset is open

Example. In any metric space X, the subset $\emptyset$ and X are open.

Lemma. *Given (X,d) a metric space $\Rightarrow B_\delta(x_0)$ is open, $\forall x_0 \in X$, $\delta > 0$*

Proof. Given $a \in B_\delta(x_0)$, choose ϵ such that

$$0 < \epsilon < \delta - d(x_0 - a)$$

To show $B_\epsilon(a) \subset B_\delta(x_0)$: $x \in B_\epsilon(a) \Rightarrow d(x,a) < \epsilon$

$$d(x,x_0) \le d(x,a) + d(a,x_0) < \epsilon + d(a,x_0) < \delta$$

which implies that $x \in B_\delta(x_0)$ □

Definition. *Given (X,d) a ,a metric space. We say that a sequence $\{x_n\}$ converges to $x \in X$ if* $\lim_{n\to\infty} d(x_n, x) = 0$

Lecture 2

Limits and Continuity

Definition. *Given (X,d) is a metric space and $\{a_n\} \subset X$ and $a \in X$, we define:*

$$\lim_{n\to\infty} a_n = a \quad \text{if} \quad \lim_{n\to\infty} d(a_n, a) = 0$$

Proposition. *If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$, then $a = b$.*

Proof. Suppose $a \neq b$. Then $d(a,b) > 0$. Let $\epsilon = \frac{d(a,b)}{3}$. Then $B_\epsilon(a) \cap B_\epsilon(b) = \emptyset$.

$x \in B_\epsilon(a) \cap B_\epsilon(b) \Rightarrow d(x,a) < \epsilon$ and $d(x,b) < \epsilon$ then $3\,\epsilon = d(a,b) \le d(x,a) + d(x,b) < 2\,\epsilon$ then $\exists N$, $\forall n > N$, $a_n \in B_\epsilon(a)$ and $\exists N'\ \forall n > N'$, $a_n \in B_\epsilon(b)$. Then fro $n > \max(N, N')$, $a_n \in B_\epsilon(a) \cap B_{\epsilon(b)} = \emptyset$, which is a contradiction. □

Definition. *Two metrics d and d' on X are equivalent if $\exists c, c' > 0$ such that:*

$$c\, d(x,y) \le d'(x,y) \le c'\, d(x,y)$$

$\forall x, y \in X$

Example. In $\mathbb{R}^n$, d_1, d_2, d_∞ are all equivalent.

$$d_\infty \le d_2 \le d_1 \le n\, d_\infty$$

However, d_0 is not equivalent to the rest

Limits of Functions

Definition. *(1) Given X, Y are metric spaces and $f: X \to Y$ is a function. We define:*

$$\lim_{x\to x_0} f(y_0)$$

if $\forall \epsilon > 0, \forall \delta > 0$ *such that for all $x \in B_\delta(x_0) - \{x_0\}$, we have $f(x) \in B_\epsilon(y_0)$*

Definition. *(2) Given X, Y are metric spaces and $f: X \to Y$ is a function. We define:*

$$\lim_{x\to x_0} f(x) = y_0$$

if for every sequence $x_n \to x_0$, $(x_n \neq x_0)$ *we have* $f(x_n) \to y_0$

Theorem. *The two definitions are equivalent.*

$\lim_{x \to x_0} f(x) = y$ *if for every sequence* $x_n \to x_0$ *we have that* $f(x_n) \to y_0$.

$$\Updownarrow$$

$\lim_{x \to x_0} f(x) = y$ *if* $\forall \epsilon > 0$, $\exists \delta > 0$ *such that for all* $x \in B_\delta(x_0) - \{x_0\}$, *we have* $f(x) \in B_\epsilon(y_0)$

Proof.

Direction 1: Suppose $\lim_{x \to x_0} f(x) = y_0$ according to definition 1. Suppose $x_n \to x_0$, $x_n \neq x_0$ $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(x) \in B_\epsilon(y_0)$ for $x \in B_\delta(x_0) - \{x_0\}$.

This implies that almost all x_n are in $B_\delta(x_0) - \{x_0\} \Rightarrow f(x_n) \in B_\epsilon(y_0)$. $\forall \epsilon > 0$, almost all $f(x_n)$ are in $B_\epsilon(y_0)$. This implies that $f(x_n) \to y_0$ so definition 2 works.

Direction 2: Suppose $\lim_{x \to x_0} f(x) = y_0$ according to definition 2. Suppose by contradiction that $\exists \epsilon > 0$, $\forall \delta > 0$, $\exists x \in B_\delta(x_0)$, $f(x) \notin B_\epsilon(y_0)$. Take $\delta = \frac{1}{n} \Rightarrow \exists x_n \neq x_0, x_n \in B_\delta(x_0)$, $f(x_n) \notin B_\epsilon(y_0)$.

Then $x_n \to x_0$ because $d(x_n, x_0) < \frac{1}{n}$; but $f(x_n) \not\to y_0$. This contradicts definition 2. Thus, definition 2 implies definition 1. □

Definition. $f: X \to Y$ *is called continuous at* $x_0 \in X$ *if* $\lim_{x \to x_0} f(x) = f(x_0)$

Definition. f *is called continuous if it is continuous at all* $x_0 \in X$

Theorem. $f: X \to Y$ *is continuous* $\Leftrightarrow$ *for every open set* $U \subset Y$, *the subset* $f^{-1}(U) \subset X$ *is open.*

Proof. $\Rightarrow$ if f is continuous. Suppose for contradiction that $f^{-1}(U)$ is not open. Then that implies that there exists a point x_0 on the boundary of $f^{-1}(U)$ such that every ball around x_0 is not contained in $f^{-1}(U)$. Now, take a sequence of balls with radius $1/n$ and take a sequence x_n that is within these balls but not in $f^{-1}(U)$. We have that $x_n \to x_0$ and since f is continuous that $f(x_n) \to f(x_0) \in U$. At the same same time by continuity we know that since $f(x_0) \in U$ and since U is open that $\exists \delta > 0$ such that $B_\delta(f(x_0)) \subset U$. Thus, $f(x_n) \in B_\delta(f(x_0)) \subset U$ but this is a contradiction.

$\Longleftarrow$ Assume preimages of open sets are open. Let $x_0 \in X$, $\epsilon > 0$. We want to show that $\exists \delta > 0$ such that $f(x) \in B_\epsilon(f(x_0))$ for $x \in B_\delta(x_0)$ i.e. $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$ and $x_0 \in f^{-1}(B_\epsilon(f(x_0)))$ because $f(x_0) \in B_\epsilon(f(x_0))$. This implies that there exists δ, $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$ □

Remark. If f is continuous and $U \subset X$ is open, the $f(U)$ is not necessarily open. For example, consider $X = Y = \mathbb{R}$, $U = (-1, 1)$ is open, $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is open, and $f(u) = [0, 1)$ not open.

Theorem. *If X,Y,Z are metric spaces and* $f: X \to Y$ *and* $g: Y \to Z$ *are both continuous, then* $g \circ f: X \to Z$ *is continuous.*

Proof. Consider $x_n \to x$. Since f is continuous, $f(x_n) \to f(x)$. Since g is continuous, $g(f(x_n)) \to g(f(x))$. Therefore, $g \circ f$ is continuous. □

Lecture 3

Definition. *Between two metric spaces* (X, d) *and* (Y, d'), *we have that* $f: X \to Y$ *is called an isometry if it is* ***bijective*** *and* $\boldsymbol{d(x, y) = d'(f(x), f(y))}$, $\forall x, y \in X$.

Example. Non-Examples:

(0,1) and (0,2) are not isometric

$\mathbb{R}$ and (0,1) are not isometric

$\mathbb{R}$ and (0,∞) are not isometric

The triangle and the square are not isometric.

Definition. *$f\colon X \to Y$ is called a homeomorphism if f is **bijective** and both **f and f^{-1} are continuous**.*

Example. Examples:

Open intervals $X = (a, b)$ and $Y = (c, d)$ as subset of $\mathbb{R}$

Real line $X = \mathbb{R}$ and the half line $Y = (0, \infty)$

Real line $X = \mathbb{R}$ and the open interval $Y = (0, 1)$

The circle and the square

Proposition. *f is an isometry $\Rightarrow$ f is a homeomorphism*

Proof. We already know that f is bijective. We want to show that f and f^{-1} are continuous.

f is continuous: We want to show that $\forall x_0 \in X$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \in B_\delta(x_0) \Rightarrow f(x) \in B_\epsilon(f(x_0))$ and $d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \epsilon$. Just choose $\delta = \epsilon$.

f^{-1} is continuous: similarly proves and use f^{-1} is an isometry as well. $\square$

Topological Properties of Metric Spaces

These properties are those that can be formulated in terms of open sets (without mentioning distance).

Example. Convergence, limits of functions, continuity, homeomorphisms.

Theorem. *(Main Properties of Open Sets). If (X, d) is a metric space, then:*

1. *The union of any collection of open sets is open*

2. *THe intersection of any finite collection of open sets is open*

Remark. 2 is false if we don't require finite: $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subset \mathbb{R}$ and $\bigcap_n U_n = \{0\}$ is not open.

Proof.

Part 1: The collection of open sets $\{U_\alpha\}_{\alpha \in I}$, where I is an index set. We want to show that $\bigcup_{\alpha \in I} U_\alpha$ is open. Let $x \in \bigcup_{\alpha \in I} U_\alpha$. This implies that $x \in U_\alpha$ for some α, which implies that $\exists \epsilon > 0$ and $B_\epsilon(x) \subset U_\alpha$, which implies that $B_\epsilon(x) \subset \bigcup_{\alpha \in I} U_\alpha$ so the union is open.

Part 2: $\{U_i\}_{i=1,...n}$, U_i is open. Let $x \in \bigcap_{i=1}^n U_i$ and $x \in U_i$ for all i, then $\exists \epsilon_i > 0$, $B_{\epsilon_i}(x) \subset U_i$. Let $\epsilon = \min_{i=1,...n} \epsilon_i > 0$. Then $B_\epsilon(x) \subset B_{\epsilon_i}(x) \subset U_i$ $\forall i$. Then $B_\epsilon(x) \subset \bigcap_{i=1}^n U_i$, so the intersection is open. ☐

Definition. *A topological space is a pair (X, T) where X i a set and T is a collection of subsets of X with the properties:*

$$
\begin{aligned}
&(1) \quad \emptyset, X \in T \\
&(2) \quad \bigcup_\alpha U_\alpha \in T \quad (\underline{\textbf{\textit{any}}} \text{ union of elements of } T \text{ is in } T) \\
&(3) \quad \bigcap_{i=1}^N U_{\alpha_i} \in T \; (\textbf{\textit{finite}} \text{ intersection of elements of } T \text{ is in } T)
\end{aligned}
$$

Remark. Equivalent metrics produce the same topology. A topology that coms from a metric space is called metrizable.

Definition. *(X,T), (X', T') are topological spaces and $f: X \to X'$ is called continuous if for every $U \subset X'$ open, the preimage $f^{-1}(U) \subset X$ is open.*

Definition. *$f: X \to X'$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.*

Example.

Trivial Topology:$T = \{\emptyset, X\}$ and it is not metrizable

Points are not closed and limits are not unique (they converge to any x since any open set around x necessarily contains all the terms of the sequence)

Discrete Topology: $T = \{\text{all subsets of X}\}$ and the corresponding metric is metric: $d_0(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

Standard Topology:$d_1, d_2, d_\infty, \ldots$ are the metrics

Trivial Topology	$T = \{\emptyset, X\}$	Non-metrizable
Discrete Topology	$T = \{\text{all subsets of X}\}$	metric: $d_0(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$
Standard Topology		$d_1, d_2, d_\infty, \ldots$

Lecture 4

More on Topological Spaces

Definition. *A set X is called countably infinite if there is a bijection $f: \{1, 2, 3, \ldots\} \to X$*

Definition. *X is countable if its either finite or countably infinite*

Example.

X is countable if it is either finite or countably infinite.

A set is countably infinite if there is a bijection from $f: \mathbb{N} \to X$.

$\mathbb{Z}, \mathbb{Q}$ are countably infinite. $\mathbb{R}$ is not countable.

Remark. Subsets of countable sets are countable, and countable unions of countable sets are countable

Definition. *Topological convergence:* $x_n \to x$ *if* $\forall U \subset X$ *open with* $x \in U$ *there exists* $N \in \mathbb{N}$ *such that* $\forall n \geq N$*, we have that* $x_n \in U$

Subspace Topology (or relative topology)

Definition. *Given* (X, T) *a topological space; and* $Y \subset X$. $T_Y = \{Y \cap U | U \in T\}$ *is a topology on* Y *called the subspace topology or relative topology.*

Theorem. *The subspace topology is a topology*

Proof.

1. $\emptyset = Y \cap \emptyset$, $Y = Y \cap X$; $\quad \emptyset, X \in T \Rightarrow \emptyset, Y \in T_y$
2. $\bigcup_{\alpha \in I} (U_\alpha \cap Y) = (\bigcup_{\alpha \in I} U_\alpha) \cap Y$ is open in Y
3. $\bigcap_{i=1}^{n} (U_i \cap Y) = (\bigcap_{i=1}^{n} U_i) \cap Y$ □

Example. Subsets of $\mathbb{R}^n$ are usually considered with the relative topology from the standard topology of $\mathbb{R}^n$

Note. If $U \subset Y \subset X$ and U is open in Y, it may not be open in X. For example $U = [0, 1] \subset Y = [0, 1] \subset X = \mathbb{R}$

Definition. *Given a topological space* (X, T)*, a subspace* $A \subset X$ *is closed if* $X - A$ *is open (i.e.* $X - A \in T$*)*

Properties of Closed Sets:

$$
\begin{array}{ll}
(1) & \emptyset, X \text{ are closed} \\
(2) & \bigcap_{\alpha} U_\alpha \quad \text{is closed} \quad (\underline{\textbf{any}} \text{ intersection of closed sets is closed}) \\
(3) & \bigcup_{i=1}^{N} U_{\alpha_i} \quad \text{is closed} \quad (\textbf{finite} \text{ union of closed sets is closed})
\end{array}
$$

Proof. Follows from the definition of closed sets at the complements of open sets and using the properties of open sets. □

Example. in $\mathbb{R}$, $[0, 1]$ and $[0, \infty)$ are closed, but $[0, 1)$ is not closed.

Topologies can be defined in terms of closed sets (just take the complements).

Example. The finite complement topology.

$T = \{$complements of finite subsets or $\emptyset\}$. In other words, take the closed sets to be all the finite subsets of X and the empty set.

Example. Zariski Topology

on $\mathbb{C}^n = \mathbb{R}^{2n}$. A set is Zariski closed if there is a collection of polynomial functions of n variables

$$A = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | f_k(z_1, \ldots z_n) = 0 \text{ for all } k = 1, \ldots, m\}$$

Definition. *Given two topologies T, T' on the same set X.*

***Finer**: T' is finer than T if $T \subset T'$ (i.e. it has more open sets)*

***Coarser:** T' is coarser than T if $T' \subset T$ (i.e. it has less open sets)*

Example. trivial topology $\subset$ finite complement $\subset$ standard $\subset$ discrete

coarsest —— ————————————$\rightarrow$finest

Bases

Definition. *Given a topological space (X, T), a family $\mathbb{B}$ of open sets in X is called a base for a topology on X if every open set in X is a union of sets in B.*

Example. (X, d) is a metric space, which implies that $\mathbb{B} = \{B_\epsilon(X) | \epsilon > 0, x \in X\}$ the balls form a base

Proposition. *$\mathbb{B}$ is a base $\Leftrightarrow$ $\forall U \subset X$ open and $x \in U$, $\exists V \in \mathbb{B}$ s.t. $x \in V \subset U$*

Proof. $\Longrightarrow$ If $\mathbb{B}$ is a base, $U \subset X$ open, and $x \in U$, and $U =$ union of sets in $\mathbb{B}$, then $x \in V \subset U$ and $V \in \mathbb{B}$.

$\Longleftarrow$ Let $U \subset X$ be open. For every $x \in U$, $\exists V_x \in \mathbb{B}$ such that $x \in V_x \subset U$. Then $U = \bigcup_{x \in U} V_x$ where $V_x \in \mathbb{B}$ and $\mathbb{B}$ is a base. ☐

Lecture 5

More on Bases

Theorem. *A base forms a topology if:*

$$\begin{array}{ll} (1) & \forall x \in X, \quad \exists B \in \mathbb{B}, s.t. \quad x \in B \\ (2) & \text{If } x \in B_1 \cap B_2, \quad B_1, B_2 \in \mathbb{B} \quad \Rightarrow \exists B_3 \in \mathbb{B} \quad s.t. \quad B_3 \subset B_1 \cap B_2 \end{array}$$

Define $T_B = \{U \subset X \quad s.t. \forall x \in U \quad \exists B \in \mathbb{B} \quad s.t. \quad x \in B \subset U\}$.

Proof. We check that T_B is a topology.

Part 1: Clearly the empty set and X are in T_B (true by (1))

Part 2: $\{U_\alpha\}_{\alpha \in I}$, $U_\alpha \in T_B$ want to show $\bigcup U_\alpha \in T_B$. If $x \in \bigcup U_\alpha$ then $x \in U_\alpha$ for some α therefore, $\exists B \in \mathbb{B}$ such that $x \in B \subset U_\alpha \subset \bigcup U_\alpha$

Part 3: Given $U_1, \ldots U_n$ want to show that $\bigcap_{i=1}^n U_i \in T_B$. We have that $x \in \bigcap_{i=1}^n U_i$, which implies that $x \in U_i$ for all i, which implies that there exists $B_i \in \mathbb{B}$ and $x \in B_i \subset U_i$.

$x \in B_1 \cap B_2 \quad \Longrightarrow$ by property 2 that $\exists B_{12} \in \mathbb{B}$, $x \in B_{12} \subset B_1 \cap B_2$

$x \in B_{12} \cap B_3 \Longrightarrow$ by property 2 that $\exists B_{123} \in \mathbb{B}$, $x \in B_{123} \subset B_{12} \cap B_{13} \subset B_1 \cap B_2 \cap B_3$

(inductively)

$\exists B_{12\ldots n} \in \mathbb{B}$ such that $x \in B_{12\ldots n} \subset B_1 \cap B_2 \cap \ldots \cap B_n \subset U_1 \cap \ldots \cap U_n$. Therefore, $U_1 \cap \ldots \cap U_n \in T_B$.

□

Example. A topology can have many bases (e.g. $\mathbb{R}^2$)

Example. Lower Limit Topology The base elements of this topology are $B = [a, b) \subset \mathbb{R}$

Example. Product Topology Given (X, T_X) and (Y, T_Y) as topological spaces. The product topology on the Cartesian product $X \times Y$ is defined as the one with base:

$$B = \{U \times V \mid \quad U \subset V \text{ open and } V \subset Y \text{ open}\}$$

Interior and Closure

Given X is a topological space and $S \subset X$ is a subset, we have:

Definition. *S is a neighborhood of x if $\exists U \subset S$ open with $x \in U \subset S$*

Definition. *Interior of S:*

$$\text{int}(S) = \{x \in S \mid S \textit{ is a neighborhood of } x\} \subset S$$

Properties:

1. int(S) is open:

   Proof. $\forall x \in \text{int}(S)$, $\exists U_x$ open and $x \in U_x \subset S$, which implies that $\text{int}(S) = \bigcup_{x \in \text{int}(S)} U_x =$ open □

2. int(S)=S ⇔ S is open

   Proof. ⇒ Follows from (1), ⟸ S is open, therefore, S is a neighborhood of any $x \in S$, thus $S = \text{int}(S)$ □

Definition. *$x \in X$ is adherent to S if every neighborhood of x intersects S.*

Definition.

$$\bar{S} = \{x \in X \mid x \textit{ is adherent to } S\}$$

or

$$\bar{S} = \{x \in X \mid \textit{every neighborhood of } x \textit{ intersects } S\}$$

Definition. *S is dense in X if* $\bar{S} = X$ *or equivalently if every open set in X intersects with S.*

Example. $\mathbb{Q}$, $\mathbb{R} - \mathbb{Q}$ are both dense in $\mathbb{R}$

Properties:

$$\begin{array}{ll} (1) & \text{int}(X - S) = X - \bar{S} \\ (2) & X - \text{int}(S) = \overline{X - S} \\ (3) & \bar{S} \text{ is closed} \\ (4) & S = \bar{S} \Leftrightarrow S \text{ is closed} \end{array}$$

Other properties:

$$\bigcup_{\alpha \in I} \bar{S}_\alpha \subset \overline{\bigcup_{\alpha \in I} S_\alpha}$$

If the collection is finite, then:

$$\bigcup_{\alpha \in I} \bar{S}_\alpha = \overline{\bigcup_{\alpha \in I} S_\alpha}$$

Definition. *A point x is called a* ***boundary point*** *of S if x is adherent to both S and* $X - S$*.*
The boundary of S is:

$$\begin{aligned} \partial S &= \bar{S} \cap (\overline{X - S}) \\ &= \bar{S} \cap (X - \text{int}(S)) \\ &= \bar{S} - \text{int}(S) \end{aligned}$$

Example. $X = \mathbb{R}$, $\partial[0,1] = [0,1] - (0,1) = \{0,1\}$

Lecture 6

Countability and Separation Axioms

What conditions can we put on a topologic space to make it "nice", e.g. metrizable?

I. Countability Axioms

Countability axioms "make sure the space is not too big. For the following definitions assume that X is a topological space.

Definition. *X is second countable if it admits a countable base.*

Examples:

$\mathbb{R}$ has base $\{(a,b) | a, b \in \mathbb{Q}\}$

$\mathbb{R}^n$ has a base $\{B_{1/n}(q) | n \in \mathbb{N}, q \in \mathbb{Q}^n\}$

Non-Examples:

$\mathbb{R}$ with the discrete topology; since each $\{x\}$ is an open set and a base must be a subset of every open set (i.e. the base is the finest)

Proof. $\forall x \in \mathbb{R}$, $\{x\}$ is an open set, which implies that X contain a base element $x \in B$ which implies that $B = \{x\}$ is the base, which implies that B is not countable ☐

Definition. *X is called first countable if* $\forall x \in X$*, there exists a countable collection* $\mathbb{B}$ *of open neighborhoods of x such that every neighborhood of x contains an element of* $\mathbb{B}$*.*

This is called the base at x (i.e. the collection $\mathbb{B}_x$ *such that very neighborhood of x contains an element of* $\mathbb{B}_x$*). First countable if the base at a single point is countable (not the entire base).*

Note. Second countable $\Rightarrow$ First countable

Examples:

The discrete topology is first countable (but not second countable)

Proposition. *All metric spaces are first countable*

Proof. Let $\mathbb{B} = \{B_{1/n}(x) | n \in \mathbb{N}\}$ □

Non-Examples:

Countable complement topology is not first countable. (note even though the discrete topology is very fine, it allows so much maneuverability, but the restriction on countable complements does not allow such maneuverability)

Definition. *X is separable if it admits a countable dense subset.*

Example:

$\mathbb{R}$ with the standard topology is separable ($\mathbb{Q}$)

Non-Example:

The discrete topology is not separable

Proposition. ***Second Countable*** $\Rightarrow$ ***Separable*** ($\not\Leftarrow$)

X is second countable, which implies that X is separable

Proof. Let $U_1, \ldots U_n$ be a countable base. Pick $x_n \in U_n$. Then every nonempty subset of X contains $x_i \in U_i$. Thus, $\{x_n\}$ is a countable dense subset. □

Example.

The lower limit topology with base $[a, b)$ is first countable and separable but not second countable.

Proof.

First countable: take $B = \left\{\left[x, x + \frac{1}{n}\right) | n \in \mathbb{N}\right\}$

Separable: $\mathbb{Q}$ is dense in $\mathbb{R}_\ell$

Not second countable: B_x for each x needed □

Definition. *An open cover of X is a collection of open subsets* $\{U_\alpha\}_{\alpha \in A}$ *such that* $\bigcup_{\alpha \in A} U_\alpha = X$

Definition. *An open sub-cover of this is a sub-collection* $\{U_\beta\}_{\beta \in B}$*,* $B \subset A$ *that still covers X.* $\bigcup_{\beta \in B} U_\beta = X$*.*

Definition. *X is Lindelof if every open cover of X admits a countable sub-cover.*

Proposition. ***Second Countable ⇒ Lindelof***

If X is second countable, then X is Lindelof

Proof. Let $\{U_\alpha\}$ be some open cover and let $\mathbb{B}$ be a countable base. Find a set C, such that it is comprised of base elements V such that $V \subset U_\alpha$ for some α. We know $\forall x \in X$ $\exists U_\alpha$ such that $x \in U_\alpha$ and $x \in V \subset U_\alpha$. Thus, the sets in C cover X. C is countable since B is countable. For each $V \in C$ pick $U_{\alpha(v)}$ such that $V \subset U_{\alpha(V)}$ thus, you have a countable sub-cover.

Other direction: counterexample consider lower limit topology □

Example. $(\mathbb{R}, \text{discrete})$ not Lindelof: the open cover $\{\{x\} | x \in \mathbb{R}\}$ does not have any countable sub-cover

II. Separation Axioms

Separation axioms basically guaranteed that open sets separate certain subsets.

Motivation: Topological spaces in general can be quite strange.

Example. $(\mathbb{R}, \text{trivial topology})$ only $\emptyset$, $\mathbb{R}$ are open. The trivial topology has very interesting properties:

- Points are not closed! $\{0\}$ is not closed
- Limits are not unique $(0, 0, \ldots)$ converges to any $x \in \mathbb{R}$ in the trivial topology, since any open set around x (necessarily $\mathbb{R}$) contains all terms of the sequence.

Definition. *A topological space X is called Hausdorff if $\forall x, y \in X$ with $x \neq y$, there exists disjoint open subsets such that $x \in U$ and $y \in V$.*

Proposition. *X is a metric space, then X is Hausdorff*

Proof. Take $U = B_\epsilon(x)$, $V = B_\epsilon(y)$, $\epsilon = \frac{d(x, y)}{3}$. $U \cap V = \emptyset$ because $z \in U \cap V$, which implies that $d(x, y) \leq d(x, z) + d(y, z) < 2\epsilon = \frac{2\,d(x, y)}{3}$ □

Proposition. ***(1) Hausdorff ⇒ {x} is closed for x∈X***

Proof. WTS $X - \{x\}$ is open. Now, since we have Hausdorff, we know that every point in $X - \{x\}$ has an open set around it that does not contain $\{x\}$. Thus, $X - \{x\}$ must be open since every point has an open set around it that is contained in $X - \{x\}$.

More formally, pick $y \in X - \{x\}$, then $y \neq x$, then $\exists$open $y \in V_y$ with $x \notin V_y$ by Hausdorff. Thus, $y \in V_y \subset X - \{x\} \Rightarrow X - \{x\} = \bigcup_{y \in X - \{x\}} V_y = \text{open}$. □

Proposition. ***(2) Hausdorff and $x_n \to x$ and $x_n \to y$ ⇒ x=y***

Proof. If $x \neq y$, choose $x \in U$, and $y \in V$, $U \cap V = \emptyset$ and $U, V -$ open

$x_n \to x \Rightarrow \exists N_1$ such that $\forall n > N_1$, $x_n \in U$

$x_n \to y \Rightarrow \exists N_2$ such that $\forall n > N_2$, $x_n \in V$

For $N > \max(N_1, N_2)$, $x_n \in U \cap V = \emptyset$ contradiction! □

Definition. *A topological space X is called T_1 if for all $x, y \in X$, $x \neq y$, $\exists V$ open s.t. $y \in V$ and $x \notin V$.*

Remark. Hausdorff$\Rightarrow T_1$

Proposition. *X is* $T_1 \Leftrightarrow$ *points are closed*

Proof. $\Rightarrow$ just like the proof of proposition 1

$\Longleftarrow$ For $x \neq y$, choose $V = X - \{x\}$ open because $\{x\}$ is closed and $y \in V$ and $x \notin V$ □

Lecture 7

Definition. *X is regular if* $\forall A \subset X$ *closed and* $x \in X - A$*, there exists* $U, V \subset X$ *such that* $x \in U$ *and* $A \subset V$ *and* $U \cap V = \emptyset$.

Definition. *X is normal if* $\forall A, B \subset X$ *closed s.t.* $A \cap B = \emptyset$*, then* $\exists U, V \subset X$ *that is open such that* $A \subset U$*,* $B \subset V$ *and* $U \cap V = \emptyset$

Definition. $T_2 = \text{Hausdorff}$
$T_1 + \text{Regular} = T_3$
$T_1 + \textit{Normal} = T_4$

Remark. $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$

Proposition. *Metric spaces satisfy all these axioms*

Proof. Last time we checked Hausdorff (T_2), thus T_1.

It suffices to check: (X, d) is a metric space $\Rightarrow$ X is normal

Let $A, B \subset X$ be closed, and $A \cap B = \emptyset$

For $a \in A \subset X - B$ (open) $\Rightarrow \exists \epsilon_a, B_{\epsilon_a}(a) \subset X - B$

Let $U = \bigcup_{a \in A} B_{\frac{\epsilon_a}{2}}(a)$ open, $A \subset U$. Similarly, $\forall b \in B \subset X - A \Rightarrow \exists \epsilon_b \quad B_{\epsilon_b}(b) \subset X - A$

Let $V = \bigcup_{b \in B} B_{\frac{\epsilon_b}{2}}(b)$ os open and $B \subset V$.

We want to show that $U \cap V = \emptyset$. If $x \in U \cap V$, $x \in B_{\frac{\epsilon_a}{2}}(a)$ for some $a \in A$ and $x \in B_{\frac{\epsilon_b}{2}}(b)$ for some $b \in B$.

$d(a, b) \leq d(a, x) + d(x, b) \leq \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} \leq \max(\epsilon_a, \epsilon_b)$, either $a \in B_{\epsilon_b}(b) \subset X - A$ or $b \in B_{\epsilon_a}(a) \subset X - B$ which is impossible, so $U \cap V = \emptyset$ □

Theorem. ***Urysohn's Metrization Theorem*** *If X is a regular topological space that is second countable and regular, then X is metrizable*

Theorem. ***Urysohn's Lemma*** *A,B* $\subset X$*,* $A \cap B = \emptyset$*, and X is normal. Then* $\exists f : X \to [0, 1]$ *continuous such that* $f = 0$ *on A and* $f = 1$ *on B.*

We will repeatedly use:

Lemma. ***(1)*** *X is normal.* $A \subset W \subset X$*, where A is closed and W is open. Then* $\exists U$ *open with* $A \subset U \subset \bar{U} \subset W$

Proof. (of Lemma 1) A and $X - W$ are closed and disjoint. Since X is normal, $\exists U, V$ open, and $A \subset U$, $X - W \subset V$ and $U \cap V = \emptyset$. Then $U \subset X - V$ (closed) $\Rightarrow \bar{U} \subset \overline{X - V} = X - V \subset W$ □

Proof. (of Urysohn's Lemma)

Let $V=(X-B)\supset A$, by Lemma 1, we know that there exists $U_{1/2}$ open, $A\subset U_{1/2}\subset \overline{U_{1/2}}\subset V/$

Lemma 1 applied to A and $U_{1/2}$ and $\overline{U_{1/2}}$ and $X-B$ implies that there exists $U_{1/4}$ and $U_{3/4}$ open such that:

$$A\subset U_{1/4}\subset \overline{U_{1/4}}\subset U_{1/2}\subset \overline{U_{1/2}}\subset U_{3/4}\subset \overline{U_{3/4}}\subset X-B$$

We keep going inductively we get $U_{1/8}, U_{3/8}, \ldots$

Eventually we get an open set U_r for every dyadic rational $r\in[0,1]$, $r=\frac{p}{2^n}$ for $p,n\in\mathbb{N}$ such that $A\subset U_r$, $\overline{U_r}\subset X-B$ whenever $r<s$.

Definition. $f\colon X\to[0,1],\ f(x)=\begin{cases} 0 & \text{if } x\in U_r, \forall r\in[0,1] \\ \sup\{r|x\notin U_r\} & \text{otherwise}\end{cases}$

Observe that $f=0$ on A because $A\subset\bigcap_r U_r$ and $f=1$ on B because $\bigcup_r u_r\subset X-B$ then $f(b)=\sup[0,1]=1\ \forall b\in B$.

Check that f is continuous at $x\in X$. There are three cases:

1. $0<f(x)<1$
2. $f(x)=0$
3. $f(x)$=1

We will do case 1, cases 2 and 3 are similar but easier.

Case 1: We want to show that $\forall\epsilon>0$, $\exists$open neighborhood W of X such that $|f(x)-f(y)|<\epsilon$, $\forall y\in W$

The dyadic rations are dene in $\mathbb{R}$. Thus, there exists dyadic rationals $r,s\in(0,1)$ such that;

$$f(x-)-\epsilon<r<f(x)<s<f(x)+\epsilon$$

$f(x)=\sup\{t|x\notin U_t\}\Rightarrow x\in U_s$

Also, for every dyadic ration $t\in(r,f(x))$, we have that $x\notin U_t$, which implies that $x\notin\overline{U_r}\subset U_t$.

Therefore, $x\in U_s-\overline{U_r}$. Let $W=U_s-\overline{U_r}$. This is an open neighborhood of x. For $y\in W$, $f(y)=\sup\{t|y\notin U_t\}\in[r,s]\Rightarrow|f(x)-f(y)|<\epsilon$ □

Lecture 8

Completeness and Compactness

For the following definitions, assume that (X,d) is a metric space.

Definition. *A sequence (x_n) of points in X is called Cauchy if $\forall\epsilon>0$, $\exists N>0$ such that $d(x_n, x_m)<\epsilon$ for all $n,m\geq N$*

Remark. If (x_n) converges to some x, it is Cauchy

Definition. (X,d) *is called complete if every Cauchy sequence is convergent*

Example. (0,1) with the d_2 Euclidean metric is not complete. Take $x_n = \frac{1}{n}$ which does not converge to anything in $(0,1)$.

$\mathbb{Q}$ is not complete

Theorem. $(\mathbb{R}^n, d_2)$ *is complete*

Theorem. $X \subset (\mathbb{R}^n, d_2)$ *is complete* $\Leftrightarrow$ X *is closed.*

Corollary. $(0, \infty)$ *and* $\mathbb{R}$ *are not isometric*

Note. In other words this corollary indicates to us that completeness is a metric property (preserved by isometries) but not a topological property (not preserved under homeomorphism).

Recall that if X is a topological space, an open cover $\{U_\alpha\}_{\alpha \in A}$ of X is a collection of open subset such that $\bigcup_{\alpha \in A} U_\alpha = X$.

Definition. *X is compact if ever open cover of X admits a finite sub-cover*

Compare this to Lindelof, where we had that "every open cover has a countable sub-cover". Note that compact $\Rightarrow$ Lindelof

Definition. *X is sequentially compact if every sequence of points in X has a convergent subsequence.*

Results from Analysis:

1. If (X, d) is a metric space, then X is compact $\Leftrightarrow$ X is sequentially compact (note this is not true in general, neither condition implies the other)

2. **Bolzano-Weierstrass Theorem:** $[a, b] \subset \mathbb{R}$ is sequentially compact

3. **Heine-Borel Theorem:** $X \subset (\mathbb{R}^n, \text{standard topology})$ is compact $\Leftrightarrow$ X is closed and bounded (i.e. $X \subset B_R(0)$ for some $R > 0$)

4. (X, d) is a metric space. X is compact $\Rightarrow$ X is complete.

   Proof. We assume that X is (sequentially) compact. Let x_n be a Cauchy sequence. Find a convergent subsequences $x_{n_1}, x_{n_2}, x_{n_3}, \ldots \to x$. Pick some $\epsilon > 0$. The for large n there exists a k such that $n_k > n$ we have that $d(x_{n_k}, x) < \frac{\epsilon}{2}$ because $(x_{n_k} \to x)$ and $d(x_n, x_{n_k}) < \frac{\epsilon}{2}$ because of Cauchiness. Therefore, together we have that $d(x_n, x) < \epsilon$. Therefore $x_n \to x$, so X is complete. □

Example.

- $\mathbb{R}^n$ is complete but not compact (because it is not bounded).
- An n-dimensional sphere $S^n = \{(x_0, x_1, \ldots x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^n x_i^2 = 1\}$ is compact

Corollary. $\mathbb{R}^n$ *(not compact) and* S^n *(compact) are not homeomorphic.*

Next we wish to find a sequentially compact topological space that is not compact. In order to do this, we need **set theory**.

Set Theory

Definition. *A well-ordered set is a set A with a binary relation $\leq$ satisfying:*

$$\begin{array}{ll} (1) & a \leq b \text{ and } b \leq a \Rightarrow a = b \\ (2) & a \leq b \text{ and } b \leq c \Rightarrow a \leq c \\ (3) & a \leq b \text{ or } b \leq a \\ (4) & \text{Every nonempty subset } S \subset A \text{ has a least element } (\exists x \in S, x \leq s, \forall s \in S) \end{array}$$

Example.

- $\{0, 1, 2, \ldots, n\}$ is well-ordered
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is well-ordered
- $\mathbb{Z}, \mathbb{R}$ is not well-ordered are not well-ordered (they have no smallest element)
- $\{0, 1\} \times \mathbb{N}$ with the dictionary order is well ordered:

$$(0,0) < (0,1) < (0,2) < \cdots < (1,0) < (1,1) < (1,2) < \cdots$$

- $\mathbb{N} \times \mathbb{N}$ with the dictionary order is well-ordered

Definition. *If A is a well-ordered set, the section of A by a is:*

$$(-\infty, a) = \{x \in A | x < a\}$$

Theorem. (Zermelo) *Every set admits a well-ordering*

Theorem. *There exists a well-ordered, uncountable set ω_1 all of whose sections are countable.*

$$\omega_1 = \text{“the first uncountable ordinal”}$$

Definition. *If X is a well-ordered set. The order topology on X is the topology with base B consisting of:*

$$\begin{aligned} (a,b) &= \{x \in X | a < x < b\} \\ (-\infty, a) &= \{x \in X | x < a\} \\ (a, \infty) &= \{x \in X | a < x\} \end{aligned}$$

Proposition. *This collection $\mathbb{B}$ is the base for a topology assuming $|X| \geq 2$*

Proof. We need to check:

1. $\forall x \in X$, $\exists B \in \mathbb{B}$, $X \subset B$. Pick $y \neq x$. If $x < y$, choose $(-\infty, y)$ and if $y > x$ choose (y, ∞)
2. $\forall x \in B_1 \cap B_2$, $B_1, B_2 \in \mathbb{B}$; $\exists B_3 \in \mathbb{B}$, $x \in B_3 \subset B_1 \cap B_2$. Choose $B_3 = B_1 \cap B_2$ $(a, b) \cap (c, d) = (\max(a, c), \min(b, d))$ where $a, b, c, d \in X \cup \{\pm\infty\}$ □

Proposition. *ω_1 with the order topology is sequentially compact but not compact*

Proof. of non-compactness:

Consider the cover consisting of $(-\infty, a)$ over all $a \in \omega_1$. This is a cover. $\forall b \in \omega_1$, $\exists a \in \omega_1$ for $a > b$. Otherwise $(-\infty, b) \cup \{b\} = \omega_1$, which is a contradiction since $(-\infty, b)$ is countable and ω_1 is not countable. Therefore, ω_1 has no maximal element.

This cover has no finite sub-cover. Otherwise, we would have $(-\infty, a_1) \cup (-\infty, a) \cup \ldots \cup (-\infty, a_n) = \omega_1$. Let $a = \max\{a_1, a_2, \ldots, a_n\}$ which implies $\omega_1 = (-\infty, a)$ and $a \notin \omega_1$ false.

Therefore, ω_1 is not compact. □

Lecture 9

Properties of Compact Topological Spaces

Recall that X is compact $\Leftrightarrow$ every open cover has a finite sub-cover. For example, a closed and bounded subsets of $\mathbb{R}^n$.

Proposition. (1) *X is a topological space, $S \subset X$ (with the subspace topology). Then, S is compact $\Leftrightarrow$ for every collection of open subset $\{U_\alpha\}_{\alpha \in A}$ of X, such that $S \subset \bigcup_{\alpha \in A} U_\alpha$, we can extract a finite sub-collection with the same property.*

Proof. The open subsets in S are those of the form $U \cap S$, $U \subset X$ is open. Apply the definition to the open cover $\{U_\alpha \cap S\}_{\alpha \in A}$ □

Proposition. (2) *If X is compact, $S \subset X$, and S is closed $\Rightarrow S$ is also compact*

Proof. Use proposition 1, let $\{U_\alpha\}_{\alpha \in A}$ be a family of open subset of X with $S \subset \bigcup_{\alpha \in A} U_\alpha$. Notice that $X - S$ is open, then the collection $\{U_\alpha\}_{\alpha \in A} \cup \{X - S\}$ is an open cover of X. X is compact implies that there exists a finite subfamily of this that still covers X, say $U_{\alpha_1}, \ldots, U_{\alpha_n}, X - S$. Then $U_{\alpha_1}, \ldots, U_{\alpha_n}$ covers S. □

Lemma. *$S \subset X$ is compact, X- Hausdorff $\Rightarrow \forall x \in X\text{-}S\ \exists U, V \subset X$ open such that $x \in U$, $S \subset V$, and $U \cap V = \emptyset$*

Proof. For every $y \in S$, by the Hausdorff condition, $\exists U_y, V_y \subset X$, $U_y \cap V_y = \emptyset$ with $x \in U_y$ and $v \in V_y$.

$\{V_y\}_{y \in S}$ covers S; S is compact, which implies by proposition 1 that there exists a finite sub-collection $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ covering S. Let $V = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$ and $S \subset V$. Let $U = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n}$ be a finite intersection of open sets, which implies it is open. We have $x \in U$ and $U \cap V = \emptyset$ since $U \cap V_{y_i} = \emptyset\ \forall i$ because $U_{y_i} \cap V_{y_i} = \emptyset$. □

Proposition. (3) *X is Hausdorff, $S \subset X$ and compact $\Rightarrow$ S is closed*

Proof. For every $x \in X - S$ by the lemma, there exists $x \in U_x$ where U_x is open such that $S \cap U_x = \emptyset$. Then $X - S = \bigcup_{x \in X - S} u_x$ is open, so S is closed. □

Corollary. *X is compact and Hausdorff. Then closed subsets are equal to compact subsets.*

For example, consider $B_1(0) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum x_i^2 \leq 1\}$

Proposition. (4) *If X is compact and Hausdorff, then X is normal (it satisfies all the separation axioms.)*

Proof. $A, B \subset X$ is disjoint closed subsets. By proposition 2, A and B are compact. The lemma tells us that $\forall x \in A$, $\exists x \in U_x$ and $B \subset V_x$, and U_x, V_x are disjoint open and $\{U_x\}_{x \in A}$ cover A.

Since A is compact, by proposition 1, we can find $x_1, \ldots x_n \in A$ such that $A \subset U_{x_1} \cup \ldots \cup U_{x_n}$.

Let $U = U_{x_1} \cup \ldots \cup U_{x_n}$. Let $V = V_{x_1} \cap \ldots \cap V_{x_n}$ be open. Then $U \cap V = \emptyset$, $A \subset U$ and $B \subset V$. □

Proposition. (5) *A finite union of compact subsets of topological spaces X is compact.*

Proof. $S_1, \ldots S_n \subset X$ with S_i compact. Then let $S = \bigcup_{i=1}^n S_i$. Use proposition 1. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open subsets of X covering S. Then $S_i \subset S \subset \bigcup_{\alpha \in A} U_\alpha$.

Since the S_i are compact, we can extract a finite sub-collection covering S_i and put all these sub-collections together to get a finite sub-collection $\{U_\alpha\}_{\alpha \in A}$ covering S. Hence S is compact. □

Proposition. (6) *If X, Y are topological space and $f: X \to Y$ is continuous. X is compact $\Rightarrow$ $f(X)$ is compact.*

Proof. $\{U_\alpha\}_{\alpha \in A}$ is a collection of open subsets of Y covering $f(x)$. Then $f^{-1}(U_\alpha)$ is open, $\forall \alpha \in A$ and $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ covers X. If X is compact, the that implies we can find a finite sub-collection $f^{-1}(f(X))$. $f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n}) = X$, then $f(X) \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$. Thus since we have a finite sub-collection covering $f(X)$, we have that $f(X)$ is compact. □

Recall: $f: X \to Y$ is a homeomorphism if f is bijective, continuous and f^{-1} is continuous.

Remark. f is bijective and continuous, then it does not necessarily imply f^{-1} is continuous. For example, take $f: [0, 2\pi] \to \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ where $f(\theta) = (\cos(\theta), \sin(\theta))$. However, f^{-1} is not continuous since it turns back on itself in a circle (the critical are is the joint).

Proposition. (7) *If X is compact, and $f: X \to Y$ is continuous and bijective and Y is Hausdorff. Then f^{-1} is also continuous (hence a homeomorphism).*

Proof. Need to show that U open in X and therefore $f(U)$ is open in Y or equivalently S is closed in X which implies that $f(S)$ is closed in Y. This is true since we know that S is closed in X and X is compact, therefore by proposition 2, we know that S is compact. Therefore by proposition 6, we know that $f(S)$ is compact. Combining this with the fact that Y is Hausdorff, we get by proposition 3 that $f(S)$ is closed. □

Lecture 10

Connectedness

Recall: $[0, 1]$ is compact and $\mathbb{R}$ is not compact, therefore they are not homeomorphic.

How about $[0, 1]$ versus $[0, 1] \cup [2, 3]$? Both are compact…

Definition. *A topological space X is called connected if the only subsets of X that are both open and closed are $\emptyset$ and X.*

Example. $[0, 1] \subset [0, 1] \cup [2, 3]$ is both open and closed in Y

Remark. X is disconnected $\Leftrightarrow$ $\exists$ separation $X = A \perp B$ where $\perp$ is the "disjoint union", A, B are both open and closed and $A, B \neq \emptyset$

Example. $\mathbb{Q}$ is disconnected:

$$\mathbb{Q} = (\mathbb{Q} \cap (-\infty, \sqrt{2})) \cup ((\sqrt{2}, \infty) \cap \mathbb{Q})$$

Proposition. $\mathbb{R}$ *in the standard topology is connected*

Proof. Assume it's not, so $\mathbb{R} = A \perp B$ and $A, B \subset \mathbb{R}$ are open, closed and nonempty. Pick $a \in A$ and $b \in B$. Let us say $a < b$. Let $c = \sup(A \cap [a, b])$. Then $a \leq c \leq b$. Then we consider two cases.

I. Suppose $c \in A$. A is open, which implies that there exists $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset A$ and $c < b$ which implies that $\exists \delta > 0$, $\delta < \min(\epsilon, b - c)$. Then we have that $c + \delta \in [a, b]$ and $c + \delta \in (c - \epsilon, c + \epsilon) \subset A$ and $c + \delta > c$. This contradicts that $c = \sup(A \cap [a, b])$.

II. Suppose $c \in B$. $c = \sup(A \cap [a, b])$ which implies that there exists a sequence $a_n \to c$ and $a_n \in A \cap [a, b]$. Since A is closed, we get that $c \in A$, which is a contradiction. Therefore, $\mathbb{R}$ is connected. □

Remark. The same proof applies to show that (a, b), [a,b], $[a, b)$, $[a, \infty)$ etc. are connected.

Corollary. *[0,1] and* $[0, 1] \cup [2, 3]$ *are not homeomorphic.*

Properties:

Proposition. (1) X is connected, $f: X \to Y$ *is continuous* $\Rightarrow$ $f(X) \subset Y$ *is connected*

Proof. Suppose that $f(X) = A \perp B$ is a separation. Then $X = f^{-1}(A) \perp f^{-1}(B)$ is a separation. A, B is open, then $f^{-1}(A), f^{-1}(B)$ is open. Thus, we have a contradiction. □

Corollary. *The circle* S^1 *is connected*

Proof. S^1 is the image of $f: \mathbb{R} \to \mathbb{R}^2$ and $f(t) = (\cos(t), \sin(t))$ □

Corollary. *[0,1] and* S^1 *are not homeomorphic. (note: even though they are both compact and connected)*

However, if you remove any point from S^1, it stays connected, which is not true for $(0, 1)$.

Proposition. (2) $\{A_\alpha\}_{\alpha \in I}$ *collection of connected subsets of* X *such that* $A_\alpha \cap A_\beta \neq \emptyset$, $\forall \alpha, \beta$. *Then* $\bigcup_{\alpha \in I} A_\alpha$ *is connected.*

Proof. Let $A = \bigcup A_\alpha$. Suppose that $U \subset A$ is open and closed and nonempty. Then $x \in U$ and $\exists \alpha_0$ such that $x \in A_{\alpha_0}$. $U \cap A_{\alpha_0} \subset A_{\alpha_0}$ which is open and closed in A_{α_0} implies that $U \cap A_{\alpha_0} = A_{\alpha_0}$. For any β, $U \cap A_\beta$ is a closed and open subset of A_β that includes $A_{\alpha_0} \cap A_\beta \neq \emptyset$ which implies that $U \cap A_\beta = A_\beta$ which further implies that $A_\beta \subset U$ $\forall \beta$. Therefore, $U = A$. Therefore, A is connected. □

Proposition. (3) *X, Y is connected, therefore,* $X \times Y$ *is connected.*

Proposition. (4) $A \subset X$ *with* A *connected implies that* $\bar{A}$ *is connected*

Connected Components

Are $[0,1]\cup[2,3]$ and $[0,1]\cup[2,3]\cup[4,5]$ homeomorphic? Both are compact and not connected...

Give a topological space X we can define an equivalence relation $x\sim y$ if there is a connected subspace $A\subset X$ with $x,y\in A$. The equivalence relation is a proper equivalence relation since it is reflexive, symmetric, and $x\sim y, y\sim z\Rightarrow x\sim z$ since if A is connected and B is connected and we have $x,y\in A$ and $y,z\in B$ then $x,z\in A\cup B$ by proposition 2 and $y\in A\cap B\neq\emptyset$.

The equivalence classes are called connected components. We have that:

$$X \text{ is connected} \Leftrightarrow X \text{ has a unique connected component}$$

$[0,1]\cup[2,3]$ has 2 connected components and $[0,1]\cup[2,3]\cup[4,5]$ has three so they are not homeomorphic.

Path Connectedness

We will define another equivalence relation on X. We have $x\sim y$ is path connected if there exists a path in X from x to y (i.e. a continuous function $f\colon[0,1]\to X$, $f(0)=x$, $f(1)=y$. The equivalence classes are called path components of X.

Proposition. *x and y are path connected implies that x and y are connected*

Proof. We have that $f\colon[0,1]\to X,\ f(0)=x,\ f(1)=y$. Take $U=f([0,1])$, we have that $[0,1]$ is connected and by proposition 1, this implies that U is connected and therefore x and y are connected ☐

Definition. *X is called path connected if it has a single path component i.e. $x\sim y\ \forall x,y\in X$*

Corollary. *A path connected space is connected*

Warning. The converse is not true. Consider the topologists's sine curve,

$S=\{(x,\sin(\frac{1}{x})|0<x\leq 1)\}\subset\mathbb{R}^2$ and $\bar{S}=S\cup(\{0\}\times[-1,1])$.

It has two path components, S and $\{0\}\times[-1,1]$, which implies that $\bar{S}$ is not path connected. S is the image of $(0,1]$ under $f\colon(0,1]\to\mathbb{R}$ and $f(x)=(x,\sin(\frac{1}{x}))$ which implies that S is connected and by proposition 4 that $\bar{S}$ is connected.

Application of Connectedness: Intermediate Value Theorem

If $f\colon X\to\mathbb{R}$ is continuous, X is connected, $a,b\in X$ and $r\in\mathbb{R}$, and $f(a)<r<f(b)$. Then there exists $x\in X$ with $f(x)=r$.

Proof. Let $Z=f(X)\subset\mathbb{R}$ and let X be connected, which implies by proposition 1 that Z is connected. Assume $r\notin Z$. Then we have that

$Z=(Z\cap(-\infty,r))\perp(Z\cap(r,\infty))$, which is a separation of Z which is a contradiction ☐

Lecture 11

Recall that given a topological spaces X, Y that the product topology on $X\times Y$ is the one generated by the base $\{U\times V\}$ over all $U\subset X$, $V\subset Y$ open.

$X_1, \ldots, X_n$ is the product topology on $X_1 \times \ldots \times X_n$ generated by $U_1 \times \ldots \times U_n$ overall $U_i \subset X_i$ open. Denote $\pi_i: X_1 \times \ldots \times X_n \to X_i$ the i^{th} projection.

Theorem. (Tychonoff's Finite Theorem) *$X_1, \ldots . X_n$ is compact $\Rightarrow X_1 \times \ldots \times X_n$ is compact.*

Lemma. *If every open cover of X consisting of base elements admits a finite sub-cover, then X is compact.*

Proof. Proof: Given a cover $\{U_\alpha\}_{\alpha \in A}$ of X, we want to extract a finite sub-cover. For all $x \in X$ we find $\exists V_x \in \mathbb{B}$ such that $x \in V_x \subset U_\alpha$ for some α. Now, each V_x covers every possible x that U_α also covered. Moreover, each V_x is inside U_α. Since V_x is a covering of base elements, we can find a finite sub-cover: $V_{x_1}, \ldots . V_{x_n}$ that covers X. Now, since each $V_x \subset U_\alpha$ we can find $U_{\alpha_1}, \ldots U_{\alpha_n}$ the corresponding open intervals in U that contain each of the V and then they must also cover X since they are supersets of V i.e. $V_{x_i} \subset U_{\alpha_i}$. Thus, we have a finite sub-cover of X. □

Proof. (of Theorem) Start with the case $n = 2$ and then proceed by induction.

Given X_1 and X_2 compact. By the lemma, we only need to show that every cover of $X_1 \times X_2$ by the base elements has a finite sub-cover.

Fix a $z \in X_2$. Then $X_1 \times \{z\}$ is compact. Which implies that that there exist finitely many base elements $U_1 \times V_1, \ldots U_m \times V_m$ covering $X_1 \times \{Z\}$. We can assume $z \in V_i$, $i = 1, \ldots, m$. Then $V(z) = V_1 \cap \ldots \cap V_m$ is open in X_2. $X_1 \times V(z) = \pi_2^{-1}(V(z))$ is covered by $U_1 \times V_1, \ldots, U_m \times V_m$.

The sets $\{V(z)\}_{z \in X_2}$ form an open cover of X_2 and since X_2 is compact, we get a finite sub-cover. There exists $z_1, \ldots z_k \in X_2$, $V(z_1) \cup \ldots \cup V(z_k) = X_2$ which implies that $X_1 \times X_2 = \pi_2^{-1}(V(z_1)) \cup \ldots \cup \pi_2^{-1}(V(z_k))$' each $\pi_2^{-1}(V(z_j))$ is covered by finitely many sets in the open cover, so is $X_1 \times X_2$. Therefore, $X_1 \times X_2$ is compact. □

Infinite Products

Consider $\{X_\alpha\}_{\alpha \in I}$ a collection of topological spaces, a cartesian product is:

$$\prod_{\alpha \in I} X_\alpha = \{f: I \to \bigcup X_\alpha \quad s.t. \quad f(\alpha) \in X_\alpha \quad \forall \alpha\}$$

A typical element is denoted as $x = \{x_\alpha\}_{\alpha \in I}$ where $x_\alpha = \pi_\alpha(x)$, $\pi_\alpha: \prod_{\alpha \in I} X_\alpha \to X_\alpha$ projection.

An example of this is $\mathbb{R}^{\mathbb{N}}$ which is the sequences of real number. When our index set I is infinite, there are two natural topologies on $\prod_{\alpha \in I} X_\alpha$.

Definition. *The product topology has the base:*

$$\mathbb{B} = \left\{ \prod_{\alpha \in I} U_\alpha | U_\alpha \subset X_\alpha \textit{ is open , and } U_\alpha = X_\alpha \textit{for all but finitely many} \alpha \right\}$$

Definition. *The box topology has the base:*

$$\mathbb{B} = \left\{ \prod_{\alpha \in I} U_\alpha | U_\alpha \subset X_\alpha \textit{ is open} \right\}$$

Properties of the Product Topology:

Proposition. *A sequence $x_n \to x$ in the product topology $\Leftrightarrow$ $\pi_\alpha(x_n) \to \pi_\alpha(x)$ in X_α for all α*

In other words, convergence in the product topology is equivalent to product topology.

Proof. $\Rightarrow$ Use π_α is continuous and the fact that $U_\alpha \subset X_\alpha$ is open, which implies that $\pi_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$ is open in the product topology.

$\Longleftarrow$ Suppose that $\pi_\alpha(x_n) \to \pi_\alpha(x) \quad \forall \alpha$. We want to show that $x_n \to x$ that is for every U open, $x \in U$ for all but finitely many x_n are in U.

Indeed, $\exists \prod_\alpha U_\alpha$ with $x \in \prod_\alpha U_\alpha \subset U$, and all but finitely many U_α are equal to X_α. Then $\prod_\alpha U_\alpha = U_{\alpha_1} \times \ldots \times U_{\alpha_k} \times \prod_{\alpha \notin \{\alpha_1, \ldots, \alpha_k\}} X_\alpha$. There exists n_1 such that $\pi_{\alpha_1}(x_n) \in U_{\alpha_1}$ for all $\quad n \geq n_1$ and there exists n_2 such that $\pi_{\alpha_2}(x_n) \in U_{\alpha_2}$ for all $n \geq n_2$ and the same up to n_k.

Then for all $n \geq \max(n_1, \ldots n_k)$, $x_n \in U_{\alpha_1} \times \ldots \times U_{\alpha_k} \times \prod_{\alpha \neq \{\alpha_1, \ldots \alpha_k\}} X_\alpha$. Therefore, $x_n \to x$ $\quad \square$

Proposition. *If Y is any topological space, then $f: Y \to \prod_{\alpha \in I} X_\alpha$ (product topology) is continuous $\Leftrightarrow$ $\pi_\alpha \circ f: Y \to X_\alpha$ is continuous for all α.*

Proposition. ***Tychonoff's General Theorem**: If $\forall a$, X_α is compact $\Rightarrow$ $\prod_{a \in I} X_a$ is compact*

Note. $\mathbb{R}^n$ in the product topology is metrizable due to this

Properties of the Box Topology

Proposition. *$\mathbb{R}^n$ in the box topology is not metrizable*

Proof. Recall that metrizable implies first countable which implies that every point in the closure of a subset S is the limit of points in S. This is false in $(\mathbb{R}^n, \text{box topology})$.

Take $S = \{(x_0, x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} | x_i > 0 \quad \forall i\}$ and $x = (0, 0, 0, \ldots)$. Then we have that $x \in \bar{S}$ because every open set around X contains a base element of the form $(a_1, b_1) \times (a_2, b_2) \times \ldots$ with $a_i < 0 < b_i$. This intersects S (e.g. at $\left(\frac{b_1}{2}, \frac{b_2}{2}, \ldots\right)$). However, there is no sequence of points in S converging to x. Suppose that $x^{(0)} = \left(x_0^{(0)}, x_1^{(0)}, \ldots\right) \in S$, $x^{(1)} = \left(x_0^{(1)}, x_1^{(1)}, \ldots\right)$, $\ldots$, $x = (0, 0, \ldots)$ in the box topology. Then choose $U = \left(-x_0^{(0)}, x_0^{(0)}\right) \times \left(-x_1^{(1)}, x_1^{(1)}\right) \times \left(-x_2^{(2)}, x_2^{(2)}\right) \times ...$ open in the box topology such that U does not contain any $x^{(i)}$ because $x_i^{(i)} \notin \left(-x_i^{(i)}, x_i^{(i)}\right)$. Hence the box topology is not metrizable. $\square$

Remark. $[0,1]^{\mathbb{N}}$ in the box topology is not compact

Proposition. *X_α is Hausdorff for all $\alpha \Rightarrow \prod_{\alpha \in I} X_\alpha$ is Hausdorff in both the product and the box topologies.*

Lecture 12

Topological Manifolds

Definition 1. *A topological space is called* <u>*locally Euclidean*</u> *if every $p \in X$ has* ***an*** *open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$*

(or equivalently an open neighborhood homeomorphic to $B^n = B_1(0) \subset \mathbb{R}^n$)

(or equivalently an open neighborhood homeomorphic to $\mathbb{R}^n$)

Remark 2. We only need to find one such open neighborhood around every $p \in X$ that is homeomorphic to some subset of $\mathbb{R}^n$.

Lemma 3. *Any open ball $B_R(x) \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$*

Proof. Choose a homeomorphism in 1 dimension of the form:

$$h: [0,1) \rightarrow [0,\infty) \quad e.g. \quad h(x) = \tan\left(\frac{2}{\pi}x\right)$$

This homeomorphism maps points in [0,1) to $[0,\infty)$. Now, we can apply this homeomorphism to $x, y \in \mathbb{R}^n$. In order to map $B_R(x) \rightarrow \mathbb{R}^n$ under a homeomorphism, we can use:

$$y = \begin{cases} \dfrac{y-x}{\|y-x\|} h\left(\dfrac{\|y-x\|}{R}\right) & y \neq x \\ 0 & y = x \end{cases}$$

Note 4. *note that the term $\frac{y-x}{\|y-x\|}$ gives the directionality in this n dimensional space (i.e. it is the unit normal pointing toward the direction of $y-x$). The term $\frac{\|y-x\|}{R}$ converts the magnitude of the difference to the interval $[0,1)$ so that we may apply our earlier 1-D homeomorphism, $h(x)$ to scale to $[0,\infty)$*

This is a homeomorphism since it is continuous in both directions and bijective. □

To connect this lemma to the equivalences in the definition of locally euclidean, notice that given

Definition 5. *An n-dimensional topological manifold (or a topological n-manifold) is a topological space with n dimension that is:*

(1) *Hausdorff*
(2) *Second Countable*
(3) *Locally Euclidean*

Example 6. Examples of Manifolds:

i. $\boldsymbol{n=0}$: the disjoint union of countably many points (must be countable)

ii. $\boldsymbol{n=1}$: 1-manifold is a curve:

- $\mathbb{R}$
- $S^1 = \text{circle}$
- Square (since it is homeomorphic to a circle)

iii. $\boldsymbol{n=2}$: 2-manifold is a surface:

- $\mathbb{R}^2$ the plane
- S^2 a sphere
- $T^2 = S^1 \times S^1$ a torus
- $\mathbb{R} \times S^1 =$ cylinder

iv. **General** $\boldsymbol{n}$:

- $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0^2 + \cdots + x_n^2 = 1\}$, "n-sphere"
- $T^n = S^1 \times S^1 \times \ldots \times S^1$, "n-torus"

Remark 7. If X is an *n-manifold* and Y is an *m-manifold*. Then $X \times Y$ is an $(n+m)$-manifold

Proposition 8. *S^n is locally Euclidean*

Proof. First of all, recall what is the definition of locally Euclidean. A locally Euclidean space is one such that every $p \in X$ has **an** open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Therefore, let us take some $q \in S^n$ and consider an open neighborhood around it. A simple choice is the following. For $\forall q \in S^n$ pick $p \neq q$. Then $q \in S^n - \{p\}$. Notice that $S^n - \{p\}$ is an open interval. Thus, all that is left to show is that $S^n - \{p\}$ is homeomorphic to $\mathbb{R}^n$.

Without loss of generality, take $p = (1, 0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ (for other p, just rotate). We will now try to find a homeomorphism $\pi: S^n - \{p\} \to \mathbb{R}^n$. Such a homeomorphism exists and is called the stereographic projection. A stereographic projections is $\pi: S^n - \{p\} \to \mathbb{R}^n$ of the form $\pi(x) = \text{line}(p, x) \cap \mathbb{R}^n$, such that:

$$\pi(x_0, x_1, \ldots x_n) = \left(\frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right)$$

Note 9. This maps points closer to p further and further away. One can think of it as a triangle. Where p is one vertex, the other vertex is the origin, and the final vertex is $\pi(x)$ which depends on x. The edge from p to $\pi(x)$ passes through the x value on the sphere.

Exercise 1. Show that this is a homeomorphism

Solution 1. one can compute $\pi(x)$ and $\pi^{-1}(y)$

$$\pi^{-1}(y_1, \ldots y_n) = \frac{1}{|y|^2 + 1}(2\, y_1, \ldots, 2\, y_n, |y|^2 - 1)$$

and see that they are both compositions of continuous functions and therefore, must be continuous (note $x_0 = 1$ is never reached). By computing $\pi^{-1}(y)$ explicitly you automatically have shown bijectivity.

Thus, since we have a homeomorphism, we have shown what was desired. □

Non-Examples:

i. **Second Countable, Hausdorff, not locally Euclidean**

1. $X = [0, 1]$. Near 0, X cannot have a neighborhood homeomorphism to $\mathbb{R}$. Suppose there existed such a homeomorphism that contained 0. Then we could remove 0 and our neighborhood would still be connected but whatever point we remove from $\mathbb{R}$ makes it disconnected.

2. $\mathbb{Q} \subset \mathbb{R}$

3. Y-shaped figure in $\mathbb{R}^2$

ii. **Hausdorff, locally Euclidean, not Second Countable:**

1. ($\mathbb{R}$, discrete topology) since there are uncountably many points

2. Long line - a connected example

iii. **Second Countable, locally Euclidean, not Hausdorff**:

Line with two origins. $X = \mathbb{R} \cup \{0'\}$ with base:

$$B = \begin{cases} \text{open intervals in } \mathbb{R} \\ \text{sets of the form } (a, 0) \cup \{0'\} \cup (0, b) \end{cases} \text{for } a < 0 < b$$

Lemma 10. *X is locally Euclidean and Second Countable but not Hausdorff*

Proof.

Locally Euclidean: Notice that $(\mathbb{R} - \{0\}) \cup \{0'\}$ is homeomorphic to $\mathbb{R}$. Thus we have a Euclidean neighborhood of $0'$ and therefore X is locally Euclidean.

Second Countable: Choose the base as above but with only rational endpoints

Not Hausdorff: 0 and $\{0\}'$ cannot be separated by disjoint open sets (i.e. since they share the same "neighborhood") □

Properties of Manifolds

1. **Dimension (n) is well-defined**

$\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$ (the proof uses algebraic topology)

2. **Path components are the same as connected components.**

Proof. Pick some $x \in X$. Its path component is $Y = \{y \in X \,|\, y \text{ is path connected to } x\}$

This is open because $y \in Y \Rightarrow$ has a neighborhood U homeomorphic to $\mathbb{R}^n$ ($f: U \to \mathbb{R}^n$). Then $f(y)$ can be joined to $f(z)$ by a linear path in $\mathbb{R}^n$ $\forall z \in U$.

Therefore, y can be joined to z in U by the preimage of that path. $\forall z \in U$ z is path connected to y, which is path connected to x. Therefore $z \in Y$.

Finally, we can conclude that Y is open.

Since $X - Y$ is the union of path components, and since each path component is open, we know that $X - Y$ is open. Thus, we can conclude that Y is also closed.

Thus, path components are connected components. □

3. **Manifolds are Metrizable**

Lecture 13

Embedding Theorem

Theorem 11. *Let X be an m-manifold. Then there exists some $N \geq m$ such that X is homeomorphic to a subset of $\mathbb{R}^N$ (in the subspace topology)*

Example 12. $S^2 \subset \mathbb{R}^3$, here $m = 2$ and $N = 3$

We will prove this theorem when X is compact, but the theorem extends to non-compact X.

Proof: For X compact

Remark 13. For X-compact it suffices to construct a continuous, injective map from $f: X \to \mathbb{R}^n$. Then the map f as a map from X to $f(X)$ is continuous and bijective (since it was injective to $\mathbb{R}^n$).

Recall the proposition, that if X is compact and Y is Hausdorff then a continuous and bijective function, then f is a homeomorphism.

Thus f in this case is a homeomorphism.

Definition 14. *A map that is a homeomorphism onto its image is called a topological embedding*

Note 15. A homeomorphism onto its image is not a regular homeomorphism, only when you consider the image

Definition 16. *The support of a function $\phi: X \to \mathbb{R}$ (X is a topological space) is:*

$$\operatorname{supp}(\phi) = \phi^{-1}(\mathbb{R} - \{0\})$$

in other words:

$$x \in \operatorname{supp}(\phi) \Leftrightarrow \exists \textit{open neighborhood of } x \textit{ where } \phi \neq 0$$

Note 17. Basically the preimage of where the function is not 0. Since $\mathbb{R} - \{0\}$ is open, it is clear that every point in the support must have an open ball around it.

Definition 18. *Let $U = \{U_1, \ldots, U_n\}$ be a finite open cover a topological space X. A family of continuous functions $\phi_i: X \to [0, 1]$ is called a partition of unity subordinate to U if*

$$\operatorname{supp}(\phi_i) \subset U_i \qquad \forall i$$

and

$$\sum_{i=1}^{n} \phi_i(x) = 1 \quad \forall x \in X$$

Proposition 19. *If X is normal and U has a finite open cover, then U admits a subordinate partition of unity*

Proof.

Recall the lemma: Given that X-normal, $A \subset U \subset X$ where A is closed and V is open $\Rightarrow \exists V \subset X$ open, $A \subset V \subset \bar{V} \subset U$.

We will prove the proposition in 2 steps:

1. Shrink the open cover $U = \{U_1, \ldots, U_n\}$ to a new open cover $\{V_1, \ldots, V_n\}$ s.t. $\bar{V_i} \subset U_i \; \forall i$

   Let $A_1 = X - (U_2 \cup \ldots \cup U_n)$. This is closed, contained in U_1 (because U is an open cover, so U_1 should be the only piece remaining).

   Apply the lemma to $A_1 \subset U_1$ to get that there exists V_1 open such that $A_1 \subset V_1 \subset \bar{V_1} \subset U_1$. Then, $\{V_1, U_2, U_3, \ldots, U_n\}$ covers X since $A_1 \subset V_1$ and $A_1 = X - (U_2 \cup \ldots \cup U_n)$.

   Next, let $A_2 = X - (V_1 \cup U_3 \cup \ldots \cup U_n)$ is closed and contained U_2. Then apply, the lemma again to get that there exists V_2 open, such that $\{V_1, V_2, U_3, \ldots U_n\}$ covers X.

   Continue this inductively until we get $\{V_1, V_2, \ldots, V_n\}$

2. Do step 1 again for $\{V_1, \ldots, V_n\}$ to get an open cover $\{W_1, \ldots, W_n\}$ with $\overline{W_i} \subset V_i$

   Then $\overline{W_i}$ and $X - V_i$ are disjoint, closed subsets of X. Since X is normal, we can now apply Urysohn's lemma to $\overline{W_i}$ and $X - V_i$ to get that there exists $\psi_i \colon X \to [0,1]$ which is continuous and $\psi_i(x) = 1 \; \forall x \in \overline{W_i}$, $\psi_i(x) = 0$, $\forall x \in X - V_i$.

   Thus, that implies that $\psi_i^{-1}(\mathbb{R} - \{0\}) \subset V_i$, which implies $\mathrm{supp}(\psi_i) = \overline{\psi_i^{-1}(\mathbb{R} - \{0\})} \subset \bar{V_i} \subset U_i$

   Thus, the support is in U_i (this is one condition in the partition of unity).

   The collection W_i covers X; $\psi_i = 1$ on W_i implies that $\psi(x) := \sum_{i=1}^{n} \psi_i(x) > 0$, $\forall x \in X$

   Now define $\phi_i(x) = \frac{\psi_i(x)}{\psi(x)} \Rightarrow \sum \phi_i(x) = \frac{\sum \psi_i(x)}{\psi(x)} = 1$ and $\mathrm{supp}(\psi_i) = \mathrm{supp}(\phi_i) \subset U_i$. Thus ϕ_i is the partition of unity.

□

Back to the proof of the embedding theorem when X is compact.

Since X is locally Euclidean that implies that it admits an open cover by subsets homeomorphic to $\mathbb{R}^m$

Since X is compact that implies that it has a finite sub-cover. So combining these two point together. We know there exists $\{U_1, \ldots, U_n\}$ with $\bigcup_{i=1}^{n} U_i = X$ and $U_i \subset X$ open and there exists $g_i \colon U_i \to \mathbb{R}^m$, which is a homeomorphism.

Since X is compact and Hausdorff that implies that X is normal.

By the proposition above (on partition of unity), we know that there exists a partition of unity $\{\phi_i\}_{i=1}^{n}$ subordinate to $\{U_1, \ldots U_n\}$. For each $i = 1, \ldots n$ define $h_i \colon X \to \mathbb{R}^m$:

$$h_i(x) = \begin{cases} \phi_i(x)\, g_i(x) & \text{for} \quad x \in U_i \\ \vec{0} & \text{for} \quad x \in X - U_i \end{cases}$$

The support of ϕ_i is a subset of U_i, and the support of h_i is the same. Since both g_i and ϕ_i are continuous so is $h_i(x)$ in U_i. Moreover, h_i is continuous outside of U_i since $h_i's$ 0 corresponds to when ϕ_i is 0.

Now, we define $f \colon X \to \mathbb{R}^N$ where $N = n + m\,n$ as follows:

$$f(x) = (\phi_1(x), \phi_2(x), \ldots. \phi_n(x), h_1(x), h_2(x), \ldots, h_n(x)) \in \mathbb{R}^n \times \mathbb{R}^{mn} = \mathbb{R}^{n+mn}$$

note that $\phi_i(x) \in \mathbb{R}$ and $h_i \in \mathbb{R}^m$, since there are n of each, we get that $\mathbb{R}^{n+mn}$. Then, we have that since ϕ_i and h_i are continuous then f is continuous.

We will now check that f is injective. If $f(x) = f(y)$, then $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ for all i. We know that $\phi_i(x) > 0$ for some i (because $\sum \phi_i(x) = 1$). This implies that there exists some i such that $\phi_i(x) = \phi_i(y) > 0$. Thus, both $x, y \in \mathrm{supp}(\phi_i) \subset U_i$. Furthermore we have that:

$$\left.\begin{array}{r}\phi_i(x)\, g_i(x) = h_i(x) = h_i(y) = \phi_i(y)\, g_i(y) \\ \phi_i(x) = \phi_i(y) > 0\end{array}\right\} \Rightarrow g_i(x) = g_i(y)$$

But $g_i: U_i \to \mathbb{R}^m$ is already a homeomorphism, so it is injective, which implies $x = y$.

Thus f is an embedding, which implies that X is homeomorphic to a subset of $\mathbb{R}^N$.

Corollary 20. *X is a compact manifold implies that X is metrizable*

Lecture 14

Quotient Topologies

Definition 21. *Let X be a topological space. Let A be a set, $p: X \to A$ a surjective map. The quotient topology on A is given by:*

$$U \subset A \textit{ is open} \quad \Leftrightarrow \quad p^{-1}(U) \textit{ is open in } X$$

Proposition 22. *The quotient topology is a topology*

Proof.

To check 1, we see that $p^{-1}(\emptyset) = \emptyset$ and $\emptyset$ is open. $p^{-1}(A) = X$ is open in X. Thus, 1 is fulfilled:

To check 2 and 3, we have that $p^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha p^{-1}(U_\alpha)$ and $p^{-1}(\bigcap_{i=1}^n U_{\alpha_i}) = \bigcap_{i=1}^n p^{-1}(U_{\alpha_i})$ □

Proposition 23. *f from $A \to Y$ is continuous if and only if $f \circ p: X \to Y$ is continuous (where A is a quotient of X by p)*

Proof.

"⇒" p is continuous by definition, and since f is assumed to be continuous, we have that the composition $f \circ p$ is continuous.

"⇐" We are given that $f \circ p$ is continuous. We want to show that $U \subset Y$ is open implies that $f^{-1}(U) \subset A$ is open. But, $f^{-1}(U)$ open ⇔ $p^{-1}(f^{-1}(U)) = (f \circ p)^{-1}(U)$ is open in X, which is true by continuity of $f \circ p$. □

Definition 24. *An Equivalence Relation on a set X is a binary relation $\sim$ such that:*

$$\begin{array}{ll}(1) & x \sim x \\ (2) & x \sim y \Rightarrow y \sim x \\ (3) & x \sim y, y \sim z \Rightarrow x \sim z\end{array}$$

This determines a splitting of X as a disjoint union of equivalence classes, X_α, each of the form $[x_0] = \{x \in X | x \sim x_0\}$ for some x_0. We denote by X^* the set of equivalence classes.

There is a natural surjective map $p: X \to X^*$, $p(x) = [x]$. If X is a topological space, we can give X^* the quotient topology.

Examples:

1. $X = \mathbb{Z}$, $x \sim y \Leftrightarrow$ x,y have the same parity, $X^* = \{[0], [1]\}$ where $[0] = \{\text{even integers}\}$, $[1] = \{\text{odd integers}\}$ in the discrete topology.

2. $X = [0, 1]$ where $x \sim y \Leftrightarrow (x = y \text{ or } \{x, y\} = \{0, 1\})$ (i.e. identify 0 and 1 as the same but everything else is separate). Then $X^* = [0, 1] / 0 \sim 1$ where this is divided by the equivalence generated by this relation (i.e. the smallest equivalence relation that satisfies all the properties of the equivalence relation and the desired equivalence). Then X^* is homeomorphic to S^1, a circle (e.g. $\cos(2\pi\theta)$, $\sin(2\pi\theta)$) and the inverse is $\frac{\arg(x + i\, y)}{2\,\pi}$).

3. Annulus: $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and $X^* = X / (0, x) \sim (1, x) \quad \forall x \in [0, 1]$ (i.e. gluing the two opposite sides of a square, for example the left edge and the right edge). You end up getting a band or what you see in between two circles in a plane. This is called an annulus and is homeomorphic to $S^1 \times [0, 1]$ (i.e. a circle and a strip) to make a cylinder.

4. Möbius Band: $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and $X^* = X / (0, x) \sim (1, 1 - x) \quad \forall x \in [0, 1]$. Notice the change in equivalence ration from example 3 (it is in the opposite direction this time).

Remark 25. The annulus and the Möbius band are not manifolds. They are Hausdorff and second countable, but they are not locally Euclidean (because the boundary is closed). These are called "manifolds with boundary". They are locally like an open subset of $H^n = \{(x_1, \ldots x_n) \in \mathbb{R}^n | x_n \geq 0\}$

Remark 26. "Manifolds with boundary" are not necessarily manifolds, but all manifolds are "manifolds with boundary", (i.e. since manifolds are locally Euclidean, they are also locally like H^n since you can put the ball in the middle of H^n)

Remark 27. The open annulus $S^1 \times (0, 1)$ and the open Möbius band $[0, 1] \times (0, 1) / (0, x) \sim (1, 1 - x)$ $\forall x \in (0, 1)$ are non-compact manifolds.

5. Torus: You have a square and identify opposite sides together. $X = [0, 1] \times [0, 1]$ and $X^* = [0, 1] \times [0, 1] / (0, x) \sim (1, x)$ and $(x, 0) \sim (x, 1)$ $\forall x \in [0, 1]$ and this is homeomorphic to $S^1 \times S^1$. This is a 2-dimensional manifold.

6. Klein Bottle: Identify opposite side of the square with one of the sides being identified in the opposite direction. $X^* = [0, 1] \times [0, 1] / (0, x) \sim (1, x)$ and $(x, 0) \sim (1 - x, 1)$ $\forall x \in [0, 1]$. Can only be seen in 4 dimensions.

7.Real Projective Plane: Identify opposite sides of the square with opposite direction. $[0, 1] \times [0, 1] / (0, x) \sim (1, 1 - x)$ and $(x, 0) \sim (1 - x, 1)$ is homeomorphic to $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} / (x, y) \sim (-x, -y)$ when $x^2 + y^2 = 1$ (i.e. a circle connected across the diameter in opposite directions).

8.Connected Sums: Let X, Y be n-manifolds. Pick open sets $U \subset X$, $V \subset Y$ whose closures are homeomorphic to the ball $B^n = \{(x_1, \ldots x_n) \in \mathbb{R}^n | \sum x_i^2 \leq 1\}$. Thus, we have a homeomorphism f: $\bar{U} \to B^n \to \bar{V}$ so we can U and V together to get the connected sum. $X \# Y = (X - U) \perp (Y - V) / x \sim f(x)$ for all $x \in \partial U$ (i.e. removing the balls and identifying the two boundaries). For example, a double torus, i.e. surface of genus 2=$\sum_2$.

Exercise 2. X#Y is still an $n -$ manifold.

9. Surface = 2-manifolds, g-genus surfaces. $\sum_g$=$T^2 \# T^2 \# \ldots . T^2$ (g times) where g = genus indicates the numbers of holes.

10. $N_g = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2$ (g times), is a "nonorientable surface of genus g" (e.g. $N_2 = \mathbb{RP}^2 \# \mathbb{RP}^2$ is a Klein bottle).

Remark 28. All of these examples can be obtained by gluing polygons

Theorem 29. *Every connected, compact surface is homeomorphic to either* Σ_g *or* N_g *for some* $g \geq 0$

Examples: $\Sigma_0 = N_0 = S^2$ (sphere), and $\Sigma_1 = T^2$, and $N_1 = \mathbb{RP}^2$

Lecture 15

Up to now, we have been studying "point-set topology". From now on, we will be doing differential topology, and our main object of study will be smooth manifolds.

We will be explore the tools of differential topology (i.e. inverse function theorem, and implicit function theorem).

Definition 30. $U \subset \mathbb{R}^n \text{open}$, $f: U \to \mathbb{R}^m$. *It's derivative is denoted* $\mathrm{Df}: U \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$

Definition 31. $U, V \subset \mathbb{R}^n$ *open, then* $f: U \to V$ *is called a diffeomorphism if it is smooth, bijective and* f^{-1} *is smooth.*

Theorem 32. *Inverse Function Theorem:* $f: U \to V$ *is smooth,* $\mathrm{Df}(a)$ *is invertible implies that there exists an open neighborhood* U_0 *of* a *and* V_0 *of* $f(a)$ *such that* $f(U_0) \to V_0$ *is a diffeomorphism.*

Proof. f^{-1} is C^∞ since it is easy to calculate the inverse using matrices. □

Corollary 33. $U \subset \mathbb{R}^n$ *open,* $f: U \to \mathbb{R}^n$ *is smooth, and* $\det(\mathrm{Df}(x)) \neq 0$, $\forall x \in U$

a) $f(U)$ *is open;* f *takes open sets to open sets.*

b) *If f is injective, then* $f: U \to f(U)$ *is a diffeomorphism (may not be globally a diffeomorphism, counterexample is* $\mathbb{R}$ *wrapping around itself, is locally a diffeomorphism but not globally).*

Theorem 34. *Implicit Function Theorem:* $U \subset \mathbb{R}^n \times \mathbb{R}^k$ *open with coordinates* $(x_1, \ldots x_n, y_1, \ldots, y_k)$ *and, let* $\phi: U \to \mathbb{R}^k$ *be a smooth function* $\phi = (\phi_1, \phi_2, \ldots, \phi_k)$ *with each* $\phi_i: U \to \mathbb{R}$. *Let* $(a, b) \in U$ *and* $(a \in \mathbb{R}^n, b \in \mathbb{R}^k)$, $c = \phi(a, b) \in \mathbb{R}^k$. *Suppose the matrix:*

$$\left(\frac{\partial \phi_i}{\partial y_j}(a, b)\right)_{i,j=1,\ldots k} \text{is non} - \text{singular } (i.e. \det \neq 0)$$

then there exists open neighborhoods $V_0 \subset \mathbb{R}^n$ *of* a, *and* $W_0 \subset \mathbb{R}^k$ *of* b *and a smooth function* f: $V_0 \to W_0$ *such that for* $(x, y) \in V_0 \times W_0$ *we have that:*

$$\phi(x, y) = c \Leftrightarrow y = f(x)$$

Remark 35. f is called an "implicit function". It gives local coordinates for the level set $\phi^{-1}(c)$.

$\phi^{-1}(c)$ near (a, b) is the graph of f. Thus, $\phi^{-1}(c)$ near (a, b) is locally Euclidean via the map $V_0 \to \phi^{-1}(c) \cap (V_0 \times W_0)$ and $x \to (x, f(x))$ and the implicit function theorem gives conditions for a level set to be a manifold.

Proof. Let us construct a different function $\psi: U \to \mathbb{R}^n \times \mathbb{R}^k$, such that $\phi(x, y) = (x, \phi(x, y))$. Now, we have that:

$$D\psi = \begin{pmatrix} I_n & 0 \\ \left(\frac{\partial \phi_i}{\partial x_j}\right)_{i,j} & \left(\frac{\partial \phi_i}{\partial y_j}\right)_{i,j} \end{pmatrix}$$

We assume that $\det\left(\left(\frac{\partial \phi_i}{\partial y_j}\right)_{i,j}\right) \neq 0$ since this is an assumption of the theorem, therefore, we have that:

$$\det(D\psi) = \det\left(\left(\frac{\partial \phi_i}{\partial y_j}\right)_{i,j}\right) \neq 0$$

Now, we can use the inverse function theorem. We define $\psi(a, b) = (a, c)$. There exists $(a, b) \in U'$ and $(a, c) \in Y'$ such that:

$$\psi|_j: U' \rightarrow Y' \text{is a diffeomorphism}$$

Pick $(a, b) \in V \times W \subset U$ base element (product of open sets)... (rest of proof in Lee Smooth Manifolds)

□

Recall: Every manifold is homeomorphic to a subset of $\mathbb{R}^n$

Question 1. Big question is which subset of $\mathbb{R}^n$ are manifolds?

e.g. shape Y is not a manifold but a circle is a manifold

Question 2. Can we give a sufficient condition for a subset to be a manifold?

Answer 1. Yes, we can with the implicit function theorem.

Proposition 36. *Suppose $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^k$ is smooth and $(D\phi)(x): \mathbb{R}^n \rightarrow \mathbb{R}^k$ is surjective for all $x \in U$. Then the preimage, $\phi^{-1}(y)$ is a manifold for all $y \in \mathbb{R}^k$.*

Proof. $D\phi(x) = k \times n$ matrix (surjective) $\Rightarrow \exists$ non-singular $k \times k$ minor matrix

Choose $y_1, \ldots y_k$ to be the coordinates for those columns and $x_1, \ldots, x_{n-k}$ to be the others.

Now, apply the implicit function theorem $\Rightarrow M = \phi^{-1}(y)$ is locally Euclidean. It's Hausdorff and second countable because it is a subset of $\mathbb{R}^n$. □

Example 37. New proof the $S^n \subset \mathbb{R}^{n+1}$ is a manifold.

Consider $\phi: \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}$, $\phi(x_1, \ldots, x_n) = x_0^2 + \cdots + x_n^2$. Then we have that $(D\phi)(x_0, \ldots, x_n) = (2x_0, \ldots, 2x_n)$ is surjective (*i.e.* $\neq (0, 0, \ldots, 0)$) whenever $(x_0, \ldots, x_n) \neq (0, 0, \ldots 0)$.

This implies that $\phi^{-1}(1) = S^n$ is a manifold.

Lecture 17

Smooth Manifolds

Assume that X is a topological manifold

Definition 38. *A chart on X is a pair (U, ϕ) where $U \subset X$ open $\phi: U \rightarrow \tilde{U} = \phi(U) \subset \mathbb{R}^n$ is a homeomorphism.*

Definition 39. *φ is called a local coordinate system*

We want to be able to say that when a function $f\colon X \to \mathbb{R}^N$ is smooth. One way to do this is through considering the composition of f with ϕ^{-1}. In other words, $f \circ \phi^{-1}$ is smooth, the f is smooth since then you can take $U \to \phi \to f(U)$ and be a smooth map

Question 3. What if we choose a different chart would it still work? Is it true that $f \circ \psi^{-1}$ is smooth if and only if $f \circ \phi^{-1}$ is smooth?

Answer 2. We can write $(f \circ \phi^{-1}) = (f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})$

Given that $(f \circ \psi^{-1})$ is smooth, then for $(f \circ \phi^{-1})$ to be smooth $(\psi \circ \phi^{-1})$ must be smooth. We have that:

$$(\psi \circ \phi^{-1})$$

is called a <u>transition function.</u>

Warning 40. There are case where $(\psi \circ \phi^{-1})$ is not smooth. Take:

$$\begin{aligned} \psi\colon \mathbb{R} \to \mathbb{R} \quad s.t. \quad \psi(x) = x \qquad \text{(identity)} \\ \phi\colon \mathbb{R} \to \mathbb{R} \quad s.t. \quad \phi(x) = x^3 \end{aligned}$$

We have that ϕ is a homeomorphism but not a diffeomorphism since $\phi^{-1}(x) = (x)^{1/3}$ is not differentiable at 0 (since the derivative goes to infinity at $x = 0$).

Note 41. a homeomorphism does not need to be a diffeomorphism as seen above

Definition 42. *Two charts (U, ϕ) and (V, ψ) on X are called <u>compatible</u> if $\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$ is a **diffeomorphism***

Note 43. Notice that we care about the places they intersect

Remark 44. We want to cover the manifold with compatible charts. This motivates what we will do next.

Note 45. Disjoint charts are compatible.

Definition 46. *An <u>atlas</u> $\mathbb{A}$ for a topological manifold X is a collection of (pairwise) compatible charts that cover X.*

$$\mathbb{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A} \qquad s.t. \quad \bigcup_{\alpha \in A} U_\alpha = X$$

Example 47. An Atlas Example

Consider the circle S^1. It can't have a single chart, but it can have a collection of charts.

Let $U = S^1 - \{1\}$ and $V = S^1 - \{-1\}$ where 1 and -1 are on the two boundaries opposite to each other.

$\phi\colon U \to (0, 2\pi) \subset \mathbb{R}$, $\phi(e^{i\theta}) = \theta$, $\theta \in (0, 2\pi)$

$$\begin{aligned} \phi\colon U \to (0, 2\pi) \subset \mathbb{R} \quad \phi(e^{i\theta}) = \theta \quad \theta \in (0, 2\pi) \\ \psi\colon V \to (\pi, 3\pi) \subset \mathbb{R} \quad \phi(e^{i\theta}) = \theta \quad \theta \in (\pi, 3\pi) \end{aligned}$$

We have that $\psi \circ \phi^{-1}: (0, \pi) \cup (\pi, 2\pi) \to (2\pi, 3\pi) \cup (\pi, 2\pi)$

and

$$\psi \circ \phi^{-1}: \begin{cases} \theta \to \theta + 2\pi & \text{for } \theta \in (0, \pi) \\ \theta \to \theta & \text{for } \theta \in (\pi, 2\pi) \end{cases}$$

Thus, (U, ϕ) and (V, ψ) form an atlas for S^1, we can make sense of a function $f: S^1 \to \mathbb{R}^N$ is smooth

Definition 48. *If A is an atlas, we let* $\mathbb{A}^{\max} =$ *<u>maximal atlas</u> containing* $\mathbb{A} = \{$*all charts compatible with the charts in* $\mathbb{A}\}$. *A is called maximal if* $\mathbb{A} = \mathbb{A}^{\max}$

Example 49. $\mathbb{A} = \{\text{all charts } (U, \phi), U \subset \mathbb{R}^n \text{ open}\}$

Definition 50. *A smooth structure on a topological manifold X is called a maximal atlas for X.*

Definition 51. *A smooth manifold* $(X, \mathbb{A})$ *is a topological manifold equipped with a smooth structure.*

Remark 52. We usually drop $\mathbb{A}$ from the notation.

Example 53. $\mathbb{R}^n$ with the maximal atlas above is $\mathbb{R}^n$ with the standard smooth structure.

If we take $\mathbb{R}$, and let $B = \{$all charts (U, ϕ), $U \subset \mathbb{R}$ open, $\phi: U \to \tilde{U} \subset \mathbb{R}$ is such that $\phi \circ \tau$ is a diffeomorphism where $\tau(x) = x^3\}$ is a different smooth structure.

But this would give you weird structure.

Definition 54. $(M, \mathbb{A})$ *- smooth manifold. Charts in* $\mathbb{A}$ *are called "smooth charts"*

Definition 55. *With M, N smooth manifold, a function* $f: M \to N$ *is <u>smooth</u> at* $x \in M$ *if for a smooth chart* (U, ϕ) *near x and* (V, ψ) *near* $f(x) \in N$, $\psi \circ f \circ \phi^{-1}$ *is smooth at φ(x).*

Note 56. More generally, $f: M \to N$ is smooth if it is smooth at all $x \in M$

Remark 57. If this true for some smooth charts $(U, \phi), (V, \psi)$, it is also true for any other smooth charts (U', ϕ') and (V', ψ'). This is since you know that the transition functions are smooth and diffeomorphism then you can add these function to the chain of compositions and will not effect the smoothness of $\psi \circ f \circ \phi^{-1}$

We can similarly define C^k function for $k \geq 0$

Notation 58. $C^k(M, N) = \{f: M \to N \quad C^k\} \quad C^k(M) = C^k(M, \mathbb{R})$

at at $k = \infty$, *we have* $C^k = \text{smooth}$

Proposition 59. $f: M \to N$, $g: N \to P$ *is smooth, then* $g \circ f: M \to P$ *is smooth.*

Definition 60. *M, N are smooth manifolds.*

$f: M \to N$ *is called a diffeomorphism if it is smooth, bijective, and* f^{-1} *is smooth.*

Example 61. $f: S^1 \to S^1$ defined by rotation by $\theta \in [0, 2\pi)$ is a diffeomorphism.

Notation 62. *S^1 as a smooth manifold, means equipped with the standard smooth structure, which means all charts compatible with the two constructed above (in the example on S^1).*

Proposition 63. *Given M, N smooth manifolds, then $M \times N$ has a smooth structure consisting of all charts compatible with those of the form $(U \times V, \phi \times \psi)$ when (U, ϕ) is a smooth chart on M and (V, ψ) is a smooth chart on N. Then $M \times N$ is a smooth manifold.*

Remark 64. IF we are given an atlas $\mathbb{A}$ for M, this determines a smooth structure on M, namely $\mathbb{A}^{\max}$.

Lecture 18

Examples of Smooth Manifolds

Remark 65. It suffices to give a topological manifold and an atlas (then we take the maximal atlas, $\mathbb{A}^{\max}$)

Example 66. Sphere $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0^2 + \cdots + x_n^2 = 1\}$

Let us take $U = S^n - \{(1, 0, 0, \ldots, 0)\}$. We know through the stereographic projection we have a homeomorphism $\phi: U \to \mathbb{R}^n$. We had that:

$$\phi(x_0, \ldots, x_n) = \left(\frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right)$$

The other stereographic projection is from $V = S^n - \{(-1, 0, 0, 0 \ldots, 0)\}$. Through a similar stereographic projection, we have a homeomorphism $\psi: V \to \mathbb{R}^n$. Such that:

$$\psi(x_0, \ldots, x_n) = \left(\frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0}, \ldots, \frac{x_3}{1 + x_0}\right)$$

We want to prove that it is smooth manifold, we have to compute the transition function $\psi \circ \phi^{-1}$. We have that:

$$\psi \circ \phi^{-1}: \phi(U \cap V) \to \psi(U \cap V)$$

In other words, the map is from $\mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ and we want to check that his map is a diffeomorphism (including smoothness). Let us call:

$$\phi = \left(\frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right) = (y_1, y_2, \ldots y_n)$$

Thus, we have that:

$$y_i = \frac{x_i}{1 - x_0}$$

Thus, we have that:

$$\begin{aligned}
\sum_i y_i^2 &= \frac{\sum_i x_i^2}{(1 - x_0)^2} \\
&= \frac{1 - x_0^2}{(1 - x_0)^2} \\
&= \frac{1 + x_0}{1 - x_0}
\end{aligned}$$

Now, we can write:

$$\begin{aligned} \frac{x_i}{1+x_0} &= \frac{1-x_0}{1+x_0}\frac{x_i}{1-x_0} \\ &= \frac{y_i}{\sum y_i^2} \end{aligned}$$

Thus, we have now have an expression for the output of ψ in terms of the output of ϕ. More explicitly, we can put this together to get:

$$(\psi \circ \phi^{-1})(y_1, \ldots, y_n) = \left(\frac{y_1}{\sum y_i^2}, \ldots, \frac{y_n}{\sum y_n^2}\right)$$

Thus implies the transition function map is:

$$\vec{y} \to \frac{\vec{y}}{\|\vec{y}\|^2}$$

This is an inversion mapping $\left(i.e. \sim \frac{1}{y}\right)$, which is a diffeomorphism from $\mathbb{R}^n - \{0\}$ to itself. Thus, we have a smooth structure on S^n.

Example 67. Real Projective Space ($\mathbb{RP}^n$) , spaces of lines $\mathbb{R}^{n+1}$ through the origin.

In other words, $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\}) / (x \sim \lambda x)\ \forall \lambda \in \mathbb{R} - \{0\}$ which is equivalent to $S^n / (x \sim (-x))$. The elements of $\mathbb{RP}^n$ are usually denoted as $[x_0: x_1: \ldots : x_n]$. This is an equivalence of class of $(x_0, x_1, \ldots x_n) \in \mathbb{R}^{n+1} - \{0\}$.

We can make an atlas of $n+1$ charts U_i such that $i = 0, \ldots, n$ where $U_i = \{[x_0: \ldots : x_n] | x_i \neq 0\}$ such that $\mathbb{RP}^n = \bigcup_{i=0}^n U_i$. We define the following homeomorphisms $\phi_i: U_i \to \mathbb{R}^n$,

$$\phi_i([x_0: \ldots : x_n]) \to \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

Now, we need to make sure the transition functions are smooth. Take functions from U_i to U_j. Consider $[x_0: \ldots : x_n] \in U_i \cap U_j$. we have that:

$$\begin{aligned} \phi_i &\to \left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) = (y_0, \ldots, y_n) \\ \phi_j &\to \left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) = \left(\frac{y_0}{y_j}, \ldots, \frac{1}{y_j}, \ldots \frac{y_n}{y_j}\right) \end{aligned}$$

where $\frac{1}{y_j}$ is the i^{th} element since we have that $\frac{x_i}{x_j} = \frac{1}{\frac{x_j}{x_i}} = \frac{1}{y_j}$. For the other coordinates, we have that $\frac{x_0}{x_j} = \frac{\frac{x_0}{x_i}}{\frac{x_j}{x_i}} = \frac{y_0}{y_j}$. Thus, since this is smooth it gives you smooth structure on $\mathbb{RP}^n$.

Example 68. $\mathbb{CP}^n$ is the same as example 2 but with complex coordinates.

Example 69. Open subsets of smooth manifolds are smooth manifolds.

Take a chart for each open set around a point.

e.g. $\mathbb{GL}_n(\mathbb{R}) = \{n \times n \,\text{invertible real matrices}\}$ is an open $\subset \mathbb{R}^{n^2} = \mathbb{M}_{n \times n}(\mathbb{R})$

Example 70. Products of smooth manifolds like $T^n = S^1 \times S^1 \times \ldots \times S^1$ is a smooth manifold.

Derivatives

Recall: $U \subset \mathbb{R}^n$ open and $f: U \to \mathbb{R}^m$ is smooth $f = (f_1, \ldots f_m)$ with :

$$D f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

and if we have a $\vec{v} \in \mathbb{R}^n$ and $p \in U$. Then the directional derivative is defined as:

$$(D_{\vec{v}} f)_p = (D f(p))(\vec{v})$$

Furthermore, we have the chain rule:

$$D(f \circ g)_p = (D f)_{g(p)} \cdot (D g)_p$$

Let us say X is a smooth manifold and $f: X \to \mathbb{R}$ is smooth

Definition 71. *X is a smooth manifold, $p \in X$. The tangent space to X at p, denoted $T_p X$ is the space of equivalence classes of pairs $((U, \phi), \vec{v})$ where (U, ϕ) is a smooth chart near p ($p \in U$) and $\vec{v} \in \mathbb{R}^n$ with the equivalence relation:*

$$((U, \phi), \vec{v}) \sim ((V, \psi), \vec{w}) \Leftrightarrow \vec{w} = D(\psi \circ \phi^{-1})_{\phi(p)}(\vec{v})$$

Definition 72. *An element of $T_p X$ is called a tangent vector at p, "direction on the manifold X starting at p".*

Remark 73. $T_p X$ is an $n-$dimensional vector space. It is isomorphic to $\mathbb{R}^n$, but the isomorphism depends on choosing a chart.

For $\vec{u} \in T_p X$ and $f: X \to \mathbb{R}$, the derivative $(D_{\vec{u}} f)_p \in \mathbb{R}$ is well-defined:

- choose a chart (U, ϕ)

- then $\vec{u} = [(U, \phi), \vec{v}]$ for some $\vec{v} \in \mathbb{R}^n$. Set:

$$(D_{\vec{u}} f)_p := D_{\vec{v}}(f \circ \phi^{-1})_{\phi(p)}$$

More generally if X, Y are smooth manifold, $f: X \to Y$ smooth, for $\vec{u} \in T_p X$, there is a derivative $(D_{\vec{u}} f)_p \in T_{f(p)} Y$.

Miscellaneous Lecture Material From Lecture 19+

Smooth Manifolds

Definition. (Topological Manifold)

- *Hausdorff*
- 2^{nd} *Countable*
- *Locally Euclidean*

Definition. (Chart) *A chart on X is a pair (U, ϕ) where $U \subset X$ is open and $\phi: U \to \tilde{U} \subset \mathbb{R}^n$ is a homeomorphism.*

Note. Basically a chart is you mapping some piece of your manifold onto $\mathbb{R}^n$ to see its "analogy" or "coordinates" on $\mathbb{R}^n$

Inverse Function Theorem and Immersions

Diffeomorphism

Definition. (Diffeomorphism) *f is a diffeomorphism if:*

- *f is smooth*
- *f is a bijection*
- *$f^{-1}$is smooth*

Definition. (Local Diffeomorphism) *f is a local diffeomorphism at $p \in X$ if there exists open neighborhood U of p and U' of $f(p)$ such that $f|_U: U \to U'$ is a diffeomorphism.*

Locality vs Globality

$$\begin{aligned} \text{Global Diffeomorphism} &\Rightarrow \text{Local Diffeomorphism Everywhere} \\ \text{Local Diffeomorphism Everywhere} &\not\Rightarrow \text{Global Diffeomorphism} \end{aligned}$$

For example, the circle is not injective.

Inverse Function Theorem

Theorem. (Inverse Function Theorem)

$$\begin{aligned} f: X \to Y \text{ is } \textbf{smooth} \quad &+ \quad Df_x \text{ at point } x \text{ is an } \textbf{isomorphism} \\ &\Downarrow \\ &\quad f \text{ is a } \textbf{local diffeomorphism} \end{aligned}$$

IF $\dim(X) = \dim(Y) \Rightarrow \mathbb{I}$

then the inverse function theorem tells us that that there exists charts ϕ and ψ such that $f = \mathbb{I}$ is locally like the identity. In other words, f is diffeomorphic to the identity map.

Theorem. (Inverse Function Theorem with $\dim(X) = \dim(Y)$)

$$\begin{aligned} f: X \to Y \text{ is } \textbf{smooth} \quad &+ \quad Df_x \text{ at point } x \text{ is an } \textbf{isomorphism} \\ &+ \quad \dim(X) = \dim(Y) \\ &\Downarrow \\ &\quad f \text{ is locally the identity} \end{aligned}$$

Proof. In more rigorous sense we can choose charts $\phi = \psi \circ f$ such that $\psi \circ f \circ \phi^{-1} = \mathbb{I}$ locally around x. □

IF $\dim(X) < \dim(Y) \Rightarrow$ Immersion

then we get that our smooth f is an immersion at the point x.

Definition. (Immersion) *$f\colon X \to Y$ is an immersion if:*

- *f is smooth at x*
- *$D\,f_x$ is* ***injective***
- $\dim(X) \leq \dim(N)$

Note that the most we can ask of $D\,f_x$ is that it be injective (it can't be isomorphic).

Theorem. (Local Immersion Theorem)

$$\left.\begin{array}{r} f\colon X \to Y \textit{ is } \textbf{\textit{smooth}} \searrow \\ D\,f_x \textit{ is } \textbf{\textit{injective}} \rightarrow \\ \mathbf{dim}(\boldsymbol{X}) \leq \mathbf{dim}(\boldsymbol{Y}) \nearrow \end{array}\right. (f \text{ is an immersion})$$

$$\Downarrow$$

$$f \textit{ is locally } i_{\text{can}}$$
$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, 0, \ldots, 0)$$

Mapping onto Manifolds

The image of a immersion need not be a manifold. The image of an immersion need not be injective and need not map open sets to open sets. These two conditions are necessary to be able to have an image that is a manifold. (Examples: figure 8, or non-intersecting figure 8).

We need to add some new conditions onto our immersions to make their images manifolds for sure.

Definition. (Topological Embedding) *A map is $f\colon X \to Y$ is called a topological embedding if f is homeomorphic onto its image (f(X))*

Definition. (Smooth Embedding) *A map is a smooth embedding if:*

topological embedding
+
immersion

Now, we have that a smooth embedding does not allow intersection or limiting intersections.

Remark. In order to make rigorous what we mean by "the image of a smooth embedding is a manifold" we must first define the idea of a sub-manifold. The idea of a sub-manifold is that it is a smooth manifold residing on a larger smooth manifold. The importance of that here is that $f(X)$ can be a smooth manifold residing on the larger manifold Y . Note in our previous definitions we were careful to say that "f is homeomorphic onto its image", etc.

Sub-manifolds

Definition. (Sub-manifold) *Let X be a smooth manifold. A sub-manifold $Y^k \subset X^n$ is a subset such that for every $p \in Y$ there exists a smooth chart (U, ϕ) for X around p $(p \in Y)$ s.t. $\phi\colon U \to \tilde{U}$ maps $U \cap Y$ homomorphically onto $\tilde{U} \cap i_{\text{can}}(\mathbb{R}^k) \subset \mathbb{R}^n$ ($\{x_1, \ldots, x_n\} \in \tilde{U} | x_{k+1} = \cdots = x_n = 0\}$)*

Note. Basically we are mapping our sub-manifold Y into a lower dimensional space in $\mathbb{R}^n$. In other words, there exists charts so that while X^n maps to all of $\mathbb{R}^n$ our subset Y^k is only projected onto the $\mathbb{R}^k$ space inside $\mathbb{R}^n$

Remark. It is clear that $Y \subset X$ is a sub-manifold $\Rightarrow$ Y is a k-dimension smooth manifold

Proof. To be a smooth manifold, you must have a smooth atlas. In this case, we can take the charts ϕ made for X and we know that they project on the smaller space smoothly, so it works $\square$

Smooth Embeddings and Sub-manifolds

Proposition.

$$f: X \to Y \textit{ is a } \textbf{\textit{smooth embedding}} \;\Rightarrow\; \begin{array}{l} f(X) \textit{ is a } \textbf{\textit{smooth sub-manifold}} \textit{ of } Y \\ f: X \to f(X) \textit{ is a } \textbf{\textit{diffeomorphism}} \end{array}$$

Proof. The local immersion theorem gives us that the map f is already locally like i_{can}. Therefore, we already know that there exists some chart ψ of $f(X)$ i.e. on Y that takes $f(X)$ locally to i_{can}. Therefore, the smooth sub-manifold pat is clear.

In order to show diffeomorphism, we need to show f^{-1} is smooth. We already know homeomorphism from topological embedding and we know f is smooth. Thus, all that is left is to show f^{-1}: $f(x) \to x$ is smooth. We realize that f^{-1} looks locally like the identity. $\square$

Identifying Sub-manifolds

Now that we showed that smooth embeddings lead to sub-manifolds. It is now important to be able to easily tell wether a given map is a smooth embedding. It is difficult to check both whether f is an immersion and f is a topological manifold. Therefore, we come up with some key characteristics that can identify smooth embeddings.

Definition. (Proper Map) *A continuous map $f: X \to Y$ is called proper if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact.*

Note. One can think of proper as a map that take ∞ to ∞

Lemma.

$$\left.\begin{array}{r} X \textit{ is } \textbf{\textit{compact}} \\ Y \textit{ is } \textbf{\textit{Hausdorff}} \\ f: X \to Y \textit{is } \textbf{\textit{continuous}} \end{array}\right. \Rightarrow \; f \textit{ is } \textbf{\textit{proper}}$$

Note. Intuitively this lemma tells us that since X is compact, there is no ∞ that needs to be mapped, so there is nothing to worry about as long f is continuous and Y is nice

Theorem.

$$\left.\begin{array}{r} f \textit{ is an } \textbf{\textit{immersion}} \\ f \textit{ is } \textbf{\textit{proper}} \\ f \textit{ is } \textbf{\textit{injective}} \end{array}\right. \Rightarrow \; f \textit{ is } \textbf{\textit{a smooth embedding}} \textit{ and } f(X) \textit{ is a } \textbf{\textit{sub-manifold}}$$

Proof. Basically we are replacing topological embedding with proper + injective. Injective just guarantees it is bijective. The proper characteristic gives that f^{-1} is continuous. Therefore, we conclude that f is a homeomorphism (i.e. a topological embedding). □

Note. The other direction of the theorem is not true. Not all smooth embeddings are proper (for example $i: (0,1) \to \mathbb{R}$ is a smooth embedding but is not proper

Submersions

In the previous cases we looked at the conditions for a the image of a map to be a manifold. Now we are looking for the conditions of the preimage of a map to be a sub-manifold. We had a very similar idea from the implicit function theorem.

Theorem. (Implicit Function Theorem)

Let $f: U \subset \mathbb{R}^{k+m} \to \mathbb{R}^m$. *Thus, we have that* $f(x,y) \to \mathbb{R}^m$ *where* $\begin{matrix} x = (x_1, \dots x_k) \in \mathbb{R}^k \\ y = (y_1, \dots, y_m) \in \mathbb{R}^m \end{matrix}$ *and we have that* $f = (f_1(x,y), f_2(x,y), \dots, f_m(x,y))$.

and

Suppose that $\left(\frac{\partial f_i}{\partial y_j}\right)_{i,j}$ *is non-singular at* (x_0, y_0) *such that* $(\det \neq 0)$.

$$\Downarrow$$

$\exists$ *neighborhoods* $\begin{matrix} V \text{ of } x_0 \text{ in } \mathbb{R}^k \\ W \text{ of } y_0 \text{ in } \mathbb{R}^m \end{matrix}$ ***and*** $\exists$ *smooth map* $g: V \to W$ *s.t.*

$$f(x,y) = c \Leftrightarrow y = g(x)$$

Corollary. *Under the same hypothesis, we have that* $f^{-1}(C) \cap (V \times W)$ *is a smooth sub-manifold of U.*