Rank varieties for modules over quantum complete intersections

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Abstract

We provide an overview of the theory of group cohomology, support varieties, and rank varieties, before discussing a generalization to the setting of quantum complete intersections $A_q^c = k\langle x_1, \ldots, x_c\rangle/(\{x_i^a\}, \{x_i x_j - q x_j x_i\}_{i<j})$. We compute rank varieties of low-dimensional indecomposable A_q^c-modules and present progress towards classifying all rank varieties in the $c = 2$ case.

Contents

1 Introduction

Group cohomology is a useful tool for studying groups which exhibits connections to algebra and topology. The object of interest is the graded commutative ring $H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$. We define support varieties V_G by taking the maximal ideal spectrum and certain subvarieties $V_G(M) \subset V_G$ corresponding to each kG-module M. Rank varieties, introduced by Carlson [Car83], can be defined when G is an elementary abelian group; these turn out to be isomorphic to support varieties but are easier to compute explicitly. This setting with kG-modules is well understood.

We now consider generalizations of support and rank varieties for quantum complete intersections, which are special k-algebras of the form

$$A_q^c = k\langle x_1, \dots, x_c\rangle / (\{x_i^a\}, \{x_i x_j - q x_j x_i\}_{i<j}).$$

Quantum complete intersections are a generalization of exterior algebras E, which appear naturally via Koszul duality, as $\operatorname{Ext}_S^*(k,k) \cong E$ for a symmetric algebra S. Rank varieties for A_q^c-modules were defined by Bergh and Erdmann [BE09], and they also been defined for modules over truncated polynomial algebras $k[x_1, \dots, x_c]/(\{x_i^a\})$ [EH06, PW09].

Our goal is to summarize the main points on this topic and provide additional examples which do not appear in the present literature. This background reading also naturally led to the work and questions in later sections.

We define group cohomology in Section 2 and give an overview of the fundamental techniques for group cohomology computations in Section 3. We then define support varieties in Section 4 and rank varieties in Section 5, before introducing quantum complete intersections and rank varieties for them Section 6. In Section 7, we compute the rank varieties of all indecomposable modules in [YZ19]. In Section 8, we pose a question about rank varieties and our partial progress, and Section 9 contains conjectures and questions for future study.

2 Group cohomology

We present an algebraic definition and a topological definition of group cohomology before proving that they are equivalent.

2.1 Algebraic definition

Let G be a finite group, and let k be a field with characteristic dividing $|G|$. The *group algebra kG* is the k-vector space $\bigoplus_{g\in G} kg$ with multiplication induced by G.

Definition 2.1. The *cohomology* of G is the graded k-vector space

$$H^*(G,k) = \operatorname{Ext}^*_{kG}(k,k).$$

It turns out that $H^*(G,k)$ is a graded commutative k-algebra, meaning that $rs = (-1)^{|r||s|} sr$ for all $r, s \in H^*(G,k)$ [Ben91a, Section 3.2]. The ring structure is derived from the Yoneda product.

Definition 2.2. The *Yoneda product* is the map

$$\operatorname{Ext}^i(B,C) \otimes \operatorname{Ext}^j(A,B) \to \operatorname{Ext}^{i+j}(A,C)$$

for modules A, B, C defined as follows. Given an element of $\operatorname{Ext}^i(B,C)$ represented by the extension

$$0 \to C \to E_1 \to \cdots \to E_i \to B \to 0$$

and an element of $\operatorname{Ext}^j(A,B)$ represented by

$$0 \to B \to F_1 \to \cdots \to F_j \to A \to 0,$$

we compose to obtain the extension

$$0 \to C \to E_1 \cdots \to E_i \to F_1 \to \cdots \to F_j \to A \to 0$$

which represents an element of $\mathrm{Ext}^{i+j}(A, C)$.

Letting $A = B = C = k$, this defines a product structure on $H^*(G, k) = \mathrm{Ext}^*(k, k)$. It turns out that the cup product for group cohomology $H^i(G, A) \otimes_k H^j(G, B) \to H^{i+j}(G, A \otimes_k B)$ with $A = B = k$ coincides with the Yoneda product in this case [BIK12, Section 2.3.1].

Example 2.3. Let $G = \mathbb{Z}/2$ and $\operatorname{char} k = 2$. We will compute that

$$H^*(G, k) \cong k[y] \text{ where } |y| = 1.$$

Letting $G = \{1, g\}$, we have $kG = k1 \oplus kg \cong k[x]/(x^2)$ by the isomorphism $g - 1 \mapsto x$. Then an infinite free resolution of k is given by

$$\cdots \to kG \xrightarrow{\cdot x} kG \xrightarrow{\cdot x} kG \xrightarrow{\cdot x} k \to 0.$$

We now take $\mathrm{Hom}_{kG}(kG, k) \cong k$ and obtain the cochain complex

$$0 \to k \xrightarrow{\cdot 0} k \xrightarrow{\cdot 0} k \xrightarrow{\cdot 0} \cdots$$

where x acts by $\cdot 0$ because g acts by $\cdot 1$ in the trivial representation. Then the cohomology of the cochain complex is

$$\mathrm{Ext}^*_R(k, k) = \bigoplus_{n \geq 0} k.$$

The following is a variation of Maschke's theorem which relates the structure of kG with the condition $\operatorname{char} k \mid |G|$ stated at the beginning of this section.

Theorem 2.4 ([BIK12, Theorem 1.33])**.** The following are equivalent:

1. kG is a semisimple ring.
2. The trivial kG-module k is projective.
3. $\operatorname{char} k \nmid |G|$.

Condition 2 implies that a projective resolution of k is simply $0 \to k \to k \to 0$, and we obtain the following corollary.

Corollary 2.5. If $\operatorname{char} k \nmid |G|$, then $H^0(G, k) \cong k$ and $H^i(G, k) \cong 0$ for $i \geq 1$.

Thus, to obtain interesting group cohomology, we include the condition $\operatorname{char} k \mid |G|$.

2.2 Topological definition

There is an equivalent topological definition of group cohomology using Eilenberg-Mac Lane spaces $K(G, 1)$ or the principal G-bundle $EG \to BG$ [Ben91b]. We use $K(G, 1)$ and BG interchangably.

Definition 2.6. The *cohomology* of G is the graded k-algebra

$$H^*(G, k) = H^*(K(G, 1); k) = H^*(BG; k).$$

Example 2.7. We revisit Example 2.3 from a topological perspective. Let $G = \mathbb{Z}/2$ and $\operatorname{char} k = 2$. Since $K(\mathbb{Z}/2, 1) \simeq \mathbb{RP}^\infty$, we have

$$H^*(\mathbb{Z}/2, k) = H^*(\mathbb{RP}^\infty; k) \cong k[x].$$

Theorem 2.8 ([Ben91b, Theorem 2.2.3])**.** Definition 2.1 and Definition 2.6 are equivalent; that is,

$$\operatorname{Ext}^*_{kG}(k, k) = H^*(K(G, 1); k).$$

Proof. Let $X = K(G, 1)$, and let $\tilde{X}$ be its contractible universal cover. As G acts freely on $\tilde{X}$ by permuting cells, we have $C_i(\tilde{X}; k)/G \cong C_i(X; k)$, or equivalently

$$C_i(\tilde{X}; k) \otimes_{kG} k \cong C_i(X; k).$$

A free resolution of k as an kG-module is given by $\cdots \to C_1(\tilde{X}; k) \to C_0(\tilde{X}; k) \to k \to 0$, so

$$\begin{aligned}
\operatorname{Ext}^i_{kG}(k, k) &= H^i(\operatorname{Hom}_{kG}(C_i(\tilde{X}; k), k)) \\
&\cong H^i(\operatorname{Hom}_k(C_i(X; k), k)) \\
&\cong H^i(C^i(X; k)) \\
&\cong H^i(X; k).
\end{aligned}$$

The second line above follows from the tensor-hom adjunction:

$$\begin{aligned}
\operatorname{Hom}_k(C_i(X; k), k) &\cong \operatorname{Hom}_k(C_i(\tilde{X}; k) \otimes_{kG} k, k) \\
&\cong \operatorname{Hom}_{kG}(C_i(\tilde{X}; k), \operatorname{Hom}_k(k, k)) \\
&\cong \operatorname{Hom}_{kG}(C_i(\tilde{X}; k), k).
\end{aligned}$$

□

3 Group cohomology computations

3.1 Low-dimensional group cohomology

We compute $H^i(G, k)$ for $i = 0, 1$ and all groups G and fields k.

Proposition 3.1. $H^0(G, k) \cong k$.

Proof. By definition (or the left-exactness of Hom),

$$\operatorname{Ext}^0_{kG}(k, k) \cong \operatorname{Hom}_{kG}(k, k) \cong k,$$

since G acts trivially on k.

Alternatively, since $K(G, 1)$ is connected, we have $H^0(K(G, 1); k) \cong k$. □

Proposition 3.2. $H^1(G, k) \cong \operatorname{Hom}_{\mathrm{gp}}(G, k)$.

Proof. Since $\pi_1(K(G, 1)) = G$, we have $H_1(BG; \mathbb{Z}) = G^{\mathrm{ab}}$. Then by the universal coefficient theorem and the universal property of the functor $(-)^{\mathrm{ab}}$,

$$\begin{aligned}
H^1(K(G, 1); k) &\cong \operatorname{Hom}_{\mathbb{Z}}(H_1(K(G, 1); \mathbb{Z}), k) \\
&\cong \operatorname{Hom}_{\mathbb{Z}}(G^{\mathrm{ab}}, k) \\
&\cong \operatorname{Hom}_{\mathrm{gp}}(G, k).
\end{aligned}$$

□

There is another proof of Proposition 3.2 using the cochain complex definition of group cohomology.

Definition 3.3. Let $C^n(G, M)$ be the group of all functions from $G^n \to M$, where $C^0(G, M)$ consists of the constant functions. Define $d^{n+1}\colon C^n(G, M) \to C^{n+1}(G, M)$ by defining $(d^{n+1}\varphi)(g_1, \dots, g_{n+1})$ as

$$g_1\varphi(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(g_1, \dots, g_{i-1}, g_ig_{i+1}, \dots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \dots, g_n).$$

Then $H^n(G, M)$ is the nth cohomology of this complex.

In particular, $(d^2\varphi)(g_1, g_2) = g_1\varphi(g_2) - \varphi(g_1g_2) + \varphi(g_1)$, and $(d^1c_m)(g) = gm - m$, so

$$\begin{aligned} \ker d^2 &= \{\varphi\colon G \to M \mid \varphi(g_1g_2) = g_1\varphi(g_2) + \varphi(g_1)\} \\ \operatorname{im} d^1 &= \{\varphi\colon G \to M \mid \varphi(g) = gm - m \text{ for some fixed } m \in M\}. \end{aligned}$$

The action of G on $M = k$ is trivial, so $\ker d^2$ consists of group homomorphisms $\varphi\colon G \to k$, and $\operatorname{im} d^1 = \{0\}$. Then

$$\begin{aligned} H^1(G, k) &= \ker d^2 / \operatorname{im} d^1 \\ &\cong \operatorname{Hom}_{\mathrm{gp}}(G, k). \end{aligned}$$

3.2 Elementary abelian groups

Elementary abelian groups play a special role in group cohomology, which we will see from the Quillen stratification theorem in Section 4.1 and the introduction of rank varieties in Section 5.

Definition 3.4. The *elementary abelian* p-group of rank r is $(\mathbb{Z}/p)^r$.

We can compute their cohomology using the Künneth theorem.

Theorem 3.5 (Künneth)**.** *For finite abelian groups G_1, G_2, there is an isomorphism of graded k-algebras*

$$H^*(G_1 \times G_2, k) \cong H^*(G_1, k) \otimes_k H^*(G_2, k).$$

Example 3.6. Let $G = (\mathbb{Z}/2)^r$ and $\operatorname{char} k = 2$. After computing that $H^*(\mathbb{Z}/2, k) \cong k[x]$ in Example 2.3, the Künneth theorem implies that

$$H^*(G, k) \cong k[x_1, \dots, x_r].$$

The cohomology of elementary abelian groups for $\operatorname{char} k > 2$ is also well known. In the following theorem, β denotes the Bockstein homomorphism.

Theorem 3.7 ([Ben91a, Corollary 3.5.7])**.** *Let $G = (\mathbb{Z}/p)^r$ and $\operatorname{char} k = p$.*

1. *For $p = 2$, $H^*(G, k) \cong k[x_1, \dots, x_r]$ for $|x_i| = 1$.*

2. *For $p > 2$, $H^*(G, k) \cong \bigwedge(x_1, \dots, x_r) \otimes_k k[y_1, \dots, y_r]$ for $|x_i| = 1$, $|y_i| = 2$, and $\beta(x_i) = y_i$.*

3.3 Periodic group cohomology

There is a technique to splice together chain complexes to form an infinite projective resolution to obtain group cohomology that is periodic. We demonstrate this with an example.

Example 3.8. The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ acts freely on

$$S^3 \cong \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}.$$

For example, if $a + bi + cj + dk = i \cdot (a + bi + cj + dk) = -b + ai - dj + ck$, then $a = b = c = d = 0$. Since S^3 is 3-dimensional, it has a CW chain complex of the form

$$0 \to \mathbb{Z} \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

where Q_8 acts freely on the cells–this is because a CW structure on S^3/Q_8 can be lifted to CW structure on S^3 for which Q_8 acts freely on the cells. We can now form the infinite splice

$$\cdots \to C_1 \to C_0 \to C_3 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

which shows that $H^*(Q_8, k)$ is periodic with period dividing 4.

This situation can be generalized to obtain the following theorem.

Theorem 3.9 ([BIK12, Theorem 2.27])**.** If G acts freely on S^{n-1}, then $H^*(G, k)$ is periodic with period dividing n.

3.4 Lyndon–Hochschild–Serre spectral sequence

The Lyndon–Hochschild–Serre spectral sequence allows us to compute more examples of group cohomology. We provide background on spectral sequences from a double complex, and sketch the construction of the Lyndon–Hochschild–Serre spectral sequence, before computing $H^*(S_3, k)$ as an example.

Definition 3.10. A *double complex* consists of abelian groups E_0^{ij} and maps d_0, d_1 as shown below such that $d_0 \circ d_0 = 0$, $d_1 \circ d_1 = 0$, and $d_0 \circ d_1 + d_1 \circ d_0 = 0$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \\
E_0^{02} & \xrightarrow{d_1} & E_0^{12} & \xrightarrow{d_1} & E_0^{22} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \\
E_0^{01} & \xrightarrow{d_1} & E_0^{11} & \xrightarrow{d_1} & E_0^{21} & \longrightarrow & \cdots \\
\uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle d_0} & & \\
E_0^{00} & \xrightarrow{d_1} & E_0^{10} & \xrightarrow{d_1} & E_0^{20} & \longrightarrow & \cdots
\end{array}$$

Definition 3.11. The *total complex* $\mathrm{Tot}(\mathbf{E})$ has graded components $\mathrm{Tot}^n(\mathbf{E}) = \bigoplus_{i+j=n} E_0^{ij}$ with differential $d := d_0 + d_1 \colon \mathrm{Tot}^n(\mathbf{E}) \to \mathrm{Tot}^{n+1}(\mathbf{E})$.

In a spectral sequence, one takes the cohomology of the ith page E_i^{**} with respect to some map d_i to obtain the successive $(i+1)$th page E_{i+1}^{**}. The abelian groups $\{E_i^{pq}\}_{i \geq 0}$ eventually stablize to something denoted by E_∞^{pq}.

Theorem 3.12 ([Ben91b, Theorem 3.4.2]). Given a double complex (E_0^{pq}, d_0, d_1), there is a spectral sequence with

$$\begin{aligned} E_1^{pq} &= H(E_0^{pq}, d_0) \\ E_2^{pq} &= H^p(H^q(\mathbf{E}, d_0), d_1) \\ E_\infty^{pq} &= F^p H^{p+q}(\mathrm{Tot}(\mathbf{E}))/F^{p+1} H^{p+q}(\mathrm{Tot}(\mathbf{E})). \end{aligned}$$

The shorthand notation for this theorem is

$$H^p(H^q(\mathbf{E}, d_0), d_1) \implies H^{p+q}(\mathrm{Tot}(\mathbf{E}), d_0 + d_1).$$

Next, we need a generalization of group cohomology with coefficients in the kG-module k to any kG-module M. For a kG-module M, let M^G denote the *invariant submodule*

$$M^G = \{m \in M \mid m = gm\} = \mathrm{Hom}_{kG}(k, M).$$

Define the *cohomology of G with coefficients in M* by $H^*(G, M) = \mathrm{Ext}^*_{kG}(k, M)$.

We now sketch the construction of the Lyndon–Hochschild–Serre spectral sequence which uses a specific choice of E_0^{pq}. Let G be a group and $N \lhd G$ be a normal subgroup. We take projective resolutions of k as an $k(G/N)$-module

$$\cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} k \to 0$$

and as an kG-module

$$\cdots \to Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\partial'_0} k \to 0.$$

Letting M be a kG-module, we define

$$E_0^{pq} = \mathrm{Hom}_{k(G/N)}(P_p, \mathrm{Hom}_{kN}(Q_q, M))$$

with differentials $d_0 = (-1)^p(\partial'_{q+1})^*$, $d_1 = (\partial_p)^*$ induced from ∂'_{q+1} and ∂_p respectively. By taking cohomology and chasing definitions, we have

$$\begin{aligned} E_1^{pq} &= \mathrm{Hom}_{k(G/N)}(P_p, H^q(N, M)) \\ E_2^{pq} &= H^p(G/N, H^q(N, M)). \end{aligned}$$

After proving that

$$\mathrm{Hom}_{k(G/N)}(P_p, \mathrm{Hom}_{kN}(Q_q, M)) \cong \mathrm{Hom}_{kG}(P_p \otimes_k Q_q, M),$$

such as in [Ben91b, Lemma 3.5.1], we can identify

$$E_\infty^{pq} \cong H^{p+q}(G, M).$$

Theorem 3.13 (Lyndon–Hochschild–Serre spectral sequence). There is a spectral sequence

$$H^p(G/N, H^q(N, M)) \implies H^{p+q}(G, M).$$

There are two cases in which the Lyndon–Hochschild–Serre spectral sequence collapses; that is, $E_2^{pq} \cong E_\infty^{pq}$.

Corollary 3.14 ([Iye04]). Let G be a finite group and $N \lhd G$ be a normal subgroup.

1. If $\operatorname{char} k \nmid [G:N]$, then
$$H^*(G,k) \cong H^*(N,k)^{G/N}.$$

2. If $\operatorname{char} k \nmid |N|$, then
$$H^*(G,k) \cong H^*(G/N,k).$$

Proof. Note that $H^0(G,M) = M^G$ by definition.

1. When $\operatorname{char} k \nmid [G:N]$, then $H^p(G/N,-) = 0$ for $p \geq 1$ by Corollary 2.5, so the spectral sequence collapses. Setting $M = k$, we have
$$H^q(G,k) \cong H^0(G/N, H^q(N,k)) \cong H^q(N,k)^{G/N}.$$

2. When $\operatorname{char} k \nmid |N|$ then $H^q(N,-) = 0$ for $q \geq 1$. Setting $M = k$, we have
$$H^p(G,k) \cong H^p(G/N, H^0(N,k)) \cong H^p(G/N,k).$$ □

Example 3.15 ([Iye04])**.** We calculate $H^*(S_3,k)$ for all fields k.

Case 1: Let $\operatorname{char} k \neq 2,3$. Then by Corollary 2.5, $H^0(S_3,k) \cong k$ and $H^i(S_3,k) \cong 0$ for $i \geq 1$.

Case 2: Let $\operatorname{char} k = 2$. Since $\operatorname{char} k \nmid |N|$ for the normal subgroup $N = \{1, b, b^2\} \lhd S_3$, Corollary 3.14 yields $H^*(S_3,k) \cong H^*(\mathbb{Z}/2,k) \cong k[x]$ with $|x| = 1$.

Case 3: Let $\operatorname{char} k = 3$. Since $\operatorname{char} k \nmid [S_3 : N]$, we have by Corollary 3.14 and Theorem 3.7 that $H^*(S_3,k) \cong H^*(N,k)^{\mathbb{Z}/2} \cong (\bigwedge(e_1) \otimes_k k[e_2])^{\mathbb{Z}/2}$. Letting y denote the generator of $\mathbb{Z}/2$, we claim that
$$y(e_1) = -e_1, \; y(e_2) = -e_2.$$
We have $H^1(N,k) \cong ke_1 \cong \operatorname{Hom}_{\mathrm{gp}}(N,k)$, where e_1 corresponds to $\varphi \in \operatorname{Hom}_{\mathrm{gp}}(N,k)$ with $\varphi(b) = 1$. The action of y on N is $y(b) = b^2$, so $(y\varphi)(b) = \varphi(b^2) = 2 = -1$ which implies $y(e_1) = -e_1$. To obtain $y(e_2) = -e_2$, we need the fact that the action of y is compatible with the Bockstein homomorphism β. Then
$$H^*(S_3,k) \cong \left(\bigwedge(e_1) \otimes_k k[e_2]\right)^{\mathbb{Z}/2} \cong \bigwedge(e_1e_2) \otimes_k k[e_2^2].$$
In the first few degrees, this ring has the following generators.

degree	0	1	2	3	4	5	6	7	8
generator	1	0	0	e_1e_2	e_2^2	0	0	$e_1e_2^3$	e_2^4

Note that $H^i(S_3,k) \cong k$ for $i \equiv 0,3 \pmod 4$, and $H^i(S_3,k) \cong 0$ otherwise.

4 Support varieties

4.1 Basic properties

Recall that $H^*(G,k)$ is a graded commutative ring: $rs = (-1)^{|r||s|}sr$ for all $r,s \in H^*(G,k)$. If $\operatorname{char} k = 2$, then $H^*(G,k)$ is commutative, and for $\operatorname{char} k \neq 2$, the subring of even degree elements $H^{2*}(G,k)$ is commutative. This motivates the following definition.

Definition 4.1. Let

$$H^{\cdot}(G,k) = \begin{cases} H^*(G,k) & \text{if } \operatorname{char} k = 2 \\ H^{2*}(G,k) & \text{else} \end{cases}$$

which is a finitely generated graded commutative ring over k.

Definition 4.2. The *support variety* V_G is the maximal ideal spectrum

$$V_G = \max H^{\cdot}(G,k).$$

The main theorem for support varieties is is the Quillen stratification theorem which divides a variety V_G into subvarieties based on the elementary abelian subgroups of G.

Definition 4.3. For a subgroup $H \le G$, there is an induced map $\operatorname{res}^*_{G,H} \colon V_H \to V_G$. For an elementary abelian subgroup $E \le G$, define the varieties

$$V_E^+ = V_E \setminus \bigcup_{E' < E} \operatorname{res}^*_{E,E'} V_{E'}$$
$$V_{G,E}^+ = \operatorname{res}^*_{G,E} V_E^+.$$

Theorem 4.4 (Quillen Stratification)**.** The variety V_G is a disjoint union of subvarieties

$$V_G = \bigsqcup_E V_{G,E}^+$$

where E ranges over representatives for the conjugacy classes of elementary abelian subgroups of G.

Example 4.5. For $G = S_3$ and $\operatorname{char} k = 2$, we have from Example 3.15 that $V_G = \operatorname{Spec} k[x]$. The elementary abelian subgroups of G are $\langle \mathrm{id} \rangle$, $\langle (12) \rangle$, $\langle (23) \rangle$, and $\langle (13) \rangle$ where the last three are conjugates. The variety V_E^+ is $\operatorname{Spec} k$ for $E = \langle \mathrm{id} \rangle$ and $\operatorname{Spec} k[x] \setminus \operatorname{Spec} k$ for $E = \langle (12) \rangle$.

For $G = S_3$ and $\operatorname{char} k = 3$, we have $V_G = \operatorname{Spec} k[e_2^2]$ using the notation from Example 3.15. The elementary abelian subgroups of G are $\langle \mathrm{id} \rangle$ and $\langle (123) \rangle$, and the respective varieties V_E^+ are $\operatorname{Spec} k$ and $\operatorname{Spec} k[e_2] \setminus \operatorname{Spec} k$.

4.2 Varieties for modules

Definition 4.6. To each kG-module M, we associate a subvariety $V_G(M) \subset V_G$ as follows. There is a natural map

$$H^{\cdot}(G,k) = \operatorname{Ext}^{\cdot}_{kG}(k,k) \xrightarrow{\otimes M} \operatorname{Ext}^*_{kG}(M,M)$$

given by mapping an extension $k \to \cdots \to k$ to the corresponding extension $k \otimes M \to \cdots \to k \otimes M$. Let the kernel of this map be $I_G(M)$. Then

$$V_G(M) := \max H^{\cdot}(G,k)/I_G(M) \subset V_G.$$

We state a few properties of the subvarieties $V_G(M)$.

Proposition 4.7 ([Ben91b, Section 5.7])**.** Let M and N be kG-modules.

1. $V_G(M) = \{0\}$ if and only if M is projective.
2. $V_G(M \oplus N) = V_G(M) \cup V_G(N)$.
3. $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.

Note that letting M be the trivial kG-module k yields $V_G(k) = V_G$. An analogue of the Quillen stratification theorem for modules was proven by Avrunin and Scott [AS82]. We define

$$V_E^+(M) = V_E^+ \cap V_E(M)$$
$$V_{G,E}^+(M) = \operatorname{res}_{G,E}^* V_E^+(M).$$

Theorem 4.8 ([AS82])**.** Let M be a finitely generated kG-module. The variety $V_G(M)$ is a disjoint union of subvarieties

$$V_G(M) = \bigsqcup_E V_{G,E}^+(M),$$

where E ranges over representatives for the conjugacy classes of elementary abelian subgroups of G.

Finally, we state two important theorems about $V_G(M)$ which will have quantum analogues in Section 6.

Theorem 4.9 ([Ben91b, Corollary 5.9.2])**.** For every closed homogeneous subvariety V of V_G, there exists a finitely generated kG-module M such that $V_G(M) = V$.

Theorem 4.10 ([Car84])**.** Let M be a finitely generated kG-module. If $V_G(M) = V_1 \cup V_2$ with $V_1 \cap V_2 = \{0\}$, then $M \cong M_1 \oplus M_2$ with $V_G(M_1) = V_1$ and $V_G(M_2) = V_2$.

5 Rank varieties

Let $\operatorname{char} k = p$ and $E = \langle g_1, \dots, g_c \rangle$ be an elementary abelian p-group of rank c. Let $x_i := g_i - 1$ so that $kE = k[x_1, \dots, x_c]/(x_1^p, \dots, x_c^p)$, and let $V_E^r = \operatorname{span}\{x_1, \dots, x_c\}$. For $\lambda = (\lambda_1, \dots, \lambda_c) \in k^c$, we define $u_\lambda \in V_E^r$ by

$$u_\lambda = \lambda_1 x_1 + \cdots + \lambda_c x_c.$$

For all $\lambda \neq 0$, $1 + u_\lambda$ is a unit of order p for all $\lambda \neq 0$, so $k\langle 1 + u_\lambda \rangle \cong k[x]/(x^p)$.

Definition 5.1. The *rank variety* $V_E^r(M)$ of a finitely generated kE-module M is

$$V_E^r(M) = \{0\} \cup \{\lambda \in k^c \mid M \text{ is not a projective } k\langle 1 + u_\lambda \rangle\text{-module}\}.$$

The following rank inequality yields a straightforward way to compute whether $\lambda \in V_E^r(M)$.

Theorem 5.2. For all $\lambda \in k^c$, we have

$$\operatorname{rank} u_\lambda \leq \tfrac{p-1}{p} \dim M,$$

where u_λ denotes the map $M \xrightarrow{\cdot u_\lambda} M$ that is multiplication by u_λ. Equality holds if and only if $\lambda \notin V_E^r(M)$.

To prove this, we use following two facts about modules over a principal ideal domain.

Lemma 5.3. A module M over a PID is projective if and only if it is free.

Lemma 5.4. For a module M over a PID R, there exists a unique sequence of ideals $(r_1) \supseteq \cdots \supseteq (r_n)$ such that $M \cong R/(r_1) \oplus \cdots \oplus R/(r_n)$.

Proof. In our case with $R = k[x]/(x^p)$, all ideals of R are of the form (x^i), so the possible summands of M are of the form $k[x]/(x^i)$. For a free R-module summand $M' = k[x]/(x^p)$, we have $\dim M' = p$ and $\dim xM' = p - 1$, so there is equality in Theorem 5.2. If M is not projective and contains a summand $k[x]/(x^i)$ for $i < p$, then we will have $\dim xM < \frac{p-1}{p} \dim M$. $\square$

Corollary 5.5. If $p \nmid \dim M$, then $V(M) = k^c$.

Proof. If $\frac{p-1}{p} \dim M$ is not an integer, then equality can not hold in Theorem 5.2. $\square$

The Avrunin–Scott theorem [AS82] states that the rank and support varieties of a kE-module M are isomorphic, which is useful as rank varieties are easier to compute.

Theorem 5.6 (Avrunin–Scott)**.** There is a natural isomorphism $V_E \cong V_E^r$ under which $V_E(M)$ is identified with $V_E^r(M)$.

There are some other properties analogous to Proposition 4.7 and Theorem 4.9.

Proposition 5.7 ([Ben91b, Section 5])**.** Let M and N be finitely generated kE-modules.

1. $V_E^r(M)$ is a closed homogeneous subvariety of k^c.
2. $V_E^r(M) = \{0\}$ if and only if M is projective.
3. $V_E^r(M \oplus N) = V_E^r(M) \cup V_E^r(N)$.
4. $V_E^r(M \otimes_k N) = V_E^r(M) \cap V_E^r(N)$.
5. For a closed homogeneous subvariety V of k^c, there exists a finitely generated kE-module M such that $V_E^r(M) = V$.

6 Quantum complete intersections

We define support and rank varieties for quantum complete intersections following [BE09], before presenting a quantum version of the Avrunin–Scott theorem which relates the two.

In the remainder of the paper, let k be an algebraically closed field. Let Λ be a finite-dimensional k-algebra, and let Λ^e be the *enveloping algebra* $\Lambda \otimes_k \Lambda^{\mathrm{op}}$. The *Hochschild cohomology ring* is $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ which is a graded k-algebra under the Yoneda product. An element of $\mathrm{HH}^n(\Lambda)$ can be represented by an extension

$$0 \to \Lambda \to E_1 \to \cdots \to E_n \to \Lambda \to 0.$$

Tensoring with a Λ-module M yields

$$0 \to M \to E_1 \otimes_\Lambda M \to \cdots \to E_n \otimes_\Lambda M \to M \to 0$$

which is exact. We then obtain a homomorphism

$$\mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_\Lambda(M, M).$$

Definition 6.1. Let H be a commutative graded subalgebra of $\mathrm{HH}^*(\Lambda)$, and let M be a Λ-module. The *support variety* $V_H(M)$ is defined by

$$V_H(M) = \{\mathfrak{m} \in \max H \mid \mathfrak{m} \supset \mathrm{Ann}_H \mathrm{Ext}^*_\Lambda(M, M)\}.$$

Next, we define a special k-algebra A_q^c that is a *quantum complete intersection*. For $\Lambda = A_q^c$, there exists a polynomial subalgebra $H \cong k[\eta_1, \ldots, \eta_c]$ of $\mathrm{HH}^{2*}(A_q^c)$ with $|\eta_i| = 2$ [BE09]. This is the H that we will take for the rest of the section.

Definition 6.2. For a fixed integer $a \geq 2$, let

$$a' := \begin{cases} a/\gcd(a, \operatorname{char} k) & \text{if } \operatorname{char} k > 0 \\ a & \text{if } \operatorname{char} k = 0 \end{cases},$$

and let $q \in k$ be a primitive a'th root of unity. The k-algebra A_q^c is defined by

$$A_q^c = k\langle x_1, \ldots, x_c\rangle / (\{x_i^a\}, \{x_i x_j - q x_j x_i\}_{i<j}).$$

Given $\lambda = (\lambda_1, \ldots, \lambda_c) \in k^c$, define $u_\lambda \in A_q^c$ by

$$u_\lambda = \lambda_1 x_1 + \cdots + \lambda_c x_c.$$

Lemma 6.3 ([BEH07])**.** We have $u_\lambda^a = 0$.

Proof. The q-analogue of multinomial coefficients is

$$\binom{r}{r_1, \ldots, r_n}_q = \frac{[r]_q!}{[r_1]_q! \cdots [r_n]_q!}$$

where $[m]_q! = [m]_q[m-1]_q \cdots [1]_q$ for $m \geq 1$, or 1 for $m = 0$, and $[m]_q = \frac{1-q^m}{1-q}$. Then

$$\binom{a'}{r_1, \ldots, r_c} = \begin{cases} 1 & \text{if } r_i = a' \text{ for some } i \\ 0 & \text{else} \end{cases}$$

because if no r_i equals a', then the numerator is $\frac{1-q^{a'}}{1-q} = 0$. Thus,

$$u_\lambda^{a'} = \lambda_1^{a'} x_1^{a'} + \cdots + \lambda_c^{a'} x_c^{a'}.$$

We are already done if $a' = a$, as $x_i^a = 0 \implies u_\lambda^a = 0$. Otherwise if $\operatorname{char} k = p$ and $a = a'p$, we use the fact that $x_i^{a'} x_j^{a'} = x_j^{a'} x_i^{a'}$ (because there are $(a')^2$ commutations and $q^{(a')^2} = 1$) to obtain

$$\begin{aligned} u_\lambda^a &= (\lambda_1^{a'} x_1^{a'} + \cdots + \lambda_c^{a'} x_c^{a'})^p \\ &= \lambda_1^a x_1^a + \cdots + \lambda_c^a x_c^a \\ &= 0. \end{aligned}$$

□

The formula for q-multinomial coefficients in the above proof also shows for $\lambda \neq 0$ that $u_\lambda^i \neq 0$ for $i < a$. Importantly, we have

$$k[u_\lambda] \cong k[x]/(x^a)$$

for all $\lambda \neq 0$. We may now define rank varieties similarly to those for kE-modules in Definition 5.1.

Definition 6.4. The *rank variety* $V_A^r(M)$ of an A_q^c-module M is

$$V_A^r(M) = \{0\} \cup \{\lambda \in k^c \mid M \text{ is not a projective } k[u_\lambda]\text{-module}\}.$$

Since u_λ has order a, the same argument as Theorem 5.2 shows that M not being a projective $k[u_\lambda]$-module is equivalent to the following rank inequality.

Theorem 6.5. For all $\lambda \in k^c$, we have

$$\operatorname{rank} u_\lambda \leq \tfrac{a-1}{a} \dim M,$$

where u_λ denotes the map $M \xrightarrow{\cdot u_\lambda} M$ that is multiplication by u_λ. Equality holds if and only if $\lambda \notin V_A^r(M)$.

This theorem shows that the rank variety is indeed a homogeneous affine variety. The rank variety is related to the support variety via a quantum version of the Avrunin–Scott theorem.

Theorem 6.6 ([BE09])**.** Let $F\colon k^c \to k^c$ be the map which sends $(\alpha_1, \dots, \alpha_c) \mapsto (\alpha_1^a, \dots, \alpha_c^a)$. For every A_q^c-module M, we have

$$F(V_A^r(M)) = V_H(M).$$

Remark 6.7. It is not possible to extend this rank variety definition to most other quantum complete intersections $k\langle x_1, \dots, x_c\rangle/(\{x_i^{a_i}\}, \{x_i x_j - q_{ij} x_j x_i\}_{i<j})$. In particular, we need u_λ to have the same order for all $\lambda \neq 0$. This is not true if the exponents a_i are not all the same. Also, Lemma 6.3 does not hold when q is not a primitive a'th root of unity.

The following two theorems are analogues of Theorem 4.9 and Theorem 4.10 for A_q^c-modules instead of kG-modules.

Theorem 6.8. For every closed homogeneous subvariety V of V_H, there exists a finitely generated A_q^c-module M such that $V_H(M) = V$.

Theorem 6.9. Let M be a finitely generated A_q^c-module. If $V_G(M) = V_1 \cup V_2$ with $V_1 \cap V_2 = \{0\}$, then $M \cong M_1 \oplus M_2$ with $V_G(M_1) = V_1$ and $V_G(M_2) = V_2$.

[EHT+04] proves these theorems for Λ-modules under two conditions:

1. There exists a commutative Noetherian graded subalgebra H of $\mathrm{HH}^*(\Lambda)$ with $H^0 = \mathrm{HH}^0(\Lambda)$.
2. $\mathrm{Ext}_\Lambda^*(M, N)$ is a finitely generated H-module for all finitely generated Λ-modules M, N.

Then [BO08, Theorem 5.5] proves that that $\Lambda := k\langle x_1, \dots, x_n\rangle/(\{x_i^{a_i}\}, \{x_j x_i - q_{ij} x_i x_j\}_{i<j})$ satisfies these conditions with respect to $H = \mathrm{HH}^{2*}(\Lambda)$ if and only if all q_{ij} are roots of unity. In particular, Theorem 4.9 and Theorem 4.10 hold for $\Lambda = A_q^c$.

7 Rank varieties for low-dimensional modules

We compute the rank varieties of the modules in [YZ19] subject to the conditions in Definition 6.2 from [BE09]. You and Zhang [YZ19] classify all indecomposable modules over the k-algebras

$$A(q, 2, 2) = k\langle x, y\rangle/(x^2, y^2, xy - qyx)$$
$$A(q, m, n) = k\langle x, y\rangle/(x^m, y^n, xy - qyx)$$

of dimensions up to 5 and 3 respectively. To apply Theorem 6.5 for $A(q, 2, 2)$ we need $q = -1$, and for $A(q, m, n)$ we need $m = n = a$ and for q to be a primitive a'th root of unity. In other words, we will compute the rank varieties of modules over the k-algebras

$$A(-1, 2, 2) = k\langle x, y\rangle/(x^2, y^2, xy + yx) \tag{1}$$
$$A(q, a, a) = k\langle x, y\rangle/(x^a, y^a, xy - qyx). \tag{2}$$

Recall from Theorem 6.5 that the rank condition for both algebras is $\operatorname{rank} u_\lambda < \frac{a-1}{a} \dim M$, where $u_\lambda = \lambda_1 x + \lambda_2 y$.

Following [YZ19], we use diagrams with vertices and labeled arrows to represent the A-modules.

Definition 7.1. The vertices are basis elements of the module, and the arrows are labeled with generators of A. A solid arrow such as

$$u \xrightarrow{x} v$$

means $xu = v$. A dashed arrow

$$u \overset{x}{\dashrightarrow} v$$

means $xu = \alpha v$ for some $\alpha \in k^*$, and two dashed arrows

$$v \overset{x}{\dashleftarrow} u \overset{x}{\dashrightarrow} w$$

with the same label means $xu = \alpha v + \beta w$ for $\alpha, \beta \in k^*$. If there does not exist an arrow from a vertex u labeled with x, then $xu = 0$.

Indecomposable modules over the algebra (1)

We first compute the rank varieties of indecomposable $A(-1, 2, 2)$-modules of dimensions up to 5. Recall from Corollary 5.5 that when $a = 2$ does not divide $\dim M$, we should obtain $V_A^r(M) = k^{\oplus 2}$.

Dimension 2

The rank condition is $\operatorname{rank} u_\lambda < \frac{1}{2} \cdot 2$, or $u_\lambda = 0$.

1.1. $u \xrightarrow{x} v$	1.2. $u \xrightarrow{y} v$	1.3. $\begin{tikzcd} u \arrow[r, bend left, "x"] \arrow[r, bend right, dashed, "y"'] & v \end{tikzcd}$

In Diagram 1.3, $yu = \alpha v$ for $\alpha \in k^*$.

1. We need $(\lambda_1 x + \lambda_2 y)u = \lambda_2 v = 0$ so $\boxed{\lambda_2 = 0}$.

2. Similar to the above case, $\boxed{\lambda_1 = 0}$.

3. We need $(\lambda_1 x + \lambda_2 y)u = (\lambda_1 + \lambda_2\alpha)v = 0$ so $\boxed{\lambda_1 + \lambda_2\alpha = 0}$.

Dimension 3

The rank condition is $\operatorname{rank} u_\lambda < \frac{1}{2} \cdot 3$, or $\operatorname{rank} u_\lambda \leq 1$.

2.1. $v_1 \xleftarrow{x} v \xrightarrow{y} v_2$	2.2. $v_1 \xrightarrow{x} v \xleftarrow{y} v_2$

1. Since v_1 and v_2 are annihilated by u_λ, the rank variety is $\boxed{k^{\oplus 2}}$.

2. Since $\operatorname{im} u_\lambda$ is spanned by v, the rank variety is $\boxed{k^{\oplus 2}}$.

Dimension 4

The condition is $\operatorname{rank} u_\lambda < \frac{1}{2} \cdot 4$, or $\operatorname{rank} u_\lambda \leq 1$.

3.1.
$$\begin{tikzcd} u_1 \arrow[r, "y"] \arrow[d, "x"'] & u_2 \arrow[d, "x"] \\ v_1 \arrow[r, dashed, "y"] & v_2 \end{tikzcd}$$

3.2.
$$\begin{tikzcd} u_1 \arrow[d, "x"'] & u_2 \arrow[dl, "y"] \arrow[d, "x"] \\ v_1 & v_2 \end{tikzcd}$$

3.3.
$$\begin{tikzcd} u_1 \arrow[d, "y"'] & u_2 \arrow[dl, "x"] \arrow[d, "y"] \\ v_1 & v_2 \end{tikzcd}$$

3.4.
$$\begin{tikzcd} u_1 \arrow[d, "x"] \arrow[d, dashed, bend right, "y"'] \arrow[dr, dashed, "y"] & u_2 \arrow[d, "x"'] \arrow[d, dashed, bend left, "y"] \\ v_1 & v_2 \end{tikzcd}$$

We have $yv_1 = q^{-1}v_2$ in Diagram 3.1 and $yu_1 = \alpha v_1 + v_2$, $yu_2 = \alpha v_2$ in Diagram 3.4.

1. Since $\operatorname{im} u_\lambda = \operatorname{span}\{\lambda_1 v_1 + \lambda_2 u_2, \lambda_1 v_2, q^{-1}\lambda_2 v_2\}$, we need $\boxed{\lambda_1 = \lambda_2 = 0}$.

2. Since $\operatorname{im} u_\lambda = \operatorname{span}\{\lambda_1 v_1, \lambda_1 v_2 + \lambda_2 v_1\}$, we need $\boxed{\lambda_1 = 0}$.

3. Similar to the above case, $\boxed{\lambda_2 = 0}$.

4. Since $\operatorname{im} u_\lambda = \operatorname{span}\{\lambda_1 v_1 + \lambda_2(\alpha v_1 + v_2), (\lambda_1 + \lambda_2\alpha)v_2\}$, we need $\boxed{\lambda_1 + \lambda_2\alpha = 0}$.

Dimension 5

The condition is $\operatorname{rank} u_\lambda < \frac{1}{2} \cdot 5$, or $\operatorname{rank} u_\lambda \leq 2$.

4.1.
$$\begin{tikzcd} u_1 \arrow[d, "x"'] & & u \arrow[dll, "y"] \arrow[drr, "x"'] & & u_2 \arrow[d, "y"] \\ v_1 & & & & v_2 \end{tikzcd}$$

4.2.
$$\begin{tikzcd} u_1 \arrow[d, "y"'] \arrow[drr, "x"] & & & & u_2 \arrow[dll, "y"'] \arrow[d, "x"] \\ v_1 & & v & & v_2 \end{tikzcd}$$

1. Since $\operatorname{im} u_\lambda$ is spanned by only v_1 and v_2, the rank variety is $\boxed{k^{\oplus 2}}$.

2. Since v_1, v, and v_2 are annihilated by u_λ, the rank variety is $\boxed{k^{\oplus 2}}$.

Indecomposable modules over the algebra (2)

We now compute the rank varieties of indecomposable $A(q, a, a)$-modules of dimensions up to 3.

Dimension 2

A 2-dimensional $A(q, a, a)$-module M for $a \geq 3$ is in fact an $A(q, 2, 2)$-module: x^2 must act by 0 on M, and similarly for y^2.

These were classified earlier. In all cases, for $a > 2$, the rank variety is $\boxed{k^{\oplus 2}}$ by Corollary 5.5 as $a \nmid \dim M$. Alternatively, we can check that $\operatorname{rank} u_\lambda < \frac{a-1}{a} \cdot 2$ or $\operatorname{rank} u_\lambda \leq 1$ is true in all examples.

Dimension 3

We need $\operatorname{rank} u_\lambda < \frac{a-1}{a} \cdot 3$. If $a > 3$, then this is satisfied for all $\lambda \in k^{\oplus 2}$ as $a \nmid \dim M$, so it remains to consider the case of $a = 3$ and $\operatorname{rank} u_\lambda \leq 1$.

5.1. $u \xrightarrow{y} v \xrightarrow{y} w$	5.2. $u \xrightarrow{x} v \xrightarrow{x} w$
5.3. $u \xrightarrow{x} v \xleftarrow{y} w$	5.4. $u \xleftarrow{x} v \xrightarrow{y} w$
5.5. $\begin{tikzcd} u \arrow[r, "y"'] \arrow[rr, dashed, bend left, "x"] & v \arrow[r, "y"'] & w \end{tikzcd}$	5.6. $\begin{tikzcd} u \arrow[r, "x"'] \arrow[rr, dashed, bend left, "y"] & v \arrow[r, "x"'] & w \end{tikzcd}$
5.7. $\begin{tikzcd} u \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & v \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & w \end{tikzcd}$	

We have $xu = \alpha w$ in Diagram 5.5, $yu = \alpha w$ in Diagram 5.6, and $yu = q\alpha v$, $yv = \alpha w$ in Diagram 5.7.

1. Since $\operatorname{im} u_\lambda = \operatorname{span}\{\lambda_2 v, \lambda_2 w\}$, we need $\boxed{\lambda_2 = 0}$.
2. Similar to the above case, $\boxed{\lambda_1 = 0}$.
3. Since $\operatorname{im} u_\lambda$ is spanned by v, the rank variety is $\boxed{k^{\oplus 2}}$.
4. Since u and w are annihilated by u_λ, the rank variety is $\boxed{k^{\oplus 2}}$.
5. Since $\operatorname{im} u_\lambda = \operatorname{span}\{\lambda_2 v + \lambda_1 \alpha w, \lambda_2 w\}$, we need $\boxed{\lambda_2 = 0}$.
6. Similar to the above case, $\boxed{\lambda_1 = 0}$.
7. Since $\operatorname{im} u_\lambda = \operatorname{span}\{\lambda_1 v + q\alpha v, \lambda_1 w + \alpha w\}$, we need $\boxed{\lambda_1 + \lambda_2 q\alpha = 0}$ or $\boxed{\lambda_1 + \lambda_2 \alpha = 0}$.

Note that the last case does not contradict Theorem 6.9: applying F to the lines $\{\lambda_1 + \lambda_2\alpha = 0\}$ and $\{\lambda_1 + \lambda_2 q\alpha = 0\}$ both yield the points $\{(\lambda^a, (-\lambda\alpha)^a) : \lambda \in k\}$ since $q^a = 1$. This is an example of how the rank variety of an A_q^c-module does not have to be connected, even though the support variety is.

8 Classifying rank varieties over quantum complete intersections

In the previous section, we computed special cases of rank varieties of modules over

$$A = k\langle x, y\rangle/(x^a, y^a, xy - qyx).$$

We now provide progress towards the following question.

Question 8.1. Characterize the attainable rank varieties $V_A^r(M)$ of indecomposable A-modules M.

The quantifier "indecomposable" is added for simplicity, as we can understand rank varieties of A-modules in general by taking unions of rank varieties of indecomposable A-modules. The support and rank varieties of an A-module are homogeneous closed subvarieties of $k^{\oplus 2}$, yielding possibilities of $\{0\}$, $k^{\oplus 2}$, and a union of lines. From Theorem 6.9, the support variety $V_H(M) = F(V_A^r(M))$ of an indecomposable A-module M is connected, which yields the cases of $\{0\}$, $k^{\oplus 2}$, and one line. We give examples for attaining $V_A^r(M) = \{0\}$ and $k^{\oplus 2}$, and a specific example in the third case which generalizes the low-dimensional examples in Section 7.

Case 1: $F(V_A^r(M)) = \{0\}$.

Then $V_A^r(M) = \{0\}$. We show this is attainable by the free A-module A. Visually, it is the following diagram which generalizes Diagram 3.1, where $xv_{ij} = q^{j-1}v_{(i+1)j}$ for all $1 \leq i \leq a-1$ and $1 \leq j \leq a$.

$$\begin{tikzcd}
v_{11} \arrow[r, "y"] \arrow[d, "x"] & v_{12} \arrow[r, "y"] \arrow[d, dashed, "x"] & \cdots \arrow[r, "y"] & v_{1a} \arrow[d, dashed, "x"] \\
v_{21} \arrow[r, "y"] \arrow[d, "x"] & v_{22} \arrow[r, "y"] \arrow[d, dashed, "x"] & \cdots \arrow[r, "y"] & v_{2a} \arrow[d, dashed, "x"] \\
\vdots \arrow[d, "x"] & \vdots \arrow[d, dashed, "x"] & \ddots & \vdots \arrow[d, dashed, "x"] \\
v_{a1} \arrow[r, "y"] & v_{a2} \arrow[r, "y"] & \cdots \arrow[r, "y"] & v_{aa}
\end{tikzcd}$$

We can find $a(a-1)$ linearly independent vectors in $\operatorname{im} u_\lambda$ which are the images of v_{ij} for $1 \leq i \leq a$ and $1 \leq j \leq a-1$, unless $\lambda_1 = \lambda_2 = 0$. Since $\dim A = a^2$, this means $\operatorname{rank} u_\lambda < \frac{a-1}{a} \dim M$ only for $\lambda = 0$, and $V_A^r(M) = \{0\}$. To see that it is indecomposable, we defer the proof to Proposition 8.2.

Case 2: $F(V_A^r(M)) = k^{\oplus 2}$.

Then $V_A^r(M) = k^{\oplus 2}$ by considering the possibilities for homogeneous subvarieties of $k^{\oplus 2}$. This is attainable whenever $a \nmid \dim M$ by Corollary 5.5. However, this is not a necessary condition, as in Diagram 5.3 with $a = 3$ and $\dim M = 3$, there were also rank varieties with $V_A^r(M) = k^{\oplus 2}$.

Case 3: $F(V_A^r(M))$ is one line.

For ease of computation, suppose $F(V_A^r(M))$ is the line $\{(\lambda^a, (-\lambda\alpha)^a) \mid \lambda \in k\}$ for some $\alpha \in k$ so that

$$V_a^r(M) \subset \bigcup_{i=0}^{a-1} \{\lambda_1 + \lambda_2 q^i \alpha = 0\}.$$

It is still an open question as to which subvarieties are attainable in this case, but we present three examples.

1. Generalizing Diagram 5.2, we can consider the following module.

   $$v_1 \xrightarrow{x} v_2 \xrightarrow{x} \cdots \xrightarrow{x} v_a$$

   In order for $\operatorname{rank} u_\lambda < \frac{a-1}{a} \cdot a$, we need $\dim \operatorname{span}\{\lambda_1 v_2, \ldots, \lambda_1 v_a\} \leq a-2$, so $\boxed{\lambda_1 = 0}$.

2. By replacing all instances of x with y in the previous module diagram, we can similarly obtain the locus $\boxed{\lambda_2 = 0}$.

3. We can construct a union of lines which are rotations by q using a generalization of Diagram 5.7. Consider the module diagram

$$\begin{tikzcd} v_1 \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & v_2 \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & \cdots \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & v_{a-2} \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & v_{a-1} \arrow[r, "x"'] \arrow[r, dashed, bend left, "y"] & v_a \end{tikzcd} \tag{3}$$

   where $yv_i = q^{a-1-i}\alpha x$ for all $1 \le i \le a-1$. The rank condition is $\operatorname{rank} u_\lambda < \frac{a-1}{a} \cdot a$, or

$$\dim \operatorname{span}\{(\lambda_1 + \lambda_2 q^{a-2}\alpha)v_2, \ldots, (\lambda_1 + \lambda_2 q\alpha)v_{a-1}, (\lambda_1 + \lambda_2\alpha)v_a\} \le a-2.$$

   This yields the locus

$$\{\lambda_1 + \lambda_2 q^{a-2}\alpha = 0\} \cup \cdots \cup \{\lambda_1 + \lambda_2 q\alpha = 0\} \cup \{\lambda_1 + \lambda_2\alpha = 0\}.$$

   which is a union of $a-1$ distinct lines if $a' = a$, or a' distinct lines if $a' < a$.

The following proposition shows that all A-modules constructed in this section are indecomposable.

Proposition 8.2. An A-module that is generated by a single element is indecomposable.

Proof. For a module M over a local ring $(A, \mathfrak{m})$, all minimal generating sets of M have the same cardinality, namely $\dim_{A/\mathfrak{m}} M/\mathfrak{m}M$ by Nakayama's lemma. As $A = k\langle x, y\rangle/(x^a, y^a, xy - qyx)$ is a local ring with maximal ideal (x, y), an A-module M which splits as a direct sum must be generated by more than one element. $\square$

9 Future directions

In addition to Question 8.1, we have the following conjectures and questions.

Conjecture 9.1. Let $c > 2$ be fixed. Not all homogeneous subvarieties $V \subset k^c$ are attainable as the rank variety $V_A^r(M)$ of some A_q^c-module M.

This is a conjectured partial answer to Question 8.1. The motivation comes from the kG-module case where the proof of Theorem 4.10 relies on a construction which implies Theorem 4.9. However, connectedness for rank varieties of indecomposable A_q^c-modules does not hold as seen from Diagram 5.7.

Note that Conjecture 9.1 is known to be true for support varieties $V_H(M)$ of A_q^c-modules as stated in Theorem 6.8. It is also true for rank varieties in the $c = 2$ case by the computations in Section 7, as all lines $\lambda_1 + \lambda_2\alpha = 0$ are attainable. In particular, we have not yet found for $c > 2$ an A_q^c-module M for which $V_A^r(M) = \{\lambda_1 + \lambda_2\alpha = 0\}$ with $\alpha \neq 0$.

For the next conjecture, we consider the $c = 2$ case and define $\varphi_{q^i} \colon A \to A$ as the automorphism with $x \mapsto x$ and $y \mapsto q^i y$. Let M_{q^i} be the "twisted" module M with the action of $\lambda \in A$ on $m \in M_{q^i}$ defined to be $\varphi(\lambda)m$. In other words, $(\alpha, q^i\beta) \in V_A^r(M)$ corresponds to $(\alpha, \beta) \in V_A^r(M_{q^i})$.

Conjecture 9.2. For the $c = 2$ case, if $(\alpha, \beta) \in V_A^r(M)$, then $(\alpha, \beta) \in V_A^r(M_{q^i})$ for all but at most one of $1 \le i \le a'$.

This encapsulates the rank variety of the module diagram (3) which is a union of $a-1$ lines if $a' = a$, or a' lines otherwise.

Question 9.3. Is it possible to generalize rank varieties to the quantum complete intersection $k\langle x_1, \dots, x_c\rangle/(\{x_i^a\}, \{x_i x_j - q_{ij} x_j x_i\}_{i<j})$ where q_{ij} are potentially different primitive a' roots of unity?

From Remark 6.7, we likely need all q_{ij} to be primitive a' roots of unity for a rank variety definition to make sense.

Question 9.4. Classify the attainable rank varieties of A_q^c-modules for $c > 2$.

This is an extension of Question 8.1 to $c > 2$ variables. We give one example of a module which naturally generalizes Diagram 3.4 to $c > 2$ variables. Consider the following module with vertices $v_1, \dots, v_r, w_1, \dots, w_r$ and arrows labeled with $x_1, \dots, x_n$ between from each u_i to each v_j.

$$\begin{tikzcd}
v_1 \arrow[d, dashed, "x_i"] \arrow[dr, dashed, "x_i"] \arrow[drrr, dashed, "x_i"] & v_2 & \cdots & v_r \\
w_1 & w_2 & \cdots & w_r
\end{tikzcd} \tag{4}$$

The rank of u_λ can be computed as follows. Construct an $r \times r$ matrix U with (i, j)th entry as

$$\lambda_1(\text{coefficient of } u_i \overset{x_1}{\dashrightarrow} v_j) + \cdots + \lambda_n(\text{coefficient of } u_i \overset{x_n}{\dashrightarrow} v_j).$$

Then $\operatorname{rank} u_\lambda = \operatorname{rank} U$, and for an $r \times r$ matrix U, we have

$$\operatorname{rank} u_\lambda = \operatorname{rank} U \le r - 1 \iff \det U = 0.$$

However, it is not possible to construct all homogeneous polynomials in $\lambda_1, \dots, \lambda_n$ as $\det U$ for some $r \times r$ matrix U with linear entries. Furthermore, this module may not be indecomposable.

Question 9.5. Is there a condition of the corresponding matrix U, such as diagonalizability, which characterizes whether the above A_q^c-module (4) is indecomposable?

For context from [YZ19], the more general form of Diagram 3.4 in the $c = 2$ case is the diagram

$$\begin{tikzcd}
u_1 \arrow[d, "x"] \arrow[d, dashed, bend right, "y"'] \arrow[dr, dashed, "y"] & u_2 \arrow[d, "x"'] \arrow[d, dashed, bend left, "y"] \arrow[dl, dashed, "y"'] \\
v_1 & v_2
\end{tikzcd} \qquad \begin{pmatrix} yu_1 \\ yu_2 \end{pmatrix} = T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

for a 2×2 invertible matrix T. From [YZ19, Lemma 2], this A-module is indecomposable if and only if T is not diagonalizable. Now assuming that k is algebraically closed, we can write T in Jordan normal form as $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ to obtain Diagram 3.4.

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