18.901, FALL 2024 — HOMEWORK 8

Each part of each main problem is worth 5 points, so this homework is graded out of 40 points. Each part of each bonus problem is worth 0.5 additional points.

Main problems

Problem 1. Let U be an open subspace of $\mathbb{R}^3$. Show that there does not exist an embedding of U into $\mathbb{R}^2$. Hint: You may use the Borsuk–Ulam theorem.

Problem 2. Let X be a topological space, let G be a group, and let $\phi : G \to \mathrm{Homeo}(X)$ be a continuous action of G on X. Suppose that every point $x \in X$ has a neighborhood U in X such that $U \cap \phi(g)(U) = \varnothing$ for all $g \in G \smallsetminus \{e\}$. Show that then the quotient map $q : X \to X/G$ is a covering map.

Problem 3. Let X and Y be topological spaces and let $p : Y \to X$ be a covering map. Assume that X is connected and that, for some $x_0 \in X$, the degree of p at x_0 is finite. Show that then the degree of p is finite at every point $x \in X$, and in fact is independent of x.

Problem 4. Let $\mathbb{Z}$ be the set of integers, regarded as a group via addition. Let $H \subseteq \mathbb{Z}$ be a subgroup. Show that there exists $n \in \mathbb{Z}$ such that $H = n\mathbb{Z}$.

Problem 5. Let $p : \mathrm{S}^1 \to \mathrm{S}^1$ be a covering map and let $z_0 \in \mathrm{S}^1$. Show that the degree of p at z_0, meaning $|p^{-1}(z_0)|$, is necessarily finite, in fact is equal to $|\deg(p)|$, where $\deg(p) \in \mathbb{Z}$ is the degree of p as a continuous map from the circle to itself.

Problem 6. Let k be a positive integer and let $\mathbb{RP}^k$ be k-dimensional real projective space, as defined in Homework 5, Problem 5. By part (c) of that problem, we have a quotient map $q : \mathrm{S}^k \to \mathbb{RP}^k$.

(a) Show that q is a covering map. Hint: You may use Problem 2.

(b) Consider the case $k = 1$. Define a homeomorphism $f : \mathbb{RP}^1 \to \mathrm{S}^1$.

(c) Let f be the homeomorphism you defined in the previous part, and consider the composite $f \circ q : \mathrm{S}^1 \to \mathrm{S}^1$. Calculate $\deg(f \circ q)$.

Problem 7. Let k be a positive integer. The k-dimensional complex projective space $\mathbb{CP}^k$ is defined analogously to $\mathbb{RP}^k$, but with the real numbers $\mathbb{R}$ replaced by the complex numbers $\mathbb{C}$.

(a) Show that there is a quotient map $q : \mathrm{S}^{2k+1} \to \mathbb{CP}^k$, such that for each $x \in \mathbb{CP}^k$, the subspace $q^{-1}(x) \subset \mathrm{S}^{2k+1}$ is homeomorphic to S^1.

(b) Consider the case $k = 1$. Define a homeomorphism $g : \mathbb{CP}^1 \to \mathrm{S}^2$.

Putting the previous two parts together gives a continuous map $h := g \circ q : \mathrm{S}^3 \to \mathrm{S}^2$ such that $h^{-1}(x)$ is homeomorphic to S^1 for each $x \in \mathrm{S}^2$. This map "exhibits S^3 as a family of circles parameterized by S^2". Another such family, "the trivial one", is exhibited by the projection map $p : \mathrm{S}^2 \times \mathrm{S}^1 \to \mathrm{S}^2$.

(c) Show that S^3 is not homotopy equivalent to $\mathrm{S}^2 \times \mathrm{S}^1$.