HOMOTOPY GROUPS OF SPHERES

SERENA AN

1. Introduction

Spheres are among the most fundamental topological spaces, and their homotopy groups help characterize the maps between spheres. Some homotopy groups of spheres are easily determined: $\pi_i(S^n) = 0$ for $i < n$ by the cellular approximation theorem, and $\pi_n(S^n) = \mathbb{Z}$ by the Hurewicz theorem. The more fascinating study is of $\pi_i(S^n)$ with $i > n$.

The Freudenthal suspension theorem, which we will prove as Theorem 2.2, implies that some homotopy groups of spheres are "stable", in the sense that $\pi_{n+k}(S^n)$ is independent of n for n sufficiently large. These stable homotopy groups are still surprisingly challenging to compute, and their study has led to the development of many new methods in algebraic topology. Recently in 2023, Isakesen, Wang, and Xu [2] have computed stable homotopy groups for $k \leq 90$.

We present a technique first introduced by Cartan and Serre [1] in 1952: forming fiber spaces from maps to Eilenberg–MacLane spaces to progressively kill higher nonzero cohomology groups. Although there are limitations to this method, it is a relatively elementary way to compute the 2-component of the first few stable homotopy groups. Our main reference is Chapter 12 of Mosher and Tangora [3].

Theorem 1.1. [3] *The 2-component of the homotopy group $\pi_{n+k}(S^n)$ for $0 \leq k \leq 7$ and $n \geq k+2$ is given by the following.*

k	0	1	2	3	4	5	6	7
$\pi_{n+k}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$

The goal of this expository paper is to compute the first two nontrivial cases of $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$, after summarizing the necessary background. We begin in Section 2 by proving the Freudenthal suspension theorem, which largely characterizes the homotopy groups of spheres. We introduce the Steenrod squares in Section 3 and the Bockstein differentials in Section 4, two natural cohomology operations which allow us to compute the cohomology of $K(\mathbb{Z}, n)$ in Section 5. We compute $\pi_{n+1}(S^n)$ in Section 6 and $\pi_{n+2}(S^n)$ in Section 7 using fibrations. Finally in Section 8, we remark on the methods used for computing $\pi_{n+k}(S^n)$ with $3 \leq k \leq 7$.

Acknowledgments

This paper was written for Harvard's Advanced Algebraic Topology (MATH 231BR) in spring 2025, taught by Dr. Andrew Senger. I would like to thank Dr. Senger and Greg Wong for helpful discussions during office hours.

Massachusetts Institute of Technology

E-mail address: `anser@mit.edu`.

Date: May 6, 2025.

2. Freudenthal Suspension Theorem

The Freudenthal suspension theorem is a powerful tool for studying homotopy groups. Let ΣX denote the suspension of X.

Definition 2.1. Let $\varphi\colon X \to \Omega\Sigma X$ be the natural map sending x to the path $t \mapsto (x,t)$. The *suspension homomorphism* $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is defined as the composition

$$\pi_i(X) \xrightarrow{\varphi_*} \pi_i(\Omega\Sigma X) \xrightarrow{\cong} \pi_{i+1}(\Sigma X).$$

We now state the Freudenthal suspension theorem and apply it to spheres, recalling that $\Sigma S^n = S^{n+1}$.

Theorem 2.2 (Freudenthal suspension theorem)**.** *If X is $(n-1)$-connected, then the suspension homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2n-1$ and an epimorphism for $i = 2n-1$.*

In particular for $X = S^n$ and $i = n+k$,

$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is an isomorphism for $n + k \leq 2n - 2$, i.e. $n \geq k+2$. In other words, $\pi_{n+k}(S^n)$ is independent of n for $n \geq k+2$, as stated in Theorem 1.1.

Proof of Theorem 2.2. Since X is $(n-1)$-connected, ΣX is n-connected and $\Omega\Sigma X$ is $(n-1)$-connected. The Serre homology exact sequence for the path space fibration

$$\begin{tikzcd} \Omega\Sigma X \arrow[r] & * \arrow[d] \\ & \Sigma X \end{tikzcd}$$

is

$$H_{2n}(*) \to H_{2n}(\Sigma X) \to H_{2n-1}(\Omega\Sigma X) \to H_{2n-1}(*) \to \cdots,$$

which yields isomorphisms $H_{i+1}(\Sigma X) \to H_i(\Omega\Sigma X)$ for all $i \leq 2n-1$. The commutativity of the diagram

$$\begin{tikzcd} H_{i+1}(\Sigma X) \arrow[r] & H_i(\Omega\Sigma X) \\ H_i(X) \arrow[u, "\cong"] \arrow[ur, "\varphi_*"'] & \end{tikzcd}$$

furthermore implies that $H_i(X) \xrightarrow{\varphi_*} H_i(\Omega\Sigma X)$ is an isomorphism for $i \leq 2n-1$.

By the Whitehead theorem, the corresponding map in homotopy $\pi_i(X) \xrightarrow{\varphi_*} \pi_i(\Omega\Sigma X)$ is an isomorphism for $i < 2n-1$ and an epimorphism for $i = 2n-1$. Composing with the isomorphism $\pi_i(\Omega\Sigma X) \to \pi_{i+1}(\Sigma X)$ shows that the suspension homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ is an isomorphism for $i < 2n-1$ and an epimorphism for $i = 2n-1$, as desired. $\square$

3. Steenrod Squares

We now introduce some definitions and theorems which will be used repeatedly in later sections for cohomology calculations.

Definition 3.1. Suppose X is $(n-1)$-connected, so the Hurewicz homomorphism $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism. The *fundamental class* $\iota_n \in H^n(X; \pi_n(X))$ corresponds to $h^{-1} \in \mathrm{Hom}(H_n(X), \pi_n(X))$ under the universal coefficient theorem isomorphism $H^n(X; \pi_n(X)) \cong \mathrm{Hom}(H_n(X), \pi_n(X))$.

The following homomorphisms are frequently applied to fundamental classes.

Definition 3.2. The *Steenrod squares* Sq^i are natural homomorphisms

$$Sq^i\colon H^n(X; \mathbb{Z}_2) \to H^{n+i}(X; \mathbb{Z}_2)$$

for $0 \le i \le n$, and $Sq^i = 0$ for $i > n$.

Refer to [3, Chapter 2] for the construction of the Steenrod squares. We instead focus on the following useful properties, to be referenced later.

Theorem 3.3 (Steenrod axioms)**.**

(1) *$Sq^i(x) = x^2$ for all $x \in H^i(K; \mathbb{Z}_2)$.*
(2) *Sq^0 is the identity.*
(3) *Sq^1 is the Bockstein homomorphism k^1 in (4.1).*
(4) *Adem relations: for $i < 2j$,*

$$Sq^i Sq^j = \sum_k \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k, \tag{3.1}$$

where the binomial coefficient is taken mod 2. In particular when $i = 1$,

$$Sq^1 Sq^j = \begin{cases} Sq^{j+1} & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}. \tag{3.2}$$

4. Bockstein Differentials

The Bockstein differentials d_r, as referenced in Theorem 3.3, are defined as follows. Our discussion will only use $\mathbb{Z}_2$ coefficients, but there is a mod p version in general.

Definition 4.1. From the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, we obtain the *Bockstein exact couple*

$$\begin{tikzcd} H^*(X;\mathbb{Z}) \arrow[rr, "i^1"] & & H^*(X;\mathbb{Z}) \arrow[dl, "j^1"] \\ & H^*(X;\mathbb{Z}_2) \arrow[ul, "k^1"] & \end{tikzcd} \tag{4.1}$$

where i^1 is from multiplication by 2 and j^1 is from reduction mod 2.

The first differential is $d_1 = j^1 k^1$, and we may form successive derived couples with differentials d_r; these are called the *Bockstein differentials*. Each Bockstein differential d_r is a map $H^*(X; \mathbb{Z}_2) \to H^{*+1}(X; \mathbb{Z}_2)$.

The following lemma helps compute Bockstein differentials; although it is not used in our calculations, it is frequently cited in the full calculation [3, Chapter 12].

Theorem 4.2 (Bockstein lemma)**.** *For a fibration $F \xrightarrow{i} E \xrightarrow{p} B$, consider the associated cohomology long exact sequence.*

$$\begin{tikzcd} H^*(F;\mathbb{Z}_2) \arrow[dr, "\tau"'] & H^*(E;\mathbb{Z}_2) \arrow[l, "i^*"'] \\ & H^*(B;\mathbb{Z}_2) \arrow[u, "p^*"'] \end{tikzcd}$$

Given $u \in H^n(F;\mathbb{Z}_2)$ and $v \in H^n(B;\mathbb{Z}_2)$ such that $d_r v = \tau(u)$, we have

$$i^* d_{r+1} p^*(v) = d_1 u$$

in $H^{n+1}(F;\mathbb{Z}_2)$.

5. Cohomology of $K(\mathbb{Z}, n)$

A first approximation for the sphere S^n is the Eilenberg–MacLane space $K(\mathbb{Z}, n)$, which agrees in homotopy and cohomology up to dimension n. We compute the higher cohomology of $K(\mathbb{Z}, n)$, which allows us to find other spaces agreeing with S^n in cohomology to a higher dimension in Section 6.

By the universal coefficient theorem

$$H^m(X;\mathbb{Z}_2) \cong \mathrm{Hom}(H_m(X), \mathbb{Z}_2) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{m-1}(X), \mathbb{Z}_2),$$

if the groups $\mathbb{Z}$ or $\mathbb{Z}_{2^r}$ are present in $H_m(X)$, then they induce the following groups in $H^m(X)$, $H^m(X;\mathbb{Z}_2)$, and $H^{m+1}(X;\mathbb{Z}_2)$.

$H_m(X)$	$H^m(X)$	$H^m(X;\mathbb{Z}_2)$	$H^{m+1}(X;\mathbb{Z}_2)$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
$\mathbb{Z}_{2^r}$	$\mathbb{Z}_{2^r}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

As a result, elements of $H^m(X;\mathbb{Z}_2)$ from a $\mathbb{Z}$-summand of $H^m(X)$ lie in the kernel of $d_r\colon H^m(X;\mathbb{Z}_2) \to H^{m+1}(X;\mathbb{Z}_2)$ for all r. Also, if the generators z, z' of $\mathbb{Z}_2$-summands in $H^m(X;\mathbb{Z}_2)$ and $H^{m+1}(X;\mathbb{Z}_2)$ respectively come from the same $\mathbb{Z}_{2^r}$-summand of $H^m(X)$, then $d_r z = z'$. Note that $d_i z = d_i z' = 0$ for all $i < r$, since z, z' do not come from a $\mathbb{Z}_{2^i}$ summand of $H_m(X)$ for $i < r$. These observations imply the following theorem.

Theorem 5.1. *Suppose $H^i(X;\mathbb{Z}_2) = 0$ for $i < m$ and $H^m(X;\mathbb{Z}_2) = \mathbb{Z}_2$, generated by z. Then excluding torsion at odd primes, we have*

$$H^m(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{if } d_i z = 0 \textit{ for all } i \\ \mathbb{Z}_{2^m} & \textit{if } d_i z = 0 \textit{ for } i < m, \textit{ and } d_m z \neq 0 \end{cases}.$$

This theorem will be relevant in determining which Eilenberg–MacLane spaces $K(\mathbb{Z}_{2^m}, n)$ to use and will be discussed more in Section 8. We now state the cohomology of $K(\mathbb{Z}_2, n)$, $K(\mathbb{Z}, n)$, and $K(\mathbb{Z}_{2^m}, n)$, which are polynomial rings in certain Steenrod squares.

Definition 5.2. From the Adem relations (3.1), any composition of Steenrod squares can be written as $Sq^{i_1} \cdots Sq^{i_r}$ with $i_j \geq 2_{j+1}$ for all $1 \leq j \leq r-1$. We say

k	$H^{n+k}(K(\mathbb{Z},n))$	$H^{n+k}(K(\mathbb{Z}_2,n+1))$	$H^{n+k}(X_1)$	$H^{n+k}(K(\mathbb{Z}_2,n+2))$
0	ι_n		ι_n	
1		ι_{n+1}		
2	$Sq^2(\iota_n)$	$Sq^1(\iota_{n+1})$		ι_{n+2}
3	$Sq^3(\iota_n)$	$Sq^2(\iota_{n+1})$	α	$Sq^1(\iota_{n+2})$
4	$Sq^4(\iota_n)$	$\vdots$	$Sq^4(\iota_n), \beta$	$\vdots$

FIGURE 1. Generators of each cohomology group (with $\mathbb{Z}_2$ coefficients) and transgressions.

that such a sequence $I = i_1 \cdots i_r$ of positive integers is *admissible*. The *excess* $e(I)$ of an admissible sequence I is $i_1 - (i_2 + \cdots + i_r)$.

In the following theorems, let Sq^I denote $Sq^{i_1} \cdots Sq^{i_r}$ for $I = i_1 \cdots i_r$.

Theorem 5.3. [3] *$H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ is the polynomial ring generated by $\{Sq^I(\iota_n)\}$ where $I = i_1 \cdots i_r$ runs through admissible sequences of excess $e(I) < n$.*

Theorem 5.4. [3] *$H^*(K(\mathbb{Z}, n); \mathbb{Z}_2)$ is the polynomial ring generated by $\{Sq^I(\iota_n)\}$ where $I = i_1 \cdots i_r$ runs through admissible sequences of excess $e(I) < n$, and $i_r \neq 1$.*

The cohomology of $K(\mathbb{Z}_{2^m}, n)$ will be necessary when larger groups, such as $\mathbb{Z}_8$, begin to show up in the 2-component of $\pi_{n+k}(S^n)$.

Theorem 5.5. [3] *$H^*(K(\mathbb{Z}_{2^m}, n); \mathbb{Z}_2)$ is the polynomial ring generated by $\{Sq^{I_m}(\iota_n)\}$ where $I = i_1 \cdots i_r$ runs through admissible sequences of excess $e(I) < n$, and Sq^{I_m} equals Sq^I if $i_r > 1$, or Sq^I with Sq^{i_r} replaced with d_m if $i_r = 1$.*

6. COMPUTATION OF $\pi_{n+1}(S^n)$

All of our computations will be towards the 2-components of homotopy groups of spheres, and *all cohomology will be with $\mathbb{Z}_2$ coefficients.* The main idea of the proof is to progressively kill the higher nonzero cohomology groups of $K(\mathbb{Z}, n)$ to obtain better approximations of S^n. We then use fiber sequences to compute the cohomology and homotopy groups of the resulting spaces, which agree with those of S^n to a higher dimension. Refer to Figure 1 for a table summarizing the relevant cohomology groups and transgressions.

Beginning with $K(\mathbb{Z}, n)$, we know all of its cohomology groups by Theorem 5.4. In particular, $H^{n+1}(K(\mathbb{Z}, n); \mathbb{Z}_2) = 0$ has no generators because if I of degree 1, then $I = i_r = 1$. Also, $H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}_2) = \mathbb{Z}_2$ is generated by $Sq^2(\iota_n)$.

We wish to replace $K(\mathbb{Z}, n)$ with a new space X_1 having the same cohomology up to dimension $n+1$, but with $H^{n+2}(X_1; \mathbb{Z}_2) = 0$ to better approximate S^n. By the isomorphism $[X, K(\pi, q)] \cong H^q(X; \pi)$ [3, Theorem 1.1], the generator $Sq^2(\iota_n) \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}_2)$ corresponds to a map $Sq^2 \colon K(\mathbb{Z}, n) \to K(\mathbb{Z}_2, n+2)$. This map Sq^2 induces a fibration $X_1 \to K(\mathbb{Z}, n)$ with fiber $\Omega K(\mathbb{Z}_2, n+2) = K(\mathbb{Z}_2, n+1)$ as illustrated below.

$$\begin{tikzcd}
K(\mathbb{Z}_2, n+1) \arrow[r, "="] \arrow[d] & K(\mathbb{Z}_2, n+1) \arrow[d] \\
X_1 \arrow[r] \arrow[d] & * \arrow[d] \\
K(\mathbb{Z}, n) \arrow[r, "Sq^2"] & K(\mathbb{Z}_2, n+2)
\end{tikzcd}$$

To determine the cohomology of X_1, we need the following lemma.

Lemma 6.1. *Given a map $B \xrightarrow{f} K$, consider the induced path space fibration.*

$$\begin{tikzcd}
\Omega K \arrow[r] \arrow[d] & \Omega K \arrow[d] \\
X \arrow[r] \arrow[d] & * \arrow[d] \\
B \arrow[r, "f"] & K
\end{tikzcd}$$

Let $\sigma\colon H^*(K) \to H^*(*, \Omega K) \cong H^*(\Sigma\Omega K)$ be the map induced from $K \to \Sigma\Omega K$ which is adjoint to the natural map $K \to \Omega\Sigma K$. Then every element of the form $\sigma z \in H^{*-1}(\Omega K)$ for $z \in H^*(K)$ is transgressive with transgression equal to $f^*(z)$.

Proof. By naturality of the Serre spectral sequence, it suffices to show for the $\Omega K \to * \to K$ case that the transgression of σz is z. From the cospan diagram

$$\begin{tikzcd}
 & H^*(*, \Omega K) & \\
H^*(K) \arrow[ur, "\sigma"] & & H^{*-1}(\Omega K) \arrow[ul, "\cong"']
\end{tikzcd}$$

we see that $\sigma z \in H^{*-1}(\Omega K)$ and $z \in H^*(K)$ are mapped to the same element in $H^*(*, \Omega K)$. Then $(\sigma z, z)$ lies in the graph of the transgression, as desired. $\square$

In particular, the transgression is precisely f^* in the range of the Serre cohomology exact sequence for which σ is an isomorphism. This is indeed the range that we work in, which shows that we can compute $H^*(X_1)$ by knowing $H^*(K(\mathbb{Z}, n))$ and $H^*(K(\mathbb{Z}_2, n+1))$, and how the transgression induced by $Sq^2\colon K(\mathbb{Z}, n) \to K(\mathbb{Z}_2, n+2)$ acts.

Proposition 6.2. $H^{n+2}(X_1) = 0$.

Proof. For brevity, we denote the fibration $K(\mathbb{Z}_2, n+1) \to X_1 \to K(\mathbb{Z}, n)$ by $F \to X_1 \to B$. Consider the associated Serre cohomology exact sequence

$$\cdots \to H^{n+1}(F) \xrightarrow{\tau} H^{n+2}(B) \xrightarrow{p^*} H^{n+2}(X_1) \xrightarrow{i^*} H^{n+2}(F) \xrightarrow{\tau} H^{n+3}(B) \to \cdots.$$

The fundamental class $\iota_{n+1} \in H^{n+1}(F)$ trangresses to the generator $Sq^2(\iota_n) \in H^{n+2}(B)$ by Lemma 6.1. Then by naturality, the generator $Sq^1(\iota_{n+1}) \in H^{n+2}(F)$ trangresses to $Sq^1Sq^2(\iota_n) = Sq^3(\iota_n) \in H^{n+3}(B)$, where we use (3.2). Consequently, τ is a surjection onto $H^{n+2}(B)$ and an injection from $H^{n+2}(F)$. By exactness, p^* and i^* are zero maps, which forces $H^{n+2}(X_1) = 0$. $\square$

From the homotopy long exact sequence for the fibration $F \to X_1 \to B$, we can easily compute that $\pi_{n+1}(X_1) = \mathbb{Z}_2$. Together with the following lemma, we will obtain

$$\pi_{n+1}(S^n) = \mathbb{Z}_2.$$

Lemma 6.3. *There exists a map $f_1 \colon S^n \to X_1$ inducing a $\mathcal{C}_2$ isomorphism on homotopy up to dimension $n+1$.*

Proof. Let $f \colon S^n \to K(\mathbb{Z}, n)$ correspond to a generator of $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$. Then $f^* \colon H^i(K(\mathbb{Z}, n); \mathbb{Z}_2) \to H^i(S^n; \mathbb{Z}_2)$ is an isomorphism for $i < n+1$ by the Hurewicz theorem. It is also an isomorphism for $i = n+1$ since $H^{n+1}(K(\mathbb{Z}, n); \mathbb{Z}_2) = 0$ from Theorem 5.4, and $H^{n+1}(S^n; \mathbb{Z}_2) = 0$ by the universal coefficient theorem.

Now the composition $S^n \xrightarrow{f} K(\mathbb{Z}, n) \xrightarrow{Sq^2} K(\mathbb{Z}_2, n+2)$ is null-homotopic because $\pi_n(K(\mathbb{Z}_2, n+2)) = 0$; then by the fibration lifting property, we can lift f to a map $f_1 \colon S^n \to X_1$.

$$\begin{tikzcd} & X_1 \arrow[d] & \\ S^n \arrow[ur, dashed, "f_1"] \arrow[r, "f"] & K(\mathbb{Z}, n) \arrow[r, "Sq^2"] & K(\mathbb{Z}_2, n+2) \end{tikzcd}$$

For $f_1^* \colon H^i(X_1; \mathbb{Z}_2) \to H^i(S^n; \mathbb{Z}_2)$, it not only induces isomorphisms for $i \leq n+1$ like f^*, but is also a monomorphism for $i = n+2$, since $H^{n+2}(X_1) = 0$ by Proposition 6.2. Thus, by the $\mathcal{C}_p$ approximation theorem [3, Theorem 10.4], f_1 induces a $\mathcal{C}_2$ isomorphism on homotopy up to dimension $n+1$. □

7. Computation of $\pi_{n+2}(S^n)$

In the previous section, we began computing the first two columns of Figure 1 and the transgression. The goal of this section is to kill $H^{n+3}(X_1)$, replacing X_1 with a space X_2 satisfying $H^{n+3}(X_2) = 0$ and with the same homotopy groups as S^n up to dimension $n+2$.

We continue computing the cohomology of X_1. Using the Serre cohomology exact sequence

$$\cdots \to H^{n-1}(F) \xrightarrow{\tau} H^n(B) \xrightarrow{p^*} H^n(X_1) \xrightarrow{i^*} H^n(F) \xrightarrow{\tau} H^{n+1}(B) \to \cdots.$$

with $H^{n-1}(F) = H^n(F) = 0$, we see that $H^n(X_1)$ is generated by its fundamental class, the image of $\iota_n \in H^n(B)$ under p^*, which we will also call ι_n. Now consider the portion of the exact sequence

$$\cdots \to H^{n+2}(F) \xrightarrow{\tau} H^{n+3}(B) \xrightarrow{p^*} H^{n+3}(X_1) \xrightarrow{i^*} H^{n+3}(F) \xrightarrow{\tau} H^{n+4}(B) \to \cdots.$$

We know how τ behaves from Figure 1. First, $\tau \colon H^{n+2}(F) \to H^{n+3}(B)$ is an isomorphism, so the shown p^* is the zero map, implying that i^* is injective. Then since $\tau \colon H^{n+3}(F) \to H^{n+4}(B)$ sends $Sq^2(\iota_{n+1}) \mapsto 0$, by exactness $H^{n+3}(X_1)$ is generated by a class α such that $i^*(\alpha) = Sq^2(\iota_{n+1})$.

We now need to kill the class $\alpha \in H^{n+3}(X_1)$, producing a space X_2 with no cohomology in dimension $n+3$. Similar to the previous section, we consider the map $\alpha \colon X_1 \to K(\mathbb{Z}_2, n+3)$ corresponding to $\alpha \in H^{n+3}(X_1)$, with fiber $\Omega K(\mathbb{Z}_2, n+3) =$

$K(\mathbb{Z}_2, n+2)$. Let X_2 be the induced fiber space, and we show this works.

$$\begin{tikzcd}
K(\mathbb{Z}_2, n+2) \arrow[r] & X_2 \arrow[d, "p_2"] & \\
& X_1 \arrow[r, "\alpha"] & K(\mathbb{Z}_2, n+3)
\end{tikzcd}$$

Omitted computation shows that in $H^{n+4}(X_1)$ is generated by a class $Sq^4(\iota_n)$ which is the image of $Sq^4(\iota_n) \in H^{n+4}(F)$ under p^*, and a class β with $i^*(\beta) = Sq^3(\iota_{n+1})$. There is no canonical choice for β, but the following calculations are true for any such β.

Lemma 7.1. *$Sq^1(\alpha) = \beta + cSq^4\iota_n$ for some unspecified coefficient c.*

Proof. Recall the definition of α as satisfying $i^*(\alpha) = Sq^2\iota_{n+1}$. Then

$$i^*(Sq^1\alpha) = Sq^1 i^*(\alpha) = Sq^1Sq^2\iota_{n+1} = Sq^3\iota_{n+1},$$

where the equalities follow from naturality, the definition of α, and the Adem relations (3.2). On the other hand $i^*(\beta) = Sq^3(\iota_{n+1})$ by definition, and $i^*(Sq^4\iota_n) = 0$ by exactness, since $Sq^4\iota_n$ was in the image of p^*. □

Proposition 7.2. *$H^{n+3}(X_2) = 0$.*

Proof. Let $F_2 \to X_1 \to X_2$ denote the fibration $K(\mathbb{Z}_2, n+2) \to X_1 \to X_2$. Refer to the third and fourth columns of Figure 1 for a visual. By Lemma 6.1, the fundamental class $\iota_{n+2} \in H^{n+2}(F_2)$ transgresses to $\alpha \in H^{n+3}(X_1)$. Then $Sq^1(\iota_{n+2}) \in H^{n+3}(F_2)$ transgresses to $Sq^1(\alpha) \in H^{n+4}(X_1)$, which equals $\beta + cSq^4\iota_n$ by Lemma 7.1 and is nonzero. This implies in the Serre cohomology exact sequence

$$\cdots \to H^{n+2}(F_2) \xrightarrow{\tau} H^{n+3}(X_1) \xrightarrow{p_2^*} H^{n+3}(X_2) \xrightarrow{i^*} H^{n+3}(F_2) \xrightarrow{\tau} H^{n+4}(X_1) \to \cdots,$$

τ is an isomorphism onto $H^{n+3}(X_1)$ and an injection from $H^{n+3}(F_2)$. By exactness, p_2^* and i^* are zero maps, so $H^{n+3}(X_2) = 0$. □

Thus, X_2 is a better approximation for S^n, with cohomology agreeing up to dimension $n+3$, and homotopy agreeing up to dimension $n+2$ by an argument similar to Lemma 6.3. From the homotopy long exact sequence for the fibration $K(\mathbb{Z}_2, n+2) \to X_2 \to X_1$, we can compute $\pi_{n+2}(X_2) = \mathbb{Z}_2$, so

$$\pi_{n+2}(S^n) = \mathbb{Z}_2.$$

8. Remarks on Further Calculations

Mosher and Tangora [3] continue the computation by constructing a space X_3 to kill $H^{n+4}(X_2)$. We remark on how the group $\mathbb{Z}_8$ appears in $\pi_{n+3}(S^n)$. After computing that d_1 and d_2 applied to the generator $p_2^*(Sq^4\iota_n) \in H^{n+4}(X_2)$ yield 0 but not d_3, Theorem 5.1 states that $p_2^*(Sq^4\iota_n)$ must be the mod 2 reduction of a class in $H^{n+4}(X_2; \mathbb{Z}) = \mathbb{Z}_8$. We should now consider a map $X_2 \to K(\mathbb{Z}_8, n+4)$ corresponding such a class and take X_3 as the induced fiber space. Calculations show that not only is $H^{n+4}(X_3)$ equal to zero, but so are $H^{n+5}(X_3)$ and $H^{n+6}(X_3)$. We can then use the homotopy long exact sequence for $K(\mathbb{Z}_8, n+4) \to X_3 \to X_2$

to obtain $\pi_{n+3}(S^n) = \mathbb{Z}_8$ and $\pi_{n+4}(S^n) = \pi_{n+5}(S^n) = 0$. Repeating this procedure twice more produces the remaining table entries of Theorem 1.1.

References

[1] Henri Cartan and Jean-Pierre Serre. Espaces fibrés et groupes d'homotopie. I. Constructions générales. *C. R. Acad. Sci. Paris*, 234:288–290, 1952.

[2] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu. Stable homotopy groups of spheres: from dimension 0 to 90. *Publications mathématiques de l'IHÉS*, 137(1):107–243, 2023.

[3] Robert E. Mosher and Martin C. Tangora. *Cohomology Operations and Applications in Homotopy Theory*. Dover Publications, 2008.