PROJECTIVE MODULES

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1. Introduction

The aim of this paper is to provide a gentle introduction to projective modules. Projective modules can be thought of as building blocks of the A-module A; they have many desirable properties and are central to fields such as representation theory and homological algebra. One of the most useful facts regarding projective modules is the bijective correspondence between *indecomposable projective* modules and *simple* modules, which is the main theorem in this paper.

We begin by introducing basic definitions relating to modules in Section 2, before defining projective modules and giving three equivalent characterizations in Section 3. In Section 4, we introduce a few helpful lemmas about indecomposable projective modules, which are used in Section 5 to prove the bijective correspondence between indecomposable projective modules P and simple modules P/M, where $M \subset P$ is a maximal submodule. Finally in Section 6, we present structural results that allow us to describe all indecomposable projective modules and simple modules. Our main references are Alperin [1] and Leinster [2].

The primary example to keep in mind throughout this paper is the algebra of upper triangular 2×2 matrices over a field k; that is,

$$A = \{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix} \right) \mid a, b, c \in k \}.$$

Letting

$$\begin{aligned} P_1 &= \{ \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix} \right) \mid a \in k \} \\ P_2 &= \{ \left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix} \right) \mid b, c \in k \} \\ M &= \{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix} \right) \mid c \in k \}, \end{aligned}$$

we will understand that the A-module A is the direct sum $P_1 \oplus P_2$ of indecomposable projective A-modules P_1 and P_2, which correspond to the simple A-modules P_1 and P_2/M, and that P_1 and P_2 are the only indecomposable projective A-modules up to isomorphism.

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2. Modules

Throughout this paper, let A be a finite-dimensional algebra over a field k. Given a ring A, an A-*module* U is an abelian group $(U, +)$ with multiplication $A \times U \to U$ by elements in A such that for all $a, b \in A$ and $u, v \in U$, we have

$$a(u + v) = au + av$$
$$(a + b)u = au + bu$$
$$a(bu) = (ab)u$$
$$1u = u.$$

A *submodule* V of an A-module U is a subgroup $V \subset U$ which is closed under multiplication by A.

Definition 2.1. A nonzero A-module U is *simple* if its only submodules are 0 and U.

Equivalently, any nonzero element $u \in U$ of a simple module is a generator, because $Au \subset U$ is a nonzero submodule that must in fact equal U.

Relatedly, an A-module U is *cyclic* if there exists some $u \in U$ such that $Au = U$. As a result, every simple module is cyclic, although not all cyclic modules are simple (e.g. the cyclic $\mathbb{Z}$-module $\mathbb{Z}/6\mathbb{Z}$ has $\mathbb{Z}/3\mathbb{Z}$ as a submodule). We will see the property of being cyclic appear in Lemma 4.3.

One way to form new A-modules from existing A-modules is via a direct sum. For A-modules U and V, their *direct sum* is

$$U \oplus V = \{(u, v) \mid u \in U, v \in V\}.$$

This is an A-module upon defining addition and scalar multiplication as

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$$
$$a(u, v) = (au, av).$$

More generally, the direct sum of the A-modules $(U_i)_{i \in I}$ for an index set I is

$$\bigoplus_{i \in I} U_i = \{(u_i)_{i \in I} \mid u_i \in U_i, \text{ finitely many } u_i \text{ nonzero}\}.$$

Example 2.2. Let A be the algebra of 2×2 upper triangular matrices over a field k; that is,

$$A = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in k\}$$

with operations given by standard matrix addition and multiplication. A can be considered as an A-module by letting the A-module multiplication $A \times A \to A$ coincide with matrix multiplication in A. Two submodules of the A-module A are

$$P_1 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid a \in k\}$$
$$P_2 = \{\left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix}\right) \mid b, c \in k\},$$

and in fact for these submodules we have $A = P_1 \oplus P_2$.

Definition 2.3. A submodule $M \subset U$ is *maximal* if $M \neq U$ and there does not exist a submodule N such that $M \subsetneq N \subsetneq U$.

By the correspondence theorem for modules, $M \subset U$ is maximal if and only if U/M is simple. We use this fact in Lemma 4.4 to show that a unique maximal submodule can be found in indecomposable projective modules.

Example 2.4. In the context of Example 2.2, one can verify that the A-module P_2 has exactly one nontrivial proper submodule, namely

$$M = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right) \mid c \in k\}.$$

As a result, M is maximal in P_2 and the only maximal submodule of P_2.

Definition 2.5. A nonzero A-module U is *indecomposable* if it can not be written as a direct sum of nontrivial submodules.

Simple modules are indecomposable, but the converse is not always true (e.g. the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$ is not simple because it has $\mathbb{Z}/2\mathbb{Z}$ as a submodule, but it is indecomposable because $\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus V$ for any submodule $V \subset \mathbb{Z}/4\mathbb{Z}$).

Example 2.6. In the context of Example 2.2, we claim that the A-modules P_1 and P_2 are indecomposable. P_1 is isomorphic to the field k, so it is simple and thus indecomposable. P_2 is indecomposable because $P_2 \ncong M \oplus V$ for any submodule $V \subset P_2$. We observe that P_2 is another example of an indecomposable module that is not simple.

3. Projective Modules

In this section, we define projective modules and give equivalent characterizations. First, we require the definition of a free module.

Definition 3.1. An A-module U is *free* if $U \cong \bigoplus_{i \in I} A$ for some index set I. Free modules are oftentimes denoted by $A^{\oplus I}$, or $A^{\oplus n}$ if $|I| = n$ is finite.

The following is an alternate characterization of free modules that will be used in our discussion of projective modules.

Proposition 3.2. *An A-module U is free if and only if there exists a k-linear subspace $X \subset U$ such that for any A-module V, any k-linear transformation $\varphi \colon X \to V$ extends uniquely to a module homomorphism $\Phi \colon U \to V$.*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\varphi"'] & U \arrow[d, dashed, "\Phi"] \\ & V \end{tikzcd}$$

Proof. If $U = A^{\oplus I}$, consider the basis elements $(e_i)_{i \in I} \subset U$, where e_i has all entries 0 except a 1 in the ith coordinate. Let $X \subset U$ be the subspace of U spanned by finite linear combinations of $(e_i)_{i \in I}$ with coefficients in k. Given a linear transformation

$\varphi\colon X \to V$, it can be extended uniquely to a module homomorphism $\Phi\colon U \to V$ as follows. For $\sum_{i\in I} a_i e_i \in U$ with $(a_i)_{i\in I} \subset A$, we have

$$\begin{aligned}\Phi\left(\sum_{i\in I} a_i e_i\right) &= \sum_{i\in I} a_i \Phi(e_i)\\ &= \sum_{i\in I} a_i \varphi(e_i),\end{aligned}$$

so Φ is uniquely determined.

For the reverse direction, suppose that W is an A-module and $Y \subset W$ a subspace with the stated property. Suppose Y has a basis indexed by $i \in I$; then letting $X \subset U = A^{\oplus I}$ be the subspace described in the first part of the proof, let $\varphi\colon X \to Y$ be an isomorphism and $\psi\colon Y \to X$ its inverse. Viewing φ as a map from X to W, there exists a unique extension $\Phi\colon U \to W$ by the assumption, and similarly there exists a unique extension $\Psi\colon W \to U$ of ψ. Then $\Phi\Psi\colon W \to W$ is a homomorphism extending $\varphi\psi = \mathrm{id}_Y$ on Y. Since the identity on W also satisfies this property, we have $\Phi\Psi = \mathrm{id}_W$ by the uniqueness property of Y. Similarly $\Psi\Phi = \mathrm{id}_U$, which shows that $W \cong U$.

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & Y \arrow[l, bend left, "\psi"] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, bend left, "\Phi"] & W \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

□

Projective modules generalize the notion of a free module. We present one definition first and will introduce two equivalent characterizations afterwards.

Definition 3.3. An A-module U is *projective* if there exists some A-module V such that $U \oplus V$ is free.

An immediate consequence of the definition is that free modules are projective. Also, direct sums of projective modules and direct summands of projective modules are projective.

Example 3.4. Recalling the direct sum $A = P_1 \oplus P_2$ from Example 2.2, P_1 and P_2 are projective A-modules. In fact, we will see that these are the only indecomposable projective A-modules (up to isomorphism) from Theorem 6.2.

The following are two additional equivalent definitions for a projective module that will be useful to work with for the remainder of the paper.

Theorem 3.5. *The following definitions of a projective A-module U are equivalent.*

(1) *U is a direct summand of a free module.*
(2) *Every surjective homomorphism $\varphi\colon V \twoheadrightarrow U$ splits; that is, $\ker\varphi$ is a direct summand of V.*
(3) *Given a surjective homomorphism $\varphi\colon V \twoheadrightarrow W$ and a homomorphism $\psi\colon U \to W$, there exists a homomorphism $\rho\colon U \to V$ such that $\varphi\rho = \psi$.*

The corresponding commutative diagram for property (3) is:

$$\begin{tikzcd} & U \arrow[dl, dashed, "\rho"'] \arrow[d, "\psi"] \\ V \arrow[r, two heads, "\varphi"] & W \end{tikzcd}$$

Before proving Theorem 3.5, we first present an equivalent characterization of the splitting condition in property (2).

Lemma 3.6 (Splitting lemma)**.** *A surjective homomorphism* $\varphi\colon V \twoheadrightarrow U$ *splits if and only if there exists a homomorphism* $\psi\colon U \to V$ *such that* $\varphi\psi = \mathrm{id}_U$*.*

Proof. Suppose φ splits, so that $V = K \oplus L$ where K denotes $\ker\varphi$. Since φ is surjective, L must be mapped onto U, as $\varphi(K) = 0$. Also, φ is injective on L, since $K \cap L = 0$. Thus, there exists a homomorphism $\psi\colon U \to V$ that is the inverse of $\varphi|_L$, i.e. $\varphi\psi = \mathrm{id}_U$.

Conversely, if there exists a homomorphism $\psi\colon U \to V$ such that $\varphi\psi = \mathrm{id}_U$, we claim that $V = K \oplus \psi(U)$, so φ splits. We have $K \cap \psi(U) = 0$ because if $\psi(u) \in K$ then $0 = \varphi(\psi(u)) = u$. Also, every $v \in V$ can be expressed as $v = (v - \psi\varphi(v)) + \psi\varphi(v)$ for $v - \psi\varphi(v) \in K$ and $\psi\varphi(v) \in \psi(U)$. □

Proof of Theorem 3.5. We first show (1) implies (3). Suppose U is a direct summand of a free module F, say $F = U \oplus U'$. We wish to use the characterizing property of free modules, so let us extend $\psi\colon U \to W$ to a homomorphism $\tilde{\psi}\colon F \to W$ that is 0 on U'. Let $X \subset F$ be a subspace as described in Proposition 3.2; since X is just a vector space and φ is surjective, we may define a linear transformation $\lambda\colon X \to V$ such that $\varphi\lambda = \tilde{\psi}$ on X (e.g. we may define λ on a basis of X and extend linearly to all of X). Then let $\tilde{\rho}\colon F \to V$ be the unique extension of λ to a homomorphism. Since $\varphi\tilde{\rho}$ and $\tilde{\psi}$ are homomorphisms from F to W that agree on X, we have $\varphi\tilde{\rho} = \tilde{\psi}$ by uniqueness. Letting $\rho\colon U \to V$ be the restriction of $\tilde{\rho}$ to U, we obtain $\varphi\rho = \psi$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\lambda"'] & F \arrow[d, "\tilde{\psi}"] \arrow[dl, dashed, "\tilde{\rho}"'] \\ V \arrow[r, two heads, "\varphi"] & W \end{tikzcd} \qquad \begin{tikzcd} & U \arrow[d, "\psi"] \arrow[dl, dashed, "\rho"'] \\ V \arrow[r, two heads, "\varphi"] & W \end{tikzcd}$$

Next we show (3) implies (2). If $\varphi\colon V \twoheadrightarrow U$ is a surjective homomorphism, (3) applied to the diagram

$$\begin{tikzcd} & U \arrow[d, "\mathrm{id}_U"] \arrow[dl, dashed, "\psi"'] \\ V \arrow[r, two heads, "\varphi"] & U \end{tikzcd}$$

implies there exists a homomorphism $\psi\colon U \to V$ such that $\varphi\psi = \mathrm{id}_U$. Then φ splits by Lemma 3.6.

Finally, we show (2) implies (1). The surjective homomorphism $\varphi\colon \bigoplus_{u\in U} A \twoheadrightarrow U$ splits by (2), so $\bigoplus_{u\in U} A \cong \ker\varphi \oplus U$. Consequently, U is isomorphic to a direct summand of a free module. □

4. Indecomposable Projective Modules

We present a few properties of indecomposable projective modules that will be used to prove the main theorem in Section 5. The key result of this section is Lemma 4.4, which states that an indecomposable projective module has exactly one maximal submodule, and the first three lemmas in this section build towards it.

Recall that an *endomorphism* φ of an A-module U is a homomorphism $\varphi\colon U \to U$. An endomorphism φ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $\varphi^n = 0$. Also, an A-module U is *finitely generated* if there exists a finite generating set $u_1, \ldots, u_n \in U$

(not necessarily a basis) such that every $u \in U$ can be written as a linear combination $u = a_1u_1 + \cdots + a_nu_n$ for $a_i \in A$.

Lemma 4.1 (Fitting's Lemma)**.** *Every endomorphism of a finitely-generated indecomposable A-module U is either nilpotent or invertible.*

Proof. Let φ be an endomorphism of U. We first show that there exists a sufficiently large $n \in \mathbb{N}$ such that $U = \ker(\varphi^n) \oplus \operatorname{im}(\varphi^n)$. Since U is finitely generated, the chain of submodules $\ker(\varphi^1) \subset \ker(\varphi^2) \subset \cdots$ must stabilize at some $\ker(\varphi^n)$. If $u \in \ker(\varphi^n) \cap \operatorname{im}(\varphi^n)$, then there exists some $v \in U$ such that $\varphi^n(v) = u$ and $\varphi^{2n}(v) = 0$; then $v \in \ker(\varphi^{2n}) = \ker(\varphi^n)$, which implies $u = 0$. Combined with the fact that $\dim \ker(\varphi^n) + \dim \operatorname{im}(\varphi^n) = \dim U$ shows that $U = \ker(\varphi^n) \oplus \operatorname{im}(\varphi^n)$.

Since U is indecomposable, $\ker(\varphi^n)$ equals 0 or U. If $\ker(\varphi^n) = U$ then φ is nilpotent. If $\ker(\varphi^n) = 0$, then φ must be injective; since U is finitely generated, this implies φ is invertible. □

The next two lemmas we state without proof but describe their usage (see Sections 4 and 5 of [2] for proofs).

Lemma 4.2. *Every nonzero projective module has a maximal submodule.*

This lemma is necessary, because in order to show that an indecomposable projective module has exactly one maximal submodule, we need to know that it has at least one maximal submodule. A general module does not necessarily have a maximal submodule, and Zorn's lemma can not be used to prove the result as the union of a chain of proper submodules may not be proper. The proof in [2] uses additional properties of the *radical* (which will be introduced in Definition 5.1).

Lemma 4.3. *Every indecomposable projective module is cyclic.*

In order to apply Lemma 4.1 to indecomposable projective modules, we need them to be finitely generated. In fact, the above stronger statement can be made, since cyclic modules are generated by only one element. The proof involves showing that every simple module is a quotient of a cyclic projective indecomposable module, and applying the uniqueness statement from the last part of the proof of Theorem 5.2 then shows that any projective indecomposable module is isomorphic to a cyclic one.

Now we prove the following lemma, which will give us half of the proof of Theorem 5.2.

Lemma 4.4. *An indecomposable projective module P has exactly one maximal submodule.*

Proof. By Lemma 4.2, P has a maximal submodule. Now suppose M and M' are maximal submodules of P, and consider the following inclusions and projections.

$$\begin{tikzcd}
M \arrow[dr, "\iota"] & & P/M \\
& P \arrow[ur, two heads, "\pi"] \arrow[dr, two heads, "\pi'"] & \\
M' \arrow[ur, "\iota'"] & & P/M'
\end{tikzcd}$$

Since P/M' is simple, the image $\operatorname{im}(\pi'\iota)$ equals 0 or P/M'. We claim $\operatorname{im}(\pi'\iota) = 0$, which would imply $M \subset M'$ and thus $M = M'$ because M is maximal. If $\operatorname{im}(\pi'\iota) = P/M'$, there exists a homomorphism ψ by property (3) of Theorem 3.5 such that the following diagram commutes.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\psi"'] \arrow[dr, two heads, "\pi'"] & \\
M \arrow[r, "\iota"] & P \arrow[r, two heads, "\pi'"] & P/M'
\end{tikzcd}$$

As $\iota\psi$ is an endomorphism of P, it must be nilpotent or invertible by Lemma 4.1. If it is nilpotent, then the fact that $\pi'(\iota\psi)^n = \pi'$ for all $n \in \mathbb{N}$ implies $\pi' = 0$ (i.e. $M' = P$), and if it is invertible, then $M = P$, both of which are contradictions. ☐

5. Projective Module Correspondence

To understand the full statement of the bijective correspondence in Theorem 5.2, we need one more definition.

Definition 5.1. The *radical* of an A-module U, denoted $\operatorname{rad}(U)$, is the intersection of the maximal submodules of U.

The radical has other characterizations, such as being the smallest submodule of $V \subset U$ such that U/V is *semisimple* (a direct sum of simple modules). At a high level, the radical helps describe the structure of a module and contains the elements which “prevent the module from being semisimple.” In the case of an indecomposable projective module P, which has exactly one maximal submodule (Lemma 4.4), $\operatorname{rad}(P)$ is equal to this maximal submodule.

Theorem 5.2. *There is a one-to-one correspondence between indecomposable projective A-modules and simple A-modules, given by $P \leftrightarrow P/\operatorname{rad}(P)$.*

This bijection is saying that

(1) For P an indecomposable projective A-module, $P/\operatorname{rad}(P)$ is a simple A-module.
(2) Every simple A-module is of the form $P/\operatorname{rad}(P)$ for some indecomposable projective A-module P.
(3) If $P/\operatorname{rad}(P) \cong Q/\operatorname{rad}(Q)$ for indecomposable projective A-modules P and Q, then $P \cong Q$.

Proof of Theorem 5.2. By Lemma 4.4, an indecomposable projective module P has exactly one maximal submodule, namely $\operatorname{rad}(P)$. Thus, $P/\operatorname{rad}(P)$ is simple, which shows (1).

To show (2), let S be a simple A-module, so there exists a surjective homomorphism $\varphi\colon A \twoheadrightarrow S$ (given by e.g. $a \mapsto as$ for a nonzero $s \in S$, since S is cyclic). Then there exists an indecomposable (projective) direct summand P of A for which the restriction homomorphism $\varphi|_P\colon P \to S$ is nonzero. Since S is simple, $\varphi|_P\colon P \twoheadrightarrow S$ must in fact be surjective, so S is isomorphic to a quotient of P. Since P has only one maximal submodule (Lemma 4.4), namely $\operatorname{rad}(P)$, we have $S \cong P/\operatorname{rad}(P)$.

To show (3), suppose P and Q are indecomposable projective A-modules such that $P/\operatorname{rad}(P) \cong Q/\operatorname{rad}(Q)$. Consider the following commutative diagram, where the vertical maps are the natural projections and the bottom map is an isomorphism. By property (3) of Theorem 3.5, there exists a homomorphism from $\varphi\colon P \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d] & Q \arrow[d] \\ P/\operatorname{rad}(P) \arrow[r, "\sim"] & Q/\operatorname{rad}(Q) \end{tikzcd}$$

The image $\varphi(P)$ in Q can not be contained in $\operatorname{rad}(Q)$, as otherwise the composition $P \to Q \to Q/\operatorname{rad}(Q)$ would be zero, while the other composition is not zero. Since $\operatorname{rad}(Q)$ is maximal in Q, this implies $\varphi\colon P \twoheadrightarrow Q$ is surjective. Then as Q is projective, Q must be isomorphic to a direct summand of P by property (2) of Theorem 3.5. Since P is indecomposable, this implies $P \cong Q$. $\square$

Example 5.3. In our running example of $A = P_1 \oplus P_2$ where

$$\begin{aligned} A &= \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in k\} \\ P_1 &= \{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid a \in k\} \\ P_2 &= \{\left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix}\right) \mid b, c \in k\}, \end{aligned}$$

the unique maximal submodule of P_1 is 0, and the unique maximal submodule of P_2 is $M = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right) \mid c \in k\}$. The correspondence in Theorem 5.2 shows that P_1 and P_2/M are simple A-modules. In fact, we will see that they are the only simple A-modules (up to isomorphism) from Theorem 6.2.

6. Characterizing Indecomposable Projective Modules

In this final section, we state a key result for indecomposable modules, namely the Krull-Schmidt theorem on the uniqueness of decomposition (see Chapter 4 of [1] for a proof). This theorem can be used to help further characterize indecomposable projective modules and thus simple modules as well.

Theorem 6.1 (Krull-Schmidt)**.** *Every finitely-generated A-module is isomorphic to a finite direct sum $M_1 \oplus \cdots \oplus M_n$ of indecomposable A-modules, unique up to reordering and isomorphism.*

In particular for the A-module A, we may write $A \cong P_1 \oplus \cdots \oplus P_n$ for indecomposable A-modules P_i, which are also projective by definition. The Krull-Schmidt theorem states that the indecomposable projective A-modules P_i are determined uniquely up to reordering and isomorphism.

Conversely, we also have the following.

Theorem 6.2. *Every indecomposable projective A-module is isomorphic to P_i for some $1 \leq i \leq n$.*

Proof. Since every indecomposable projective A-module P is cyclic by Lemma 4.3, there exists a surjective homomorphism $\varphi\colon A \twoheadrightarrow P$ (e.g. the map $a \mapsto au$ for a generator $u \in P$). Then φ splits by property (2) of Theorem 3.5, and $A \cong P \oplus \ker\varphi$. Separating $\ker\varphi$ into its indecomposable modules and applying the uniqueness part of the Krull-Schmidt theorem shows that P is isomorphic to some P_i. $\square$

Another consequence of this theorem is that there are only finitely many isomorphism classes of projective A-modules, and only finitely many isomorphism classes of simple A-modules.

Example 6.3. Since $A = P_1 \oplus P_2$ in the context of Example 5.3, P_1 and P_2 are the only indecomposable projective A-modules up to isomorphism. Consequently, the correspondence in Theorem 5.2 shows that P_1 and P_2/M are the only simple A-modules up to isomorphism.

References

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