DYSON BROWNIAN MOTION AS NONCOLLIDING BROWNIAN MOTION

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Abstract. In this expository paper, we discuss Dyson Brownian motion as noncolliding Brownian motions via Doob's h-transform. We additionally outline the usage of Doob's h-transform for describing Dyson Brownian motion with a boundary at the origin.

1. Introduction

Dyson Brownian motion was initially described as the eigenvalues of an $n \times n$ Hermitian random matrix with entries performing independent complex-valued Brownian motions [Dys62]. These n (real) eigenvalues $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n$ satisfy the stochastic differential equations

$$dX_i = \sum_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \frac{dt}{X_i - X_j} + dB_i \qquad (1 \leqslant i \leqslant n)$$

for independent one-dimensional Brownian motions B_i $(1 \leqslant i \leqslant n)$. The process also satisfies $X_1(t) < X_2(t) < \cdots < X_n(t)$ for all times $t > 0$ a.s. [Sas11]. From the SDE, the eigenvalues can be thought of as evolving according to a Brownian motion and a "repulsion force" between nearby eigenvalues that is inversely proportional to their separation.

The fact that the eigenvalues are distinct motivates viewing Dyson Brownian motion as n processes conditioned to never collide. In fact, this leads to the main theorem of our paper: an equivalent definition of Dyson Brownian motion is n *independent Brownian motions conditioned to never collide.*

The aim of this expository paper is to introduce Dyson Brownian motion from the perspective of noncolliding Brownian motions. We begin in Section 2 by defining Doob's h-transform, a technique of rescaling transition probabilities that will be useful in the later sections; our primary reference is Bloemendal [Blo10]. In Theorem 3.1, we state the aforementioned main theorem, that the eigenvalues in Dyson Brownian motion are equal in distribution to n independent Brownian motions conditioned to never collide. In the remainder of Section 3, we present one proof following Li [Li21] which uses clever algebraic identities and martingale properties. In Section 4, we outline another proof by Sasamoto [Sas11] which uses a more general method with transition densities. Finally in Section 5, we outline how Sasamoto [Sas11] uses this general method to construct Dyson Brownian motion with a boundary at the origin.

2. Doob's h-transform

2.1. **Definitions.**

Informally, Doob's h-transform is used to rescale the transition probabilities of a Markov process in order to condition on an event. We can then study the conditioned Markov process and compute its modified SDE.

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Let $(X_t)_{t\geqslant 0}$ be a Markov process on a state space $(E, \mathcal{E})$ with transition semigroup $P^t(x, dy)$; for our purposes, we will only use $E = \mathbb{R}^n$. Let $\mathbb{P}_x[\cdot]$ denote $\mathbb{P}[\cdot \mid X_0 = x]$, so that

$$P^t(x, dy) = \mathbb{P}_x[X_t \in dy].$$

Now consider a *shift-invariant* event A of positive probability, meaning that

$$\mathbb{P}[(X_{t+s})_{s\geqslant 0} \in A \mid X_t = y] = \mathbb{P}[(X_s)_{s\geqslant 0} \in A \mid X_0 = y] \tag{2.1}$$

for all $t \geqslant 0$. We are interested the conditional probability

$$\tilde{P}^t(x, dy) := \mathbb{P}_x[X_t \in dy \mid A], \tag{2.2}$$

which can be computed in terms of $P^t(x, dy)$ by Bayes' theorem as follows. Letting $h(x) = \mathbb{P}_x[A]$, we have

$$\tilde{P}^t(x, dy) = \mathbb{P}_x[X_t \in dy \mid A] = \frac{\mathbb{P}_x[A \mid X_t = y]\mathbb{P}_x[X_t \in dy]}{\mathbb{P}_x[A]} = \frac{h(y)}{h(x)} P^t(x, dy), \tag{2.3}$$

where the equality $\mathbb{P}_x[A \mid X_t = y] = h(y)$ is from the shift-invariance of A.

Let $\tilde{E} = \{x \in E \mid \mathbb{P}_x[A] > 0\}$ denote the set of states from which A is accessible. Since for all $x \in \tilde{E}$ we have

$$\int_{\tilde{E}} \tilde{P}^t(x, dy) = \int_{\tilde{E}} \mathbb{P}_x[X_t \in dy \mid A] = 1,$$

$\tilde{P}^t(x, dy)$ is a transition semigroup on $\tilde{E}$. Restated using (2.3), we have the following theorem.

Theorem 2.1 (Doob's h-transform). *$(X_t)_{t\geqslant 0}$ is a Markov process on $\tilde{E}$ with transition semigroup*

$$\tilde{P}^t(x, dy) = \frac{h(y)}{h(x)} P^t(x, dy). \tag{2.4}$$

More generally, Theorem 2.1 holds for the following class of functions h, for which integrating $\tilde{P}^t(x, dy)$ in (2.4) also yields a probability measure.

Definition 2.2. A *harmonic* function h satisfies $P^t h = h$ for all $t \geqslant 0$; written out, that is

$$h(x) = \int_E h(y) P^t(x, dy)$$

for all $x \in E$.

Let H be a *shift-invariant* function so that $H = H \circ \theta_t$, where $\theta_t : X_s \mapsto X_{t+s}$ is the time shift. One can show that $h(x) := \mathbb{E}_x[H]$ is a harmonic function, and that all harmonic functions are of this form [Blo10]. The aforementioned case of a shift-invariant event $A = \theta_t^{-1} A$ in (2.1) is the special case $H = \mathbb{1}_A$.

In summary, Doob's h-transform (2.4) can be thought of as reweighting transition probabilities $P^t(x, dy)$ by some function $h(y)$, and dividing by a normalization factor $h(x)$. For the following sections in which we condition a process to be confined to a certain region of $\mathbb{R}^n$, the key is to find the appropriate harmonic function $h(x)$ that has the correct behavior at the boundaries of the region.

2.2. **Effect on an SDE.** We now study the Markov process associated with $\tilde{P}^t(x, dy)$, deriving its infinitesimal generator and SDE in terms of those for the original Markov process.

Recall the infinitesimal generator L is given by

$$Lf(x) = \lim_{t\downarrow 0} \frac{P^t f(x) - f(x)}{t} = \lim_{t\downarrow 0} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t}.$$

We may compute the conditioned generator $\tilde{L}$ in terms of L as follows:

$$\begin{aligned}
\tilde{L}f(x) &:= \lim_{t\downarrow 0} \frac{\mathbb{E}_x[f(X_t) \mid A] - f(x)}{t} \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left(\int_E f(y)\tilde{P}^t(x, dy) - f(x)\right) \\
&= \lim_{t\downarrow 0} \frac{1}{t}\left(\int_E f(y)\frac{h(y)}{h(x)}P^t(x, dy) - f(x)\right) \\
&= \frac{1}{h(x)} \lim_{t\downarrow 0} \frac{1}{t}\left(\int_E f(y)h(y)P^t(x, dy) - f(x)h(x)\right) \\
&= \frac{1}{h(x)} Lfh(x). \qquad (2.5)
\end{aligned}$$

Let us now consider specifically the case of an n-dimensional diffusion process $\underline{X}_t$ satisfying

$$d\underline{X}_t = \mu(\underline{X}_t)\,dt + d\underline{B}_t \qquad (2.6)$$

for $\underline{B}_t$ an n-dimensional Brownian motion. For such a process, Itô's formula yields

$$Lf(x) = \langle \mu(x), \nabla f(x)\rangle + \frac{1}{2}\Delta f(x).$$

We wish to explicitly compute the conditioned generator $\tilde{L}$ for (2.6) as well, using (2.5). A fact that simplifies the computation is the following: for harmonic functions h satisfying $P^t h = h$, we have by the definition of L that

$$0 = Lh(x) = \langle \mu(x), \nabla h(x)\rangle + \frac{1}{2}\Delta h(x).$$

Now we compute that

$$\begin{aligned}
\tilde{L}f &= \frac{1}{h}Lfh \\
&= \frac{1}{h}\left(\langle \mu, \nabla(fh)\rangle + \frac{1}{2}\Delta(fh)\right) \\
&= \frac{1}{h}\left(\langle \mu, (\nabla f)h + f(\nabla h)\rangle + \frac{1}{2}(f(\Delta h) + 2\langle \nabla f, \nabla h\rangle + (\Delta f)h)\right) \\
&= \langle \mu, \nabla f\rangle + \frac{1}{h}\langle \nabla h, \nabla f\rangle + \frac{1}{2}\Delta f \\
&= \langle \mu + \nabla \log h, \nabla f\rangle + \frac{1}{2}\Delta f.
\end{aligned}$$

Written out in entirety, the conditioned generator $\tilde{L}$ for (2.6) is

$$\tilde{L}f(x) = \langle \mu(x) + \nabla \log h(x), \nabla f(x)\rangle + \frac{1}{2}\Delta f(x),$$

which corresponds to the diffusion process

$$d\underline{\tilde{X}}_t = (\mu(\underline{\tilde{X}}_t) + \nabla \log h(\underline{\tilde{X}}_t))\, dt + d\underline{B}_t. \tag{2.7}$$

Comparing (2.6) and (2.7) shows that Doob's h-transform *adds a drift term* $\nabla \log h(\underline{\tilde{X}}_t)\, dt$.

The resulting SDE in (2.7) is nice in terms of the function h, but oftentimes h will be intractable. Fortunately, Doob's h-transform will work nicely in the following sections because the functions h we consider will have special structures and clean partial derivatives.

3. Dyson Brownian Motion as Noncolliding Brownian Motion

The main theorem in this section is an alternate characterization of Dyson Brownian motion as n independent Brownian motions conditioned to never collide.

Theorem 3.1. *Let $(X_1, \ldots, X_n)$ be a Dyson Brownian motion with initial conditions satisfying $X_1(0) < \cdots < X_n(0)$. Then for independent standard Brownian motions $(B_1, \ldots, B_n)$ with the same initial conditions, we have the following equality in distribution:*

$$(X_1, \ldots, X_n) \stackrel{d}{=} (B_1, \ldots, B_n) \mid A,$$

where A denotes the event that the B_i never intersect.

The proof in the remainder this section follows Li [Li21]. Since A has probability 0, we cannot use the construction in (2.3) with $h(x) = \mathbb{P}_x[A] = 0$. However, we introduce the following definition to write A as the limit of events with positive probability, in order to use Doob's h-transform.

Definition 3.2. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the *Vandermonde determinant* $V_n(x)$ is defined as

$$V_n(x) = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

The initial condition in Theorem 3.1 implies that $V_n(\underline{B}_0) > 0$. Furthermore, we have $V_n(\underline{B}_t) > 0$ for all $t \geqslant 0$ if and only if the B_i never intersect. This motivates defining for $c > 0$ the events

$$A_c = \{V_n(\underline{B}_t) \text{ hits } c \text{ before } 0\},$$

which satisfy

$$A = \lim_{c \to \infty} A_c.$$

We now show that $V_n(\underline{B}_t)$ is a martingale, so the h-transform $h_c(x) := \mathbb{P}_x[A_c]$ can be easily calculated from the optional stopping theorem: the probability that a martingale $V_n(\underline{B}_t)$ started from $V_n(\underline{B}_0) = V_n(x)$ hits c before 0 is $\frac{V_n(x)}{c}$.

Lemma 3.3. *$V_n(\underline{B}_t)$ is a martingale.*

Proof. By Itô's formula, we have

$$dV_n(\underline{B}_t) = \frac{1}{2} \sum_{i=1}^{n} \partial_{ii} V_n(\underline{B}_t)\, dt + \nabla V_n(\underline{B}_t) \cdot d\underline{B}_t,$$

where the second term is a martingale. We will compute that the coefficient of dt above is in fact 0, so $V_n(\underline{B}_t)$ will be a true martingale.

For $x \in \mathbb{R}^n$, we compute

$$\begin{aligned}\partial_i V_n(x) &= \sum_{j\neq i} \frac{V_n(x)}{x_i - x_j},\\ \partial_{ii} V_n(x) &= \sum_{j\neq i}\sum_{k\neq i,j} \frac{V_n(x)}{(x_i-x_j)(x_i-x_k)}.\end{aligned}$$

Then by the identity

$$\frac{1}{(x_i-x_j)(x_i-x_k)} + \frac{1}{(x_j-x_i)(x_j-x_k)} + \frac{1}{(x_k-x_i)(x_k-x_j)} = 0 \tag{3.1}$$

for distinct x_i, x_j, x_k, we have

$$\begin{aligned}\sum_{i=1}^n \partial_{ii} V_n(x) &= V_n(x) \sum_{i=1}^n \sum_{j\neq i}\sum_{k\neq i,j} \frac{1}{(x_i-x_j)(x_i-x_k)}\\ &= 0,\end{aligned}$$

as desired. □

Proof of Theorem 3.1. By Lemma 3.3 and the optional stopping theorem, the h-transform corresponding to the event A_c is

$$h_c(x) := \mathbb{P}_x[A_c] = \frac{V_n(x)}{c}.$$

Now we may explicitly compute the $\partial_i \log h_c(x)$ term that appears in the conditioned SDE (2.7):

$$\begin{aligned}\partial_i \log h_c(x) &= \partial_i(\log V_n(x) - \log c)\\ &= \frac{1}{V_n(x)} \partial_i V_n(x)\\ &= \sum_{j\neq i} \frac{1}{x_i - x_j}. \end{aligned}\tag{3.2}$$

Returning to the theorem statement, we begin with a standard Brownian motion $(B_1, \ldots, B_n)$ which satisfies $d\underline{B}_t = d\underline{B}_t$. From (2.7) and (3.2), the process conditioned on A_c satisfies

$$d\tilde{B}_i = \sum_{j\neq i} \frac{dt}{\tilde{B}_i - \tilde{B}_j} + dB_i, \qquad (1 \leqslant i \leqslant n)$$

which is the SDE for a Dyson Brownian motion $(X_1, \ldots, X_n)$. Observing that the conditioned SDE is independent of c, we may take $c \to \infty$ to obtain $(X_1, \ldots, X_n) \overset{d}{=} (B_1, \ldots, B_n) \mid A$. □

The proof of Theorem 3.1 in this section uses only the special case of Doob's h-transform from (2.2) with the events of positive probability A_c. It also relies on almost-magical properties of the Vandermonde determinant for Lemma 3.3 to hold and give a clean closed form to $\mathbb{P}_x[A_c]$. In the next section, we outline a more reliable proof method that uses the full generality of Doob's h-transform from (2.4).

4. Transition Densities

In this section, we outline the computations that Sasamoto [Sas11] uses to prove Theorem 3.1 via transition densities. By the Kolmogorov backward equation, the transition density $p_t^+(x,y)$ for Dyson Brownian motion satisfies

$$\frac{\partial}{\partial t}p_t^+ = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}p_t^+ + \sum_{i=1}^n\sum_{j\neq i}\frac{1}{x_i - x_j}\cdot\frac{\partial}{\partial x_i}p_t^+. \tag{4.1}$$

On the other hand, the Karlin-McGregor formula [KM59] gives the transition density $p_t(x,y)$ of n noncolliding Brownian motions as the determinant

$$p_t(x,y) = \det\left(\tfrac{1}{\sqrt{2\pi t}}e^{-(x_i-y_j)^2/2t}\right)_{1\leqslant i,j\leqslant n}$$

where each entry is a 1-dimensional Brownian transition density. We remark that $p_t(x,y)$ satisfies the heat equation

$$\frac{\partial}{\partial t}p_t = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}p_t, \tag{4.2}$$

since each 1-dimensional Brownian transition density in the determinant expansion satisfies the heat equation.

The function we use for Doob's h-transform is, unsurprisingly, the Vandermonde determinant:

$$h(x) := \prod_{1\leqslant i<j\leqslant n}(x_j - x_i).$$

One can show that h is harmonic via an integral computation [Sas11]. Then Theorem 3.1 can be rephrased as follows.

Theorem 4.1. *The h-transformed transition probability*

$$\frac{h(y)}{h(x)}p_t(x,y)$$

satisfies (4.1) *in place of* $p_t^+(x,y)$.

We color-code groups of expressions in the following proof for ease of verification. The key to the computation is the fact that

$$\sum_{i=1}^n\left(-\frac{\partial^2}{\partial x_i^2}\log h(x) - \left(\frac{\partial}{\partial x_i}\log h(x)\right)^2\right) = 0 \tag{4.3}$$

for this particular function $h(x)$.

Proof of Theorem 4.1. Writing $\sum_{j\neq i}\frac{1}{x_i-x_j}$ as $\frac{\partial}{\partial x_i}\log h(x)$, the equation we wish to show is

$$\frac{\partial}{\partial t}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) = \frac{1}{2}\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) + \sum_{i=1}^n\frac{\partial}{\partial x_i}\log h(x)\cdot\frac{\partial}{\partial x_i}\left(\frac{h(y)}{h(x)}p_t(x,y)\right).$$

From (4.2), we know that the left hand side equals

$$\frac{\partial}{\partial t}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) = \frac{1}{2}\sum_{i=1}^n\frac{h(y)}{h(x)}\cdot\frac{\partial^2}{\partial x_i^2}p_t(x,y). \tag{4.4}$$

For the last term on the right hand side, applying the product rule and the fact that

$$\frac{\partial}{\partial x_i}\frac{1}{h(x)} = \frac{1}{h(x)} \cdot -\frac{\partial}{\partial x_i}\log h(x) \tag{4.5}$$

yields

$$\frac{\partial}{\partial x_i}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) = -\frac{\partial}{\partial x_i}\log h(x) \cdot \frac{h(y)}{h(x)}p_t(x,y) + \frac{h(y)}{h(x)} \cdot \frac{\partial}{\partial x_i}p_t(x,y). \tag{4.6}$$

Using (4.5) and (4.6), we also obtain the equation

$$\begin{aligned}\frac{\partial}{\partial x_i}\frac{h(y)}{h(x)} \cdot \frac{\partial}{\partial x_i}p_t(x,y) &= -\frac{\partial}{\partial x_i}\log h(x) \cdot \frac{h(y)}{h(x)} \cdot \frac{\partial}{\partial x_i}p_t(x,y)\\ &= -\frac{\partial}{\partial x_i}\log h(x) \cdot \left(\frac{\partial}{\partial x_i}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) + \frac{\partial}{\partial x_i}\log h(x) \cdot \frac{h(y)}{h(x)}p_t(x,y)\right).\end{aligned}$$

Then we compute the second partial derivative by the product rule on (4.6):

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) &= -\frac{\partial^2}{\partial x_i^2}\log h(x) \cdot \frac{h(y)}{h(x)}p_t(x,y) - \frac{\partial}{\partial x_i}\log h(x) \cdot \frac{\partial}{\partial x_i}\left(\frac{h(y)}{h(x)}p_t(x,y)\right)\\ &\quad + \frac{\partial}{\partial x_i}\frac{h(y)}{h(x)} \cdot \frac{\partial}{\partial x_i}p_t(x,y) + \frac{h(y)}{h(x)} \cdot \frac{\partial^2}{\partial x_i^2}p_t(x,y)\\ &= \frac{h(y)}{h(x)} \cdot \frac{\partial^2}{\partial x_i^2}p_t(x,y) - 2\frac{\partial}{\partial x_i}\log h(x) \cdot \frac{\partial}{\partial x_i}\left(\frac{h(y)}{h(x)}p_t(x,y)\right) \qquad (4.7)\\ &\quad + \left(-\frac{\partial^2}{\partial x_i^2}\log h(x) - \left(\frac{\partial}{\partial x_i}\log h(x)\right)^2\right)\frac{h(y)}{h(x)}p_t(x,y).\end{aligned}$$

One may verify that

$$-\frac{\partial^2}{\partial x_i^2}\log h(x) - \left(\frac{\partial}{\partial x_i}\log h(x)\right)^2 = -2\sum_{j,k\neq i}\frac{1}{(x_i-x_j)(x_i-x_k)},$$

so upon summing from $1 \leqslant i \leqslant n$, this term disappears by the identity (3.1). Comparing the right hand sides of (4.4), (4.6), and (4.7) yields the desired equation. □

This technique of considering partial differential equations satisfied by the transition densities of certain processes will also be applied in the next section when considering Dyson Brownian motion with a boundary at the origin.

5. Dyson Brownian Motion with a Boundary

In this section, we outline how Sasamoto [Sas11] uses Doob's h-transform to construct Dyson Brownian motion of types C and D, which can be thought of as Dyson Brownian motion with a wall at the origin that is absorbing or reflecting, respectively. Their SDEs are as follows.

Definition 5.1. In Dyson Brownian motion *of type* C, the stochastic dynamics of the particles $0 < X_1 < X_2 < \cdots < X_n$ are described by the stochastic differential equations

$$dX_i = dB_i + \frac{dt}{X_i} + \sum_{j\neq i}\left(\frac{1}{X_i - X_j} + \frac{1}{X_i + X_j}\right)dt.$$

for $1 \leqslant i \leqslant n$.

Definition 5.2. In Dyson Brownian motion *of type* D, the stochastic dynamics of the particles $0 \leqslant X_1 < X_2 < \cdots < X_n$ are described by the stochastic differential equations

$$dX_i = dB_i + \frac{1}{2}\mathbb{1}_{\{i=1\}} dL_t + \sum_{j \neq i} \left(\frac{1}{X_i - X_j} + \frac{1}{X_i + X_j} \right) dt$$

for $1 \leqslant i \leqslant n$, where L_t denotes the local time of X_1 at the origin.

By the Kolmogorov backward equation, the transition densities $p_t^+(x, y)$ and $q_t^+(x, y)$ for Dyson Brownian motion of type C and type D satisfy, respectively,

$$\frac{\partial}{\partial t} p_t^+ = \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} p_t^+ + \sum_{i=1}^n \left(\frac{1}{x_i} + \sum_{j \neq i} \left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right) \right) \cdot \frac{\partial}{\partial x_i} p_t^+ \tag{5.1}$$

$$\frac{\partial}{\partial t} q_t^+ = \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} q_t^+ + \sum_{i=1}^n \sum_{j \neq i} \left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right) \cdot \frac{\partial}{\partial x_i} q_t^+ . \tag{5.2}$$

On the other hand, a consequence of the Karlin-McGregor formula described by Grabiner [Gra99] is that the transition densities $p_t(x, y)$ and $q_t(x, y)$ for n noncolliding Brownian motions on the regions $\{x \in \mathbb{R}^n \mid 0 < x_1 < x_2 < \cdots < x_n\}$ and $\{x \in \mathbb{R}^n \mid 0 \leqslant x_1 < x_2 < \cdots < x_n\}$ are, respectively,

$$p_t(x, y) = \det \left(\tfrac{1}{\sqrt{2\pi t}} e^{-(x_i - y_j)^2/2t} - \tfrac{1}{\sqrt{2\pi t}} e^{-(x_i + y_j)^2/2t} \right)_{1 \leqslant i, j \leqslant n}$$

$$q_t(x, y) = \det \left(\tfrac{1}{\sqrt{2\pi t}} e^{-(x_i - y_j)^2/2t} + \tfrac{1}{\sqrt{2\pi t}} e^{-(x_i + y_j)^2/2t} \right)_{1 \leqslant i, j \leqslant n} .$$

The functions we will use for Doob's h-transform are

$$h^{(C)}(x) := \prod_{i=1}^n x_i \prod_{1 \leqslant i < j \leqslant n} (x_j^2 - x_i^2)$$

$$h^{(D)}(x) := \prod_{1 \leqslant i < j \leqslant n} (x_j^2 - x_i^2).$$

After a computation to verify that they are harmonic [Sas11], we have the following theorem.

Theorem 5.3. *The h-transforms*

$$\frac{h^{(C)}(y)}{h^{(C)}(x)} p_t(x, y) \quad \text{and} \quad \frac{h^{(D)}(y)}{h^{(D)}(x)} q_t(x, y)$$

satisfy (5.1) *and* (5.2) *in place of* $p_t^+(x, y)$ *and* $q_t^+(x, y)$, *respectively.*

Proof. We may rewrite (4.1) in the more general form

$$\frac{\partial}{\partial t} p_t^+ = \frac{1}{2} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} p_t^+ + \sum_{i=1}^n \frac{\partial}{\partial x_i} \log h(x) \cdot \frac{\partial}{\partial x_i} p_t^+ ,$$

such that replacing $h(x)$ by $h^{(C)}(x)$ and $h^{(D)}(x)$ yield (5.1) and (5.2), respectively. The proof then becomes identical to the proof of Theorem 4.1, with $h(x)$ replaced by $h^{(C)}(x)$ and $h^{(D)}(x)$. In particular, a straightforward computation shows that $h^{(C)}(x)$ and $h^{(D)}(x)$ also satisfy (4.3). □

This section demonstrated the versatility of Doob's h-transform to describe Dyson Brownian motion in the presence of a boundary. For Dyson Brownian motion in Theorem 3.1, we implicitly considered the region of $\mathbb{R}^n$ given by

$$\{x \in \mathbb{R}^n \mid x_1 < x_2 < \cdots < x_n\},$$

while for Dyson Brownian motion of types C and D in Theorem 5.3, we considered the regions

$$\{x \in \mathbb{R}^n \mid 0 < x_1 < x_2 < \cdots < x_n\}$$
$$\{x \in \mathbb{R}^n \mid 0 \leqslant x_1 < x_2 < \cdots < x_n\}.$$

Conditioning n independent Brownian motions to stay within such a region for all time corresponded to the respective harmonic functions used for Doob's h-transform.

6. Conclusions

In this paper, we discussed Doob's h-transform and proved in two ways that Dyson Brownian motion is equal in distribution to n independent Brownian motions conditioned to never intersect. We also considered Dyson Brownian motion with an absorbing or reflecting boundary, adapting our previous proof to show that Doob's h-transform yields the desired SDEs.

There are many more interesting examples of Doob's h-transform applied to Brownian motion and other Markov processes, as given by Bloemendal [Blo10]. Additional properties of Brownian motion related to boundaries or non-intersection are described by Sasamoto [Sas11] and Grabiner [Gra99]. For example, Sasamoto [Sas11] uses similar techniques to prove that interlacing $n+1$ particles between an n-particle Dyson Brownian motion yields an $(n+1)$-particle Dyson Brownian motion.

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