Complex cobordism: geometry & algebra

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See, it's never either/or now:
stigmata ampersand.
– Pinegrove

1. Introduction

I recently gained a new layer of appreciation for some mathematical entities and theorems that were already quite close to my heart, in the form of a more geometric perspective on them. The emotional results of this are that they seem to me more mysterious and incredible than ever, and that I want more than ever for everybody to know them. The practical result is that I have a lighter and more relatable version of the story to tell, which is the content of today's talk. The goal is to convey to you my modest understanding and, at the end, overwhelming bewilderment.

The following notion was introduced by René Thom in the early 1950s [Tho54].

1.1. Definition. Let Y_0 and Y_1 be two compact smooth manifolds. A *cobordism from* Y_0 *to* Y_1 consists of a compact smooth manifold B with boundary together with a diffeomorphism $\partial\mathrm{B} \simeq \mathrm{Y}_0 \amalg \mathrm{Y}_1$.

1.1.1. Examples. (a) A pair of pants may be viewed as a cobordism from $\mathrm{Y}_0 := \mathrm{S}^1 \amalg \mathrm{S}^1$ to $\mathrm{Y}_1 := \mathrm{S}^1$.

(b) Given any compact smooth manifolds Y_0 and Y_1 and a diffeomorphism $\alpha\colon \mathrm{Y}_0 \simeq \mathrm{Y}_1$, the cylinder $\mathrm{B} := \mathrm{Y}_0 \times [0,1]$ can be viewed as a cobordism from Y_0 to Y_1 via the identification

$$\partial\mathrm{B} \simeq (\mathrm{Y}_0 \times \{0\}) \amalg (\mathrm{Y}_0 \times \{1\}) \overset{\alpha}{\simeq} \mathrm{Y}_0 \amalg \mathrm{Y}_1.$$

1.1.2. Remark. Cobordism defines an equivalence relation. This is a weaker form of equivalence than diffeomorphism, by (1.1.1)(b), and certainly a strictly weaker one, as demonstrated by (1.1.1)(a).

It will be convenient later to use an equivalent formulation of cobordism, which does not refer to manifolds with boundary. Let me just state it now to avoid a break in the flow later. For this, and in fact throughout this talk, we will need following the standard notion.

1.2. Definition. A map of manifolds $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ is said to be *proper* if for any compact subset $\mathrm{Z} \subseteq \mathrm{X}$, the preimage $f^{-1}(\mathrm{Z}) \subseteq \mathrm{Y}$ is also compact.

1.3. Definition. A *cobordism from* Y_0 *to* Y_1 consists of a smooth manifold Y (without boundary), a proper map $f\colon \mathrm{Y} \longrightarrow \mathbb{R}$ for which $0, 1 \in \mathbb{R}$ are regular values, and diffeomorphisms $\mathrm{Y}_0 \simeq f^{-1}(0)$ and $\mathrm{Y}_1 \simeq f^{-1}(1)$.

1.3.1. Remark. As indicated above, this notion of cobordism is equivalent to the one stated in (1.1):

– To go from (1.3) to (1.1), we take the map $f\colon \mathrm{Y} \longrightarrow \mathbb{R}$ and set $\mathrm{B} := f^{-1}([0,1])$; this is compact since f is proper, and will have boundary $f^{-1}(0) \amalg f^{-1}(1) \simeq \mathrm{Y}_0 \amalg \mathrm{Y}_1$.

– To go from (1.1) to (1.3), take the manifold with boundary B and form Y by gluing on infinite cylinders $Y_0 \times \mathbb{R}$ and $Y_1 \times \mathbb{R}$ to the boundary components Y_0 and Y_1; it is then straightforward to produce an appropriate proper map $f\colon Y \longrightarrow \mathbb{R}$.

We can also consider cobordism for manifolds with extra structure. For example, there is a natural notion of *oriented cobordism* between oriented manifolds. Another natural structure one might be interested in is that of a complex manifold. But this—or even an almost-complex structure, i.e. a complex structure on the tangent bundle—makes sense only in even dimension, while the notion of cobordism necessarily involves manifolds of both even and odd dimension simultaneously. Thus, to contemplate cobordism in this setting, we make the following definition.

1.4. Definition. Let X be a smooth manifold, and let τ_X denote the tangent bundle on X. A *stable almost-complex structure* on X is a complex structure on $\tau_X \oplus \underline{\mathbb{R}^m}$ (the second factor denoting the trivial bundle of rank m) for some $m \in \mathbb{N}$. A *stably almost-complex manifold* is a smooth manifold equipped with a stable almost-complex structure.

Now, there is a natural notion of (stably almost-complex) cobordism between stably almost-complex manifolds.

1.5. Definition. For $k \in \mathbb{N}$, let Ω_k denote the set of cobordism classes of stably almost-complex manifolds of dimension k. The graded object Ω_* carries the structure of a graded-commutative ring, with addition given by disjoint union and multiplication given by cartesian product. This is called the *complex cobordism ring*.

Thom demonstrated that these types of "cobordism rings" could be studied using the methods of homotopy theory. For example, he was able to completely compute the *unoriented cobordism ring* (i.e. the analogous ring for manifolds with no extra structure), giving a complete classification of unoriented manifolds up to cobordism.

Quillen later gave a more refined perspective on Thom's computation, relating it to a certain notion that first arose in algebraic number theory, that of a *formal group law*. There is an analogous perspective on and computation of the complex cobordism ring Ω_*, which turns out to be significantly more interesting and have powerful consequences in stable homotopy theory.

The rest of this talk will be devoted to explaining (but not proving) this characterization of Ω_*, which describes a classification of stably-almost complex manifolds up to cobordism (as well as certain features of the geometry of such manifolds) in terms of the algebra of formal group laws.

2. Cohomology theories

One of the main ideas from Thom's and Quillen's work is that cobordism rings can be extended to (and then studied in the framework of) cohomology theories, in the sense of algebraic topology. I understand that many people will feel the urge to immediately turn off at this phrase, but please shrug off the impulse! These are cohomology theories with concrete, geometric descriptions and consequences, and that is the perspective I want to focus on today.

The purpose of this first section is to recall what a cohomology theory is, as well as a couple of fundamental examples. To keep things as geometric as I can throughout this talk, I am going to set up a restricted framework in which we only ever apply our cohomology theories to *manifolds*.

2.1. Notation. Let Mfld denote the category whose objects are smooth manifolds without boundary that admit a finite good open cover (i.e. an open cover $\{U_i\}_{1 \leqslant i \leqslant n}$ such that any intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ $(1 \leqslant k \leqslant n)$ is diffemorphic to a Euclidean space $\mathbb{R}^m$) and whose morphisms are smooth maps between such. From now on, whenever I use the word *manifold*, I mean an object of this category.

2.2. Definition. A *collar-gluing* consists of:

- a manifold X;
- two open submanifolds $U, V \subseteq X$ covering X, i.e. such that $X = U \cup V$;
- a properly[1] embedded submanifold $Z \hookrightarrow U \cap V$;
- a diffemorphism $Z \times \mathbb{R} \xrightarrow{\sim} U \cap V$, which restricts to the given embedding on $Z \times \{0\} \subseteq Z \times \mathbb{R}$.

We denote such a gluing by $(X; U, V, Z)$, leaving the diffeomorphism implicit.

2.2.1. Example. We can take X to be a sphere S^n with $n \geqslant 1$, the open cover U, V to be given by two overlapping hemispheres, and Z to be an equatorial S^{n-1}.

2.3. Definition. A *multiplicative cohomology theory on* Mfld consists of the following data:

- *values*: a functor E^* from $\mathrm{Mfld}^{\mathrm{op}}$ to the category of ($\mathbb{Z}$-)graded commutative rings; the superscript $*$ denotes the grading, so for $k \in \mathbb{Z}$, we let E^k denote the k-th graded piece of E^*.
- *boundary maps*: for each collar gluing $(X; U, V, Z)$, a map of graded $E^*(\mathrm{pt})$-modules $\partial\colon E^*(Z) \longrightarrow E^{*+1}(X)$, which is natural with respect to maps of collar-gluings;

subject to the following conditions:

- *homotopy invariance*: for any manifold X, the projection $X \times \mathbb{R} \longrightarrow X$ is carried to an isomorphism $E^*(X) \xrightarrow{\sim} E^*(X \times \mathbb{R})$.
- *locality*: for each collar-gluing $(X; U, V, Z)$ in Mfld, the sequence

$$\cdots \longrightarrow E^k(X) \longrightarrow E^k(U) \oplus E^k(V) \longrightarrow E^k(Z) \xrightarrow{\partial} E^{k+1}(X) \longrightarrow \cdots$$

  is exact.

2.3.1. Remark. I will denote cohomology theories just by their underlying functors E^*, leaving the boundary maps ∂ implicit. Also, all cohomology theories appearing today will be multiplicative and on Mfld, so I will mostly omit these modifiers and use just the phrase *cohomology theory* to refer to the above notion.

2.3.2. Remark. By the homotopy-invariance axiom and the definition of a collar-gluing, note that for any collar-gluing $(X; U, V, Z)$ and cohomology theory E^* we have a canonical isomorphism $E^*(Z) \simeq E^*(U \cap V)$. In particular, we could have equivalently stated the boundary map data and locality axiom above with $U \cap V$ replacing Z. The sequence may then look a bit more familiar, as the so-called *Mayer-Vietoris sequence.*

2.3.3. Notation. If E^* is a cohomology theory and $f\colon Y \longrightarrow X$ is a smooth map of manifolds, I will use the notation $f^\star\colon E^*(X) \longrightarrow E^*(Y)$ to denote the induced map $E^*(f)$.

2.3.4. Remark. The homotopy invariance axiom has the following two equivalent formulations:

(a) If $f\colon Y \longrightarrow X$ is a smooth homotopy equivalence of manifolds, then the induced map $f^\star\colon E^*(X) \longrightarrow E^*(Y)$ is an isomorphism.

(b) If $f, g\colon X \longrightarrow Y$ are homotopic smooth maps[2], then the maps $f^\star, g^\star\colon E^*(Y) \longrightarrow$

[1] This just means that the image of the embedding is a closed subset of the target.

[2] It's a fact that if two smooth maps are homotopic through continuous maps, then they are homotopic through smooth maps.

$E^*(X)$ are equal.

2.3.5. Remark. Any manifold admits an open cover by Euclidean spaces, which are contractible, so the homotopy invariance and locality axioms combine to give some kind of recipe for computing the cohomology of any manifold (given the cohomology of a point).

As a warm-up example, we may use the collar-gluing of the sphere S^n described in (2.2.1). For any cohomology theory E^*, this allows us to inductively compute that $E^*(S^n) \simeq E^*(\mathrm{pt})[x]/(x^2)$, where $x \in E^n(S^n)$.

2.4. I'll now recall a few examples of cohomology theories on Mfld. I'm not going to explain in any detail why these are in fact cohomology theories, but I will make some general comments on this point after stating all of the examples.

2.4.1. Example (de Rham cohomology). Perhaps the first topological cohomology theory we encounter in our education is *de Rham cohomology*. For a manifold X, we have a complex of differential forms,

$$0 \longrightarrow \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim(X)}(X) \longrightarrow 0,$$

and we define the de Rham cohomology $H^*_{dR}(X)$ as the homology of this complex, i.e.

$$H^k_{dR}(X) := \frac{\ker(d\colon \Omega^k(X) \longrightarrow \Omega^{k+1}(X))}{\mathrm{im}(d\colon \Omega^{k-1}(X) \longrightarrow \Omega^k(X))}.$$

This defines a cohomology theory on Mfld.

2.4.2. Example (ordinary cohomology). Let A be a commutative ring. For any topological space T, we may form $H^*_{sing}(T; A)$, its *singular cohomology with coefficients in* A. This in particular defines a cohomology theory on Mfld. For manifolds X, this cohomology theory is naturally isomorphic to $H^*_{sh}(X; \underline{A})$, the *sheaf cohomology of the constant sheaf* $\underline{A}$ *on* X.

From now on, we'll just use the notation $H^*(X; A)$ for either of these things, and refer to it as *ordinary* A*-cohomology*, or just *ordinary cohomology* in the case $A = \mathbb{Z}$. In fact, this cohomology theory is uniquely characterized by its value on a point: $H^*(\mathrm{pt}; A) \simeq A$, concentrated in degree 0.[3] A special case of this uniqueness is exhibited by the de Rham theorem, which supplies a canonical isomorphism $H^*(X; \mathbb{R}) \simeq H^*_{dR}(X)$.

2.4.3. Example (K-theory). Given a manifold X, we may consider the set Vect(X) of isomorphism classes of complex vector bundles on X. Direct sum and tensor product of vector bundles gives Vect(X) the structure of a commutative semiring. Formally adjoining additive inverses gives us a commutative ring $\mathrm{Vect}(X)^+$, the additive group completion (sometimes called the Grothendieck group or ring) of Vect(X).

There is a cohomology theory K^* on Mfld, called *(topological, complex) K-theory*, satisfying the following properties:

(a) For $X \in$ Mfld, we have $K^0(X) \simeq \mathrm{Vect}(X)^+$. (NB: our implicit finiteness hypothesis on all manifolds (2.1) is necessary for this statement to be true.)

(b) The theory is 2-periodic: for any manifold X, we have a canonical isomorphism of graded abelian groups $K^*(X) \simeq K^{*+2}(X)$.

(c) The K-theory of a point is given by $K^*(\mathrm{pt}) \simeq \mathbb{Z}[\beta^{\pm 1}]$, where $\beta \in K^2(\mathrm{pt})$. (Note that this is consistent with the previous two properties.)

2.4.4. Remark. Why do de Rham cohomology, constant-coefficient sheaf cohomology, and K-theory actually determine cohomology theories as defined in (2.3)? There are two key points:

[3] Warning: it is not true in general that a cohomology theory is uniquely characterized by its value on a point.

(a) *Locality*: All of these theories arise from locally specified geometric data on manifolds: differential forms, sections of sheaves, and vector bundles. This is what underlies the definition of boundary maps satisfying the locality axiom of (2.3) in these examples.

(b) *Homotopy invariance*: We have fundamental results telling us that these constructions are indeed homotopy-invariant, e.g. $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{R}) \simeq 0$ for $k \neq 0$, and $\mathrm{Vect}(\mathrm{X}) \simeq \mathrm{Vect}(\mathrm{X} \times \mathbb{R})$.

3. Fundamental classes and orientations

The example cohomology theories given in (2.4) by definition record certain types of geometric data on manifolds. But there is another sense in which they hold geometric information, namely through the theory of fundamental classes. The idea is that, in any of these theories E^* and for any manifold X, there are cohomology classes $[\mathrm{Y}] \in \mathrm{E}^*(\mathrm{X})$ "representing" certain compact submanifolds $\mathrm{Y} \hookrightarrow \mathrm{X}$, or more generally certain proper maps $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$. Moreover, the algebra of the cohomology theory (e.g. the formation of products of such classes) encodes geometry of these objects (e.g. their intersection theory).

3.1. Example. Let me briefly recall how the most basic version of fundamental classes works in de Rham cohomology. Let X be a compact oriented manifold of dimension n, and let $\mathrm{Y} \hookrightarrow \mathrm{X}$ be a compact oriented submanifold of dimension $m \leqslant n$. There are integration maps $\int_{\mathrm{X}}\colon \mathrm{H}^n_{\mathrm{dR}}(\mathrm{X}) \longrightarrow \mathbb{R}$ and $\int_{\mathrm{Y}}\colon \mathrm{H}^m_{\mathrm{dR}}(\mathrm{Y}) \longrightarrow \mathbb{R}$. The fundamental class $[\mathrm{Y}] \in \mathrm{H}^{n-m}_{\mathrm{dR}}(\mathrm{X})$ is uniquely characterized by the property that

$$\int_{\mathrm{X}} \omega \wedge [\mathrm{Y}] = \int_{\mathrm{Y}} \omega|_{\mathrm{Y}} \qquad \text{for all } \omega \in \mathrm{H}^m_{\mathrm{dR}}(\mathrm{X}).$$

3.1.1. Remark. In the special case $\mathrm{Y} = \mathrm{X}$, it's clear from the above characterization that $[\mathrm{X}] = 1 \in \mathrm{H}^0_{\mathrm{dR}}(\mathrm{X})$, i.e. the unit in the graded-commutative ring $\mathrm{H}^*_{\mathrm{dR}}(\mathrm{X})$. In the general case, we may think of $[\mathrm{Y}]$ as a "pushforward" of $1 \in \mathrm{H}^*_{\mathrm{dR}}(\mathrm{Y})$ along the inclusion $f\colon \mathrm{Y} \hookrightarrow \mathrm{X}$ to a class in $\mathrm{H}^*_{\mathrm{dR}}(\mathrm{X})$. Indeed, we may define a *pushforward map* $f_*\colon \mathrm{H}^*_{\mathrm{dR}}(\mathrm{Y}) \longrightarrow \mathrm{H}^{*+n-m}_{\mathrm{dR}}(\mathrm{X})$ by a straightforward generalization of the above formula for the fundamental class, namely

$$\int_{\mathrm{X}} \omega \wedge f_*(\eta) = \int_{\mathrm{Y}} f^*(\omega) \wedge \eta,$$

which indeed specializes to give $[\mathrm{Y}] = f_*(1)$.

Such pushforwards exist not just for embeddings of closed submanifolds, but for any proper map (with an appropriate notion of orientation). Another typical example is the case that $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ is a smooth fiber bundle with compact fibers. If the fibers are oriented, and say of dimension k, then "integration along fibers" defines a pushforward map $f_*\colon \mathrm{H}^*_{\mathrm{dR}}(\mathrm{Y}) \longrightarrow \mathrm{H}^{*-k}_{\mathrm{dR}}(\mathrm{X})$.

As I mentioned at the beginning of this section, a similar formalism of fundamental classes and pushforwards also exists in other cohomology theories, e.g. K-theory. I next want to make a precise axiomatic definition of this kind of formalism. This will require a couple of preliminary definitions.

3.2. Definition. We say a map $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ in Mfld has *codimension k* for $k \in \mathbb{Z}$ if for any point $y \in \mathrm{Y}$ we have

$$\dim(\mathrm{Y}, y) = \dim(\mathrm{X}, f(y)) - k$$

(the notation here refers to the dimension of a manifold at a given point; we have not required our manifolds to be equidimensional). Note that k is allowed to be negative here.

For the next definition, note the importance of *orientations* in the discussion (3.1) of fundamental classes and pushforwards in de Rham cohomology. For today's story, we need a stricter notion of orientation, one that for instance is sufficient to define fundamental classes also in K-theory (the usual notion of orientation is not sufficient in this case).

3.3. Definition. Let $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ be a map in Mfld. An *unstable complex-orientation of* f is a factorization

$$\begin{array}{ccc} \mathrm{Y} & \xrightarrow{\ i\ } & \mathrm{X} \times \mathbb{R}^{\mathrm{N}} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi_1} \\ & & \mathrm{X} \end{array}$$

of f in which i is a closed embedding whose normal bundle ν_i is equipped with a complex structure. There is an evident notion of two unstable complex-orientations of f being *stably equivalent*. A *complex-orientation of* f is a stable equivalence class of unstable complex-orientations. A *complex-oriented map* in Mfld means a map equipped with a complex-orientation.

3.3.1. Remark. One can make similar definitions with other kinds of bundle structures required on the normal bundle ν_i, e.g. an orientation in the usual sense.

3.3.2. Remark. If you are comfortable with stable vector bundles and structures on such, then a complex-orientation on $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ is equivalent to a complex structure on the *stable normal bundle* of f, defined to be the stable vector bundle $\nu_f := \tau_{\mathrm{Y}} - f^\star(\tau_{\mathrm{X}})$ on Y, where $\tau_{\mathrm{X}}, \tau_{\mathrm{Y}}$ denote the respective (stable) tangent bundles.

3.3.3. Examples. In the following cases, f admits a canonical complex-orientation:

(a) f is a closed embedding of (almost-)complex manifolds;

(b) f is a fiber bundle whose fibers are (almost-)complex manifolds.

3.3.4. Remark. (a) Given two complex-oriented maps $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ and $g\colon \mathrm{Z} \longrightarrow \mathrm{Y}$, there is a canonical *composite complex-orientation* on the composite map $fg\colon \mathrm{Z} \longrightarrow \mathrm{X}$.

(b) Given a pullback diagram in Mfld

$$\begin{array}{ccc} \mathrm{Y}' & \xrightarrow{\ q\ } & \mathrm{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{X}' & \xrightarrow{\ p\ } & \mathrm{X} \end{array}$$

with f transverse to p and f complex-oriented, there is a canonical *pulled-back complex-orientation* on g.

We may now state the definition of a cohomology theory with pushforwards in the complex-oriented setting (due to Quillen [Qui71]).

3.4. Definition. A *complex-oriented cohomology theory on* Mfld is a multiplicative cohomology theory E^* on Mfld equipped with the following extra data:

– for each proper complex-oriented map $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ in Mfld of codimension k, a map of $\mathrm{E}^*(\mathrm{X})$-modules $f_\star\colon \mathrm{E}^*(\mathrm{Y}) \longrightarrow \mathrm{E}^{*+k}(\mathrm{X})$ of $\mathrm{E}^*(\mathrm{X})$-modules[4] (note that this map can depend on the complex-orientation on f);

subject to the following conditions:

– given a collar gluing $(\mathrm{X}; \mathrm{U}, \mathrm{V}, \mathrm{Z})$, if we equip the embedding $i\colon \mathrm{Z} \hookrightarrow \mathrm{X}$ (which is proper of codimension 1) with the complex-orientation coming from the given trivialization of its normal bundle, then $i_\star = \partial\colon \mathrm{E}^*(\mathrm{Z}) \longrightarrow \mathrm{E}^{*+1}(\mathrm{X})$;

– for any identity map $\mathrm{id}_{\mathrm{X}}\colon \mathrm{X} \longrightarrow \mathrm{X}$ equipped with its canonical complex-orientation, we have $(\mathrm{id}_{\mathrm{X}})_\star = \mathrm{id}_{\mathrm{E}^*(\mathrm{X})}\colon \mathrm{E}^*(\mathrm{X}) \longrightarrow \mathrm{E}^*(\mathrm{X})$.

[4]Here $\mathrm{E}^*(\mathrm{Y})$ is viewed as an $\mathrm{E}^*(\mathrm{X})$-module via the ring map $f^\star\colon \mathrm{E}^*(\mathrm{X}) \longrightarrow \mathrm{E}^*(\mathrm{Y})$.

– for two proper complex-oriented maps $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ and $g\colon \mathrm{Z} \longrightarrow \mathrm{Y}$ of codimensions k and l respectively, if we equip $fg\colon \mathrm{Z} \longrightarrow \mathrm{X}$ with the composite complex-orientation, then $f_\star g_\star = (fg)_\star\colon \mathrm{E}^*(\mathrm{Z}) \longrightarrow \mathrm{E}^{*+k+l}(\mathrm{X})$;

– for any pullback diagram in Mfld

$$\begin{array}{ccc} \mathrm{Y} \times_{\mathrm{X}} \mathrm{Z} & \xrightarrow{q} & \mathrm{Z} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Y} & \xrightarrow{f} & \mathrm{X}. \end{array}$$

with f transverse to g and g proper and complex-oriented of codimension k, if we equip p with the pulled-back complex-orientation, then $f^\star g_\star = p_\star q^\star\colon \mathrm{E}^*(\mathrm{Z}) \longrightarrow \mathrm{E}^{*+k}(\mathrm{Y})$.

As indicated in (3.1.1), a theory of pushforwards gives rise to a theory of fundamental classes:

3.4.1. Definition. Let E^* be a complex-oriented cohomology theory. Let $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ be a proper complex-oriented map of codimension k. We define $[\mathrm{Y}] := f_\star(1) \in \mathrm{E}^k(\mathrm{X})$, where $1 \in \mathrm{E}^0(\mathrm{Y})$ denotes the unit of the graded commutative ring $\mathrm{E}^*(\mathrm{Y})$. We call $[\mathrm{Y}]$ the *fundamental class of* Y. (The notation and terminology are sloppy: the manifold X and map f are meant to be understood from context.)

The following two results illustrate how the above axiomatics encode naturally desirable features for a theory of fundamental classes.

3.4.2. Proposition. Let $f\colon \mathrm{Y} \longrightarrow \mathrm{X}$ and $g\colon \mathrm{Z} \longrightarrow \mathrm{X}$ be proper complex-oriented maps of codimensions k and l respectively, and suppose f and g are transverse. Let $h\colon \mathrm{Y} \times_{\mathrm{X}} \mathrm{Z} \longrightarrow \mathrm{X}$ denote the fiber product in Mfld. Then in any complex-oriented cohomology theory E^*, we have $[\mathrm{Y} \times_{\mathrm{X}} \mathrm{Z}] = [\mathrm{Y}][\mathrm{Z}] \in \mathrm{E}^{k+l}(\mathrm{X})$.

Proof. Consider the pullback diagram

$$\begin{array}{ccc} \mathrm{Y} \times_{\mathrm{X}} \mathrm{Z} & \xrightarrow{q} & \mathrm{Z} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ \mathrm{Y} & \xrightarrow{f} & \mathrm{X}. \end{array}$$

In any complex-oriented cohomology theory E^*, we by definition have $f^\star g_\star = p_\star q^\star$, and hence

$$f_\star f^\star g_\star = f_\star p_\star q^\star = h_\star q^\star.$$

Applying this to $1 \in \mathrm{E}^*(\mathrm{Z})$, we obtain

$$f_\star f^\star[\mathrm{Z}] = [\mathrm{Y} \times_{\mathrm{X}} \mathrm{Z}].$$

Finally, $f_\star$ being $\mathrm{E}^*(\mathrm{X})$-linear implies that $f_\star f^\star[\mathrm{Z}] = [\mathrm{Y}][\mathrm{Z}]$, and thus the claim is proven. □

3.4.3. Proposition. Let $f, f'\colon \mathrm{Y} \longrightarrow \mathrm{X}$ be proper complex-oriented maps of codimension k. Suppose f and f' are homotopic. Then in any complex-oriented cohomology theory E^* we have $f_\star = f'_\star\colon \mathrm{E}^*(\mathrm{Y}) \longrightarrow \mathrm{E}^{*+k}(\mathrm{X})$. In particular, the fundamental class $[\mathrm{Y}]$ only depends on the homotopy class of the map f.

I'll postpone the proof of (3.4.3) to the next section, where we'll actually prove something slightly more general.

3.4.4. Remark. Let E^* be a complex-oriented cohomology theory. For $n \geqslant 1$, let $i_n\colon \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ denote the inclusion of a hyperplane (the choice of hyperplane does not matter up to homotopy). Define $t_n := [\mathbb{CP}^{n-1}] \in \mathrm{E}^2(\mathbb{CP}^n)$. These classes satisfy the following properties:

(a) Since $\mathbb{CP}^1$ is homotopy equivalent to S^2, we have by (2.3.5) that $E^2(\mathbb{CP}^1) \simeq E^2(S^2) \simeq E^0(\mathrm{pt})$. Under this isomorphism, the class $t_1 \in E^2(\mathbb{CP}^1)$ is sent to the unit $1 \in E^0(\mathrm{pt})$.

(b) We have $i_{n+1}^*(t_{n+1}) = t_n$.

Remarkably, it turns out that conversely the entire structure of a complex-orientation on a cohomology theory E^* can be extended uniquely from the data of classes $t_n \in E^2(\mathbb{CP}^n)$ for $n \geqslant 1$ satisfying the above two properties.

3.4.5. Examples. Both ordinary cohomology $H^*(-;\mathbb{Z})$ and K-theory K^* admit complex-orientations. This can be seen in terms of the characterization of complex-orientations given in (3.4.4). For example, in K-theory, the class $t_n \in K^2(\mathbb{CP}^n) \simeq K^0(\mathbb{CP}^n) \simeq \mathrm{Vect}(\mathbb{CP}^n)^+$ is given by the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$.

I'll finish this section by describing an example of how a basic situation in projective geometry is encoded in the algebra of fundamental classes. We will need the following fundamental computation (which will also play an important role later on).

3.5. Proposition. Let E^* be a complex-oriented cohomology theory. Then for all $n \in \mathbb{N}$,

$$E^*(\mathbb{CP}^n) \simeq E^*(\mathrm{pt})[t_n]/(t_n^{n+1}).$$

Here $t_n := [\mathbb{CP}^{n-1}]$ as in (3.4.4), and more generally we have $t_n^k = [\mathbb{CP}^{n-k}]$ for $0 \leqslant k \leqslant n$.

Proof. We proceed inductively, the case $n = 0$ being tautological. For $n \geqslant 1$, consider the inclusion of a hyperplane $i\colon \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$. Let $U \subseteq \mathbb{CP}^n$ denote a tubular neighborhood of this submanifold, diffeomorphic to its normal bundle ν_i and hence homotopy equivalent to $\mathbb{CP}^{n-1}$. Letting $V \subseteq \mathbb{CP}^n$ denote the affine space complementary to i, we have an open cover $\mathbb{CP}^n = U \cup V$.

The intersection $U \cap V$ is diffeomorphic to the complement of the zero-section in ν_i, which in turn is diffeomorphic to the product of the sphere bundle $S(\nu_i) \hookrightarrow \nu_i$ and $\mathbb{R}$. In other words, we have a collar gluing $(\mathbb{CP}^n; U, V, S(\nu_i))$.

We now use the fact that the bundle ν_i is isomorphic to the tautological bundle $\mathcal{O}(1)$ on $\mathbb{CP}^{n-1}$, whose sphere bundle is given by the canonical projection $S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$ (which indeed is an S^1-fiber bundle). So we conclude that we have a collar gluing $(\mathbb{CP}^n; U; V; Z)$ in which U is equivalent to $\mathbb{CP}^{n-1}$, V is contractible, and Z is equivalent to S^{2n-1}.

From this we obtain the Mayer-Vietoris sequence

$$\cdots \longrightarrow E^k(\mathbb{CP}^n) \longrightarrow E^k(\mathbb{CP}^{n-1}) \oplus E^k(\mathrm{pt}) \longrightarrow E^k(S^{2n-1}) \longrightarrow \cdots,$$

which then gives a long exact sequence

$$\cdots \longrightarrow E^k(\mathbb{CP}^n) \longrightarrow E^k(\mathbb{CP}^{n-1}) \longrightarrow \widetilde{E}^k(S^{2n-1}) \longrightarrow \cdots$$

where $\widetilde{E}^*(X)$ denotes *reduced cohomology*, i.e. the cokernel of the map $E^*(\mathrm{pt}) \longrightarrow E^*(X)$ induced by the projection $X \longrightarrow \mathrm{pt}$.

Now, we know that $i^*(t_n) = t_{n-1}$, so the inductive hypothesis implies that the map $i^*\colon E^*(\mathbb{CP}^n) \longrightarrow E^*(\mathbb{CP}^{n-1})$ is surjective. Thus the previous long exact sequence breaks up into short exact sequences

$$0 \longrightarrow \widetilde{E}^{k-1}(S^{2n-1}) \longrightarrow E^k(\mathbb{CP}^n) \longrightarrow E^k(\mathbb{CP}^{n-1}) \longrightarrow 0.$$

By (2.3.5) we know that $\widetilde{E}^{k-1}(S^{2n-1}) \simeq E^{k-2n}(\mathrm{pt})$. So in total we see that we have a short-exact sequence of $E^*(\mathrm{pt})$-modules

$$0 \longrightarrow E^{*-2n}(\mathrm{pt}) \longrightarrow E^*(\mathbb{CP}^n) \longrightarrow E^*(\mathbb{CP}^{n-1}) \longrightarrow 0.$$

To finish, it suffices to show that the image of the unit $1 \in E^0(\mathrm{pt})$ under the above map $E^{*-2n}(\mathrm{pt}) \longrightarrow E^*(\mathbb{CP}^n)$ is $t_n^n \in E^{2n}(\mathbb{CP}^n)$. Chasing back through how we arrived at that map, we see that the image of 1 is the image of the generator $x \in E^{2n-1}(S^{2n-1})$ under the map $\partial\colon E^*(S^{2n-1}) \longrightarrow E^{*+1}(\mathbb{CP}^n)$. But now in the complex-oriented setting we have that ∂ is given by pushforward along the embedding. Using this same fact, one can check that $x = [\mathrm{pt}] \in E^{2n-1}(S^{2n-1})$. It follows that its image is $[\mathrm{pt}] = [\mathbb{CP}^0] = t_n^n \in E^{2n}(\mathbb{CP}^n)$, as desired. □

3.6. Examples. Let $Z \hookrightarrow \mathbb{CP}^n$ be a complex hypersurface of degree d. This is a closed embedding of complex manifolds, hence is canonically complex-oriented (3.3.3), so we get a fundamental class $[Z] \in E^2(\mathbb{CP}^n)$ for any complex-oriented cohomology theory E^*. For example, I can say what happens in ordinary cohomology and in K-theory:

- *Ordinary cohomology*: By (3.5), we have $H^2(\mathbb{CP}^n;\mathbb{Z}) \simeq \mathbb{Z}\cdot[\mathbb{CP}^{n-1}]$. One finds that
$$[Z] = d\cdot[\mathbb{CP}^{n-1}].$$
- *K-theory*: Recall that K-theory is 2-periodic. So we can view $[Z]$, as well as the classes $t_n^k = [\mathbb{CP}^{n-k}]$, all as degree-0 classes in $K^0(\mathbb{CP}^n)$. With this modified notation, (3.5) tells us that $K^0(\mathbb{CP}^n;\mathbb{Z}) \simeq \bigoplus_{k=0}^n \mathbb{Z}\cdot[\mathbb{CP}^{n-k}]$. One finds that
$$[Z] = d\cdot[\mathbb{CP}^{n-1}] - \binom{d}{2}\cdot[\mathbb{CP}^{n-2}] + \binom{d}{3}\cdot[\mathbb{CP}^{n-3}] - \cdots.$$

Now let $Z' \hookrightarrow \mathbb{CP}^n$ be another hypersurface, of degree e and intersecting Z transversely. Then by (3.4.2) we have the formula $[Z\cap Z'] = [Z][Z']$, so from the above we can also compute the fundamental class $[Z\cap Z']$ in ordinary cohomology and K-theory:

- *Ordinary cohomology*: We have $[Z\cap Z'] = de\cdot[\mathbb{CP}^{n-2}]$.
- *K-theory*: We have $[Z\cap Z'] = de\cdot[\mathbb{CP}^{n-2}] - \frac{1}{2}de(d+e-2)\cdot[\mathbb{CP}^{n-3}] + \cdots$.

Observe that, when $n \geqslant 3$, one can extract the individual degrees d, e from the coefficients in the K-theory fundamental class of the intersection, while one only sees their product de in ordinary cohomology.[5]

4. Cobordism

Various questions naturally arise from the above discussion of fundamental classes in complex-oriented cohomology theories. For example:

4.1. Questions. (a) Given a complex-oriented cohomology theory E^* and a class $t \in E^*(X)$ for some manifold X, can one find a proper complex-oriented map $f\colon Y \longrightarrow X$ such that $t = [Y]$?

(b) Can one characterize the sorts of formulae that appear for the fundamental classes of complete intersections in projective space in a general complex-oriented cohomology theory E^*?

When one has some general class of objects, like complex-oriented cohomology theories, many times a useful strategy for understanding these objects is to produce a *universal example* in this class and to just study that one particular object. It turns out that this strategy is indeed viable and fruitful in the current scenario, and leads us directly to today's protagonist: complex cobordism.

The perspective I want to emphasize here is that complex cobordism is a cohomology theory on Mfld characterized by having essentially tautological fundamental classes for proper complex-oriented maps. That is, we will design complex cobordism as a cohomology theory Ω^* so that classes in $\Omega^*(X)$ are *by definition* represented by such maps $Y \longrightarrow X$. Its universality will then be a result of all complex-oriented cohomology theories E^* having their own interpretations of these classes, namely as their own fundamental classes.

[5] If you're interested in learning more about K-theory from this kind of perspective, I recommend looking at [Dug14].

4.2. Remark. A naïve first guess is to define $\Omega^k(\mathrm{X})$ to be the set of isomorphism classes of proper complex-oriented maps $\mathrm{Y} \longrightarrow \mathrm{X}$ of codimension k, and for a map $p\colon \mathrm{X}' \longrightarrow \mathrm{X}$ to define $p^\star\colon \Omega^k(\mathrm{X}) \longrightarrow \Omega^k(\mathrm{X}')$ by pullback, $\mathrm{Y} \longmapsto \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}'$. We note two related issues with this guess:

- If we want the pullback to remain a manifold, we will in general need to perturb the situation to a transverse one, and then we need to worry about dependence on the choice of perturbation.
- A cohomology theory needs to be homotopy-invariant. For example, all inclusions $\mathrm{X} \xrightarrow{\sim} \mathrm{X} \times \{t\} \hookrightarrow \mathrm{X} \times \mathbb{R}$ must induce the same map on pullback (note this is related to the well-definedness concern in the previous point).

These issues bring us immediately back to the notion of cobordism, in a slightly generalized form:

4.3. Definition. Let $\mathrm{X} \in \mathrm{Mfld}$ and let $f_0\colon \mathrm{Y}_0 \longrightarrow \mathrm{X}$ and $f_1\colon \mathrm{Y}_1 \longrightarrow \mathrm{X}$ be two proper complex-oriented maps. A *cobordism* from f_0 to f_1 is a proper complex-oriented map $f\colon \mathrm{Y} \longrightarrow \mathrm{X} \times \mathbb{R}$ transverse to the embeddings $\mathrm{X} \times \{0\} \hookrightarrow \mathrm{X} \times \mathbb{R}$ and $\mathrm{X} \times \{1\} \hookrightarrow \mathrm{X} \times \mathbb{R}$ together with diffeomorphisms $f^{-1}(\mathrm{X} \times \{0\}) \simeq \mathrm{Y}_0$ and $f^{-1}(\mathrm{X} \times \{1\}) \simeq \mathrm{Y}_1$ respecting the maps to X and complex-orientations. Diagramatically this looks as follows:

$$\begin{array}{ccccc} \mathrm{Y}_0 & \hookrightarrow & \mathrm{Y} & \hookleftarrow & \mathrm{Y}_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} \\ \mathrm{X} \times \{0\} & \hookrightarrow & \mathrm{X} \times \mathbb{R} & \hookleftarrow & \mathrm{X} \times \{1\} \end{array}$$

where the two squares are cartesian.

4.3.1. Remark. Again, this notion of cobordism defines an equivalence relation.

The concerns of (4.2) can now be articulated precisely in the form of the following result:

4.4. Proposition. Let E^* be a complex-oriented cohomology theory. Let $f_0\colon \mathrm{Y}_0 \longrightarrow \mathrm{X}$ and $f_1\colon \mathrm{Y}_1 \longrightarrow \mathrm{X}$ are proper complex-oriented maps of manifolds. If there is a cobordism from f_0 to f_1, then $[\mathrm{Y}_0] = [\mathrm{Y}_1] \in \mathrm{E}^*(\mathrm{X})$.

Proof. Let $f\colon \mathrm{Y} \longrightarrow \mathrm{X} \times \mathbb{R}$ be a cobordism from f_0 to f_1. For $t \in \mathbb{R}$ form the pullback square

$$\begin{array}{ccc} \mathrm{Y}_t & \xrightarrow{j_t} & \mathrm{Y} \\ \downarrow{\scriptstyle f_t} & & \downarrow{\scriptstyle f} \\ \mathrm{X} \times \{t\} & \xrightarrow{i_t} & \mathrm{X} \times \mathbb{R}. \end{array}$$

Assuming f is transverse to the inclusion i_t (it is by hypothesis for $t \in \{0,1\}$), in the cohomology theory E^* we have

$$[\mathrm{Y}_t] = (f_t)_\star(1) = (f_t)_\star(j_t)^\star(1) = (i_t)^\star f_\star(1).$$

It follows from homotopy-invariance that the map $(i_t)^\star\colon \mathrm{E}^*(\mathrm{X} \times \mathbb{R}) \longrightarrow \mathrm{E}^*(\mathrm{X})$ is independent of t (it is necessarily the inverse to the isomorphism $\mathrm{E}^*(\mathrm{X}) \xrightarrow{\sim} \mathrm{E}^*(\mathrm{X} \times \mathbb{R})$ induced by projection), and hence $[\mathrm{Y}_t]$ is independent of t. □

This leads us to the following definition of complex cobordism as a cohomology theory.

4.5. Definition. For $\mathrm{X} \in \mathrm{Mfld}$ and $k \in \mathbb{Z}$, let $\Omega^k(\mathrm{X})$ denote the set of cobordism classes of proper complex-oriented maps $\mathrm{Y} \longrightarrow \mathrm{X}$ of codimension k.

4.5.1. Remark. Recalling the notation from (1.5), we have $\Omega^*(\mathrm{pt}) \simeq \Omega_{-*}$.

4.5.2. Construction. Let me just sketch how to construct some of the structure of a complex-oriented cohomology theory on the above assignment $X \rightsquigarrow \Omega^*(X)$:

- We need a graded commutative ring structure on $\Omega^*(X)$. The sum of two cobordism classes of proper complex-oriented maps $f\colon Y \longrightarrow X$ and $f'\colon Y' \longrightarrow X$ is given by the cobordism class of the disjoint union $Y \amalg Y' \longrightarrow X$. To form the product of two such cobordism classes, we may assume f and f' are transverse (we can always homotope one of them to assure this is true, and this does not affect its cobordism class), and then we form the fiber product $Y \times_X Y' \longrightarrow X$, whose cobordism class is well-defined.
- Given a map $p\colon X' \longrightarrow X$, we must define a pullback map $p^*\colon \Omega^*(X) \longrightarrow \Omega^*(X')$. This is similar to the formation of products just discussed: we may represent an element of $\Omega^*(X)$ by a proper complex-oriented map $f\colon Y \longrightarrow X$ that is transverse to p, and the pullback is defined to be the fiber product $Y \times_X X' \longrightarrow X'$, which has well-defined cobordism class.
- Given a proper complex-oriented map $f\colon Y \longrightarrow X$, we must define a pushforward map $f_*\colon \Omega^*(Y) \longrightarrow \Omega^*(X)$. This is easy: for an element of $\Omega^*(Y)$ represented by a proper complex-oriented map $g\colon Z \longrightarrow Y$, its pushforward is defined to be the cobordism class of the composite $fg\colon Z \longrightarrow X$.
- The homotopy-invariance axiom is essentially designed to hold via the definition of cobordism, which implies that the pullback map induced by the inclusion $i_t\colon X \simeq X \times \{t\} \hookrightarrow X \times \mathbb{R}$ is independent of $t \in \mathbb{R}$.
- The locality/exactness axiom requires perhaps the most thought; allow me to skip over this.

I'll end this section by stating precisely the result that complex cobordism is the universal complex-oriented cohomology theory.

4.6. Proposition. Let E^* be a complex-oriented cohomology theory on Mfld. Then there exists a unique map of complex-oriented cohomology theories $\Omega^* \longrightarrow E^*$.

Proof sketch. I haven't actually defined what a map of complex-oriented cohomology theories is, but you can surely reconstruct the definition for yourself. In any case, it must send fundamental classes to fundamental classes, and this immediately tells us how to define the desired map $\Omega^* \longrightarrow E^*$: we send an element of $\Omega^*(X)$ represented by a proper complex-oriented map $f\colon Y \longrightarrow X$ to the fundamental class $[Y] \in E^*(X)$. Note that (4.4) shows that this is well-defined. $\square$

5. Characteristic classes and formal group laws

I will now finally get to explaining the main theorem, relating complex cobordism with the theory of formal group laws. This relationship is mediated by the theory of characteristic classes of vector bundles.

5.1. Construction. Let E^* be a complex-oriented cohomology theory. Let $X \in$ Mfld and let $p\colon L \longrightarrow X$ be a complex line bundle. Let $s\colon X \longrightarrow E$ denote the zero-section. The *first Chern class (of* L *in* E^**-cohomology)* is defined to be

$$c_1(L) := s^* s_*(1) \in E^2(X).$$

5.1.1. Remark. Note that any two sections of a vector bundle are homotopic (by a straight-line homotopy), so we may replace either or both appearances of s in this formula by any other section. For example, choose a section $s'\colon X \longrightarrow L$ that is transverse to s. If we let $i\colon \mathrm{Z} \hookrightarrow \mathrm{X}$ denote the vanishing locus of s', then we have a pullback square

$$\begin{array}{ccc} \mathrm{Z} & \xhookrightarrow{i} & \mathrm{X} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle s'} \\ \mathrm{X} & \xhookrightarrow{s} & \mathrm{L} \end{array}$$

from which we get

$$\mathrm{c}_1(\mathrm{L}) = s^\star s'_\star(1) = s^\star s'_\star(1) = i_\star i^\star(1) = [\mathrm{Z}],$$

so that the first Chern class is identified with the fundamental class of Z, the vanishing locus of any such section s'.

5.1.2. Example. Suppose L is a trivial line bundle. Then we may choose a *non-vanishing* section $s'\colon \mathrm{X} \longrightarrow \mathrm{L}$, so that in (5.1.1) we have $\mathrm{Z} = \varnothing$, and hence $\mathrm{c}_1(\mathrm{L}) = [\varnothing] = 0$.

5.1.3. Example. Let $n \in \mathbb{N}$ and consider the tautological bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$. A section of $\mathcal{O}(1)$ is given by a homogenous linear polynomial, and thus the vanishing locus of a nonzero section is a hyperplane in $\mathbb{CP}^n$. It thus follows from (5.1.1) that $\mathrm{c}_1(\mathcal{O}(1)) = [\mathbb{CP}^{n-1}] = t_n$. Two remarks on this example:

(a) Comparing with (5.1.2), we see that the first Chern class has the ability to demonstrate non-triviality of line bundles.

(b) This is the universal example: for any complex line bundle $\mathrm{L} \longrightarrow \mathrm{X}$, there is a map $f\colon \mathrm{X} \longrightarrow \mathbb{CP}^n$ for some $n \in \mathbb{N}$ such that $\mathrm{L} \simeq f^\star\mathcal{O}(1)$, and it is straightforward to check that the first Chern class constructed above is natural in the sense that we then have $\mathrm{c}_1(\mathrm{L}) = f^\star(\mathrm{c}_1(\mathcal{O}(1))) = f^\star(t_n)$.

5.1.4. Question. What is $\mathrm{c}_1(\mathcal{O}(d))$ for $d > 1$? Recall that $\mathcal{O}(d)$ is the complex line bundle on $\mathbb{CP}^n$ whose sections are given by homogenous polynomials of degree d. Thus, analogous to (5.1.3), it follows from (5.1.1) that this question is equivalent to asking: what is the fundamental class of a degree-d hypersurface in $\mathbb{CP}^n$?

In (3.6), I stated what the answer to (5.1.4) looks like in ordinary cohomology and in K-theory, and we saw that there was different behavior in these two cases. So, if you believe me, there must be something nontrivial here to be understood. Let's pursue the question.

5.2. We continue working with a given complex-oriented cohomology theory E^*. Now, note that $\mathcal{O}(d) \simeq \mathcal{O}(1)^{\otimes d}$, so we might hope to be able to use our understanding of $\mathcal{O}(1)$ from (5.1.3) to get at $\mathcal{O}(d)$. We are led to ask the following finer question:

5.2.1. Question. Given two complex lines bundles L, L' on $\mathrm{X} \in \mathrm{Mfld}$, can one compute $\mathrm{c}_1(\mathrm{L} \otimes \mathrm{L}')$ in terms of $\mathrm{c}_1(\mathrm{L})$ and $\mathrm{c}_1(\mathrm{L}')$?

To address this question, we will again use the strategy of contemplating the universal example. Consider the product $\mathbb{CP}^n \times \mathbb{CP}^n$ for $n \in \mathbb{N}$. Let $\pi_1, \pi_2\colon \mathbb{CP}^n \times \mathbb{CP}^n \rightrightarrows \mathbb{CP}^n$ be the two projections. These give us two line bundles $\pi_1^\star\mathcal{O}(1), \pi_2^\star\mathcal{O}(1)$ on $\mathbb{CP}^n \times \mathbb{CP}^n$, and we may form the tensor product $\pi_1^\star\mathcal{O}(1) \otimes \pi_2^\star\mathcal{O}(1)$.

This is the "universal example of a tensor product of two line bundles", in the following sense: given two line bundles L, L' on $\mathrm{X} \in \mathrm{Mfld}$, there are some $n \in \mathbb{N}$ and maps $f, f'\colon \mathrm{X} \rightrightarrows \mathbb{CP}^n$ such that $\mathrm{L} \simeq f^\star\mathcal{O}(1)$ and $\mathrm{L}' \simeq (f')^\star\mathcal{O}(1)$. The product of these two maps gives a map $f := (f, f')\colon \mathrm{X} \longrightarrow \mathbb{CP}^n \times \mathbb{CP}^n$, and we have

$$\mathrm{L} \otimes \mathrm{L}' \simeq f^\star(\pi_1^\star\mathcal{O}(1) \otimes \pi_2^\star\mathcal{O}(1)).$$

In light of this, and the naturality of the Chern class, to answer our question (5.2.1) it essentially suffices to understand how to express $\mathrm{c}_1(\pi_1^\star\mathcal{O}(1) \otimes \pi_2^\star\mathcal{O}(1))$ in terms of

$c_1(\pi_1^\star \mathcal{O}(1))$ and $c_1(\pi_2^\star \mathcal{O}(1))$ on $\mathbb{CP}^n \times \mathbb{CP}^n$ (for all $n \in \mathbb{N}$). The first step towards understanding this is to know what $E^*(\mathbb{CP}^n \times \mathbb{CP}^n)$ looks like. In this complex-oriented situation we have a Kunneth formula:

5.2.2. Proposition. For all $n \in \mathbb{N}$, the two projections $\pi_1, \pi_2 \colon \mathbb{CP}^n \times \mathbb{CP}^n \rightrightarrows \mathbb{CP}^n$ induce an isomorphism

$$E^*(\mathbb{CP}^n) \otimes_{E^*(\mathrm{pt})} E^*(\mathbb{CP}^n) \xrightarrow{\sim} E^*(\mathbb{CP}^n \times \mathbb{CP}^n).$$

From this and (3.5), we deduce that we have an isomorphism

$$E^*(\mathrm{pt})[u_n, v_n]/(u_n^{n+1}, v_n^{n+1}) \xrightarrow{\sim} E^*(\mathbb{CP}^n \times \mathbb{CP}^n),$$

under which $u_n = \pi_1^\star(t_n)$ and $v_n = \pi_2^\star(t_n)$.

Proof. Omitted. ☐

An immediate consequence of (5.2.2) is that on $\mathbb{CP}^n \times \mathbb{CP}^n$ we have

$$c_1(\pi_1^\star \mathcal{O}(1) \otimes \pi_2^\star \mathcal{O}(1)) = f_n(u_n, v_n)$$

where f_n is a polynomial of degree n with coefficients in $E^*(\mathrm{pt})$. It's moreover straightforward to see that the polynomials f_n are compatible as n varies, in the the sense that the degree-n truncation of f_{n+1} is f_n. By the reasoning following (5.2.1), we deduce the following.

5.2.3. Corollary. There is a canonical power series in two variables $f_E \in E^*(\mathrm{pt})[\![u, v]\!]$ such that for any two line bundles L, L' on a manifold X, we have

$$c_1(L \otimes L') = f_E(c_1(L), c_2(L')) \in E^*(X).$$

5.2.4. Remark. The fact that tensor product of line bundles is an associative and commutative operation with unit given by the trivial line bundle immediately implies that the power series f_E of (5.2.3) enjoys the following properties:

(a) $f_E(0, v) = v$;

(b) $f_E(u, v) = f_E(v, u)$;

(c) $f_E(u, f_E(v, w)) = f_E(f_E(u, v), w)$.

We embark on a quick algebraic digression motivated by this last result.

5.3. Definition. Let R be a commutative ring. A power series $f \in R[\![u, v]\!]$ satisfying the properties (a–c) of (5.2.4) is called a *formal group law* over the ring R.

5.3.1. Examples. Remarkably, one obtains the two simplest examples of formal group laws in the examples of ordinary cohomology and K-theory:

(a) When E^* is taken to be ordinary cohomology, we obtain the *additive formal group law*: $f_E(u, v) = u + v$.

(b) When E^* is taken to be K-theory, we obtain the *multiplicative formal group law*: $f_E(u, v) = u + v + uv = (1 + u)(1 + v) - 1$ (here we are being abusive with the grading just as we were in (3.6)).

These computations (which are not very difficult) are how one makes the computations of fundamental classes of hypersurfaces described in (3.6), using the logic that motivated the question (5.2.1).

5.3.2. Remark. Let R be a commutative ring and $f \in R[\![u, v]\!]$ a formal group law. Given a map of commutative rings $\phi \colon R \longrightarrow S$, we get an induced map $\phi \colon R[\![u, v]\!] \longrightarrow S[\![u, v]\!]$, and it is easy to see that $\phi(f) \in S[\![u, v]\!]$ is a formal group law over S.

We return yet again to the theme of universal examples:

5.4. Construction. The *Lazard ring* $\mathbb{L}$ is defined to be the quotient of the ring $\mathbb{Z}[\{a_{ij}\}_{i,j\in\mathbb{N}}]$ by the smallest ideal of this ring such that in the quotient the power series

$$f_{\text{univ}} := \sum_{i,j\in\mathbb{N}} a_{ij}u^i v^j$$

is a formal group law. (The ideal is generated by the relations $a_{00} = a_{01} = 0$ (unitality), $a_{ij} = a_{ji}$ (commutativity), and a more complicated set of relations encoding the associativity condition).

It is then clear that the formal group law f_{univ} over $\mathbb{L}$ is universal, in the sense that given a formal group law over a ring R there exists a unique ring homomorphism $\phi\colon \mathbb{L} \longrightarrow \mathrm{R}$ such that $f = \phi(f_{\text{univ}})$.

From our construction, the Lazard ring looks quite unwieldy and inaccessible. The following result (which is not easy) gives us a better idea of what it's like.

5.4.1. Theorem (Lazard). The ring $\mathbb{L}$ is isomorphic to a polynomial ring on countably many generators $\mathbb{Z}[x_1, x_2, \ldots]$.

We shall now finish the story by bringing cohomology theories back on stage. Our discussion in (5.2) can now be rephrased as follows: given a complex-oriented cohomology theory E^*, there is a formal group law f_{E} over $\mathrm{E}^*(\mathrm{pt})$ describing the Chern class of a tensor product of line bundles, classified by a ring homomorphism $\phi_{\mathrm{E}}\colon \mathbb{L} \longrightarrow \mathrm{E}^*(\mathrm{pt})$.

Thus, if our aim is to best understand the possible behavior of Chern classes under tensor product, we should ask: is there any restriction on the kinds of formal group law that can appear as f_{E}? Since we have a universal complex-oriented cohomology theory, namely complex cobordism Ω^*, answering this question amounts to understanding the formal group law f_Ω. This brings us to the following deep result.

5.5. Theorem (Quillen). The map $\phi_\Omega\colon \mathbb{L} \longrightarrow \Omega^*(\mathrm{pt}) \simeq \Omega_{-*}$ is an isomorphism. In other words, the formal group law f_Ω is the universal one.

The above is the punchline of the talk. Let me end by summarizing why I think this result is so amazing:

- At a high level, Quillen's theorem (5.5) connects the classification of complex manifolds (incarnated as the complex cobordism ring Ω_*) with a universal object in algebra.
- Combining this with the Lazard's (purely algebraic) theorem (5.4.1) tells us that the complex cobordism ring Ω_* is a polynomial ring! (Though we do not currently know explicit generators from the manifold point of view.)
- I pointed out above that determining the formal group law associated to ordinary cohomology and K-theory allowed us to compute the fundamental classes of projective hypersurfaces and complete intersections in these cohomology theories. The same logic can of course be applied in complex cobordism, and we deduce that there must be some completely insane formula arising from the universal formal group law for the cobordism class of hypersurfaces and complete intersections in terms of the (unknown) generators of Ω_* and the hyperplane classes $[\mathbb{CP}^n]$. I find this bewildering (this is the "new layer of appreciation" I mentioned in the introduction).
- This theory supplies a strategy for constructing new cohomology theories: given a formal group law over a ring R we obtain a classifying ring homomorphism $\Omega^*(\mathrm{pt}) \simeq \mathbb{L} \longrightarrow \mathrm{R}$, and we can consider the functor

  $$\mathrm{E}^*(\mathrm{X}) := \Omega^*(\mathrm{X}) \otimes_{\Omega^*(\mathrm{pt})} \mathrm{R}.$$

  It turns out that this functor remains a cohomology theory (i.e. the requisite exactness axioms are satisfied) under mild algebraic assumptions on the formal group law. These ideas are the beginning of an fertile connection between the theory of cohomology theories and the theory of formal group laws, which has ended up being a powerful approach to understanding stable homotopy theory.

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