Flip Graphs on Self-Complementary Ideals of Chain Products

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Motivation

A family of sets is *intersecting* if every pair of sets share an element.

Example

The family $\{\{1,2\},\{1,3\},\{2,3\}\}$ is intersecting.

A family of subsets of $\{1,\ldots,n\}$ is *maximally intersecting* if adding any other subset to the family makes it no longer intersecting.

Example

For $n=3$, the family $\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$ is maximally intersecting.

Our project stems from a generalization of maximal intersecting families. Flip graphs on maximally intersecting families have been studied before, and our goal is to generalize these results.

Ideals

Let $\ell_1, \ldots, \ell_d$ be a sequence of positive integers. Define

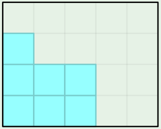
$$P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}.$$

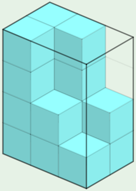
Definition

A subset $I \subseteq P$ is an *ideal* if

$$(a_1, \ldots, a_d) \in I \text{ and } b_1 \le a_1, \ldots, b_d \le a_d \implies (b_1, \ldots, b_d) \in I.$$

Example





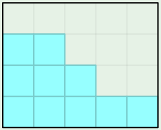
Self-Complementary Ideals

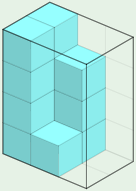
Let $P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}$.

Definition

An ideal $I \subset P$ is *self-complementary* if for every $(a_1, \ldots, a_d) \in P$, exactly one of $(a_1, \ldots, a_d)$ or $(\ell_1 + 1 - a_1, \ldots, \ell_d + 1 - a_d)$ lies in I.

Example





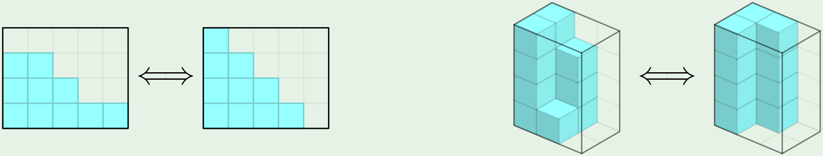
Flips

Let $P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}$, and let I and J be two self-complementary ideals of P.

Definition

I and J differ by a *flip* if $|I \setminus J| = |J \setminus I| = 1$.

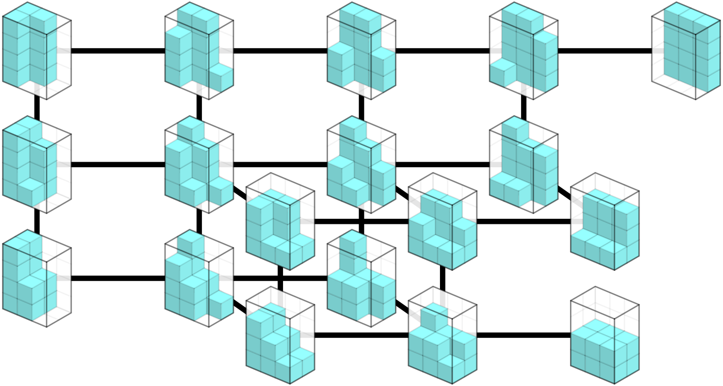
Example



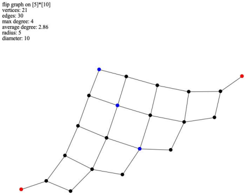
Flip Graphs on Self-Complementary Ideals

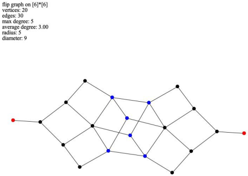
Definition

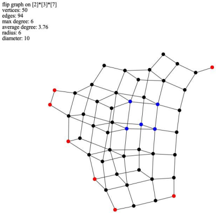
The *flip graph on self-complementary ideals of P* is the graph whose vertices are the self-complementary ideals of P, and whose edges connect pairs of ideals that differ by a flip.

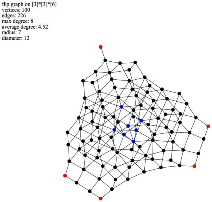


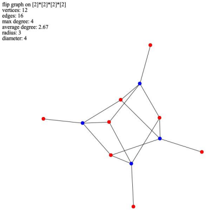
Flip Graph Examples

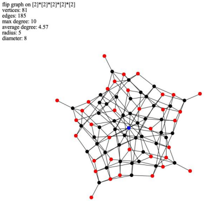








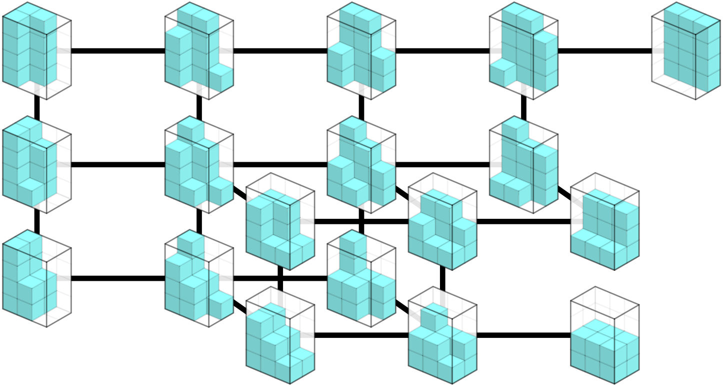




Graph Terminology

Let G be a connected graph.

- The *eccentricity* of a vertex v is the maximum distance from v to another vertex.
- The *diameter* of G is the maximum eccentricity of a vertex.
- The *radius* of G is the minimum eccentricity of a vertex.



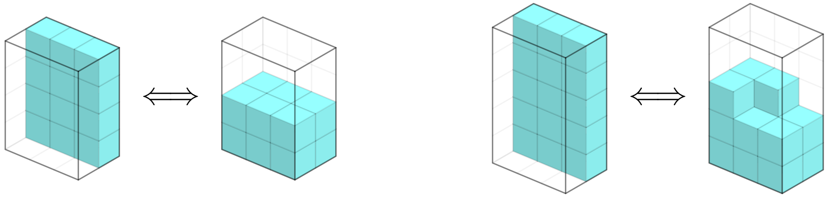
Diameter of Flip Graphs on Self-Complementary Ideals

Let $P = \{1, \ldots, \ell_1\} \times \cdots \times \{1, \ldots, \ell_d\}$, and let G denote the flip graph on self-complementary ideals of P.

Theorem

The diameter of G is

$$\begin{cases} 0 & \text{if all of } \ell_1, \ldots, \ell_d \text{ are odd,} \\ \frac{1}{4}\,|P| & \text{if at least two of } \ell_1, \ldots, \ell_d \text{ are even, and} \\ \frac{1}{4}(|P| - \ell_k) & \text{if } \ell_k \text{ is even and the rest are odd.} \end{cases}$$



Radius of Flip Graphs on Self-Complementary Ideals

Let $P = \{1, \dots, \ell_1\} \times \cdots \times \{1, \dots, \ell_d\}$, and let G denote the flip graph on self-complementary ideals of P.

Theorem

Suppose $\ell_1, \dots, \ell_d$ are even. Assuming Chvátal's conjecture, G's radius is

$$\left\lceil \left(\frac{1}{4} - \frac{1}{2^{d+1}} \binom{d-1}{\lfloor \frac{1}{2}(d-1) \rfloor} \right) |P| \right\rceil .$$

Cyclically Symmetric Self-Complementary Ideals

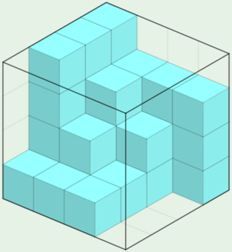
Let $P = \{1, \ldots, 2r\} \times \{1, \ldots, 2r\} \times \{1, \ldots, 2r\}$.

Definition

A self-complementary ideal $I \subset P$ is *cyclically symmetric* if

$$(a_1, a_2, a_3) \in I \implies (a_2, a_3, a_1) \in I \text{ and } (a_3, a_1, a_2) \in I$$

Example



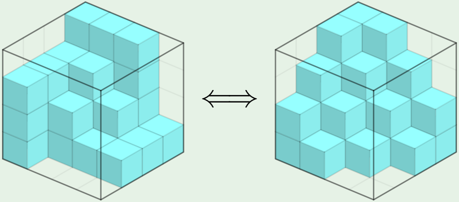
CSSC Flips

Let $P = \{1, \ldots, 2r\}^3$, and let I and J be two CSSC ideals of P.

Definition

I and J differ by a *CSSC flip* if $|I \setminus J| = |J \setminus I| = 3$.

Example

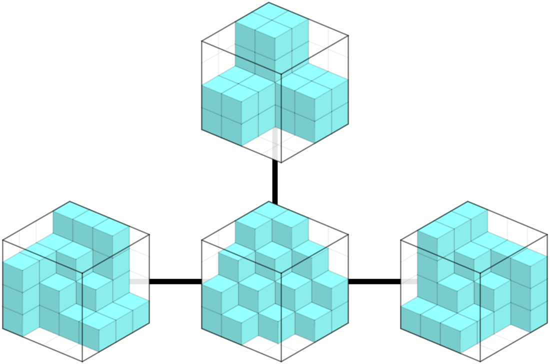


Flip Graphs on CSSC Ideals

Let $P = \{1, \ldots, 2r\}^3$.

Definition

The *flip graph on CSSC ideals of P* is the graph whose vertices are the CSSC ideals of P, and whose edges connect pairs of ideals that differ by a CSSC flip.



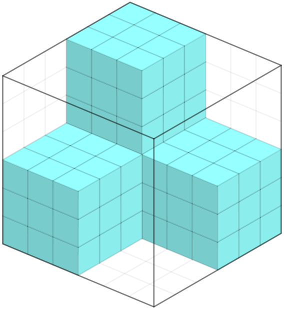
Diameter of Flip Graphs on CSSC Ideals

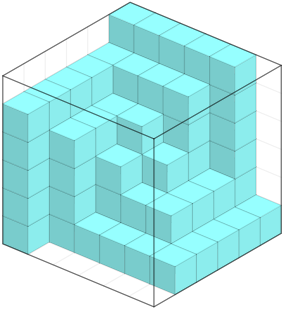
Let $P = \{1, \ldots, 2r\}^3$, and let G denote the flip graph on CSSC ideals of P.

Theorem

The diameter of G is

$$\frac{1}{3}(r-1)(r)(r+1).$$





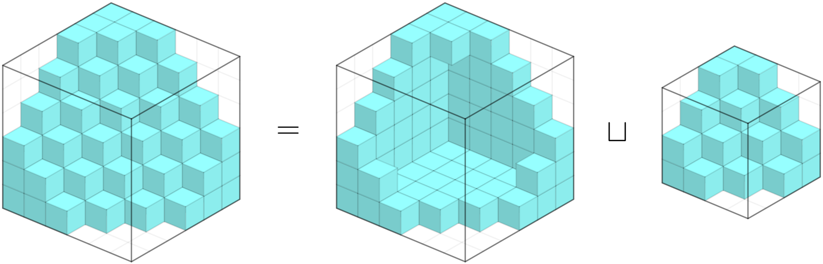
Radius of Flip Graphs on CSSC Ideals

Let $P = \{1, \ldots, 2r\}^3$, and let G denote the flip graph on CSSC ideals of P.

Theorem

The radius of G is

$$\frac{1}{6}(r-1)(r)(r+1).$$



Totally Symmetric Self-Complementary Ideals

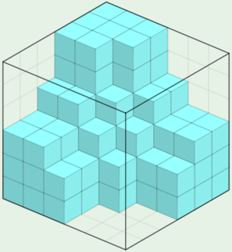
Let $P = \{1, \ldots, 2r\}^3$.

Definition

A self-complementary ideal $I \subset P$ is *totally symmetric* if for every permutation $\sigma \in S_3$,

$$(a_1, a_2, a_3) \in I \implies (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) \in I.$$

Example



Properties of Flip Graphs on TSSC Ideals

Let $P = \{1, \ldots, 2r\}^3$. It is possible to define a flip graph G on TSSC ideals of P.

Theorem

The diameter of G is

$$\frac{1}{6}(r-1)(r)(2r-1).$$

Conjecture

The radius of G is

$$\left\lceil \frac{1}{12}(r-1)(r)(2r-1) \right\rceil.$$

Future Directions

What we studied:

- vertex count
- diameter
- radius

Other properties of interest:

- maximum degree
- edge count and average degree
- set of vertices with minimum eccentricity (center)
- set of vertices with maximum eccentricity (perimeter)

Acknowledgments

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