FAITHFUL SPECIALIZATIONS OF THE BURAU REPRESENTATION

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1. Introduction

Braids are a useful algebraic tool for studying knots and links, as all links can be realized as the closure of some braid. Braids on n strands form a group B_n under composition, and we can understand the structure of B_n via its representations.

The Burau representation of B_n, first introduced by Werner Burau [2] in 1935, is a well-studied matrix representation of the braid group with connections to the Alexander polynomial. It is well-known that the Burau representation of B_n is faithful for $n \leq 3$, and Bigelow [1] proved in 1999 that it is unfaithful for $n \geq 5$. Meanwhile, the case of $n = 4$ is a longstanding open problem.

In this paper, we prove faithfulness for $n \leq 3$ before asking a more interesting question for the $n = 3$ case: is the Burau representation $\psi_3 \colon B_3 \to GL_2(\mathbb{Z}[t, t^{-1}])$ still faithful when t is replaced with a real number? Such a map is called a *specialization* of the Burau representation, and Theorem 5.2 states that it is faithful for all $t < 0$ with $t \neq -1$. Following Scherich [7], we give a self-contained proof which takes a surprising detour into hyperbolic geometry!

We begin in Section 2 by introducing the braid group B_n and its connection to links and the Alexander polynomial. We define the Burau representation of B_n in Section 3 and show that it is faithful for $n \leq 3$ in Section 4. Beginning in Section 5, we discuss specializations of the Burau representation and reduce the question of faithfulness to an application of the ping-pong lemma. In Section 6, we introduce tools from hyperbolic geometry to complete the proof of faithfulness in Section 7.

Our main references are Ohtsuki [5] on the Burau representation, Scherich [7] on specializations of the Burau representation, and Mangahas [4] on hyperbolic geometry.

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2. Braid Groups

Intuitively, a braid is a collection of strands which weave around each other. Formally, we present the below definition for completeness.

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Definition 2.1. A *(geometric) braid* is a union of n strands embedded in $\mathbb{R}^2 \times [0,1]$ with boundary $\{1,2,\dots,n\} \times \{0\} \times \{0,1\}$ such that no strand has a critical point with respect to the vertical coordinate.

Up to isotopy, any braid on n strands is obtained by attaching vertically a combination of σ_i's and σ_i^{-1}'s for $1 \leq i \leq n-1$, where σ_i is a positive crossing of the ith and $(i+1)$th strands, and σ_i^{-1} is a negative crossing.

σ_i σ_i^{-1}

Thus, isotopy classes of braids on n strands form a group under vertical composition, called the *braid group* B_n. The braid group is known to have the following presentation.

Theorem 2.2. *The braid group B_n is presented by generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ with relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } |i-j| \geq 2 \tag{2.1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{for } 1 \leq i \leq n-2. \tag{2.2}$$

There is a natural connection between braids and links, which we explain as follows.

Definition 2.3. The *closure* of a braid is the link obtained by connecting each lower end of the braid with its respective upper end, as shown in Figure 1.

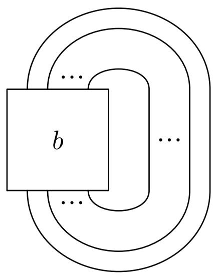


FIGURE 1. The closure of a braid b.

Taking the closure of any braid forms a link, and the converse is also true by Alexander's theorem: any oriented link is isotopic to the closure of some braid. However, note that this correspondence is not bijective.

3. The Burau Representation

In this section, we define the Burau representation of the braid group B_n. We then further the connection between braids and links in Theorem 3.4, which derives the Alexander polynomial from the (reduced) Burau representation.

Definition 3.1. An n-dimensional *matrix representation* of a group G is a homomorphism $\rho\colon G \to GL_n(k)$ for a field k.

We slightly generalize matrix representations by working over the ring $\mathbb{Z}[t, t^{-1}]$ where t is an indeterminate. From the relations (2.1) and (2.2), our homomorphism $\psi\colon B_n \to GL_n(\mathbb{Z}[t, t^{-1}])$ must satisfy

$$\psi(\sigma_i\sigma_j) = \psi(\sigma_j\sigma_i) \qquad \text{for } |i-j| \geq 2 \tag{3.1}$$

$$\psi(\sigma_i\sigma_{i+1}\sigma_i) = \psi(\sigma_{i+1}\sigma_i\sigma_{i+1}) \qquad \text{for } 1 \leq i \leq n-2. \tag{3.2}$$

Definition 3.2. The *unreduced Burau representation* $\tilde{\psi}_n\colon B_n \to GL_n(\mathbb{Z}[t, t^{-1}])$ of B_n is given on generators by

$$\tilde{\psi}_n(\sigma_i) = I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

for $1 \leq i \leq n-1$.

Omitted computation shows that $\tilde{\psi}_n$ indeed satisfies (3.1) and (3.2), and is thus well-defined on B_n. The braid group B_n acts on $\mathbb{Z}[t, t^{-1}]^{\oplus n}$ via the matrices from $\tilde{\psi}_n$, and one can verify that this action preserves the submodule

$$\{(f_1(t), \ldots, f_n(t)) \in \mathbb{Z}[t, t^{-1}]^{\oplus n} \mid f_1(t) + \cdots + f_n(t) = 0\}.$$

The restriction of $\tilde{\psi}_n$ to this submodule is the *reduced Burau representation* ψ_n, given explicitly by the following matrices.

Definition 3.3. The *reduced Burau representation* $\psi_n\colon B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$ of B_n for $n \geq 3$ is given on generators by

$$\psi_n(\sigma_i) = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

for $2 \leq i \leq n-2$, and

$$\psi_n(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}$$

$$\psi_n(\sigma_{n-1}) = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}.$$

In the case of $n = 2$, we have $\psi_2(\sigma_1) = \begin{pmatrix} -t \end{pmatrix}$.

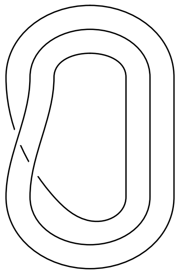
For the remainder of this paper, we work with the reduced Burau representation ψ_n. The following theorem connects ψ_n to the Alexander polynomial–see [5, Chapter 2.3] for a proof. It is especially useful for efficiently computing the Alexander polynomial of a complicated braid on only a few strands, as the sizes of the matrices involved depend on the number of strands rather than the number of crossings.

Theorem 3.4. [5] *Let L be oriented link with n components, and let b be a braid with closure isotopic to L. The Alexander polynomial $\Delta_L(t)$ of L is determined by the reduced Burau representation ψ_n:*

$$\Delta_L(t) \sim \frac{1-t}{1-t^n} \det(I_{n-1} - \psi_n(b)),$$

where "$\sim$" means equality up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$.

Example 3.5. Consider $b = \sigma_1\sigma_2$ in B_3, whose closure L is isotopic to the unknot, as shown below.



Then $\psi_n(b) = \left(\begin{smallmatrix} -t & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ t & -t \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -t \\ t & -t \end{smallmatrix}\right)$, and $\det(I_{n-1} - \psi_n(b)) = \left|\begin{smallmatrix} 1 & t \\ -t & 1+t \end{smallmatrix}\right| = 1+t+t^2$. Indeed, the Alexander polynomial for the unknot is $\frac{1-t}{1-t^3}(1+t+t^2) = 1$.

4. Faithfulness of the Burau Representation

Representations of a group give us information about its structure. A natural question is whether a given representation is injective, so that group elements can be distinguished by their actions.

Definition 4.1. A representation is *faithful* if it is injective.

We prove the faithfulness of the Burau representation for $n \leq 3$, following [6].

Theorem 4.2. *The Burau representation $\psi_n \colon B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$ is faithful for $n \leq 3$.*

Proof. The $n = 1$ case is clear because B_1 is the trivial group.

For $n = 2$, we note that $B_2 \cong \mathbb{Z}$ is freely generated by σ_1. As $\psi_2(\sigma_1) = \begin{pmatrix} -t \end{pmatrix}$, no nonzero power of $\psi_2(\sigma_1)$ equals I_1.

In the $n = 3$ case, we have

$$\psi_3(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \psi_3(\sigma_2) = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}. \tag{4.1}$$

Substitute $t = -1$ to obtain the matrices $a_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $a_2 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$. Now we compute the kernel of the composite map

$$\rho \colon B_3 \xrightarrow{\psi_3} GL_2(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t \mapsto -1} SL_2(\mathbb{Z})$$

where $\sigma_1 \mapsto a_1$ and $\sigma_2 \mapsto a_2$, noting that $\ker \psi_3 \subset \ker \rho$.

The matrices a_1, a_2 in fact generate $SL_2(\mathbb{Z})$ [6, Theorem 3.7], and we can write the presentations of B_3 and $SL_2(\mathbb{Z})$ as

$$B_3 \cong \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$
$$SL_2(\mathbb{Z}) \cong \langle a_1, a_2 \mid a_1a_2a_1 = a_2a_1a_2, (a_1a_2a_1)^4 = 1 \rangle.$$

We claim that the kernel of $\rho \colon B_3 \to SL_2(\mathbb{Z})$ is $\langle (\sigma_1\sigma_2\sigma_1)^4 \rangle$. This will finish from combining the facts

$$\ker \psi_3 \subset \ker \rho = \langle (\sigma_1\sigma_2\sigma_1)^4 \rangle,$$

and $\psi_3((\sigma_1\sigma_2\sigma_1)^4) = \begin{pmatrix} t^6 & 0 \\ 0 & t^6 \end{pmatrix}$, so no nonzero power of $\psi_3((\sigma_1\sigma_2\sigma_1)^4)$ equals I_2.

It is clear from the presentation of $SL_2(\mathbb{Z})$ that $(\sigma_1\sigma_2\sigma_1)^4 \in \ker \rho$. For the reverse direction, suppose the expression $p(\sigma_1, \sigma_2) \in B_3$ is in the kernel of ρ, so $p(a_1, a_2) \in SL_2(\mathbb{Z})$ is the identity.

Let F_{a_1,a_2} denote the free group generated by a_1, a_2. For a subset S of a group G, let $\langle\langle S \rangle\rangle$ denote the the smallest normal subgroup of G containing of S. From the presentation of $SL_2(\mathbb{Z})$, we have by definition that $SL_2(\mathbb{Z}) \cong F_{a_1,a_2}/\langle\langle a_1a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}, (a_1a_2a_1)^4 \rangle\rangle$. Then $p(a_1, a_2)$ viewed in F_{a_1,a_2} lies in $\langle\langle a_1a_2a_1a_2^{-1}a_1^{-1}a_2^{-1}, (a_1a_2a_1)^4 \rangle\rangle$. Since the relation $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ is already in B_3, we can thus assume $p(\sigma_1, \sigma_2) \in \langle\langle (\sigma_1\sigma_2\sigma_1)^4 \rangle\rangle$ in B_3. Upon verifying the relations

$$\begin{aligned} (\sigma_1\sigma_2\sigma_1)^2\sigma_1 &= \sigma_1(\sigma_1\sigma_2\sigma_1)^2 \\ (\sigma_1\sigma_2\sigma_1)^2\sigma_2 &= \sigma_2(\sigma_1\sigma_2\sigma_1)^2, \end{aligned} \tag{4.2}$$

we see that $(\sigma_1\sigma_2\sigma_1)^2$ commutes with both σ_1 and σ_2, so $(\sigma_1\sigma_2\sigma_1)^4$ does as well. Then $\langle\langle (\sigma_1\sigma_2\sigma_1)^4 \rangle\rangle = \langle (\sigma_1\sigma_2\sigma_1)^4 \rangle$ and $p(\sigma_1, \sigma_2) \in \langle (\sigma_1\sigma_2\sigma_1)^4 \rangle$, as desired. □

5. Specializations of the Burau Representation

In the previous section, we showed that the Burau representation ψ_3 of B_3 is faithful. As an intermediate step, we evaluated the output of ψ_3 at $t = -1$; although the resulting representation had a nontrivial kernel, upon returning t to an indeterminate, the representation became faithful. This leads to the natural question: for which real numbers t will the resulting representation still be faithful?

Definition 5.1. A *specialization* ρ of the Burau representation ψ_3 is a composition

$$\rho\colon B_3 \xrightarrow{\psi_3} GL_2(\mathbb{Z}[t, t^{-1}]) \xrightarrow{\tau} GL_2(\mathbb{R}),$$

where τ is the evaluation map sending $t \mapsto t_0$ for some fixed $t_0 \in \mathbb{R}$.

The proof of Theorem 4.2 implies the specialization ρ is faithful for all transcendental $t \in \mathbb{R}$. A complete classification of the faithful specializations can be found in [7]. We prove the following illustrative case which nicely utilizes hyperbolic isometries, to be introduced in Section 6.

Theorem 5.2. *The specialization of ψ_3 is faithful for all $t < 0$ except $t = -1$.*

We use the following key lemma which gives a condition for a representation of B_n to be faithful.

Lemma 5.3. [3] *Let $\rho\colon B_n \to GL(V)$ be a representation, and suppose $N \lhd B_n$ is a normal subgroup which is nontrivial and noncentral. If $\rho|_N$ is faithful, then ρ is faithful, except possibly on the center of B_n.*

To apply Lemma 5.3, we use the following facts about B_3; the proofs are omitted, but they are standard exercises in group theory. Let F_2 denote the free group on two generators.

Lemma 5.4.

(1) *The center of B_3 is $\langle (\sigma_1\sigma_2)^3 \rangle$.*

(2) *The subgroup* $N = \langle \sigma_1^{-1}\sigma_2, \sigma_2\sigma_1^{-1} \rangle$ *of* B_3 *is normal and isomorphic to* F_2*.*

Remark 5.5. In (4.2), we checked that $(\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)^2$ lies in the center of B_3. See Figure 2 for a visual interpretation of this computation: $\sigma_1\sigma_2$ twists the three strands by one notch counterclockwise, and $(\sigma_1\sigma_2)^3$ twists the three strands such that they return to their original positions. Intuitively, this twist can be done before or after any braid.

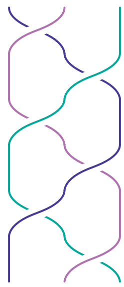


FIGURE 2. The element $(\sigma_1\sigma_2)^3$ in B_3.

Now in light of Lemma 5.3 and Lemma 5.4, we need to show that $\rho|_N$ is faithful and that ρ is faithful on the center of B_3. The latter is easy: using (4.1), ρ sends

$$(\sigma_1\sigma_2)^3 \mapsto \begin{pmatrix} t^3 & 0 \\ 0 & t^3 \end{pmatrix},$$

and no power of $\left(\begin{smallmatrix} t^3 & 0 \\ 0 & t^3 \end{smallmatrix}\right)$ equals I_2 when $t < 0$ and $t \neq -1$ (in fact for all $t \neq 0, 1, -1$).

It remains to show that $\rho(N) = \langle \rho(\sigma_1^{-1}\sigma_2), \rho(\sigma_2\sigma_1^{-1}) \rangle$ is isomorphic to F_2 inside of $GL_2(\mathbb{R})$, so $\rho|_N$ is faithful. We will apply the following trick for showing that a group is free of rank 2. There is a generalization for free groups of any rank n, but we only need the case $n = 2$ in this paper.

Lemma 5.6 (Ping-pong lemma)**.** *Let* $\langle a, b \rangle$ *be a group acting on a set* X*. Suppose there exist disjoint nonempty subsets* X_a *and* X_b *of* X *such that* $a^k(X_b) \subset X_a$ *and* $b^k(X_a) \subset X_b$ *for all integers* $k \neq 0$*. Then* $\langle a, b \rangle$ *is isomorphic to* F_2*.*

Proof. Consider an element in $\langle a, b \rangle$ of the form $a^{i_1}b^{i_2}a^{i_3}\cdots b^{i_{m-1}}a^{i_m}$ for some nonzero exponents $i_1, \ldots, i_m$. Applying this element to any $x \in X_b$, we see that first $a^{i_m}(x) \in X_a$, then $b^{i_{m-1}}(a^{i_m}x) \in X_b$, and so on until finally $a^{i_1}(b^{i_2}\cdots a^{i_m}x) \in X_a$. As X_a and X_b are disjoint, this action is nontrivial and $a^{i_1}b^{i_2}a^{i_3}\cdots b^{i_{m-1}}a^{i_m}$ does not equal the identity. Furthermore, any nonempty expression in $\langle a, b \rangle$ is conjugate to an element of that form, upon conjugating by a sufficiently large power of a, and thus does not equal the identity. □

The name "ping-pong" comes from the alternating nature of the sets X_a and X_b. We do a simple example of the ping-pong lemma to demonstrate its effectiveness.

Example 5.7. We use the ping-pong lemma to show that the subgroup generated by $a = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ inside $SL_2(\mathbb{Z})$ is isomorphic to F_2. By induction, we have $a^k = \left(\begin{smallmatrix} 1 & 2k \\ 0 & 1 \end{smallmatrix}\right)$ and $b^k = \left(\begin{smallmatrix} 1 & 0 \\ 2k & 1 \end{smallmatrix}\right)$ for all integers k. Consider the disjoint subsets of $\mathbb{R}^2$ given by

$$X_a = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2 \mid |x| > |y|\}, \quad X_b = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^2 \mid |x| < |y|\}.$$

We check that $a^k(X_b) \subset X_a$ for all integers $k \neq 0$: from $\left(\begin{smallmatrix} 1 & 2k \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+2ky \\ y \end{smallmatrix}\right)$, if $|x| < |y|$ then $|x + 2ky| \geq |2ky| - |x| > |2ky| - |y| \geq |y|$ for all integers $k \neq 0$. Similarly, one can verify that $b^k(X_a) \subset X_b$ for all integers $k \neq 0$, so $\langle a, b\rangle \cong F_2$.

However, it is challenging to find by inspection two appropriate ping-pong sets in $\mathbb{R}^2$ for the generators of $\rho(N)$

$$\rho(\sigma_1^{-1}\sigma_2) = \begin{pmatrix} \frac{t-1}{t} & -1 \\ t & -t \end{pmatrix}, \quad \rho(\sigma_2\sigma_1^{-1}) = \begin{pmatrix} -\frac{1}{t} & \frac{1}{t} \\ -1 & 1-t \end{pmatrix}, \tag{5.1}$$

especially when t is an arbitrary real number. Instead of working with $\mathbb{R}^2$, we turn to techniques from hyperbolic geometry to find the right subsets X_a and X_b for this problem.

6. Hyperbolic Geometry

We introduce the concepts from hyperbolic geometry needed to prove Theorem 5.2. Our goal is to find a set on which the matrices from (5.1) can act.

Definition 6.1. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. A *Möbius transformation* is a map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of the form $z \mapsto \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

Remark 6.2. The condition $ad - bc \neq 0$ ensures that the map is nonconstant. One can verify that Möbius transformations form a group under composition.

It is a fact that Möbius transformations are precisely the bijective *conformal* maps $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, meaning that they preserve oriented angles between line segments; in particular, they send generalized circles to generalized circles.

Definition 6.3. Let n be a positive integer and k be a field. The *projective special linear group* $PSL(n, k)$ is the quotient $SL(n, k)/\{I_n, -I_n\}$.

Proposition 6.4. *There is an isomorphism between* $PSL(2, \mathbb{C})$ *and the group of Möbius transformations given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad z \mapsto \frac{az+b}{cz+d}.$$

Although this proposition is a starting point, we wish to work with matrices over $\mathbb{R}$ instead of $\mathbb{C}$. To obtain a similar theorem with $PSL(2, \mathbb{R})$, we introduce the *hyperbolic plane* $\mathbb{H}^2$. We present two equivalent definitions, first using the upper half-plane model $\mathbb{U}$, and then the Poincaré disk model $\mathbb{D}$.

Definition 6.5. Let $\mathbb{U} = \{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0\}$ denote the *upper half-plane*.

We put a metric on $\mathbb{U}$ given by

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$

which allows us to compute lengths: the length of a path $f(t) = x(t) + iy(t)$ parameterized by $t \in [0, 1]$ is thus

$$\int_0^1 \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{6.1}$$

Note that at each point $x + iy \in \mathbb{C}$, the metric on $\mathbb{U}$ is a scalar multiple of the Euclidean metric $\sqrt{(dx)^2 + (dy)^2}$. Intuitively, when y is close to 0, the length of a path under this metric is much larger than with the Euclidean metric, and conversely as y increases.

Definition 6.6. A *geodesic* is a shortest path between two points. An *infinite geodesic* is an infinite path such that any finite subpath is a geodesic.

For two points with the same real coordinate in $\mathbb{U}$, their unique geodesic is the vertical line segment between them, so that the quantity $(\frac{dx}{dt})^2$ in the length formula (6.1) is minimized. In general, the unique geodesic between two points of $\mathbb{U}$ is the arc along the circle with center on the real axis which passes through the two points. This can be shown by using an appropriate Möbius transformation and the fact that Möbius transformations are conformal.

Now we are ready to state the desired theorem involving $PSL(2, \mathbb{R})$.

Theorem 6.7. *Every Möbius transformation that preserves $\mathbb{U}$ is an isometry of $\mathbb{U}$ with its hyperbolic metric. The group of such Möbius transformations is isomorphic to $PSL(2, \mathbb{R})$.*

The proof involves showing that such a Möbius transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ must preserve the extended real line $\mathbb{R} \cup \{\infty\}$, implying that $a, b, c, d \in \mathbb{R}$; then $\mathbb{U}$ is preserved if and only if $ad - bc > 0$ by considering where $z = i$ is sent.

We now introduce a second model of the hyperbolic plane which will help us visualize the ping-pong lemma in our eventual proof.

Definition 6.8. The *Poincaré disk model* $\mathbb{D}$ is the image of $\mathbb{U}$ under the Möbius transformation $z \mapsto \frac{i-z}{i+z}$.

This maps the extended real line $\mathbb{R} \cup \{\infty\}$ to the unit disk in $\mathbb{C}$, and the upper half-plane into the interior. As Möbius transformations are isometries under the hyperbolic metric, the geodesics in $\mathbb{U}$ are mapped to the geodesics in $\mathbb{D}$. Consequently, the infinite geodesics in $\mathbb{D}$ are the circles and lines orthogonal to the unit circle at two points, as illustrated in Figure 3.

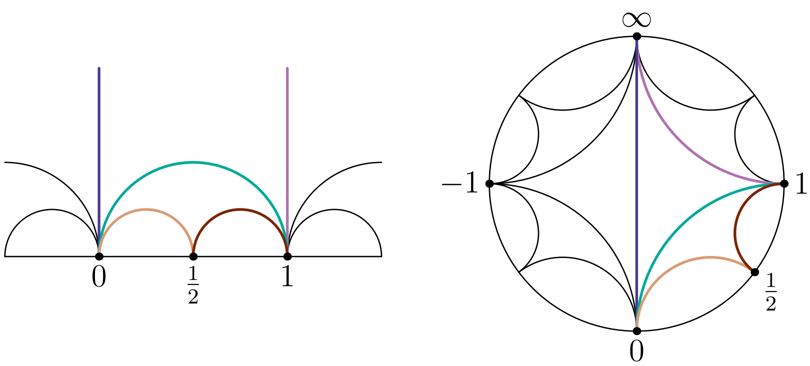


FIGURE 3. Examples of infinite geodesics in $\mathbb{U}$ and $\mathbb{D}$.

Finally, we need to understand how Möbius transformations act on $\mathbb{D}$. The isometries of $\mathbb{H}^2$ from by Möbius transformations can be classified into three types: elliptic, parabolic, and hyperbolic.

Definition 6.9. A *hyperbolic* isometry fixes exactly two points on the boundary of $\mathbb{D}$.

A hyperbolic isometry has an *axis*, which is a (unique) geodesic in $\mathbb{D}$ on which it acts as a translation. It is a fact that the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ is hyperbolic if and only if $|a+d| > 2$.

7. Faithfulness of the Specialization

We are now ready to prove the faithfulness of the specialization in Theorem 5.2 using the ping-pong lemma and tools from hyperbolic geometry.

Proof of Theorem 5.2. Let

$$a \coloneqq \rho(\sigma_1^{-1}\sigma_2) = \begin{pmatrix} \frac{t-1}{t} & -1 \\ t & -t \end{pmatrix}$$
$$b \coloneqq \rho(\sigma_2\sigma_1^{-1}) = \begin{pmatrix} -\frac{1}{t} & \frac{1}{t} \\ -1 & 1-t \end{pmatrix}.$$

We show that $\langle a, b\rangle \cong F_2$ by applying the ping-pong lemma to the action of $\langle a, b\rangle$ on $\mathbb{D}$. We see that that a and b are hyperbolic isometries on $\mathbb{D}$, because both of their traces are $1 - t - \frac{1}{t} \geq 3$ by AM-GM (using $t < 0$).

For the remainder of this proof, refer to Figure 4 for a visual. From the computations $a(0) = \frac{1}{t}$ and $a(1) = \infty$, the action of a on $\mathbb{D}$ sends the infinite geodesic connecting 0 and 1 to the infinite geodesic connecting $\frac{1}{t}$ to ∞. Let X_a^- and X_a^+ denote the interiors of the two regions bounded by these geodesics respectively. By the definition of a hyperbolic isometry, a maps $\mathbb{D} \setminus X_a^-$ inside X_a^+. Conversely, a^{-1} maps $\mathbb{D} \setminus X_a^+$ inside X_a^-.

Similarly, from the computations $b(\infty) = \frac{1}{t}$ and $b(1) = 0$, the action of b sends the infinite geodesic connecting 1 and ∞ to the infinite geodesic connecting 0 and $\frac{1}{t}$. Defining X_b^- and X_b^+ analogously, b maps $\mathbb{D} \setminus X_b^-$ inside X_b^+, and b^{-1} maps $\mathbb{D} \setminus X_b^+$ inside X_b^-.

In particular, the sets $X_a = X_a^+ \cup X_a^-$ and $X_b = X_b^+ \cup X_b^-$ are disjoint and satisfy the ping-pong lemma by the above computations: $a^k(X_b)$ is contained in X_a^+ if $k > 0$ or X_a^- if $k < 0$. Similarly, $b^k(X_a) \subset X_b^+ \cup X_b^-$ for all $k \neq 0$. □

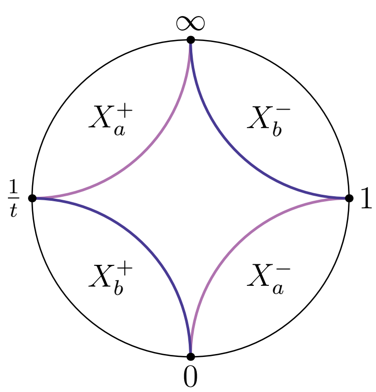


Figure 4. The ping-pong sets in the proof of Theorem 5.2.

Remark 7.1. In addition to the hyperbolic isometries, another special aspect of the $t < 0$ case is that $\frac{1}{t} < 0$, so we obtain the nice picture in Figure 4 of four nonintersecting geodesics. Otherwise, the regions X_a and X_b, as we defined them, would intersect.

The full classification by Scherich [7] compares $|\operatorname{tr}(a)| = |\operatorname{tr}(b)| = |1 - t - \frac{1}{t}|$ to 2 to determine whether a and b are elliptic, parabolic, or hyperbolic isometries. The specialization is faithful in the the parabolic case of $t = \frac{3\pm\sqrt{5}}{2}$ and the hyperbolic case of $t < \frac{3-\sqrt{5}}{2}$ or $t > \frac{3+\sqrt{5}}{2}$ (except at $t = 0, -1$). The only unfaithful specializations come from the elliptic case of $\frac{3-\sqrt{5}}{2} < t < \frac{3+\sqrt{5}}{2}$.

The faithfulness of these specializations is a much stronger result than the faithfulness of the Burau representation ψ_3 alone. The study of the specializations of B_3 has led to constructions of unfaithful specializations of B_4 [7], providing insight on the question of faithfulness for the Burau representation of B_4.

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