Projective Modules

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Introduction

- Projective modules can be thought of as building blocks of the A-module A.
- They have many desirable properties and are central to fields such as representation theory and homological algebra.
- The main theorem of this presentation is the bijective correspondence between *indecomposable projective* modules and *simple* modules.

Definition of a Module

Definition

Given a ring A, an *A-module* U is an abelian group $(U, +)$ with multiplication $A \times U \to U$ by elements in A. A *submodule* V of an A-module U is a subgroup $V \subset U$ which is closed under multiplication by A.

Example

For

$$A = \{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in k \},$$

we can consider the A-module A with operations of standard matrix addition and multiplication. Two submodules of the A-module A are

$$P_1 = \{ \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid a \in k \} \text{ and } P_2 = \{ \left(\begin{smallmatrix} 0 & b \\ 0 & c \end{smallmatrix}\right) \mid b, c \in k \},$$

and in fact for these submodules we have $A = P_1 \oplus P_2$.

Types of Modules

Definition

A submodule $M \subset U$ is *maximal* if $M \neq U$ and there does not exist a submodule N such that $M \subsetneq N \subsetneq U$.

Definition

A nonzero A-module U is *simple* if its only submodules are 0 and U.

By the correspondence theorem for modules, $M \subset U$ is maximal if and only if U/M is simple.

Definition

A nonzero A-module U is *indecomposable* if it can not be written as a direct sum of nontrivial submodules.

Simple modules are indecomposable, but the converse is not always true (e.g. the $\mathbb{Z}$-module $\mathbb{Z}/4\mathbb{Z}$).

Returning to Our Example

Example

- P_1 is simple and has 0 as a maximal submodule.
- P_2 has exactly one nontrivial proper submodule, namely

$$M = \{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix} \right) \mid c \in k \}.$$

Thus, M is maximal in P_2 and the only maximal submodule of P_2.

P_1 and P_2 are both indecomposable: P_1 is simple and thus indecomposable; P_2 is indecomposable because $P_2 \ncong M \oplus V$ for any submodule $V \subset P_2$.

Projective Modules

Definition

An A-module U is *free* if $U \cong \underbrace{A \oplus \cdots \oplus A}_{n}$ for some $n \in \mathbb{N}$.

Definition

An A-module U is *projective* if there exists some A-module V such that $U \oplus V$ is free.

Example

Since $A = P_1 \oplus P_2$, P_1 and P_2 are projective.

Indecomposable Projective Modules

Definition

The *radical* of an A-module U, denoted $\text{rad}(U)$, is the intersection of the maximal submodules of U.

At a high level, the radical helps describe the structure of a module and contains the elements which "prevent the module from being *semisimple*" (a direct sum of simple modules).

Lemma

An indecomposable projective module P has exactly one maximal submodule, namely $\text{rad}(U)$.

Bijective Correspondence

Theorem

There is a one-to-one correspondence between indecomposable projective A-modules and simple A-modules, given by $P \leftrightarrow P/\operatorname{rad}(P)$.

Example

In our running example, the unique maximal submodule of P_1 is 0, and the unique maximal submodule of P_2 is $M = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right) \mid c \in k\}$. The bijective correspondence implies that P_1 and P_2/M are simple A-modules.

In fact, we will see that they are the only simple A-modules (up to isomorphism).

Structural Results

Theorem (Krull-Schmidt)

Every finitely-generated A-module is isomorphic to a finite direct sum $M_1 \oplus \cdots \oplus M_n$ of indecomposable A-modules, unique up to reordering and isomorphism.

In particular for the A-module A, we may write $A \cong P_1 \oplus \cdots \oplus P_n$ for indecomposable A-modules P_i, which are also projective by definition.

Theorem

Every indecomposable projective A-module is isomorphic to P_i for some $1 \leq i \leq n$.

Example

Since $A = P_1 \oplus P_2$ in our running example, P_1 and P_2 are the only indecomposable projective A-modules up to isomorphism. Consequently, P_1 and P_2/M are the only simple A-modules up to isomorphism.