18.965 Geometry of Manifolds I

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Contents

1 Smooth Manifolds

1.1 Topological spaces

A *topological space* is a set X with a notion of *open sets* satisfying 3 properties:

1. X and $\emptyset$ are both open.
2. Any union (even infinite) of open sets is also an open set.
3. If U and V are open, then so is $U \cap V$ (or any finite intersection).

An example is $\mathbb{R}^n$, where a set U is open if $\forall y \in U$, $\exists \delta > 0$ such that $B_\delta(y) \subset U$, where

$$B_\delta(y) = \{z \in \mathbb{R}^n \mid |y - z| < \delta\}.$$

A subset $S \subset X$ is *closed* if $X \setminus S$ is open. The *closure* of a set S is the intersection of all closed sets containing S. A set is *connected* if the only subsets that are both open and closed are itself and $\emptyset$.

X is *Hausdorff* if for all points $x_1 \neq x_2$, there exist open sets $U_1 \ni x_1$ and $U_2 \ni x_2$ with $U_1 \cap U_2 = \emptyset$.

X is *second countable* if there is a countable basis of open sets. A *basis* is a collection $\mathcal{B}$ of open sets such that for all U open and $x \in U$, there exists $V \in \mathcal{B}$ such that $x \in V \subset U$. Every open set is thus a (likely infinite) union of open sets in $\mathcal{B}$.

Example 1.1

$\mathbb{R}^n$ is second countable by taking B to be all open balls with rational centers and rational radii.

A map $f\colon X \to Y$ is *continuous* if for all $V \subset Y$ open, $f^{-1}(V) \subset X$ is open. A map f is a *homeomorphism* if it is a continuous bijection, and f^{-1} is continuous. The identity map $\mathrm{id}\colon X \to X$ is always continuous, and so is any constant map.

1.2 Differentiable manifolds

Intuitively, a manifold is a topological space that is locally homeomorphic to subsets of $\mathbb{R}^n$.

Definition 1.2 (manifold, chart, atlas). M is an *n-dimensional (topological) manifold* if

1. M is a second countable and Hausdorff topological space.
2. For all $x \in M$, there exists an open set $U \ni x$ and a homeomorphism $\phi\colon U \to V \subset \mathbb{R}^n$.

Note that V is open because homeomorphisms push open sets. The pair (U, ϕ) is a *chart*, and a collection of pairs is an *atlas*.

The open sets U in an atlas form an open cover of M. An atlas is not necessarily unique, as there can be many open covers of a space. For $\mathbb{R}^n$, there is an atlas with just one chart using the standard coordinates.

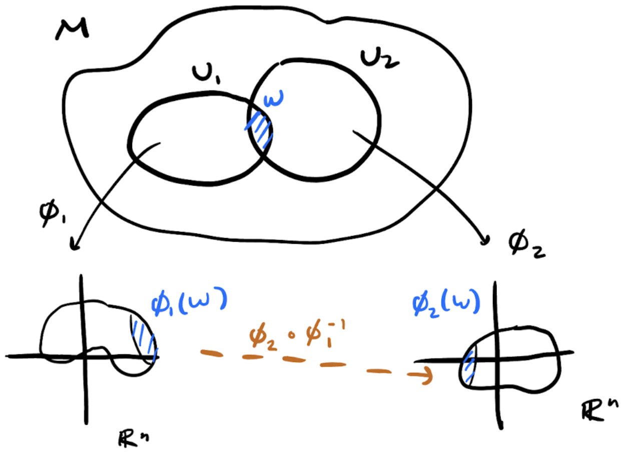
It does not make sense to say a map from $M \to \mathbb{R}^n$ is differentiable, because M may not have a differential structure. We work around this as follows.

Definition 1.3 (differentiable manifold). M is *differentiable* if for any pair of charts (U_1, ϕ_1) and (U_2, ϕ_2) with $U_1 \cap U_2 = W \neq \emptyset$, the map

$$\phi_2 \circ \phi_1^{-1}\colon \phi_1(W) \to \phi_2(W)$$

is differentiable. Note that $\phi_1(W)$ and $\phi_2(W)$ are open subsets of $\mathbb{R}^n$.

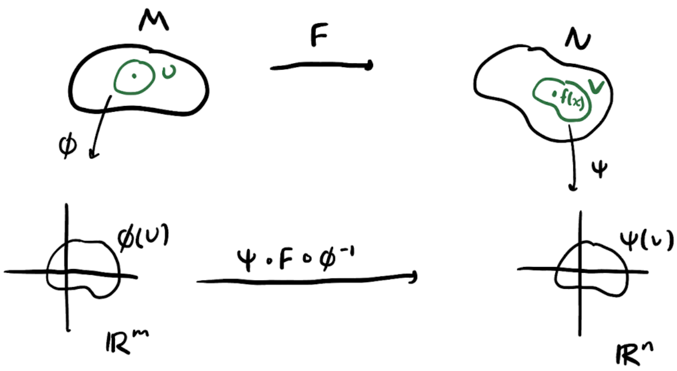
The map $\phi_2 \circ \phi_1^{-1}$ is called a *transition map*.



We will work with *smooth manifolds* where the transition maps $\phi_2 \circ \phi_1^{-1}$ are smooth (infinitely differentiable).

Given a topological manifold, the question of if there exists a differential structure (or multiple) was the beginning of the subject of differential topology.

Definition 1.4 (smooth map). For smooth manifolds M and N, a map $F\colon M \to N$ is *smooth* if for all $x \in M$, there exist charts (U, ϕ) and (V, ψ) with $x \in U$ and $F(x) \in V$ such that $\psi \circ F \circ \phi^{-1}\colon \phi(x) \to \psi(x)$ is smooth.



We only need to find one pair of charts, because it will then be true for all pairs of charts by transition functions.

Example 1.5 (smooth maps)

Letting $N = \mathbb{R}$, we have *smooth functions* $f\colon M \to \mathbb{R}$, the set of which is denoted by $C(M)$.

Letting $M = I \subset \mathbb{R}$ be an interval, we have *smooth curves* $\gamma\colon I \to N$.

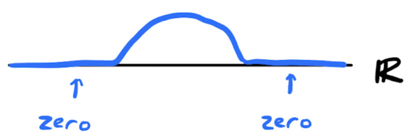
Definition 1.6 (diffeomorphism). A *diffeomorphism* $F\colon M \to N$ is a smooth bijection such that F^{-1} is smooth.

1.3 Tangent spaces

The intuition for a bump function is that it is 0 in most places and positive on an interval. For example, start with the smooth function

$$f(x) = \begin{cases} e^{-1/x^2}x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

which is infinitely differentiable at 0 because the e^{-1/x^2} term dominates any polynomial terms from the derivatives. Reflect horizontally and translate $f(x)$ to the right to get some function $g(x)$, and then the product $f(x)g(x)$ can be taken as a bump function. Note that bump functions can not be analytic (as it would have to be 0 everywhere), but can be smooth.



Lemma 1.7 (bump function)

Given $x \in M$ and $U \ni x$ open, there exists a *bump function* $u \in C(M)$ on M such that

1. $u \equiv 1$ in an open set containing x.
2. The support of u is contained in U.
3. $0 \le u \le 1$ everywhere (i.e. the image of u is in $[0, 1]$).

We can modify the example with $e^{-1/x^2}x$ to make it 0 on $x \le 0$, indeterminate for some time, and 1 after some time, and extend it to $\mathbb{R}^n$ by rotation.

Definition 1.8 (tangent vector)**.** A *tangent vector* V at $p \in M$ is an $\mathbb{R}$-linear map $V : C(M) \to \mathbb{R}$ which satisfies the Leibniz rule

$$V(fg) = f(p)V(g) + g(p)V(f).$$

This property is being a *derivation*. Let T_pM denote the set of all tangent vectors at p.

$\mathbb{R}$-*linear* means that $V(f+g) = V(f) + V(g)$ and $V(af) = aV(f)$ for all $a \in \mathbb{R}$ and $f, g \in C(M)$. It is clear T_pM is an $\mathbb{R}$-vector space, but we will see that it is n-dimensional when M is n-dimensional.

Example 1.9

Directional derivatives are tangent vectors in $M = \mathbb{R}^n$.

There is always the zero tangent vector, and we will prove that there exists a nontrivial tangent vector.

Lemma 1.10

Suppose $V \in T_pM$.

1. If $u, w \in C(M)$ are equal in a neighborhood of p, then $V(u) = V(w)$. (Locality)
2. If u is constant, then $V(u) = 0$.

Proof. By $\mathbb{R}$-linearity, $V(0) = V(0+0) = V(0) + V(0)$, so $V(0) = 0$.

1. Let η be a bump function with support in an open set where $u = w$, and $\eta(p) = 1$. Then the product $\eta(u-w) \equiv 0$, because either $u-w \equiv 0$, or $\eta \equiv 0$ when $u-w \not\equiv 0$. Then $V(\eta(u-w)) = 0$, and applying the Leibniz rule shows

$$\begin{aligned} 0 &= \eta(p)V(u-w) + (u-w)(p)V(\eta) \\ &= V(u-w) \\ &= V(u) - V(w) \end{aligned}$$

because $\eta(p) = 1$ and $(u-w)(p) = 0$. Thus, $V(u) = V(w)$.

2. By the Leibniz rule,

$$V(1) = V(1 \cdot 1) = 1 \cdot V(1) + 1 \cdot V(1),$$

so $V(1) = 0$. Then also $V(c) = cV(1) = 0$ for any constant function $c \in \mathbb{R}$. □

1.4 Basis theorem

Lemma 1.11

Given a derivation V on $\mathbb{R}$ at 0, we have for all $u \in C(\mathbb{R})$ that

$$V(u) = V(x)u'(0).$$

In other words, V is a constant times the ordinary derivative, and that the space of derivations on $\mathbb{R}$ is 1-dimensional.

Proof. By Taylor expansion, we may write $u(x) = u(0) + xw(x)$ for some $w \in C(\mathbb{R})$. Then by $\mathbb{R}$-linearity, $V(u) = V(u(0)) + V(xw) = V(xw)$, since $u(0)$ is a constant function. Then by the derivation property, $V(xw) = V(x)w(0) + x(0)V(w)$, but $x(0) = 0$ so we obtain $V(u) = V(x)w(0)$. Note that $u'(0) = w(0)$. □

What happens if V is a derivation on $\mathbb{R}$ at $p \in \mathbb{R}$? An essentially identical proof works: write $u(x) = u(p) + (x-p)w(x)$ and apply V to obtain $V(u) = (x-p)(p)V(w) + w(p)V(x-p) = w(p)V(x-p)$. Once again, $\frac{\partial u}{\partial x}(p) = w(p)\frac{\partial}{\partial x}(x-p) = w(p)$, so $V(u) = u'(p)V(x-p)$.

Lemma 1.12

Generalizing to a derivation V on $\mathbb{R}^n$ at $p \in \mathbb{R}^n$, we have for all $u \in C(\mathbb{R}^n)$ that

$$V(u) = \sum_{i=1}^{n} V(x_i)\frac{\partial u}{\partial x_i}(p).$$

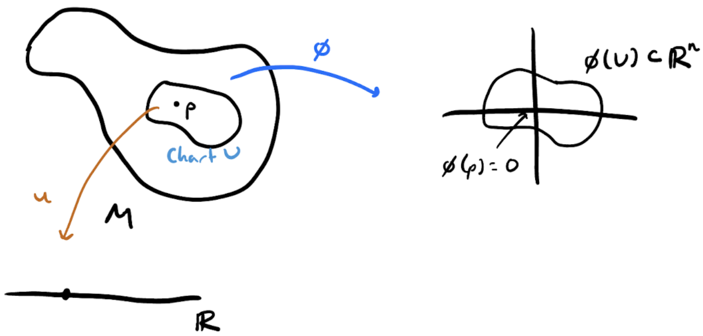
We see that $\frac{\partial}{\partial x_i}$ forms a basis for derivations on $\mathbb{R}^n$ at a point: they span by Lemma 1.12 with coefficients $V(x_i)$, and they are linearly independent as $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$.

We next show the existence of nonzero vector fields in T_pM in general.

Definition 1.13 (coordinate system). An n-tuple of functions $x_1, \ldots, x_n \in C(M)$ is a *coordinate system* at $p \in M$ if there is an open set $U \ni p$ such that $(x_1, \ldots, x_n)\colon U \to \mathbb{R}^n$ is a diffeomorphism to an open subset of $\mathbb{R}^n$.

Definition 1.14. Let (U, ϕ) be a chart with $p \in U$ and $\phi(p) = 0$, and pick a coordinate system $x_1, \ldots, x_n$ of $\mathbb{R}^n$. Corresponding to each x_i, there is a tangent vector $\partial_i|_p \in T_pM$ defined by

$$\partial_i|_p(u) := \frac{\partial(u \circ \phi^{-1})}{\partial x_i}(0).$$



Note $u \circ \phi^{-1}$ is a map $\mathbb{R}^n \to \mathbb{R}$. This is a derivation of p because $\frac{\partial}{\partial x_i}$ is and $\phi^{-1}(0) = p$:

$$\begin{aligned}
\partial_i|_p(uv) &= \frac{\partial(uv \circ \phi^{-1})}{\partial x_i}(0) \\
&= (u \circ \phi^{-1})(0)\frac{\partial(v \circ \phi^{-1})}{\partial x_i}(0) + (v \circ \phi^{-1})(0)\frac{\partial(u \circ \phi^{-1})}{\partial x_i}(0) \\
&= u(p)\partial_i|_p(v) + v(p)\partial_i|_p(u).
\end{aligned}$$

Theorem 1.15 (Basis Theorem)

Any tangent vector $V \in T_p(M)$ can be written as

$$V = \sum_{i=1}^{n} V(x_i)\partial_i|_p.$$

In particular, $\dim T_p(M) = n$, and $\partial_i|_p$ for $1 \le i \le n$ form a basis of T_pM as an $\mathbb{R}$-vector space.

Similar to before, $\partial_i|_p$ span by Theorem 1.15, and they are linearly independent as $\partial_i|_p(x_j) \equiv \delta_{ij}$. Strictly speaking, we need $V(x_i\eta)$ where η is an appropriate bump function so that things are defined, but $V(x_i\eta) = V(x_i)$ by locality (Lemma 1.10).

Proof (sketch). We first write a Taylor expansion

$$w(x) = w(0) + \sum_{i=1}^{n} x_i w_i(x).$$

Let $w = u \circ \phi^{-1}$ so that $u = w \circ \phi$. Then applying V to both sides yields

$$\begin{aligned} V(u) &= V\Big(w(0) + \sum_{i=1}^{n} (x_i \circ \phi)(w_i \circ \phi)\Big) \\ &= \sum_{i=1}^{n} (w_i \circ \phi(p)) V(x_i \circ \phi) \\ &= \sum_{i=1}^{n} w_i(0) V(x_i \circ \phi) \\ &= \sum_{i=1}^{n} V(x_i \circ \phi) \frac{\partial w}{\partial x_i}(0), \end{aligned}$$

where the second line is by the Leibniz rule and $x_i \circ \phi(p) = x_i(0) = 0$. □

There's a trick for showing this Taylor expansion exists. Fix x, and define a function $F(t) = w(tx)$. Then

$$\begin{aligned} F(1) - w(0) = w(x) - w(0) &= \int_0^1 \frac{dF}{dt}\, dt \\ &= \int_0^1 \sum_{i=1}^{n} x_i \frac{\partial w}{\partial x_i}(tx)\, dt \\ &= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial w}{\partial x_i}(tx)\, dt, \end{aligned}$$

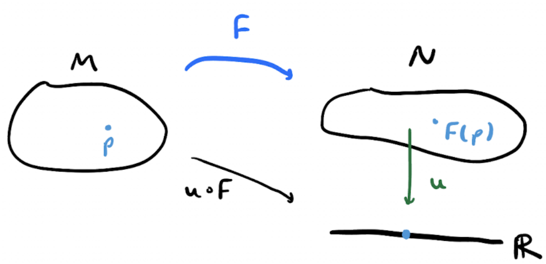
so we can let $w_i(x) \coloneqq \int_0^1 \frac{\partial w}{\partial x_i}(tx)\, dt$.

1.5 Differential of a map

Definition 1.16 (differential)**.** A smooth map $F \colon M \to N$ induces a linear map

$$dF_p \colon T_pM \to T_{F(p)}N$$

at each $p \in M$ given by $dF_p(V)(u) = V(u \circ F)$. The map dF_p is the *differential* of F at p.

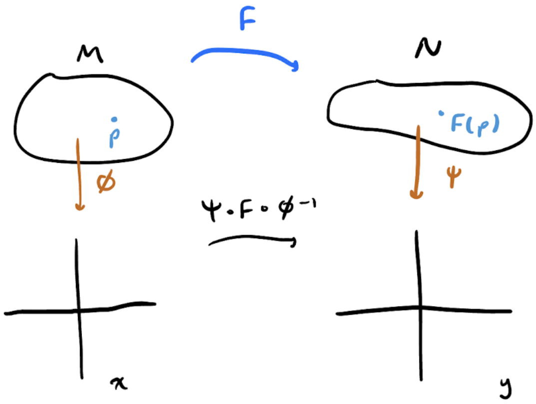


Lemma 1.17

Let x_i be a coordinate chart near $p \in M$ and y_j be a coordinate chart near $F(p) \in N$. Then

$$dF_p(\partial_{x_i}|_p) = \sum_j \left(\frac{\partial(y_j \circ F)}{\partial x_i}(p) \right) \partial_{y_j}|_{F(p)}.$$

(It's the usual Jacobian.)



Definition 1.18 (tangent vector of curve). Let $\gamma\colon I \to N$ be a differentiable curve where $I \subset \mathbb{R}$ is parameterized by s. For each $p \in I$, there is a natural tangent vector $\gamma'(p) \in T_{\gamma(p)}N$ by defining

$$\gamma'(p)(u) := d\gamma_p(\partial_s)(u) = \frac{\partial(u \circ \gamma)}{\partial s}(p).$$

Definition 1.19 (immersion, embedding). F is an *immersion* if dF_p is injective for all p.

F is an *embedding* it if is a one-to-one immersion and a homeomorphism onto its image.

Example 1.20

The figure 8 curve in $\mathbb{R}^2$ is an immersion but not an embedding.

If $\phi\colon M \to N$ is an immersion, then $\dim M \leq \dim N$. The difference $\dim N - \dim M$ is the *codimension.*

1.6 Vector fields

Definition 1.21 (vector field). A *vector field* V on M is a smooth choice of tangent vector $V(p) \in T_pM$ for every $p \in M$. (This is saying that V is globally defined, and the coefficients $V(x_i)$ are smooth in every chart.) Let $\Gamma(M)$ denote the space of vector fields on M.

Equivalently, a *vector field* V is a derivation $V\colon C(M) \to C(M)$, meaning that it is $\mathbb{R}$-linear and satisfies the Leibniz rule $V(fg) = fV(g) + gV(f)$.

The fact that these are equivalent is pset 1.5.

Example 1.22

Let U be a chart and ∂_i be a coordinate vector field on U. If η is a bump function with support inside U, then $\eta\partial_i$ is a vector field (that is 0 outside U).

Definition 1.23 (Lie bracket)**.** Given two vector fields $V, W \in \Gamma(M)$, the *Lie bracket* $[V, W] \in \Gamma(M)$ is

$$[V, W](u) = V(W(u)) - W(V(u)).$$

Intuitively, this measures how the vector field changes? It's clear that $[V, W]$ is $\mathbb{R}$-linear, but it's not obvious that the Leibniz rule holds. It turns out that the second derivatives cancel:

$$\begin{aligned}
[V, W](fg) &= V(W(fg)) - W(V(fg)) \\
&= V(fW(g) + gW(f)) - W(fV(g) + gV(f)) \\
&= V(f)W(g) + fV(W(g)) + V(g)W(f) + g(V(W(f)) \\
&\quad - W(f)V(g) - fW(V(g)) - W(g)V(f) - gW(V(f)) \\
&= f\Big(V(W(g)) - W(V(g))\Big) + g\Big(V(W(f)) - W(V(f))\Big) \\
&= f[V, W](g) + g[V, W](f).
\end{aligned}$$

Lemma 1.24

Given $f \in C(M)$ and $X, Y \in \Gamma(M)$, we have

$$[fX, Y] = f[X, Y] - Y(f)X.$$

Proof. We have

$$\begin{aligned}
[fX, Y](u) &= fX(Y(u)) - Y(fX(u)) \\
&= fX(Y(u)) - Y(f)X(u) - fY(X(u)) \\
&= f[X, Y](u) - Y(f)X(u).
\end{aligned}$$

□

1.7 Tensor algebras

For a vector space V, its *dual vector space* is

$$V^* = \{\text{linear maps } \theta\colon V \to \mathbb{R}\}.$$

If $\dim V$ is finite, then $\dim V = \dim V^*$, because the only choices for an element of V^* are what it does to basis elements of V. Letting v_i be a basis for V, there is a *dual basis* θ_j for V^* given by $\theta_j(v_i) = \delta_{ij}$.

Definition 1.25 (1-form)**.** A *1-form* α is a $C(M)$-linear map $\alpha\colon \Gamma(M) \to C(M)$. The set of all 1-forms is denoted $\Gamma^*(M)$.

Equivalently (more locally), let $T_p^*M = (T_pM)^* = \{\text{linear maps } T_pM \to \mathbb{R}\}$. We can define a 1-form α to be a smooth family of $\alpha(p) \in T_p^*M$ as p varies.

In a chart, we have a basis ∂_i for T_pM, and let dx_i be a dual basis for T_p^*M. We can write

$$\alpha = \sum_{i=1}^{n} \alpha_i dx_i$$

for some functions α_i with $\alpha_i = \alpha(\partial_i)$. Then applying α to $V = \sum_{i=1}^n V_i \partial_i$ (Theorem 1.15), we have by $C(M)$-linearity that

$$\alpha(V) = \sum_{i=1}^n V_i \alpha(\partial_i) = \sum_{i=1}^n \alpha_i V_i. \tag{1.1}$$

This is the *basis theorem for 1-forms.*

Definition 1.26 (tensor)**.** An (r,s)-*tensor* A is a $C(M)$-multilinear map from r 1-forms $\alpha_1, \ldots, \alpha_r$ and s vector fields $V_1, \ldots, V_s$ to $C(M)$. The set of (r,s)-tensors is denoted by $\Gamma^{r,s}(M)$.

Example 1.27

- A $(0,1)$-tensor is a map $\Gamma(M) \to C(M)$, which is a 1-form.
- A $(1,0)$-tensor is a vector field, with V taking α to $\alpha(V)$.
- A $(1,1)$-tensor A locally takes $V = \sum_{i=1}^n V^i \partial_i$ and $\alpha = \sum_{j=1}^n \alpha_j dx_j$, so

  $$A(V, \alpha) = A\Big(\sum_{i=1}^n V^i \partial_i, \sum_{j=1}^n \alpha_j dx_j\Big) = \sum_{i,j} A(\partial_i, dx_j) V^i \alpha_j.$$

  Let us define $A_i^j := A(\partial_i, dx_j)$, so

  $$A = \sum_{i,j} A_i^j dx_i \otimes \partial_j,$$

  and $dx_i \otimes \partial_j$ is a "basis."
- A $(0,2)$-tensor is a map from 2 vector fields to $C(M)$, so in a chart we get $\sum_{ij} g_{ij} dx_i \otimes dx_j$.
- A $(2,0)$-tensor is a map $H = \sum_{i,j} H^{ij} \partial_i \otimes \partial_j$.

1.8 Traces

Definition 1.28 (trace)**.** In a chart, the *trace* of a $(1,1)$-tensor B is

$$\mathrm{Tr}(B) = \sum_{i=1}^n B(\partial_i, dx_i).$$

It turns out this does not depend on the chart! Let $\{y_j\}$ be an overlapping chart; we have

$$dx_i = \sum_j dx_i(\partial_{y_j}) dy_j, \qquad \partial_{x_i} = \sum_k \partial_{x_i}(y_k) \partial_{y_k}$$

by the dual basis and basis theorems. Then

$$\begin{aligned}
\sum_i B(\partial_{x_i}, dx_i) &= \sum_{i,j,k} B\left(\frac{\partial y_k}{\partial x_i} \partial_{y_k}, dx_i(\partial_{y_j}) dy_j\right) \\
&= \sum_{i,j,k} B(\partial_{y_k}, dy_j) \frac{\partial y_k}{\partial x_i} \frac{\partial x_i}{\partial y_j} \\
&= \sum_j B(\partial_{y_j}, dy_j),
\end{aligned}$$

where the last equality is by the chain rule:

$$\sum_i \frac{\partial y_k}{\partial x_i}\frac{\partial x_i}{\partial y_j} = \frac{\partial y_k}{\partial y_j} = \delta_{jk}.$$

We see that the trace of a $(1,1)$-tensor gives a $(0,0)$-tensor. In general, the trace of (r,s)-tensor gives an $(r-1,s-1)$-tensor (but you have rs choices for which indices to trace over).

1.9 Metrics

Definition 1.29 (metric)**.** A *metric* g is a $(0,2)$-tensor such that

1. $g(V,W) = g(W,V)$ (symmetric).
2. $g(V,V)(p) \geq 0$, with equality if and only if $V(p) = 0$ (positive definite).

Definition 1.30 (Riemannian manifold)**.** A *Riemannian manifold* (M^n, g) is a smooth manifold M^n with a metric g.

In a chart, we have $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$, where $g_{ij} = g(\partial_i, \partial_j)$. At each point p, the matrix $\{g_{ij}(p)\}$ must be positive definite and symmetric ($g_{ij} = g_{ji}$).

Example 1.31

$\mathbb{R}^n$ with metric $g_{ij} = \delta_{ij}$ is Euclidean space. The identity matrix is positive definite and symmetric.

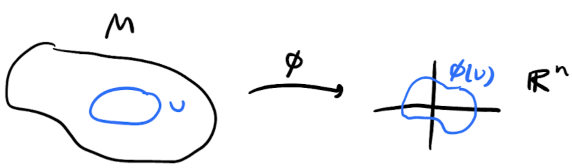
Definition 1.32 (length)**.** We can now define the *length* of a curve $\gamma\colon [a,b] \to M$ by

$$L(\gamma) = \int_a^b \sqrt{g(\gamma'(s), \gamma'(s))}\, ds,$$

where $g(\gamma'(s), \gamma'(s))$ is the norm of the tangent vector.

Definition 1.33 (volume)**.** For an open set U in a chart, we can define its *volume* by

$$\mathrm{vol}(U) = \int_{\phi(U)} \sqrt{|\det g_{ij}|}\, dx_1 \cdots dx_n.$$



1.10 Induced metrics on tensors

Metrics can measure tensors too. For $V \in \Gamma(M)$, define $|V|^2 = g(V,V)$. In a chart, we have

$$|V|^2 = \sum_{i,j} g_{ij} V^i V^j.$$

For $\alpha \in \Gamma^*(M)$, we similarly define

$$|\alpha|^2 = \sum_{i,j} g^{ij}\alpha_i\alpha_j,$$

where $\{g^{ij}\}$ is the inverse matrix of $\{g_{ij}\}$. It turns out this is a $(2,0)$-tensor called g^{-1} (pset 1.4).

For a $(2,0)$-tensor B, we define

$$|B|^2 = \sum_{i,j,p,q} B^{ij}B^{pq}g_{ip}g_{jq}$$

which does not depend on the chart because it is the 4-time trace of the $(4,4)$-tensor $B \otimes B \otimes g \otimes g$.

Definition 1.34 (pullback metric)**.** Let $F\colon M \to N$ be an immersion where N has a metric g. The *pullback metric* F^*g on M is given by

$$(F^*g)(V,W)(p) = g(dF(V), dF(W))(F(p)).$$

1.11 Raising and lowering indices

Given a Riemannian metric g, there is a canonical 1-form $\alpha_V \in \Gamma^*(M)$ associated to each vector field $V \in \Gamma(M)$ given by

$$\alpha_V(W) = g(V,W).$$

In local coordinates, let $V = \sum_i V^i\partial_{x_i}$, $W = \sum_j W^j\partial_{x_j}$, and $g = \sum_{k,\ell} g_{k\ell}dx_k \otimes dx_\ell$. Then

$$\begin{aligned}\alpha_V(W) &= g\Big(\sum_i V^i\partial_{x_i}, \sum_j W^j\partial_{x_j}\Big)\\ &= \sum_{i,j} V^iW^jg(\partial_{x_i}, \partial_{x_j})\\ &= \sum_{i,j} V^iW^jg_{ij},\end{aligned}$$

as expected. Thinking of it as a tensor contraction of $V \otimes W \otimes g$ is another way to see that it is well defined.

Remark 1.35. Technically, V^i and W^j are not in $C(M)$ as they are only defined locally, so it is somewhat illegal to use the $C(M)$-linearity of g above. However, we can multiply them by bump functions so they become $C(M)$ functions, as we are working locally. For the rest of the course, we will not mention this.

On the other hand from the basis theorem for 1-forms (1.1), we have

$$\alpha_V(W) = \sum_j W^j\alpha_V(\partial_{x_j}) = \sum_j W^j(\alpha_V)_j,$$

so

$$(\alpha_V)_j = \sum_i g_{ij}V^i.$$

This process is called *lowering the index.*

The process of *raising the index* takes a 1-form ($(0,1)$-tensor) to a vector field ($(1,0)$-tensor). An example is the *gradient* ∇f which locally is

$$\nabla f = \sum_{i,j} g^{ij}\frac{\partial f}{\partial x_i}\partial_{x_j}$$

with jth coordinate

$$\nabla f^j = \sum_i g^{ij} \frac{\partial f}{\partial x_i}.$$

We see that the norm is preserved after raising the index:

$$\begin{aligned}
|\nabla f|^2 &= \sum_{i,j} g_{ij} \nabla f^i \nabla f^j \\
&= \sum_{i,j,\ell,m} g_{ij} \Big(g^{i\ell} \frac{\partial f}{\partial x_\ell} \Big) \Big(g^{jm} \frac{\partial f}{\partial x_m} \Big) \\
&= \sum_{j,\ell,m} \delta_{j\ell} \frac{\partial f}{\partial x_\ell} g^{jm} \frac{\partial f}{\partial x_m} \\
&= \sum_{\ell,m} g^{\ell m} \frac{\partial f}{\partial x_\ell} \frac{\partial f}{\partial x_m} \\
&= |df|^2 .
\end{aligned}$$

In general, we can go between any (r, s)-tensors as long as $r + s$ is kept constant. The same proof shows that raising or lowering indices in general preserves the norm.

Example 1.36

We know that every vector field on S^2 vanishes at some point (hairy ball theorem). Similarly, every 1-form must vanish at some point, because lowering the index preserves the norm.

2 Connections and covariant derivatives

2.1 Affine connections

Definition 2.1 (affine connection)**.** An *affine connection* is a map $\nabla_\bullet(\bullet)\colon \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ sending $X, Y \mapsto \nabla_X Y$ such that

1. $\nabla_X Y$ is $C(M)$-linear in X
2. $\nabla_X Y$ is $\mathbb{R}$-linear in Y
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ (Leibniz rule for Y).

Think of it as differentiating Y with respect to X. Note that if $\nabla_X Y$ were $C(M)$-linear in Y, then it would be a $(1, 2)$-tensor. Intuitively, multiplying Y by a function should have some Leibniz rule.

Lemma 2.2

If $V \in \Gamma(M)$, then $\nabla_\bullet V$ is a $(1, 1)$-tensor with $\nabla_\bullet V(W, \alpha) \coloneqq \alpha(\nabla_W V)$.

Proof. It is $C(M)$-linear in $W \in \Gamma(M)$ by definition and in $\alpha \in \Gamma^*(M)$ because α is $C(M)$-linear. □

We can use the Leibniz rule to define ∇ for all tensors. Given $\alpha \in \Gamma^*(M)$, we want a 1-form $\nabla_X \alpha$ satisfying $X(\alpha(V)) = (\nabla_X \alpha)(V) + \alpha(\nabla_X V)$, so we define

$$(\nabla_X \alpha)(V) \equiv X(\alpha(V)) - \alpha(\nabla_X V).$$

Similarly given $g \in \Gamma^{0,2}(M)$, we define $\nabla_X g$ by

$$(\nabla_X g)(V, W) \equiv X(g(V, W)) - g(\nabla_X V, W) - g(V, \nabla_X W). \tag{2.1}$$

For $B(V, \alpha) \in \Gamma^{1,1}(M)$, we define $\nabla_X B$ by

$$(\nabla_X B)(V, \alpha) \equiv X(B(V, \alpha)) - B(\nabla_X V, \alpha) - B(V, \nabla_X \alpha).$$

2.2 Christoffel symbols

Definition 2.3 (Christoffel symbols)**.** The *Christoffel symbols* are locally defined functions Γ_{ij}^k satisfying

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k.$$

Remark 2.4. The Γ_{ij}^k are only locally defined and depend on the choice of coordinates. They do not vary algebraically (i.e. via some matrix multiplication) because partials are involved.

If we know the Γ_{ij}^k, then we can compute $\nabla_X Y$ locally:

$$\begin{aligned}
\nabla_X Y &= \sum_{i,j} \nabla_{X^i \partial_i}(Y^j \partial_j) \\
&= \sum_{i,j} X^i \nabla_{\partial_i}(Y^j \partial_j) \\
&= \sum_{i,j} X^i [Y^j \nabla_{\partial_i} \partial_j + \partial_i(Y^j) \partial_j] \\
&= \sum_{i,j} X^i \Big[Y^j \sum_k \Gamma_{ij}^k \partial_k + \partial_i(Y^j) \partial_j \Big].
\end{aligned}$$

where the second equality uses linearity in the bottom entry, and the third equality is by the Leibniz rule. There is a dual identity with

$$\nabla_{\partial_i} dx_j = -\sum_k \Gamma_{ik}^j dx_k.$$

This follows from

$$\begin{aligned}
(\nabla_{\partial_i} dx_j)(\partial_k) &= \partial_i(dx_j(\partial_k)) - dx_j(\nabla_{\partial_i} \partial_k) \\
&= -dx_j \Big(\sum_\ell \Gamma_{ik}^\ell \partial_\ell \Big) \\
&= \sum_\ell \delta_{j\ell} \Gamma_{ik}^\ell \\
&= -\Gamma_{ik}^j,
\end{aligned}$$

where the first equality is by the Leibniz rule, and the second equality is from $\partial_i(\mathrm{Id}) = 0$.

2.3 Levi-Civita connection

If we impose two additional conditions on $\nabla_X Y$, there will be a canonical affine connection.

1. *Metric compatibility*: $\nabla_X g = 0$ for any X. Then from (2.1), $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

2. *Symmetry*: $[X, Y] = \nabla_X Y - \nabla_Y X$. In local coordinates, it is equivalent to $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all i, j, k, as

$$\begin{aligned} 0 &= [\partial_i, \partial_j] \\ &= \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i \\ &= \sum_k (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k. \end{aligned}$$

Theorem 2.5 (Levi-Civita connection)

There exists a unique connection for each g, called the *Levi-Civita connection.*

From now on, let $\langle X, Y\rangle$ denote $g(X, Y)$.

Proof. We will find a formula for $\nabla_X Y$ to show uniqueness. From metric compatibility, we get

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{2.2}$$
$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \tag{2.3}$$
$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \tag{2.4}$$

From (2.2) + (2.3) − (2.4), we have

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \langle \nabla_X Y, Z\rangle + \langle Z, \nabla_Y X\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle + \langle X, \nabla_Y Z - \nabla_Z Y\rangle \\ &= 2\langle \nabla_X Y, Z\rangle - \langle \nabla_X Y, Z\rangle + \langle \nabla_Y X, Z\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle \\ &= 2\langle \nabla_X Y, Z\rangle + \langle Z, [Y, X]\rangle + \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle \end{aligned}$$

so we know $\langle \nabla_X Y, Z\rangle$ for all Z. As g is positive definite and in particular invertible, this uniquely determines $\nabla_Y X$.

It remains to check that this definition of $\nabla_X Y$ satisfies Definition 2.1, which is pset 2.2. □

2.4 Christoffel symbols in terms of the metric

Applying the formula from Theorem 2.5 with $X = \partial_i, Y = \partial_j, Z = \partial_k$ yields

$$\begin{aligned} \partial_i g_{ij} + \partial_j g_{ik} - \partial_k g_{ij} &= 2\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle \\ &= 2\Big\langle \sum_\ell \Gamma^\ell_{ij}\partial_\ell, \partial_k\Big\rangle \\ &= 2\sum_\ell \Gamma^\ell_{ij} g_{\ell k}. \end{aligned}$$

Multiplying by g^{mk} and summing over k, we obtain

$$2\Gamma^m_{ij} = \sum_k g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}).$$

Example 2.6

On $\mathbb{R}^n$, $\Gamma \equiv 0$ because the metric is δ_{ij}.

Remark 2.7. We will begin to use the Einstein summation notation which omits the summation $\sum$ if there is an index variable appearing as an upper and a lower index. For example, we would write $2\Gamma_{ij}^m = g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$

Remark 2.8. Vector fields use upper indices, 1-forms use lower indices, and tensors use an appropriate combination of upper and lower indices. When tracing over an index, it should appear in both upper and lower indices.

2.5 Parallel transport

Let γ be a curve with tangent vector γ'. In local coordinates, γ is given by its components γ^i for $1 \le i \le n$. Let

$$\gamma_s^i(s) = \frac{\partial \gamma^i}{\partial s}(s), \qquad \gamma'(s) = \sum_i \gamma_s^i(s)\partial_i.$$

Let V be a vector field along the curve and write $V = \sum_j V^j(s)\partial_j$. We want to define a derivative along the curve by

$$\begin{aligned}\nabla_{\gamma'} V &\overset{?}{=} \sum_{i,j} \gamma_s^i \nabla_{\partial_i}(V^j\partial_j) \\ &\overset{?}{=} \sum_{i,j} \gamma_s^i(\partial_i(V^j)\partial_j + V^j\nabla_{\partial_i}\partial_j) \\ &\overset{?}{=} \sum_j V_s^j\partial_j + \sum_{i,j,k}\gamma_s^i V^j \Gamma_{ij}^k \partial_k.\end{aligned}$$

where

$$V_s^j(s) = \frac{d}{ds}V^j(\gamma(s)).$$

Note V^j is only defined along the curve so terms like $\nabla_{d_i}(V^j\partial_j)$ do not actually make sense, but this is the motivation for the actual definition.

Definition 2.9 (derivative along curve)**.** For a curve γ and a vector field V, we define

$$\nabla_{\gamma'} V = \sum_j V_s^j\partial_j + \sum_{i,j,k}\gamma_s^i V^j \Gamma_{ij}^k \partial_k.$$

This function takes in a point on γ and outputs a tangent vector at that point.

Definition 2.10 (parallel along curve)**.** V is *parallel along* γ if $\nabla_{\gamma'} V \equiv 0$.

The parallel condition is a 1st order system of ODEs in s. Existence and uniqueness of ODEs implies that each initial condition has a unique solution, called *parallel transport* along γ.

Example 2.11

On Euclidean space, parallel transport leaves the vector constant. From $\Gamma_{ij}^k \equiv 0$, we need $V_s^j \equiv 0$.

Definition 2.12 (parallel)**.** A tensor A is *parallel* if $\nabla_X A \equiv 0$ for any vector field X. A parallel vector field $A = V$ on the entire space is also parallel along any curve γ.

Claim 2.13 — The length $|V|^2$ is preserved under parallel transport.

Proof. By the chain rule, we have

$$\frac{d}{ds}|V|^2(\gamma(s)) = \gamma'(|V|^2) = 2\langle \nabla_{\gamma'} V, V\rangle = 2\langle 0, V\rangle = 0.$$

□

2.6 Killing fields

Definition 2.14 (Lie derivative)**.** For a vector field V and an (r,s)-tensor A, we define the Lie derivative $\mathcal{L}_V A$, which is an (r,s)-tensor, as follows. Let $\mathcal{L}_V W = [V, W]$ and $\mathcal{L}_V f = V(f)$. Then we can define $\mathcal{L}_V$ on any tensor by the Leibniz rule.

For example, the Lie derivative of a Riemannian metric g is

$$\begin{aligned}(\mathcal{L}_V g)(X,Y) &= V\langle X, Y\rangle - \langle \mathcal{L}_V X, Y\rangle - \langle X, \mathcal{L}_V Y\rangle \\ &= \langle \nabla_V X, Y\rangle + \langle X, \nabla_V Y\rangle - \langle [V,X], Y\rangle - \langle X, [V,Y]\rangle && \text{(metric compatibility)} \\ &= \langle \nabla_V X, Y\rangle + \langle X, \nabla_V Y\rangle - \langle \nabla_V X - \nabla_X V, Y\rangle - \langle X, \nabla_V Y - \nabla_Y V\rangle && \text{(symmetry)} \\ &= \langle \nabla_X V, Y\rangle + \langle X, \nabla_Y V\rangle.\end{aligned}$$

Definition 2.15 (Killing fields)**.** A vector field V is *Killing* if $\mathcal{L}_V g = 0$. Equivalently,

$$\langle \nabla_X V, Y\rangle + \langle X, \nabla_Y V\rangle = 0,$$

meaning that $\langle \nabla_\bullet V, \bullet\rangle$ is skew.

Example 2.16 (rotation)

On $(\mathbb{R}^2, \delta_{ij})$, consider the vector field $V = x_1\partial_2 - x_2\partial_1$. Then $\nabla_V = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ by computing

$$\langle \nabla_{\partial_1} V, \partial_1\rangle = 0, \quad \langle \nabla_{\partial_2} V, \partial_2\rangle = 0, \quad \langle \nabla_{\partial_1} V, \partial_2\rangle = 1, \quad \langle \nabla_{\partial_2} V, \partial_1\rangle = -1,$$

so ∇_V is skew-symmetric.

The Killing field V generates a one-parameter family of isometries (rotations) with

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}.$$

We have $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta\end{pmatrix}$. Taking the derivative and restricting to $\theta = 0$ yields

$$\frac{\partial}{\partial\theta}R(\theta)\begin{pmatrix} x_1 \\ x_2\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix} -x_1\sin\theta - x_2\cos\theta \\ x_1\cos\theta - x_2\sin\theta\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix} -x_2 \\ x_1\end{pmatrix}$$

which gives $V = -x_2\partial_1 + x_1\partial_2$.

Pset 2.4 involves finding all Killing forms on Euclidean space (consider translations and rotations at least). There will be few solutions because it is a first-order system.

3 Curvature

3.1 Riemann curvature tensor

Definition 3.1 (Riemann tensor)**.** A *Riemann tensor* R is a $(1,3)$-tensor with $R(X,Y)Z \in \Gamma(M)$ defined by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

It is canonically the same as a $(0,4)$-tensor with

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle.$$

We will prove that R is a tensor. It is clear that R is $\mathbb{R}$-linear in each slot, but we need to show that it is $C(M)$-linear.

We apply Lemma 1.24 which states $[fX,Y] = f[X,Y] - Y(f)X$.

$$\begin{aligned}
R(fX,Y)Z &\equiv \nabla_Y \nabla_{fX} Z - \nabla_{fX} \nabla_Y Z + \nabla_{[fX,Y]} Z \\
&= \nabla_Y (f \nabla_X Z) - f \nabla_X \nabla_Y Z + \nabla_{f[X,Y]-Y(f)X}(Z) \\
&= Y(f)\nabla_X Z + f \nabla_Y \nabla_X - f \nabla_X \nabla_Y Z + f \nabla_{[X,Y]} Z - Y(f) \nabla_X Z \\
&= f \nabla_Y \nabla_X - f \nabla_X \nabla_Y Z + f \nabla_{[X,Y]} Z \\
&= f R(X,Y)Z.
\end{aligned}$$

This shows $C(M)$-linearity in X, and by skew-symmetry $R(X,Y)Z = -R(Y,X)Z$ it is also $C(M)$-linear in Y. Finally we check $C(M)$-linearity in Z:

$$\begin{aligned}
R(X,Y)(fZ) &\equiv \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X,Y]}(fZ) \\
&= \nabla_Y (f \nabla_X Z + X(f)Z) - \nabla_X (Y(f)Z + f \nabla_Y Z) + [X,Y](f)Z + f \nabla_{[X,Y]} Z \\
&= Y(f)\nabla_X Z + f \nabla_Y \nabla_X Z + Y(X(f))Z + X(f) \nabla_Y Z \\
&\quad - X(Y(f))Z - Y(f)\nabla_X Z - f \nabla_X \nabla_Y Z - X(f) \nabla_Y Z + [X,Y](f)Z + f \nabla_{[X,Y]} Z \\
&= f \nabla_Y \nabla_X Z - f \nabla_X \nabla_Y Z + f \nabla_{[X,Y]} Z \\
&= f R(X,Y)Z.
\end{aligned}$$

The Riemann tensor has many symmetries in addition to skew-symmetry $R(X,Y)Z = -R(Y,X)Z$.

Lemma 3.2 (first Bianchi identity)
$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$

The key is showing that it equals the Jacobi identity $[Y,[X,Z]] + [X,[Z,Y]] + [Z,[Y,X]] = 0$ from pset 1.

Proof.

$$\begin{aligned}R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z \\ &\quad + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X \\ &\quad + \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y \\ &= \nabla_Y[X,Z] + \nabla_X[Z,Y] + \nabla_Z[Y,X] \\ &\quad + \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[Z,X]}Y \\ &= [Y,[X,Z]] + [X,[Z,Y]] + [Z,[Y,X]] \\ &= 0\end{aligned}$$

□

Recall that the $(0,4)$-tensor $R(X,Y,Z,W)$ from lowering an index is given by $\langle R(X,Y)Z, W\rangle$.

Lemma 3.3 (skew in last two)
$R(X,Y,Z,W) = -R(X,Y,W,Z)$

Proof. Using polarization, we just need to show that $R(X,Y,V,V) = 0$ for all V.

$$\begin{aligned}R(X,Y,V,V) &= \langle\nabla_Y\nabla_X V, V\rangle - \langle\nabla_X\nabla_Y V, V\rangle + \langle\nabla_{[X,Y]}V, V\rangle \\ &= \Big(Y\langle\nabla_X V, V\rangle - \langle\nabla_X V, \nabla_Y V\rangle\Big) - \Big(X\langle\nabla_Y V, V\rangle - \langle\nabla_Y V, \nabla_X V\rangle\Big) + \frac{1}{2}[X,Y]\langle V,V\rangle \\ &= \frac{1}{2}YX\langle V,V\rangle - \frac{1}{2}XY\langle V,V\rangle + \frac{1}{2}[X,Y]\langle V,V\rangle \\ &= 0\end{aligned}$$

where we use metric compatibility. □

Lemma 3.4 (symmetry in pairs)
$R(X,Y,Z,W) = R(Z,W,X,Y)$.

Proof. Using the first Bianchi identity, we have

$$0 = R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) \tag{3.1}$$
$$0 = R(X,Z,W,Y) + R(Z,W,X,Y) + R(W,X,Z,Y) \tag{3.2}$$
$$0 = R(X,Y,W,Z) + R(Y,W,X,Z) + R(W,X,Y,Z) \tag{3.3}$$
$$0 = R(Y,Z,W,X) + R(Z,W,Y,X) + R(W,Y,Z,X). \tag{3.4}$$

Taking (3.1) - (3.3) and using Lemma 3.3, we get

$$0 = 2R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) - R(Y,W,X,Z) - R(W,X,Y,Z).$$

Taking (3.2) - (3.4) and using Lemma 3.3, we get

$$0 = 2R(Z,W,X,Y) + R(X,Z,W,Y) + R(W,X,Z,Y) - R(Y,Z,W,X) - R(W,Y,Z,X).$$

The colored terms match up, so the remaining terms $2R(X,Y,Z,W)$ and $2R(Z,W,X,Y)$ are equal. □

We did the only thing that we could do to prove these identities.

Lemma 3.5 (second Bianchi identity)
$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0.$

Recall $\nabla_X R$ is defined by the Leibniz rule:

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z) - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W.$$

3.2 Ricci curvature

Definition 3.6 (Ricci tensor). The *Ricci tensor* is a $(0,2)$-tensor from tracing R in an orthonormal frame:

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^n R(X, e_i, Y, e_i)$$

where $\langle e_i, e_j \rangle = \delta_{ij}$.

There are three possible ways to trace, but $R(X,Y,e_i,e_i)$ gives 0, and $R(X,e_i,e_i,Y)$ is the negative. Note $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$ by symmetry in pairs (Lemma 3.4).

Definition 3.7 (Einstein). A manifold is *Einstein* if $\mathrm{Ric} = \lambda g$ for a constant $\lambda \in \mathbb{R}$.

We say $\mathrm{Ric} \geq \lambda$ for $\lambda \in \mathbb{R}$ if $\mathrm{Ric}(V,V) \geq \lambda |V|^2$. $\mathrm{Ric} \geq 0$ is a common condition.

Example 3.8
Euclidean space is Einstein because $R = 0$ so $\mathrm{Ric} = 0$.

Definition 3.9 (scalar curvature). Tracing the Ricci tensor gives the *scalar curvature* $S \in C(M)$ defined by

$$S = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$$

where e_i is an orthonormal frame.

In a general frame, $S = \sum_{i,k=1}^n \sum_{j=1}^n g^{ij}\,\mathrm{Ric}_{jk}$ (raise an index and then trace, because we can't directly trace a $(0,2)$-tensor).

Definition 3.10 (sectional curvature). Given orthonormal vectors V, W, the *sectional curvature* is

$$K_{VW} = R(V,W,V,W).$$

Lemma 3.11 (Schur)
$dS = 2\,\mathrm{div}(\mathrm{Ric})$

$\nabla\,\mathrm{Ric}$ is $(0,3)$-tensor, which we raise to get $(1,2)$-tensor, and then trace to get a $(0,1)$-tensor, i.e. a 1-form.

In an orthonormal frame,

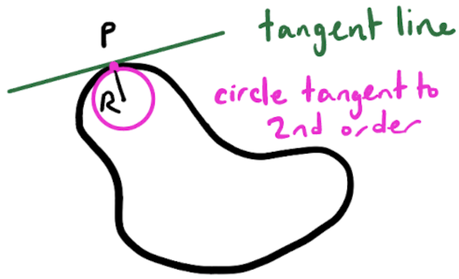
$$\begin{aligned}\operatorname{div}(\operatorname{Ric})(V) &= \sum_{i=1}^{n} (\nabla_{e_i} \operatorname{Ric})(e_i, V) \\ &= \sum_{i=1}^{n} \Big(e_i \operatorname{Ric}(e_i, V) - \operatorname{Ric}(\nabla_{e_i} e_i, V) - \operatorname{Ric}(e_i, \nabla_{e_i} V \Big) \\ &= \sum_{i,j=1}^{n} \Big(e_i R(e_i, e_j, V, e_j) - R(\nabla_{e_i} e_i, e_j, V, e_j) - R(e_i, e_j, \nabla_{e_i} V, e_j) \Big) \\ &= \sum_{i,j=1}^{n} \Big((\nabla_{e_i} R)(e_i, e_j, V, e_j) + R(e_i, \nabla_{e_i} e_j, V, e_j) + R(e_i, e_j, V, \nabla_{e_i} e_j) \Big)\end{aligned}$$

2nd Bianchi says that $(\nabla_{e_i} R)(e_i, e_j) + (\nabla_{e_i} R)(e_j, e_i) + (\nabla_{e_j} R)(e_i, e_i) = 0$.

4 Submanifolds

A circle in Euclidean space of radius R has curvature $\frac{1}{R}$.

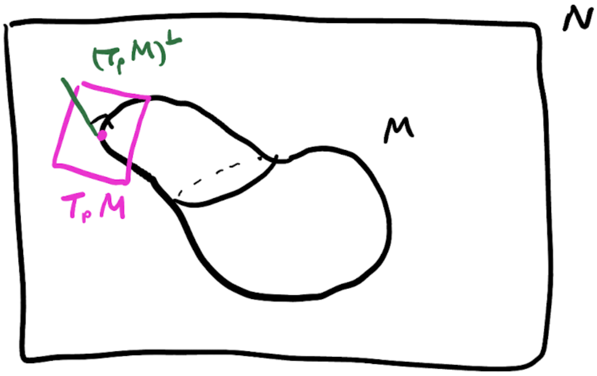
Definition 4.1 (geodesic curvature)**.** The *geodesic curvature* is $k_g(p) = \pm\frac{1}{R}$ where the $\pm$ is depending on whether the circle is inside or outside.



The intrinsic curvature of a curve viewed as a manifold is 0, but the extrinsic curvature when viewed as a submanifold is more interesting.

4.1 Induced structure on submanifolds

Consider an immersion $F\colon M^m \hookrightarrow N^n$, which means dF is injective and $m \leq n$.



At each point $p \in F(M)$, we view T_pM as a subspace of T_pN and write

$$T_pN = (T_pM) \oplus (T_pM)^\perp.$$

In particular, any V can be written as

$$V = V^\top + V^\perp$$

where $V^\top \in T_pM$ is the *tangential part* and $V^\perp \in (T_pM)^\perp$ is the *normal part*.

The structure of N induces various structures on M. For now think of F as the identity map, but these definitions can be modified for any immersion F.

- **Induced metric**: g_N induces a metric g_M on M with

$$g_M(V, W) \equiv g_N(dF(V), dF(V)).$$

  It also induces a metric on $(T_pM)^\perp$.

- **Induced connections**: An affine connection ∇ on N induces an affine connection $\overline{\nabla}$ on M by

$$\overline{\nabla}_V W \equiv (\nabla_V W)^\top$$

  for V, W vector fields tangent to M. To check that it is a connection, we first see that it is $C(M)$-linear in the lower entry. Also for $f \in C(M)$, we have

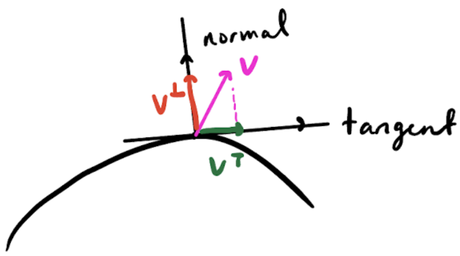
$$\begin{aligned}\overline{\nabla}_V(fW) &= (\nabla_V(fW))^\top \\ &= (f\nabla_V W + V(f)W)^\top \\ &= f(\nabla_V W)^\top + V(f)W.\end{aligned}$$

- **Induced connection on normal bundle**: If X normal vector field (i.e. a vector field that is perpendicular at each point to T_pM) and V is a tangent vector field, then we can define

$$\overline{\nabla}^\perp_V X \equiv (\nabla_V X)^\perp.$$

The normal part that we throw away in the induced connection is called the second fundamental form.

> **Definition 4.2** (second fundamental form). The *second fundamental form* is $A(V, W) = (\nabla_V W)^\perp$ for tangent vector fields V, W.
>
> A is in $\Gamma^{0,2}(M) \otimes \Gamma(TM^\perp)$ as it takes in two tangent vector fields and outputs a normal vector field.



Example 4.3 ($S^1 \hookrightarrow \mathbb{R}^2$)

Consider the embedding $S^1 \hookrightarrow \mathbb{R}^2$. Then

$$\nabla_\theta(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta).$$

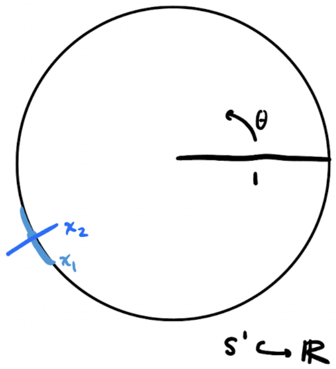
Since the Levi-Civita connection on $\mathbb{R}^2$ is the directional derivative, we have

$$\nabla_{\partial_\theta}\partial_\theta = (-\cos\theta, -\sin\theta)$$

which is purely normal and equal to $(\nabla_{\partial_\theta}\partial_\theta)^\perp$. Then $(\nabla_{\partial_\theta}\partial_\theta)^\top = 0$ and

$$A(\partial_\theta, \partial_\theta) = (-\cos\theta, -\sin\theta),$$

so the second fundamental form is the inward-pointing vector.



For general $V = f(\theta)\partial_\theta$, we have $\overline{\nabla}_{\partial_\theta}(f(\theta)\partial_\theta) = f'(\theta)\partial_\theta$.

Proposition 4.4

The second fundamental form is symmetric: $A(V, W) = A(W, V)$. (In the two tangent slots, although it also has a normal slot.)

Proof. We have

$$\begin{aligned} A(V,W) - A(W,V) &= (\nabla_V W)^\perp - (\nabla_W V)^\perp \\ &= (\nabla_V W - \nabla_W V)^\perp \\ &= ([V,W])^\perp \\ &= 0 \end{aligned}$$

where the last equation is by the general fact that if V, W are tangent to M then so is $[V, W]$. To show this, work in a local coordinate chart and choose coordinates $(x_1, \ldots, x_n)$ on N such that M is given by $\{x_{m+1} = \cdots = x_n = 0\}$. We can do this by implicit function theorem.

Then $V = \sum_{i=1}^m V^i \partial_i$ and $W = \sum_{j=1}^m W^j \partial_j$ are linear combinations of only $\partial_1, \ldots, \partial_m$. We compute

$$\begin{aligned} V(W(u)) &= \sum_{i,j} V^i \partial_i (W^j u_j) = \sum_{i,j} V^i (\partial_i W^j) u_j + V^i W^j u_{ji} \\ W(V(u)) &= \sum_{i,j} W^j \partial_j (V^i u_i) = \sum_{i,j} W^j (\partial_j V^i) u_i + V^i W^j u_{ji} \\ [V, W](u) &= \sum_{i,j} V^i (\partial_i W^j) u_j - W^j (\partial_j V^i) u_i. \end{aligned}$$

(Here $u_i = \frac{\partial}{\partial x_i} u$.) Note that the u_{ji} terms cancel, and we are left with just a linear combination of $\partial_1, \ldots, \partial_m$, implying that $[V, W]$ is tangent to M. □

Since A is symmetric, all eigenvalues are real and are called *principal curvatures*. The sum of all principal curvatures is the *mean curvature*.

Definition 4.5 (mean curvature)**.** The *mean curvature* is the trace $\operatorname{Tr} A = \vec{H}$.

If $\vec{H} = \vec{0}$ then it is called *minimal*.

4.2 Gauss equation

Lemma 4.6 (Gauss equation)

Let $M \subset N$ be a submanifold, and let $\overline{R}$ and R be Riemann curvature tensors on M and N. Then for any vector fields W, X, Y, Z tangent to M, we have

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + \langle A(X, Z), A(Y, W)\rangle - \langle A(Y, Z), A(X, W)\rangle.$$

This is saying that the curvature is determined by the curvature R in ambient space and how M sits in the ambient space via A.

Proof.

$$\begin{aligned} \overline{R}(X, Y, Z, W) &= \langle W, \nabla_Y^\top (\nabla_X^\top Z) - \nabla_X^\top (\nabla_Y^\top Z) + \nabla_{[X,Y]}^\top Z\rangle \\ &= \langle W, \nabla_Y (\nabla_X^\top Z) - \nabla_X (\nabla_Y^\top Z) + \nabla_{[X,Y]} Z\rangle \\ &= \langle W, \nabla_Y (\nabla_X Z - A(X, Z)) - \nabla_X (\nabla_Y Z - A(Y, Z)) + \nabla_{[X,Y]} Z\rangle \\ &= R(X, Y, Z, W) - \langle W, \nabla_Y A(X, Z)\rangle + \langle W, \nabla_X A(Y, Z)\rangle \\ &= R(X, Y, Z, W) + \langle \nabla_Y W, A(X, Z)\rangle - \langle \nabla_X W, A(Y, Z)\rangle \\ &= R(X, Y, Z, W) + \langle \nabla_Y^\perp W, A(X, Z)\rangle - \langle \nabla_X^\perp W, A(Y, Z)\rangle \\ &= R(X, Y, Z, W) + \langle A(Y, W), A(X, Z)\rangle + \langle A(X, W), A(Y, Z)\rangle. \end{aligned}$$

Since W is tangential, we can get rid of the $\top$'s in the second equation. We use metric compatibility in the fifth equation and add back the $\perp$'s in the sixth equation because $A(\bullet, \bullet)$ is normal. □

Example 4.7

Consider $S^n \hookrightarrow \mathbb{R}^{n+1}$ where $S^n = \{|\vec{x}|^2 = 1\}$. If V, W are tangent vectors to S^n, then at the point x we have

$$A(V,W) = \langle \nabla_V W, \vec{x} \rangle \vec{x} = -\langle W, \nabla_V \vec{x} \rangle \vec{x}$$

because $A(V,W)$ is the normal part of $\nabla_V W$. The second equation is by metric compatibility because $\langle W, \vec{x} \rangle = 0$.

The Euclidean space miracle is that $\nabla_V \vec{x} = V$ for any vector V. For example in $\mathbb{R}^2$, taking the derivative of the vector field (x,y) with respect to (a,b) gives

$$\begin{aligned} \nabla_{(a,b)}(x,y) &= (\partial_{(a,b)}x, \partial_{(a,b)}y) \\ &= (a\partial_x x + b\partial_y x, a\partial_x y + b\partial_y y) \\ &= (a,b) \end{aligned}$$

since $\partial_x x = \partial_y y = 1$, $\partial_y x = \partial_x y = 1$.

So from the above, we have

$$A(V,W) = -\langle W, V \rangle \vec{x}.$$

Now by Gauss's equation, we have

$$\overline{R}(X,Y,Z,W) = \langle X, Z \rangle \langle Y, W \rangle - \langle Y, Z \rangle \langle X, W \rangle.$$

If V, W are orthonormal, then the sectional curvature is constant and equal to 1:

$$\overline{K}_{VW} = \overline{R}(V,W,V,W) = \langle V, V \rangle \langle W, W \rangle - \langle W, V \rangle \langle V, W \rangle = 1 \cdot 1 - 0 \cdot 0 = 1.$$

Then $\overline{\text{Ric}} = (n-1)g$ which is Einstein, and $\overline{S} = n(n-1)$.

4.3 Codazzi equation

Lemma 4.8 (Codazzi)

If $U, V, W \in \Gamma(M)$, then

$$(R(U,V)W)^\perp = (\overline{\nabla}_V A)(U,W) - (\overline{\nabla}_U A)(V,W).$$

Proof. The LHS is

$$\begin{aligned} (R(U,V)W)^\perp &= \nabla_V^\perp \nabla_U W - \nabla_U^\perp \nabla_V W + \nabla_{[U,V]}^\perp W \\ &= \nabla_V^\perp(\overline{\nabla}_U W + A(U,W)) - \nabla_U^\perp(\overline{\nabla}_V W + A(V,W)) + A([U,V],W) \\ &= A(V, \overline{\nabla}_U W) + \nabla_V^\perp(A(U,W)) - A(U, \overline{\nabla}_V W) - \nabla_U^\perp(A(V,W)) + A([U,V],W) \end{aligned}$$

For the RHS, we use the Leibniz rule:

$$\begin{aligned} (\overline{\nabla}_V A)(U,W) &= \overline{\nabla}_V(A(U,W)) - A(\overline{\nabla}_V U, W) - A(U, \overline{\nabla}_V W) \\ &= \overline{\nabla}_V^\perp(A(U,W)) - A(\overline{\nabla}_V U, W) - A(U, \overline{\nabla}_V W) \\ (\overline{\nabla}_U A)(V,W) &= \overline{\nabla}_U^\perp(A(V,W)) - A(\overline{\nabla}_U V, W) - A(V, \overline{\nabla}_U W). \end{aligned}$$

They are equal, noting that $A([U,V],W) = A(\overline{\nabla}_U V, W) - A(\overline{\nabla}_V U, W)$ by symmetry (Section 2.3). □

The Codazzi equation is the most useful when the LHS is zero, such as when $R = 0$ in Euclidean space $\mathbb{R}^n$. Then

$$(\overline{\nabla}_V A)(U, W) = (\overline{\nabla}_U A)(V, W)$$

and $\overline{\nabla} A$ is fully symmetric in U, V, W.

Example 4.9

Let Ω be an open subset of $\mathbb{R}^2$ and consider the graph of $u \in C(\mathbb{R}^2)$

$$\Omega' = \{(x, y, u(x, y)) : (x, y) \in \Omega\}.$$

For example, the upper hemisphere is $u = \sqrt{1 - x^2 - y^2}$.

On $\mathbb{R}^2$, we have local coordinate vector fields ∂_x, ∂_y (actually global in this case). The induced metric from this immersion is

$$g_{xx} = g(\partial_x, \partial_x) = \langle dF(\partial_x), dF(\partial_x)\rangle = 1 + u_x^2$$

because $dF(\partial_x) = (1, 0, u_x)$. Similarly $dF(\partial_y) = (0, 1, u_y)$ so

$$\begin{aligned} g_{xy} &= u_x u_y \\ g_{yy} &= 1 + u_y^2. \end{aligned}$$

The unit normal vector is a unit vector perpendicular to $dF(\partial_x)$ and $dF(\partial_y)$, so it is

$$\vec{n} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + u_x^2 + u_y^2}}.$$

Example 4.10

Intuitively, a path in $\mathbb{R}^2$ with this metric has the same length as if we were going along the surface of the sphere. Going from $(0, 0)$ to $(0, 1)$ should have length $\frac{\pi}{2}$.

Consider the curve $\gamma(s) = (0, s)$ in $\mathbb{R}^2$. Since $\gamma' = (0, 1) = \partial_y$, we have

$$\left|\gamma'\right|^2 = g_{yy} = 1 + u_y^2(0, s)$$

Since $u = \sqrt{1 - x^2 - y^2}$,

$$\begin{aligned} u_y &= \frac{-y}{\sqrt{1 - x^2 - y^2}} \\ \implies 1 + u_y^2 &= \frac{1 - x^2}{1 - x^2 - y^2}. \end{aligned}$$

At $(x, y) = (0, s)$, we have

$$\left|\gamma'\right|^2(s) = \frac{1}{1 - s^2}$$

so the arc length is

$$L(\gamma) = \int_0^1 \frac{1}{\sqrt{1 - s^2}}\, ds = \arcsin s\Big|_0^1 = \frac{\pi}{2}.$$

There are two ways to find the Levi-Civita connection for this g. The first way is to compute Γ_{ij}^m from the formula $2\Gamma^m_{ij} = \sum_k g^{mk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$. The second is using submanifold geometry:

$$\nabla_{F_x} F_x = \nabla_{F_x}((1,0,u_x)) = (0,0,u_{xx})$$

since F_x is just moving in the x direction, and the second equality is by the chain rule. Then

$$\begin{aligned}(\nabla_{F_x} F_x)^\perp &= (0,0,u_{xx})^\perp \\ &= \langle (0,0,u_{xx}), \vec{n}\rangle \vec{n} \\ &= \left\langle (0,0,u_{xx}), \frac{(-u_x,-u_y,1)}{\sqrt{1+u_x^2+u_y^2}} \right\rangle \vec{n} \\ &= \frac{u_{xx}}{\sqrt{1+u_x^2+u_y^2}} \vec{n}\end{aligned}$$

so we can compute the induced Levi-Civita connection $\overline{\nabla}_{F_x} F_x = (\nabla_{F_x} F_x)^\top$.

4.4 Umbilic submanifolds

Assume there is a unit normal $\vec{n}$ (existing locally). The *shape operator* $S\colon T_pM \to T_pM$ is defined by

$$\begin{aligned}\langle S(V), W\rangle &:= \langle A(V,W), \vec{n}\rangle \\ &= \langle \nabla_V W, \vec{n}\rangle \\ &= V\langle W, \vec{n}\rangle - \langle W, \nabla_V \vec{n}\rangle && \text{(metric compatibility)} \\ &= -\langle W, \nabla_V \vec{n}\rangle,\end{aligned}$$

where the last equation is because W is $\top$ and $\vec{n}$ is $\perp$. The RHS is some number because $A(V,W)$ and $\vec{n}$ are both normal. Note that $\langle \nabla_V \vec{n}, \vec{n}\rangle = \frac{1}{2} V\langle \vec{n}, \vec{n}\rangle = 0$.

In particular if $S \equiv 0$, then $\vec{n}$ is constant.

Corollary 4.11

Let M be a sub-n-manifold of $\mathbb{R}^{n+1}$. If $A \equiv 0$ and M is connected, then M is a hyperplane.

Proof. Taking $p \in M$, there is a neighborhood around p where $\vec{n}$ is defined. Choose coordinates such that $\vec{n}(p) = \partial_{n+1}$ and $x_{n+1}(p) = 0$ (rotate and translate). We claim that $M \subseteq \{x_{n+1} = 0\}$ which implies equality.

The set $M \cap \{x_{n+1} = 0\}$ is automatically closed (because x_{n+1} is continuous) and nonempty (as p is in there). By connectedness, it remains to show that is is also open.

Take a curve γ in M with $\gamma(0) = p$. Let

$$f(s) = \langle \vec{n}(\gamma(s)), \partial_{n+1}\rangle.$$

so $f(0) = 1$. By the chain rule,

$$\begin{aligned}f'(s) &= \gamma'\langle \vec{n}(\gamma(s)), \partial_{n+1}\rangle \\ &= \langle \nabla_{\gamma'} \vec{n}, \partial_{n+1}\rangle \\ &= \langle \nabla^\top_{\gamma'} \vec{n}, \partial_{n+1}\rangle + \langle \nabla^\perp_{\gamma'} \vec{n}, \partial_{n+1}\rangle \\ &= \langle \nabla_{\gamma'} \vec{n}, \partial^\top_{n+1}\rangle + 0 \\ &= -\langle \vec{n}, \nabla_{\gamma'} \partial^\top_{n+1}\rangle && \text{(metric compatibility)} \\ &= -\langle \vec{n}, A(\gamma', \partial^\top_{n+1})\rangle \\ &= 0. && (A \equiv 0)\end{aligned}$$

Note in the fourth equation that $\nabla^\perp_{\gamma'} \vec{n} = 0$ because $\langle \nabla_{\gamma'} \vec{n}, \vec{n}\rangle = 0$. ☐

$S\colon T_pM \to T_pM$ is a $(1,1)$-tensor that is "symmetric", meaning

$$\langle S(V), W\rangle = \langle S(W), V\rangle.$$

Thus it has real eigenvalues $\lambda_1, \dots, \lambda_n$ which we call the *principal curvatures*.

Definition 4.12 (umbilic)**.** An n-manifold M is *umbilic* if $\lambda_1 = \cdots = \lambda_n$ at every point.

In other words, S is multiplication by a constant at each point, but not necessarily the same constant.

Example 4.13

$\mathbb{R}^n$ and S^n are umbilic. It turns out that these are the only two examples.

Proposition 4.14

Umbilic implies flat in $\mathbb{R}^n$ for $n \geq 3$.

Proof. Since S is scalar multiplication (a diagonal matrix with the same entry along the diagonal), we can locally find a function f such that $S(V) = fV$. Then

$$f\langle V, W\rangle = \langle S(V), W\rangle = \langle \vec{n}, A(V, W)\rangle.$$

We compute the divergence

$$\operatorname{div}(S) = \operatorname{Tr}(\nabla_\bullet S)$$

which is a $(0,1)$-tensor (i.e. 1-form) because S is a $(1,1)$-tensor and the covariant derivative $\nabla_\bullet S$ is a $(1,2)$-tensor.

Let $S = \sum_{i,j} S^i_j \partial_i \otimes dx_j$ and $\nabla_\bullet S = \sum_{i,j,k} S^i_{j,k}(\partial_i \otimes dx_j \otimes dx_k)$ where we write $S^i_{j,k}$ for $(\nabla_\bullet S)^i_{j,k}$. Then

$$\begin{aligned} S^i_{j,k} &= (\nabla_{\partial_k} S)(dx_i, \partial_j) \\ &= \partial_k(S(dx_i, \partial_j)) - S(\nabla_{\partial_k} dx_i, \partial_j) - S(dx_i, \nabla_{\partial_k}\partial_j). && \text{(Leibniz rule)} \end{aligned}$$

To compute $S(dx_i, \partial_j)$, we use

$$\sum_i S^i_j g_{i\ell} = \left\langle \sum_i S^i_j \partial_i, \partial_\ell \right\rangle = \langle S(\partial_j), \partial_\ell\rangle = \langle A_{j\ell}, \vec{n}\rangle.$$

Now multiply both sides by $g^{\ell k}$ and sum over ℓ to get

$$\begin{aligned} \sum_\ell g^{\ell k}\langle A_{j\ell}, \vec{n}\rangle &= \sum_{i,\ell} g^{\ell k} S^i_j g_{i\ell} \\ &= \sum_i \delta_{ik} S^i_j \\ &= S^k_j, \end{aligned}$$

so

$$S(dx_i, \partial_j) = S^j_i = \sum_\ell g^{\ell j}\langle A_{i\ell}, \vec{n}\rangle.$$

Taking ∂_k, some terms cancel with $S(\nabla_{\partial_k} dx_i, \partial_j)$ and $S(dx_i, \nabla_{\partial_k}\partial_j)$, and we end up with

$$S^i_{j,k} = \sum_\ell g^{\ell i}\langle A_{j\ell,k}, \vec{n}\rangle.$$

$A_{j\ell,k}$ is fully symmetric in j, ℓ, k by the Codazzi equation.

We now compute $\operatorname{div}(S)(V)$ in two ways. On one hand,

$$\begin{aligned}\operatorname{div}(S)(V) &= \sum_{i,j} S^i_{j,i} V^j \\ &= \sum_i (\nabla_{e_i} S(e_i))(V) - S(\nabla_{e_i} e_i)(V) \\ &= \sum_i (\nabla_{e_i}(f e_i))(V) - f(\nabla_{e_i} e_i)(V) \\ &= \sum_i (\nabla_{e_i} f) e_i(V) \\ &= V(f).\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{div}(S)(V) &= \sum_i (\nabla_V S)(e_i)(e_i) \\ &= \sum_i (\nabla_V(f e_i) - f \nabla_V e_i)(e_i) \\ &= \sum_i V(f)\langle e_i, e_i\rangle \\ &= (n-1)V(f).\end{aligned}$$

Assuming $n > 2$, this means $V(f) = 0$ for all V, and f is constant. There are two possibilities:

- If $f \equiv 0$, then $A \equiv 0$ which implies A is flat.
- If $f \equiv \lambda$ for $\lambda \neq 0$, then define a map $\phi \colon \Sigma \to \mathbb{R}^n$ by

  $$\phi(x) = x + \frac{1}{\lambda}\vec{n}(x).$$

  For any tangent vector V, we have

  $$V(\phi) = \nabla_V X + \frac{1}{\lambda}\nabla_V \vec{n} = V - \frac{1}{\lambda} S(V) = 0$$

  from assuming $S(V) = \lambda V$. This implies ϕ is constant, there is some fixed point $p = \phi(x)$ for all x. Then Σ has to be contained in the $(n-1)$-sphere with radius $\frac{1}{|\lambda|}$ and center p.

□

5 Geodesics

5.1 Geodesic definition

Definition 5.1 (geodesic). A curve $\gamma \colon [a, b] \to M$ is a *geodesic* if $\nabla_{\gamma'}\gamma' = 0$.

Corollary 5.2

The length $|\gamma'|$ is constant.

Proof. The derivative of $|\gamma'|^2$ is
$$\frac{d}{dt}\left|\gamma'\right|^2 = 2\langle\nabla_{\gamma'}\gamma', \gamma'\rangle = 2\langle 0, \gamma'\rangle = 0. \qquad \square$$

Locally, write $\gamma(t) = (\gamma^i(t))_{1\le i\le n}$ so that $\gamma' = \sum_i \gamma_t^i\partial_i$ (where $\gamma_t^i = \frac{d}{dt}\gamma^i$). Then
$$\begin{aligned}\nabla_{\gamma'}\gamma' &= \sum_{i,j}\gamma_t^i\nabla_{\partial_i}(\gamma_t^j\partial_j)\\ &= \sum_{i,j}\gamma_t^i(\partial_i(\gamma_t^j)\partial_j + \gamma_t^j\nabla_{\partial_i}\partial_j)\\ &= \sum_j \gamma_{tt}^j\partial_j + \sum_{i,j,k}\gamma_t^i\gamma_t^j\Gamma_{ij}^k\partial_k,\end{aligned}$$
which must equal 0 for a geodesic.

Example 5.3

In Euclidean space, we need $\gamma_{tt}^j = 0$ for all j, so the geodesics γ are straight lines parameterized at constant speeds.

In general, a geodesic is a second order system. Being given an initial position and velocity determines the geodesic.

Consider a curve γ in M which is a submanifold of N. Then
$$\begin{aligned}0 &= \nabla_{\gamma'}^{\top}\gamma'\\ &= \nabla_{\gamma'}\gamma' - \nabla_{\gamma'}^{\perp}\gamma'\end{aligned}$$
so
$$\nabla_{\gamma'}\gamma' = A(\gamma', \gamma').$$

Example 5.4

Let $M = S^n$ be the unit sphere, so $A(\gamma', \gamma') = -\left|\gamma'\right|^2\gamma$. Spherical geodesics have to satisfy $\gamma_{tt} = -\left|\gamma_t\right|^2\gamma$ where $|\gamma_t|$ is constant by Corollary 5.2. If $|\gamma_t| = 1$ (unit speed), then each component of γ has $\gamma_{tt}^j = -\gamma^j$, and the functions that satisfy this are sin and cos. Geodesics in S^n are great circles (see pset).

5.2 Exponential map

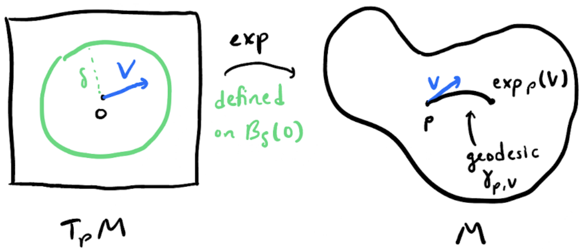
Given $p \in M$, we define a map $\exp_p\colon T_pM \to M$ with $\exp_p(0) = p$ as follows. Let $\gamma(p, V, t)$ be the geodesic with
$$\gamma(p, V, 0) = p, \quad \gamma_t(p, V, 0) = V.$$
By existence and uniqueness of ODEs, this is defined for $|t| \le C$ for some constant C. Note for $a > 0$ that
$$\gamma(p, V, at) = \gamma(p, aV, t)$$
if $\gamma(t)$ is a geodesic, then $\sigma(t) := \gamma(at)$ is also a geodesic (drawing γ at a faster speed). Alternatively from the original equation, we see $\nabla_{a\gamma'}a\gamma' = a^2\nabla_{\gamma'}\gamma' = 0$.

Definition 5.5. The map $\exp_p\colon T_pM \to M$ is defined by $\exp_p(V) = \gamma(p, V, 1)$.

Example 5.6

In $\mathbb{R}^n$, the geodesic is $\gamma(p, V, t) = p + tV$, and the exponential map is $\exp_p(V) = p + V$.

Given initial conditions $\gamma(0) = p$ and $\gamma'(0) = V \in T_pM$, we get $\gamma\colon [0, t_0] \to M$. We call γ by $\gamma_{p,V}$ to show the initial conditions.



Given V, a solution $\gamma_{p,V}$ exists up to some time t_0. Using $\gamma_{p,V}(at) = \gamma_{p,aV}(t)$ and setting $a = t_0$, then $\gamma_{p,aV}(1)$ is defined. For $|V| < \delta$ fixed, $\exp_p(V)$ is defined.

We determine the differential $(d\exp_p)_V$ of the exponent map. Since the tangent space of T_pM is itself, the map $\exp_p\colon T_pM \to M$ induces

$$(d\exp_p)_V\colon T_pM \to T_{\exp_p(V)}M.$$

We first consider $V = 0$, so $(d\exp_p)_0$ is a map $T_pM \to T_pM$.

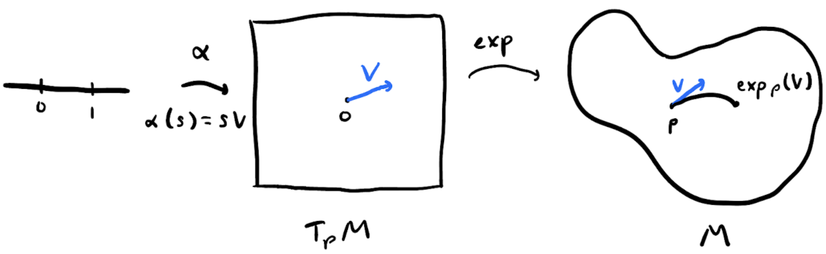
Lemma 5.7

$(d\exp_p)_0 = \text{Id}$.

Proof. Let α be a curve with $\alpha(s) = sV$. Consider $\sigma\colon [0,1] \to M$ given by $\sigma(s) = \exp_p(sV) = \gamma_{p,sV}(1) = \gamma_{p,V}(s)$. By the chain rule, we have

$$\sigma'(0) = (d\exp_p)_0(d\alpha(\partial_s)) = (d\exp_p)_0(V).$$

Also explicitly, $\sigma'(0) = \gamma'_{p,V}(0) = V$, so $(d\exp_p)_0(V) = V$. $\square$

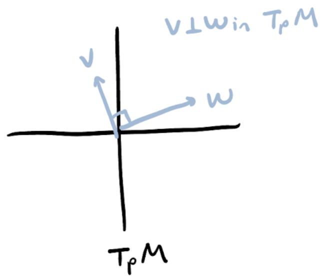


The inverse function theorem implies $\exp_p$ is a diffeomorphism in some ball $B_\delta(0)$, called a *normal neighborhood*. For a set $U \subset M$, a *totally normal neighborhood* is a ball $B_{\delta'}(0)$ in which $\exp_q$ is a diffeomorphism for all $q \in U$.

5.3 Gauss lemma

Definition 5.8. For $V, W \in T_pM$ perpendicular, define $F : \mathbb{R}^2 \to M$ by

$$F(s,t) = \exp_p(t(V + sW)).$$



When only t varies, it is a geodesic. When only s varies, it traces out different geodesics. There are two nice vector fields along the image, namely $F_s = dF(\partial_s)$ and $F_t = dF(\partial_t)$.

By the chain rule (the s derivative of $t(V + sW)$ is tW, and the t derivative is $V + sW$),

$$F_s(s,t) = (d\exp_p)_{t(V+sW)}(tW)$$
$$F_t(s,t) = (d\exp_p)_{t(V+sW)}(V + sW).$$

At $t = 0$, $(d\exp_p)_0 = \mathrm{Id}$ so

$$F_s(s,0) = \mathrm{Id}(0W) = 0$$
$$F_t(s,0) = \mathrm{Id}(V + sW) = V + sW.$$

Lemma 5.9 (Gauss)

(a) $|F_t(s,t)|^2 = |V|^2 + s^2|W|^2 = \left|(d\exp_p)_{t(V+sW)}(V + sW)\right|^2$.

(b) $\langle F_s(0,t), F_t(0,t)\rangle = 0 = \langle (d\exp_p)_{tV}(V), (d\exp_p)_{tV}(W)\rangle$.

Proof. We can check that (a) is true for $t = 0$ by using $F_t(s,0) = V + sW$, and cross terms disappear because V, W orthogonal. Also (b) is true from $F_s(0,0) = 0$. Now we consider the derivatives of both sides. We use these facts:

(i) $[F_s, F_t] = 0$ because it equals $dF([\partial_s, \partial_t]) = dF(0) = 0$.

(ii) $\nabla_{F_t}(F_t) = 0$ for any (s,t) because the image of F_t is a geodesic.

(iii) As a consequence of (ii), $\partial_t |F_t|^2 = 0$ for any geodesic (Corollary 5.2).

Now (a) is obvious from (iii). To prove (b), We know $\langle F_s, F_t\rangle = 0$ at $t = 0$. Differentiating, we get

$$\begin{aligned}
dF(\partial_t\langle F_s, F_t\rangle) &= F_t(\langle F_s, F_t\rangle) \\
&= \langle \nabla_{F_t} F_s, F_t\rangle + \langle F_s, \nabla_{F_t} F_t\rangle && \text{(metric compatibility)} \\
&= \langle \nabla_{F_t} F_s, F_t\rangle \\
&= \langle \nabla_{F_s} F_t, F_t\rangle && \text{(using (i))} \\
&= \frac{1}{2}\partial_s |F_t|^2 \\
&= s|W|^2 && \text{(using (a))} \\
&= 0, && (s = 0)
\end{aligned}$$

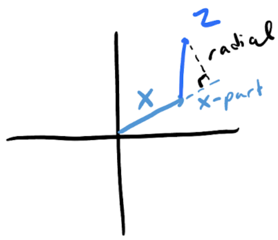
as desired. □

Corollary 5.10

Given any vector fields $Z, X \in T_pM$, we decompose Z as the part $Y = \frac{\langle Z,X\rangle}{\langle X,X\rangle}X$ along X, and the normal part $Z - Y$. Then

$$\begin{aligned}\left|(d\exp_p)_X(Z)\right|^2 &= \left|(d\exp_p)_X(Y)\right|^2 + \left|(d\exp_p)_X(Z-Y)\right|^2 \\ &= |Y|^2 + \left|(d\exp_p)_X(Z-Y)\right|^2.\end{aligned}$$

We can make the radial part V and the angular part W, and then apply Gauss.



5.4 Riemannian distance

Definition 5.11 (Riemannian distance). For points $p, q \in M$ connected, define

$$d(p,q) = \inf_\gamma L(\gamma)$$

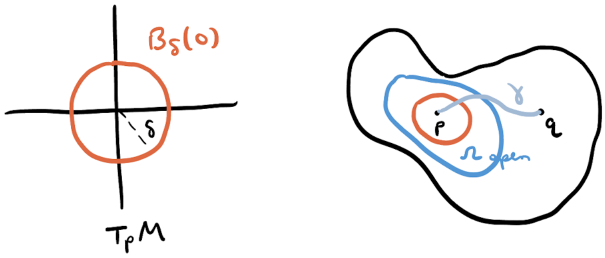
as γ ranges over curves from p to q that are piecewise smooth (finitely many breaks).

To check this is a metric, we need to check that it is

- Symmetric: $d(p,q) = d(q,p)$.
- Positive definite: $d(p,q) \geq 0$ with equality if and only if $p = q$.
- Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$.

Symmetric follows by going along the curve backwards. The triangle inequality follows from pasting two curves together. Positive definiteness is not obvious–although each curve has positive length, there could be curves with length converging to 0.

We want to find a positive lower bound for $L(\gamma)$ for $\gamma\colon p \to q$ with $p \neq q$. By the Hausdorff condition, there exists some open neighborhood Ω of p not containing q. There exists some δ such that $\exp_p$ is a diffeomorphism in $B_\delta(0)$, and the image $\exp_p(\partial B_\delta(0))$ is contained in Ω and isomorphic to S^{n-1}. Note $q \notin \exp_p(\overline{B_\delta(0)})$. If γ is any curve from p to q, it must pass through $\exp_p(\partial B_\delta(0))$.

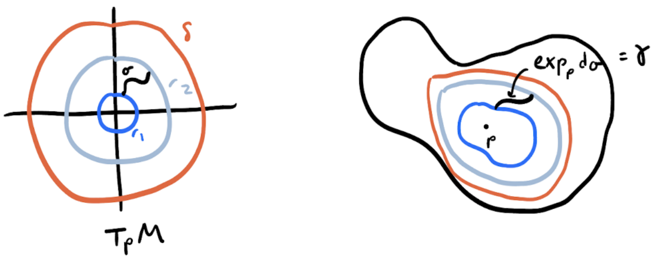


Lemma 5.12

Suppose that $\exp_p$ is a diffeomorphism on $B_\delta(0)$. Let $0 < r_1 < r_2 \le \delta$ and σ be a curve in T_pM. Given that $\sigma(0) \in \partial B_{r_1}(0)$, $\sigma(1) \in \partial B_{r_2}(0)$ and $\sigma \subset \overline{B}_{r_2}(0) \setminus B_{r_1}(0)$ then $L(\exp_p(\sigma)) \ge r_2 - r_1$.

Equality holds if and only if σ is a monotone ray, meaning $\sigma/|\sigma|$ is a constant vector and $|\sigma|$ is monotone.

The first part of the lemma implies δ is a lower bound. Later we'll see why monotone ray part is useful.



Proof. For $Z, X \in T_pM$, we decompose Z as the part $Y := \frac{\langle Z,X\rangle}{\langle X,X\rangle}X$ along X, and the normal part $Z - Y$. As in Corollary 5.10, we have by the Gauss lemma on Y and $Z - Y$ that

$$\begin{aligned} \left|(d\exp_p)_X(Z)\right|^2 &= \left|(d\exp_p)_X(Y)\right|^2 + \left|(d\exp_p)_X(Z-Y)\right|^2 \\ &= |Y|^2 + \left|(d\exp_p)_X(Z-Y)\right|^2 \\ &= \frac{\langle Z,X\rangle^2}{|X|^2} + \left|(d\exp_p)_X(Z-Y)\right|^2 \\ &\ge \frac{\langle Z,X\rangle^2}{|X|^2} \end{aligned}$$

with equality if and only if $(d\exp_p)_X(Z - Y) = 0$, or $Z = Y$ if $\exp_p$ is a diffeomorphism. The first equation is by Gauss lemma (b) which says the cross term is 0, and the second equation is by Gauss lemma (a).

Let $\gamma = \exp_p(\sigma)$. Applying the above inequality to $Z = \sigma'$ and $X = \sigma$, we obtain

$$\begin{aligned} |\gamma'|^2 &= \left|(d\exp_p)_{\sigma(t)}(\sigma')\right|^2 \\ &\ge \frac{\langle \sigma',\sigma\rangle^2}{|\sigma|^2} \end{aligned}$$

with equality if and only if $\sigma' = \frac{\langle \sigma', \sigma \rangle}{|\sigma|^2}\sigma$. Integrating yields

$$\begin{aligned} L(\gamma) &= \int_a^b |\gamma'|(t)\,dt \\ &\geq \int_a^b \frac{|\langle \sigma', \sigma\rangle|}{|\sigma|}(t)\,dt \\ &= \int_a^b \left|\frac{d}{dt}|\sigma|\right| dt \\ &\geq \int_a^b \frac{d}{dt}|\sigma|\,dt \\ &= |\sigma|(b) - |\sigma|(a) \\ &= r_2 - r_1 \end{aligned}$$

The third line is by

$$2|\sigma|\,|\sigma|' = \frac{d}{dt}|\sigma(t)|^2 = 2\langle \sigma', \sigma\rangle,$$

so $|\sigma|' = \frac{\langle \sigma', \sigma\rangle}{|\sigma|}$. (As a concrete example, we can compute the derivative of $|x|$.)
The second inequality is sharp if and only if σ is monotonic. The first inequality is sharp if and only if $\sigma' = \frac{\langle \sigma', \sigma\rangle}{|\sigma|^2}\sigma$, which we claim is equivalent to $\frac{\sigma}{|\sigma|}$ constant. Geometrically, it's clear you should stay along the ray from that equation if it's a multiple of the ray pointing to the origin. Algebraically,

$$\left(\frac{\sigma}{|\sigma|}\right)' = \frac{\sigma'}{|\sigma|} - \frac{\sigma|\sigma|'}{|\sigma|^2} = \frac{|\sigma|\sigma' - \sigma\frac{\langle \sigma', \sigma\rangle}{|\sigma|}}{|\sigma|^2} = \frac{|\sigma|^2\sigma' - \sigma\langle \sigma', \sigma\rangle}{|\sigma|^3} = 0$$

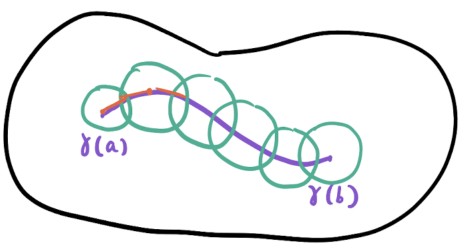
using $\sigma' = a\sigma$ for $a \in \mathbb{R}$. □

Definition 5.13. A curve $\gamma\colon [a,b] \to M$ is *(length) minimizing* if $L(\gamma) = d(\gamma(a), \gamma(b))$.

It is not necessarily unique, e.g. on a sphere between two antipodal points. It also may not exist, e.g. on a disk missing a point.

Corollary 5.14 (of Lemma 5.12)
If γ is length minimizing, then we can (monotonically) reparameterize γ such that it is a geodesic.

Proof. We know that any subcurve of γ is also minimizing, and that γ is monotone. Reparameterize γ such that the speed $|\gamma'| = 1$ is constant, which is still length minimizing because reparameterizing doesn't change the length.



By compactness, we can cover γ by a finite collection of totally normal neighborhoods. In each totally normal neighborhood (centered at p), the inverse image of γ in $\exp_p$ must be a monotone ray, so γ must be a monotone geodesic ray. (We are using the fact that in a normal neighborhood, a length minimizing curve is a geodesic.) In the regions where two geodesics overlap, it is the same curve and with the same speed, so γ is a geodesic. □

5.5 Hopf–Rinow theorem

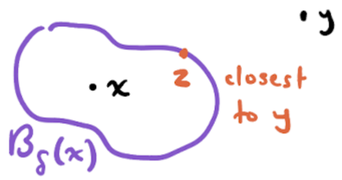
Theorem 5.15 (Hopf–Rinow)
Let M be a connected manifold with Riemannian distance d. TFAE:

1. There exists $p \in M$ such that $\exp_p$ is defined on all of T_pM.
2. Any subset of M that is closed and bounded is also compact.
3. (M, d) is complete (every Cauchy sequence converges).
4. M is *geodesically complete* (for all $p \in M$, $\exp_p$ is defined on all of T_pM).

Any of these imply that for all $p, q \in M$ there exists a length minimizing geodesic γ from p to q.

Lemma 5.16
Let $x \in M$ such that $\exp_x$ is a diffeomorphism on $B_{\delta'}(0)$ for $\delta' > \delta$. For $y \notin \overline{B_\delta(x)}$,

1. There exists $z \in \partial B_\delta(x)$ such that $d(z, y) = \inf_{w \in \partial B_\delta(x)} d(w, y)$.
2. Any such z satisfies $d(x, y) = d(x, z) + d(z, y) = \delta + d(z, y)$.



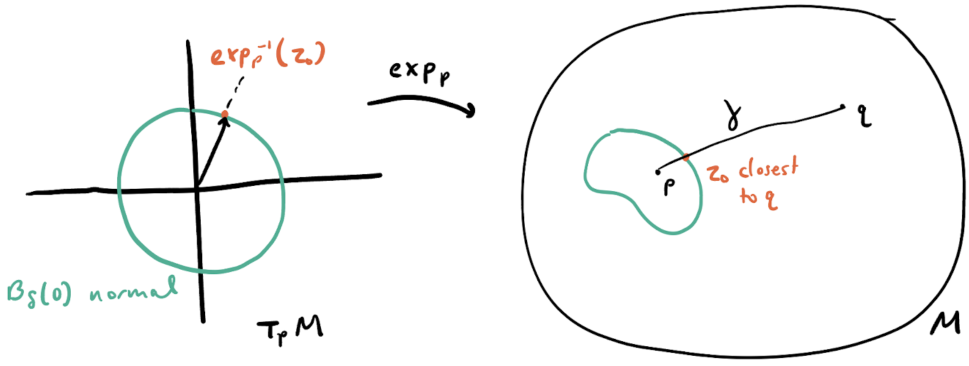
Proof. 1. $\partial B_\delta(x) = \exp_x(\partial B_\delta(0))$ is the image of a sphere so it is compact. The distance function is Lipschitz by the triangle inequality and has a minimum on the compact set $\partial B_\delta(x)$.

2. The fact that $d(x, z) = \delta$ for $z \in \partial B_\delta(x)$ is by the Gauss lemma. For the first equality, the $\leq$ direction is by the triangle inequality. For the $\geq$ direction, any curve γ from x to y must hit the boundary $\partial B_\delta(x)$, say at z'. Then $L(\gamma) \geq d(x, z') + d(z', y) \geq \delta + d(z, y)$. Taking the infimum over all γ yields $d(x, y) \geq \delta + d(y, z)$. □

Proposition 5.17
Condition 1 in Theorem 5.15 implies for all q that there exists a minimizing geodesic γ from p to q.

This implies $\exp_p$ is onto.

Proof. By Lemma 5.16, choose $z_0 \in \partial B_\delta(p)$ which minimizes the distance to q. Choose a geodesic γ such that $|\gamma'| \equiv 1$, $\gamma(0) = p$, and $\gamma(\delta) = z_0$. This γ is defined for all time t by condition 1.



Let $r = d(p,q)$ and we show that $\gamma(r) = q$. Define

$$A \coloneqq \{t \in [0,r] : r \geq t + d(\gamma(t), q)\}.$$

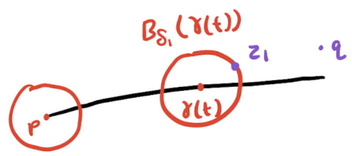
To see that $r \in A$, we note

- $0 \in A$.
- A is closed.
- If $t \in A$ and $s < t$, then $s \in A$. By the triangle inequality, we have

$$\begin{aligned} d(\gamma(s), q) &\leq d(\gamma(t), \gamma(s)) + d(\gamma(t), q) \\ &\leq (t-s) + (r-t) \\ &= r - s. \end{aligned}$$

- A is open. Say $t \in [0,r)$ is in A so that $d(\gamma(t), q) \leq r - t$. In fact $d(\gamma(t), q) = r - t$ because the reverse inequality is the triangle inequality.

  Choose $0 < \delta_1 < r - t$ such that $B_{2\delta_1}(\gamma(t))$ is a normal neighborhood. Let $z_1 \in \partial B_{\delta_1}(\gamma(t))$ be closest to q. (Note this image is misleading because z_1 and q will be on γ, but we don't know that yet.)



  Lemma 5.16 part 2 says $d(\gamma(t), q) = \delta_1 + d(z_1, q)$. Then

$$r - t = \delta_1 + d(z_1, q) \implies d(z_1, q) = r - (t + \delta_1).$$

  Then by the triangle inequality,

$$r = d(p,q) \leq d(p, \gamma(t)) + d(\gamma(t), z_1) + d(z_1, q) = t + \delta_1 + (r - t + \delta_1) = r,$$

  so $d(p, z_1) = t + \delta_1$. The curve $p \to \gamma(t) \to z_1$ is length minimizing and has to be a geodesic by Corollary 5.14. Uniqueness of ODEs says that it has to be γ because they agrees up to time t. Thus $\gamma(t + \delta_1) = z_1$, which implies $t + \delta_1 \in A$. □

Recall that the metric is $d(p,q) = \inf\{L(\gamma) \mid \gamma \colon p \to q\}$. The Gauss lemma showed that given $\exp_p$ is a diffeomorphism on $B_\delta(0)$, if $q = \exp_p(V)$ with $|V| < \delta$, then $d(p,q) = |V|$ and any $\gamma \colon p \to q$ of length $|V|$ is a "monotone ray."

Proof of Theorem 5.15 (Hopf–Rinow). $4 \Rightarrow 1$: This is clear.

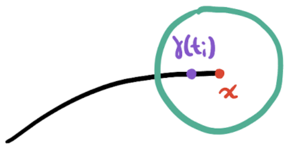
$1 \Rightarrow 2$: Let $p \in M$ be the point for which $\exp_p$ is defined on all of T_pM. Ω being closed and bounded means there exists R such that $d(p,x) \le R$ for all $x \in \Omega$. By Proposition 5.17, $\Omega \subset \exp_p(\overline{B_R(0)})$ since any $x \in \Omega$ has a geodesic leaving p to x with length $d(p,x) \le R$. $\overline{B_R(0)}$ is compact in $T_pM \simeq \mathbb{R}^n$ and $\exp_p$ is continuous, so $\exp_p(\overline{B_R(0)})$ is compact. Since Ω is a closed subset of $\exp_p(\overline{B_R(0)})$, Ω is compact.

$2 \Rightarrow 3$: A Cauchy sequence (p_n) is bounded, so it lies inside a compact set by our assumption. Then there exists a convergent subsequence, which implies convergence for a Cauchy sequence.

$3 \Rightarrow 4$: We are given that (M,d) is complete. We show that a geodesic γ starting at $p \in M$ can be extended for all time t. By the normal neighborhood, we can do this for $t \in [0,\delta]$. Let

$$A := \{t \in [0,\infty) : \gamma(t) \text{ is defined}\}.$$

A is automatically open, as we can consider a normal neighborhood around $\gamma(t)$ (e.g. consider a geodesic from the origin to the surface of the unit ball).

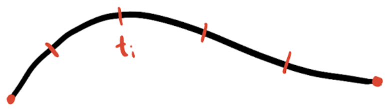


It remains to show that A is closed: given that $\gamma(t)$ is defined for all $t \in [0,T)$, we wish to extend it to T. Let t_i be an increasing sequence converging to T. The sequence $\gamma(t_i)$ is Cauchy because

$$d(\gamma(t_i),\gamma(t_j)) = L(\gamma|_{[t_i,t_j]}) \le |\gamma'|\,|t_j - t_i| < |\gamma'|\,\epsilon$$

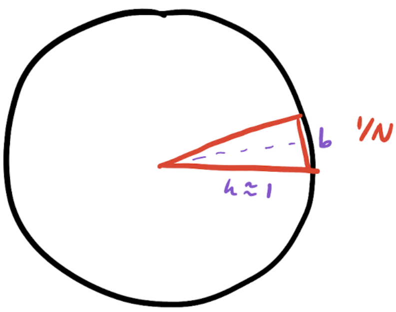
when $|t_i - t_j| < \epsilon$. By completeness, $\gamma(t_i)$ converges to some $x \in M$. Looking at a totally normal neighborhood about x, we have $\gamma(T) = x$ by uniqueness of ODEs. ☐

To find the length of a curve in a metric space, we partition the curve at points t_i, consider $\sum d(\gamma(t_i),\gamma(t_{i+1}))$, and take the size of the partition to 0.



Example 5.18

Consider partitioning the perimeter P of the circle. Given that π is the area of the circle, the area of one $\frac{1}{N}$ sector is $\frac{\pi}{N} \approx \frac{1}{2}bh \approx \frac{1}{2}\cdot 1 \cdot \frac{P}{N}$. Then $P = 2\pi$.



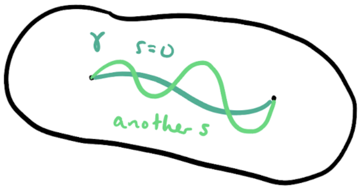
6 Variational theory of geodesics

Recall that a curve $\gamma\colon [a,b] \to M$ is a *geodesic* if $\nabla_{\gamma'}\gamma' = 0$. These have constant speed $|\gamma'|$. A curve $\sigma\colon [a,b] \to M$ is *minimizing* if $L(\sigma) = d(\sigma(a), \sigma(b))$. We showed that such a σ is a monotone reparameterization of a geodesic.

6.1 Jacobi equation

Definition 6.1 (variation, proper). Given a curve $\gamma\colon [a,b] \to M$, a *variation of* γ is a continuous (usually piecewise smooth) map $F\colon (-\epsilon,\epsilon) \times [a,b] \to M$ such that $F(0,t) = \gamma(t)$.

A variation F is *proper* if the endpoints are fixed, meaning $F(s,a) = \gamma(a)$ and $F(s,b) = \gamma(b)$ for all s.



For F, there are two natural vector fields along the image:

- $F_s = dF(\partial_s)$, called the *variation vector field*.
- $F_t = dF(\partial_t)$, which gives the tangent along each curve for each s.

At $s = 0$, $F_t = \gamma' = \gamma_t$. If F is proper, then $F_s = 0$ at the endpoints.

Corollary 6.2
$\nabla_{F_s}\nabla_{F_t}F_t = \nabla_{F_t}\nabla_{F_t}F_s + R(F_t, F_s)F_t$.

Proof. Recall that $[F_s, F_t] = 0$ because $[\partial_s, \partial_t] = 0$ in $\mathbb{R}^2$ and dF preserves the Lie bracket. Then

$$\begin{aligned}\nabla_{F_s}\nabla_{F_t}F_t &= R(F_t, F_s)F_t + \nabla_{F_t}\nabla_{F_s}F_t - \nabla_{[F_t,F_s]}F_t \\ &= R(F_t, F_s)F_t + \nabla_{F_t}\nabla_{F_t}F_s\end{aligned}$$

where $[F_t, F_s] = 0$ and symmetry implies $\nabla_{F_s}F_t = \nabla_{F_t}F_s$. □

If F_t is a geodesic, then $\nabla_{F_t}F_t = 0$, and we get that the RHS above is 0.

Definition 6.3 (Jacobi field). Let γ be a geodesic. A vector field J along γ is a *Jacobi field* if

$$\nabla_{\gamma'}\nabla_{\gamma'}J + R(\gamma', J)\gamma' = 0.$$

If $t \to F(s,t)$ is a geodesic for every s, then F_s is a Jacobi field.

Example 6.4
On $\mathbb{R}^n$, geodesics γ are straight lines. Since $R \equiv 0$, J is a Jacobi field if and only if $\nabla_{\gamma'}\nabla_{\gamma'}J = 0$. Let $e_1, \ldots, e_{n-1}, e_n = \gamma'$ be a parallel orthonormal frame along γ. Writing $J = \sum_{i=1}^n f_i(t)e_i(t)$, we have $\nabla_{\gamma'}J = \sum_{i=1}^n f_i'(t)e_i(t)$ and $\nabla_{\gamma'}\nabla_{\gamma'}J = \sum_i f_i''(t)e_i(t)$ because the e_i are parallel. Thus, J is Jacobi when $f_i(t)$ is affine for all i, i.e. $f_i(t) = a_i + b_it$ for constants a_i, b_i.

Jacobi fields are supposed to be infinitesimal generators of nearby geodesics, so it makes sense that they are linear on $\mathbb{R}^n$.

Lemma 6.5

Let $\gamma\colon [a,b] \to M^n$ be a geodesic. Then

1. The space of Jacobi fields along γ is $2n$-dimensional and is uniquely determined by $J(a)$ and $J'(a) = (\nabla_{\gamma'} J)(a)$.
2. If $J(a) = 0$, then $J(t) = (d\exp_{\gamma(a)})_{(t-a)\gamma'(a)}(tJ'(a))$.

Proof. 1. The Jacobi equation is a 2nd order ODE, so it is determined by J and J' at a. There are n choices for the coefficients of $J(a)$, and similarly for $J'(a)$.

2. Define a variation

$$F(s,t) = \exp_{\gamma(a)}\Big((t-a)(\gamma'(a) + sJ'(a))\Big)$$

where $\gamma'(a)$ is a fixed constant vector. For s fixed, this is a geodesic. Then $F_s|_{s=0}$ is a Jacobi field. Note that $F_s(s,a) = 0$.

What is $\nabla_{F_t}(F_s|_{s=0}) = \nabla_{F_t} F_s$?

$$\begin{aligned} F_s(0,t) &= (d\exp_{\gamma(a)})_{(t-a)\gamma'(a)}((t-a)J'(a)) \\ &= (t-a)(d\exp_{\gamma(a)})_{(t-a)\gamma'(a)}(J'(a)). \end{aligned}$$

When we compute $\nabla_{\gamma'}$ of this at $t = a$, the term with $(t-a)$ after the chain rule vanishes, and the other term is

$$\begin{aligned} \nabla_{F_t} F_s &= (d\exp_{\gamma(a)})_{(t-a)\gamma'(a)}(J'(a))|_{t=a} \\ &= (d\exp_{\gamma(a)})_0(J'(a)) \\ &= J'(a) \end{aligned}$$

at $s = 0$ and $t = a$, using $(d\exp_{\gamma(a)})_0 = \mathrm{Id}$ by Lemma 5.7. The uniqueness in part (a) implies that $F_s = J$. □

Example 6.6

There are always two simple Jacobi fields

- $J = \gamma'$, because $\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = 0$ and $R(\gamma',\gamma')\gamma' = 0$ (R is skew).
- $J = t\gamma'$, because $\nabla_{\gamma'}\nabla_{\gamma'}(t\gamma') = \nabla_{\gamma'}(t'\gamma' + t\nabla_{\gamma'}\gamma') = \nabla_{\gamma'}\gamma' = 0$ and $R(\gamma', t\gamma')\gamma' = 0$ (pull out t by linearity).

We show that the other $2n-2$ Jacobi fields must be normal.

Proposition 6.7

If J is a Jacobi field with $J(0)$ and $J'(0)$ normal to $\gamma'(0)$, then J is normal to γ' for all t.

Proof.

$$\begin{aligned}
\partial_t\langle J,\gamma'\rangle &= \langle J',\gamma'\rangle + \langle J,\nabla_{\gamma'}\gamma'\rangle \\
&= \langle J',\gamma'\rangle \\
\implies \partial_t^2\langle J,\gamma'\rangle &= \partial_t\langle J',\gamma'\rangle \\
&= \langle \nabla_{\gamma'}\nabla_{\gamma'}J,\gamma'\rangle \\
&= -\langle R(\gamma',J)\gamma',\gamma'\rangle && \text{(Jacobi equation)} \\
&= -R(\gamma',J,\gamma',\gamma') \\
&= 0 && \text{(Lemma 3.3)}
\end{aligned}$$

Then $\langle J,\gamma'\rangle = a+bt$ for some $a,b\in\mathbb{R}$. As J and J' are both perpendicular to γ' at 0, we must have $a=b=0$. Then $\langle J,\gamma'\rangle\equiv 0$. ☐

6.1.1 Normal Jacobi fields on constant curvature spaces

Suppose M has constant sectional curvature κ. Assume $|\gamma'|=1$. If V is a parallel normal vector field along γ (so $\nabla_{\gamma'}V=0$), then fV is a Jacobi field if and only if

$$\begin{aligned}
0 &= \nabla_{\gamma'}\nabla_{\gamma'}(fV) + R(\gamma',fV)\gamma' \\
&= f''V + fR(\gamma',V)\gamma' && (\nabla_{\gamma'}V=0) \\
&= f''V + \kappa fV \\
&= (f''+\kappa f)V.
\end{aligned}$$

For the third equation, recall that $R(X,Y,Z,W)=\kappa(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle)$ for all tangent vectors X,Y,Z,W (pset 3.3). Then $\langle R(\gamma',V)\gamma',W\rangle = \kappa\langle V,W\rangle$ for all W, so $R(\gamma',V)\gamma'=\kappa V$.

This is a differential equation that we can solve:

- If $\kappa=0$, then $f''=0$ and $f=a+bt$.
- If $\kappa=1$ (e.g. S^n), then $f''=-f$ and $f=a\sin t+b\cos t$.
- If $\kappa=-1$ (e.g. hyperbolic space) then $f''=f$ and $f=a\cosh t+b\sinh t$.

6.2 Conjugate points

Geodesics on S^n get farther apart until they get to the equator, then they get closer together until they coincide at the antipode. In hyperbolic space, then initially grow linearly, but then exponentially.

> **Definition 6.8** (conjugate points). Let γ be a geodesic. Then $\gamma(a)$ and $\gamma(b)$ are *conjugate* if there exists a nonzero Jacobi field J such that $J(a)=J(b)=0$.

Conjugate points correspond to places where exp is not a diffeomorphism? There are no conjugate points in $\mathbb{R}^n$, as if $f=a+bt=0$ and $f'=0$ then $a=b=0$. There are also no conjugate points in hyperbolic space. However, antipodal points are conjugate on S^n.

6.3 Energy

> **Definition 6.9** (energy). The *energy* $E(\gamma)$ of a curve $\gamma\colon[a,b]\to M$ is
>
> $$E(\gamma)=\int_a^b |\gamma'|^2(t)\,dt.$$

If it were $|\gamma'|$ instead of $|\gamma'|^2$, that would be the length of the curve.

Lemma 6.10
We have
$$d(\gamma(a),\gamma(b))^2 \leq L(\gamma)^2 \leq (b-a)E(\gamma)$$
with equality if and only if γ is a minimizing geodesic for both L and E.

Recall the Cauchy–Schwarz inequality
$$\Big(\int_a^b uv\Big)^2 \leq \Big(\int_a^b u^2\Big)\Big(\int_a^b v^2\Big)$$
with equality if and only if u, v are multiples of each other. It comes from the L^2-norm inequality
$$\langle u,v\rangle_{L^2}^2 \leq ||u||_{L^2}^2\,||v||_{L^2}^2$$
which is always true in an inner product space. A special case is that
$$\Big(\int_a^b |u|\Big)^2 \leq \Big(\int_a^b 1\Big)\Big(\int_a^b u^2\Big) = (b-a)\int_a^b u^2$$
with equality if and only if u is constant.

Proof. The first inequality is by the definition of d. The second inequality is by Cauchy–Schwarz:
$$L(\gamma)^2 = \Big(\int_a^b |\gamma'|\Big)^2 \leq \Big(\int_a^b 1\Big)\Big(\int_a^b |\gamma'|^2\Big) = (b-a)E(\gamma).$$
If the first inequality is an equality, then γ is a monotone reparameterization of a length minimizing geodesic by Corollary 5.14. The second equality implies $|\gamma'|$ is constant, so γ is a length minimizing geodesic. □

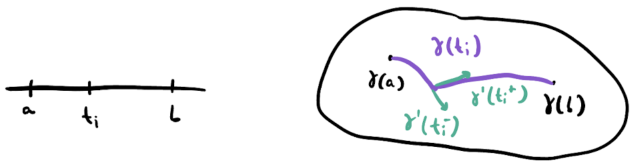
There are too many choices for curves that minimize length, so we consider one with minimal energy. Given a length minimizing curve, we can reparameterize it to be energy minimizing. A length minimizing geodesic is the same as an energy minimizing curve.

Remark 6.11. Historical aside: Poincaré asked when given a complete manifold, whether there is always a closed geodesic (a map from $S^1 \to M$). For example on the sphere, we can take an equator. For surfaces with genus ≥ 1, we can take a nontrivial loop and find a variation that minimizes energy, which will then be a geodesic.

On simply connected surfaces such as the sphere, this doesn't work because all loops are null homotopic. Birkhoff used sweepouts, or a family of curves which start and end at a point curve. He considered a curve with maximal length, which turned out to be a geodesic.

6.3.1 First variation of energy

Let $\gamma\colon [a,b] \to M$ be continuous and piecewise smooth with breaks at $a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$. Let $\gamma'(t_i^-)$ be the incoming tangent vector, and let $\gamma'(t_i^+)$ be the outgoing tangent vector.



Let $F(s,t)$ be a proper, piecewise smooth, continuous variation so $F(s,a) \equiv \gamma(a)$ and $F(s,b) \equiv \gamma(b)$ for all s. We compute the variation $\frac{dE}{ds}$, where $E(s)$ is the energy of $F(s)$:

$$E(s) = \int_a^b |F_t(s,t)|^2 \, dt = \sum_i \int_{t_i}^{t_{i+1}} |F_t(s,t)|^2 \, dt.$$

Differentiating,

$$\begin{aligned}
E'(s) &= \sum_i \int_{t_i}^{t_{i+1}} F_s \langle F_t, F_t \rangle \, dt \\
&= 2 \sum_i \int_{t_i}^{t_{i+1}} \langle \nabla_{F_s} F_t, F_t \rangle \, dt && \text{(metric compatibility)} \\
&= 2 \sum_i \int_{t_i}^{t_{i+1}} \langle \nabla_{F_t} F_s, F_t \rangle dt && \text{([}F_s, F_t\text{] = 0 and symmetry)} \\
&= 2 \sum_i \int_{t_i}^{t_{i+1}} \left(F_t \langle F_s, F_t \rangle - \langle F_s, \nabla_{F_t} F_t \rangle \right) dt && \text{(metric compatibility)} \\
&= 2 \sum_i \langle F_s, F_t \rangle \Big|_{t_i}^{t_{i+1}} - 2 \int_a^b \langle F_s, \nabla_{F_t} F_t \rangle \, dt && \text{(FTC)} \\
&= -2 \int_a^b \langle F_s, \nabla_{F_t} F_t \rangle \, dt + 2 \sum_i \left(\langle F_t, F_s \rangle (t_{i+1}^-) - \langle F_t, F_s \rangle (t_i^+) \right) \\
&= -2 \int_a^b \langle F_s, \nabla_{F_t} F_t \rangle \, dt + 2 \sum_i \langle F_s, F_t(t_i^-) - F_t(t_i^+) \rangle && (6.1)
\end{aligned}$$

The last sum would telescope if γ were smooth to yield

$$E'(s) = -2 \int_a^b \langle F_s, \nabla_{F_t} F_t \rangle \, dt + 2 \langle F_s, F_t(b) - F_t(a) \rangle. \tag{6.2}$$

Proposition 6.12

γ is a smooth geodesic if and only if $E'(0) = 0$ for all proper variations of γ.

Proof. ($\Rightarrow$) The first term in (6.1) vanishes because $F_s = 0$ for a geodesic, and the second term vanishes when γ is smooth.

($\Leftarrow$) Every vector field V on γ gives a variation $F(s,t) = \exp_{\gamma(t)}(sV(t))$, where $F_s(0,t) = V(t)$.

We first show that γ is a broken geodesic. We let V be 0 at the breakpoints, so the summation term in (6.1) is 0. Then the integral term is 0, but F_s can be arbitrary so $\nabla_{F_t} F_t = 0$.

Next we show that γ is smooth. If γ has a break point at t_i, then we let V jump at t_i, so $F_s(0, t_i) = F_t(t_i^-) - F_t(t_i^+)$, and $F_s(0, t_j) = 0$ for all $j \neq i$. Then there would be a nonzero contribution in the second term, which is a contradiction. □

6.3.2 Second variation of energy

Proposition 6.13

If $\gamma\colon [a, b] \to M$ is a smooth geodesic, then

$$\frac{1}{2}E''(0) = \int_a^b |\nabla_{F_t} F_s|^2 - R(F_t, F_s, F_t, F_s)\, dt.$$

Note that $\nabla_{F_t}\nabla_{F_t} F_s + R(F_t, F_s)F_t$ is the Jacobi operator and equals 0 when F_s is a Jacobi field.

Proof. We have

$$\begin{aligned}
\frac{1}{2}E''(0) &= -\int_a^b F_s(\langle F_s, \nabla_{F_t} F_t\rangle)\, dt \\
&= -\int_a^b \langle \nabla_{F_s} F_s, \nabla_{F_t} F_t\rangle + \langle F_s, \nabla_{F_s}\nabla_{F_t} F_t\rangle\, dt \\
&= -\int_a^b \langle F_s, \nabla_{F_s}\nabla_{F_t} F_t\rangle\, dt \\
&= -\int_a^b \langle F_s, \nabla_{F_t}\nabla_{F_t} F_s + R(F_t, F_s)F_t\rangle\, dt && \text{(Corollary 6.2)} \\
&= -\int_a^b \langle F_s, \nabla_{F_t}\nabla_{F_t} F_s\rangle + R(F_t, F_s, F_t, F_s)\, dt \\
&= -\int_a^b F_t(\langle F_s, \nabla_{F_t} F_s\rangle) - \langle \nabla_{F_t} F_s, \nabla_{F_t} F_s\rangle + R(F_t, F_s, F_t, F_s)\, dt \\
&= \langle F_s, \nabla_{F_t} F_s\rangle\Big|_a^b + \int_a^b |\nabla_{F_t} F_s|^2 - R(F_t, F_s, F_t, F_s)\, dt \\
&= \int_a^b |\nabla_{F_t} F_s|^2 - R(F_t, F_s, F_t, F_s)\, dt.
\end{aligned}$$

The third equation is because $\nabla_{F_t} F_t = 0$ at $s = 0$ since γ is a geodesic. □

The RHS $\int_a^b |\nabla_{F_t} F_s|^2 - R(F_t, F_s, F_t, F_s)\, dt \equiv I(F_s, F_s)$ is called an *index form*.

Definition 6.14. An *index form* I satisfies

$$I(V, V) = \int_a^b |\nabla_{F_t} V|^2 - R(F_t, V, F_t, V)\, dt.$$

This motivates Bonnet–Myers, which says that a length minimizing geodesic can not be too long.

Recall that if γ is a minimizing geodesic, then

$$0 \le I(V, V) = \int \left|\nabla_{\gamma'} V\right|^2 - R(\gamma', V, \gamma', V)$$

for all V along γ that vanish at the endpoints.

6.4 Bonnet–Myers

Theorem 6.15 (Bonnet–Myers)
Suppose (M^n, g) is complete and $\mathrm{Ric} \geq c$ for some constant $c > 0$. Then

$$\mathrm{diam}(M)^2 \leq (n-1)\frac{\pi^2}{c}.$$

As a corollary, M is compact.

Proof. By Hopf–Rinow, there is a minimizing geodesic between any two points. It then suffices to show for any minimizing geodesic γ of length L that $L^2 \leq (n-1)\frac{\pi^2}{c}$.

Say $|\gamma'| = 1$, so $\gamma\colon [0, L] \to M$. Choose a parallel orthonormal frame $e_1, \ldots, e_{n-1}, e_n = \gamma'$ along γ; note γ' is parallel because $\nabla_{\gamma'}\gamma' = 0$. Define

$$V_j = \sin\left(\frac{\pi t}{L}\right)e_j,$$

which makes $V_j = 0$ at the endpoints $t = 0, L$. Then

$$\begin{aligned}
0 &\leq I(V_j, V_j) \\
&= \int_0^L \left|\nabla_{\gamma'}\left(\sin\left(\frac{\pi t}{L}\right)e_j\right)\right|^2 - \sin^2\left(\frac{\pi t}{L}\right)R(\gamma', e_j, \gamma', e_j)\,dt \\
&= \int_0^L \left|\frac{\pi}{L}\cos\left(\frac{\pi t}{L}\right)e_j\right|^2 - \sin^2\left(\frac{\pi t}{L}\right)R(\gamma', e_j, \gamma', e_j)\,dt \\
&= \int_0^L \frac{\pi^2}{L^2}\cos^2\left(\frac{\pi t}{L}\right) - \sin^2\left(\frac{\pi t}{L}\right)R(\gamma', e_j, \gamma', e_j)\,dt.
\end{aligned}$$

Summing over $1 \leq j \leq n-1$ yields

$$\begin{aligned}
0 &\leq (n-1)\frac{\pi^2}{L^2}\int_0^L \cos^2\left(\frac{\pi t}{L}\right)dt - \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\sum_{j=1}^{n-1} R(\gamma', e_j, \gamma', e_j)\,dt \\
&= (n-1)\frac{\pi^2}{L^2}\int_0^L \cos^2\left(\frac{\pi t}{L}\right)dt - \int_0^L \sin^2\left(\frac{\pi t}{L}\right)\mathrm{Ric}(\gamma', \gamma')\,dt \\
&\leq (n-1)\frac{\pi^2}{L^2}\int_0^L \cos^2\left(\frac{\pi t}{L}\right)dt - c\int_0^L \sin^2\left(\frac{\pi t}{L}\right)dt.
\end{aligned}$$

The summation equals $\mathrm{Ric}(\gamma', \gamma')$ because the missing term is $R(\gamma', \gamma', \gamma', \gamma) = 0$. The two integrals are equal, so dividing out yields

$$c \leq (n-1)\frac{\pi^2}{L^2}.$$

□

Remark 6.16. Taking $V = \sin(\frac{\pi t}{L})e_j$ results in the optimal bound. If $V = fe_j$ with $f = 0$ at the endpoints and $\kappa \equiv 1$, then

$$\begin{aligned} 0 &\le I(V,V) \\ &= \int_0^L (f')^2 - f^2 R(\gamma', e_j, \gamma', e_j)\, dt \\ &= \int_0^L (f')^2 - f^2\, dt. \end{aligned}$$

We want to minimize the Raleigh quotient

$$\frac{\int_0^L (f')^2\, dt}{\int_0^L f^2\, dt}.$$

Letting $L = \pi$ for simplicity and writing the Fourier series $f = \sum_{n=1}^{\infty} a_n \sin(nx)$, the quotient becomes

$$\frac{\sum_{n=1}^{\infty} a_n^2 n^2}{\sum_{n=1}^{\infty} a_n^2},$$

which is minimized when $a_1 = 1$ and $a_2 = a_3 = \cdots = 0$.

Remark 6.17. Cheng's maximal diameter theorem states that if equality occurs, then M is a round sphere.

Example 6.18

There is no complete metric on $\mathbb{R}^2$ with $\mathrm{Ric} \ge c > 0$ because it is not compact.

Corollary 6.19

If (M^n, g) is complete with $\mathrm{Ric} \ge c > 0$, then $\pi_1(M)$ is finite.

Proof. Let $\widetilde{M}$ be the universal cover of M. $\widetilde{M}$ is also complete and satisfies the same diameter bound because it has a natural metric by lifting the metric on M. By Bonnet–Myers, $\widetilde{M}$ is compact, so $\pi_1(M)$ is finite. □

Example 6.20

$T^2 = S^1 \times S^1$ has no such metric, because $\pi_1(T^2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is infinite.

7 Laplacian

7.1 Harmonic functions and eigenvalues

Let $u \in C(M)$ be a function.

Definition 7.1 (gradient)**.** The *gradient* of u is the vector field obtained by raising the index:

$$\nabla u = \sum_{i,j} g^{ij} u_j \partial_{x_i}.$$

Definition 7.2 (Hessian)**.** The *Hessian* of u is a $(1,1)$-tensor and the covariant derivative of u:

$$\operatorname{Hess} u = \nabla(\nabla u).$$

Definition 7.3 (Laplacian)**.** The *Laplacian* of u is the trace of the Hessian:

$$\Delta u = \operatorname{Tr} \nabla(\nabla u).$$

Example 7.4

On $\mathbb{R}^n$, $\nabla u = u_i$, $\nabla(\nabla u) = u_{ij}$, and

$$\Delta u = \sum_{j=1}^{n} u_{jj} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

Definition 7.5 (divergence)**.** The *divergence* of a vector field V is the function

$$\operatorname{div} V = \operatorname{Tr} \nabla V.$$

Note that $\Delta u = \operatorname{div}(\nabla u)$. In an orthonormal frame e_i, we have

$$\operatorname{div} V = \sum_{i=1}^{n} \langle \nabla_{e_i} V, e_i \rangle.$$

In local coordinates, the divergence of $V = \sum_j V^j \partial_j$ is

$$\operatorname{div} V = \sum_i \partial_i V^i + \sum_{i,j} \Gamma^j_{ij} V^i.$$

By the Leibniz rule, we have $[\nabla(uV)]^i_j = u_j V^i + u V^i_j$, so

$$\operatorname{div}(uV) = \sum_i (u_i V^i + u V^i_i) = u \operatorname{div} V + V(u) = u \operatorname{div} V + \langle \nabla u, V \rangle.$$

Applying this when $V = \nabla v$, we have

$$\operatorname{div}(u \nabla v) = u \Delta v + \langle \nabla u, \nabla v \rangle.$$

Theorem 7.6 (Riemann divergence)

Let $\Omega \subset M$ be a domain with boundary $\partial\Omega$. Then $\int_{\partial\Omega} \operatorname{Flux} V = \int_\Omega \operatorname{div} V$.

Corollary 7.7

If M is compact (no boundary), then $\int_M \Delta u = 0$ for any u.

Corollary 7.8

If $u, v \equiv 0$ on $\partial\Omega$, then

$$\int_\Omega u\Delta v = \int_\Omega v\Delta u = -\int_\Omega \langle \nabla u, \nabla v \rangle.$$

The Laplacian is self-adjoint.

Definition 7.9 (harmonic)**.** A function u is *harmonic* if $\Delta u = 0$.

Example 7.10

On $\mathbb{R}^2$, some harmonic functions are

$$x, y, xy, x^2 - y^2, x^3 - 3xy^2, y^3 - 3x^2y, \dots.$$

In general, $\operatorname{Re}(x+yi)^k$ and $\operatorname{Im}(x+yi)^k$ are harmonic by the Cauchy–Riemann equation.

Lemma 7.11

Suppose $\Delta u \equiv 0$ on Ω and $v \equiv 0$ on $\partial\Omega$. Then

$$\int_\Omega |\nabla(u+v)|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla v|^2.$$

In other words, the energy of $u+v$ is the sum of the energies of u and v. In particular, a harmonic function u minimizes the energy across all functions that are the same as u on $\partial\Omega$.

Proof. The difference between the two sides is

$$\begin{aligned} 2\int_\Omega \langle \nabla u, \nabla v \rangle &= 2\int_\Omega \operatorname{div}(v\nabla u) - v\Delta u \\ &= 0 \end{aligned}$$

because $v\nabla u \equiv 0$ on $\partial\Omega$, and $\Delta u \equiv 0$. □

Lemma 7.12 (Reverse Poincaré inequality)

Let u be a harmonic function and ϕ be a cutoff function (i.e. compact support). Then

$$\int |\nabla u|^2 \phi^2 \le 4\int u^2 |\nabla \phi|^2.$$

Proof. Since ϕ has compact support, $\operatorname{div}(\phi^2 u \nabla u)$ integrates to 0. This evaluates to

$$\phi^2 u\Delta u + \phi^2 |\nabla u|^2 + 2\phi u \langle \nabla\phi, \nabla u \rangle,$$

so

$$\int \phi^2 |\nabla u|^2 = -2\int \phi u \langle \nabla \phi, \nabla u \rangle.$$

Then by the absorbing inequality (AM-GM) $2ab \le \epsilon a^2 + \frac{b^2}{\epsilon}$ with $\epsilon = \frac{1}{2}$, $a = \phi\,|\nabla u|$, and $b = u\,|\nabla \phi|$, we have

$$\begin{aligned} \int \phi^2 |\nabla u|^2 &\le 2 \int |\phi u \langle \nabla \phi, \nabla u\rangle| \\ &\le \frac{1}{2}\int \phi^2 |\nabla u|^2 + 2\int u^2 |\nabla \phi|^2 \\ \implies \int \phi^2 |\nabla u|^2 &\le 4 \int u^2 |\nabla \phi|^2 . \end{aligned}$$

□

Corollary 7.13 (Yau)
An L^2-harmonic function on a complete (connected) manifold must be constant.

Proof. Fix any point $p \in M$. Let ϕ_j be a sequence of cutoff functions with $\phi_j \equiv 1$ on the ball B_j that cuts off on $B_{j+1} \setminus B_j$ linearly in distance:

$$\phi_j(x) = \begin{cases} 1 & d(x,p) \le j \\ j+1-d(x,p) & j \le d(x,p) \le j+1 \\ 0 & d(x,p) \ge j+1 \end{cases} .$$

Applying the reverse Poincaré inequality on ϕ_j, the LHS converges to $\int |\nabla u|^2$, while the RHS converges to 0 (because u is in L^2 and the dominated convergence theorem). □

7.2 Bochner formula

For a function $u \in C(M)$, recall that

- The gradient ∇u is a $(1,0)$-tensor, and the differential du is a $(0,1)$-tensor.
- The Hessian $\nabla\nabla u$ is a $(1,1)$-tensor, and ∇du is a $(0,2)$-tensor.
- The Laplacian $\Delta u = \mathrm{Tr}(\nabla\nabla u) = \mathrm{div}(\nabla u)$ is a function.

Lemma 7.14 (Bochner)
$\frac{1}{2}\Delta |\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u)$.

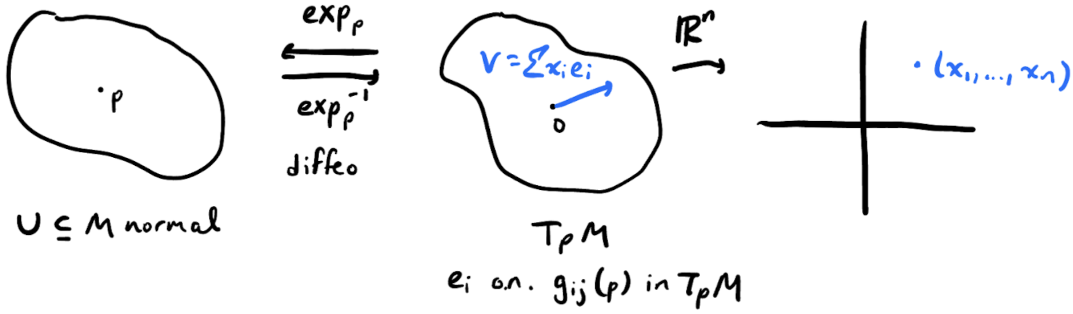
The formula is most useful when $\Delta u = 0$, so the middle term vanishes.

Corollary 7.15
If $\Delta u \equiv 0$ and $\mathrm{Ric} \ge 0$, then $\Delta |\nabla u|^2 \ge 0$.

The Bochner formula is an equality of functions, so choose coordinates to make the computation easier.

Definition 7.16 (geodesic normal coordinates). *Geodesic normal coordinates* about a fixed $p \in M$ satisfy

- $g_{ij}(p) = \delta_{ij}$, so the ∂_i are orthonormal at p.
- $\Gamma^k_{ij}(p) = 0$, so $\nabla_\bullet \partial_i(p) \equiv 0$ and $\nabla_\bullet dx_i(p) \equiv 0$.



We show that such a coordinate system exists. Let e_i be an orthonormal of T_pM. Let x_i denote the ith coordinate of V under this basis. Then $y_i = x_i \circ \exp_p^{-1}$ is a coordinate system in some neighborhood $U \subset M$ of p. They are orthonormal (only) at p because $(d\exp_p)_0 = \mathrm{Id}$ and the e_i are orthonormal by construction.

The curve $\gamma(t) = t(a_1, \ldots, a_n)$ for some constants a_i is a geodesic. Then from $\gamma' = \sum_i a_i \partial_i$, we have

$$0 = \nabla_{\gamma'}\gamma' = \sum_i a_i \nabla_{\partial_i}(a_j \partial_j) = \sum_{i,j} a_i a_j \nabla_{\partial_i} \partial_j = \sum_{i,j,k} a_i a_j \Gamma_{ij}^k \partial_k.$$

At p, this implies

$$0 = \sum_{i,j,k} a_i a_j \Gamma_{ij}^k(p) \partial_k,$$

for any a_i, so $\Gamma_{ij}^k(p) = 0$.

This implies $\nabla \partial_i(p) = 0$ and $\nabla dx_i(p) = 0$, because they can be written in terms of $\Gamma_{ij}^k(p)$.

Proof of Lemma 7.14. The facts we will use are

1. Symmetry of the Hessian: $\langle \nabla_V \nabla f, W \rangle = \langle \nabla_W \nabla f, V \rangle$.

   The LHS is $V\langle \nabla f, W \rangle - \langle \nabla f, \nabla_V W \rangle$, while the RHS is $W\langle \nabla f, V \rangle - \langle \nabla f, \nabla_W V \rangle$. Recalling that $\langle \nabla f, W \rangle = W(f)$, the difference is $V(W(f)) - W(V(f)) - \langle \nabla f, [V, W] \rangle = 0$.
2. At p, $\Delta f(p) = \sum_i \langle \nabla_{\partial_i} \nabla f, \partial_i \rangle(p) = \sum_i \partial_i \langle \nabla f, \partial_i \rangle(p)$. The first equality is the definition of trace, and the second equality is by metric compatibility and how $\nabla_{\partial_i} \partial_i(p) = 0$ by the geodesic normal coordinates.
3. $\nabla_{\partial_i} \nabla_{\nabla u} \nabla u = R(\nabla u, \partial_i) \nabla u + \nabla_{\nabla u} \nabla_{\partial_i} u - \nabla_{[\nabla u, \partial_i]} \nabla u$ by the definition of R.
4. If $f = \frac{1}{2} |\nabla u|^2$, then $\langle \nabla f, V \rangle = V(f) = \frac{1}{2} V \langle \nabla u, \nabla u \rangle = \langle \nabla_V \nabla u, \nabla u \rangle = \langle \nabla_{\nabla u} \nabla u, V \rangle$, where the last equality is by the symmetry of the Hessian.

Now we compute

$$\begin{aligned}
\Delta f(p) &= \sum_i \partial_i \langle \nabla f, \partial_i \rangle && \text{(by 2)}\\
&= \sum_i \partial_i \langle \nabla_{\nabla u} \nabla u, \partial_i \rangle && \text{(by 4)}\\
&= \sum_i \langle \nabla_{\partial_i} \nabla_{\nabla u} \nabla u, \partial_i \rangle && \text{(metric compatibility, } \nabla \partial_i(p) = 0)\\
&= \sum_i R(\nabla u, \partial_i, \nabla u, \partial_i) + \langle \nabla_{\nabla u} \nabla_{\partial_i} \nabla u - \nabla_{[\nabla u, \partial_i]} \nabla u, \partial_i \rangle && \text{(by 3)}\\
&= \mathrm{Ric}(\nabla u, \nabla u) + \sum_i \nabla u \langle \nabla_{\partial_i} \nabla u, \partial_i \rangle - \sum_i \langle \nabla_{\partial_i} \nabla u, [\nabla u, \partial_i] \rangle && \text{(by 1)}\\
&= \mathrm{Ric}(\nabla u, \nabla u) + \sum_i \nabla u \langle \nabla_{\partial_i} \nabla u, \partial_i \rangle + \sum_i \langle \nabla_{\partial_i} \nabla u, \nabla_{\partial_i} \nabla u \rangle\\
&= \mathrm{Ric}(\nabla u, \nabla u) + \sum_i \nabla u \langle \nabla_{\partial_i} \nabla u, \partial_i \rangle + |\mathrm{Hess}_u|^2
\end{aligned}$$

where $[\nabla u, \partial_i] = -\nabla_{\partial_i} \nabla u$ because $\nabla_\bullet \partial_i = 0$ at p.

It remains to show that $\langle \nabla u, \nabla \Delta u \rangle = \sum_i \nabla u \langle \nabla_{\partial_i} \nabla u, \partial_i \rangle$ at p, which is true because

$$\begin{aligned}
\langle \nabla u, \nabla \Delta u \rangle &= \nabla u (\Delta u)\\
&= \nabla u \Big(\sum_i dx_i (\nabla_{\partial_i} \nabla u) \Big)\\
&= \sum_i (\nabla_{\nabla u} dx_i)(\nabla_{\partial_i} \nabla u) + dx_i (\nabla_{\nabla u} \nabla_{\partial_i} \nabla u) && \text{(Leibniz rule)}\\
&= \sum_i 0 + \langle \nabla_{\nabla u} \nabla_{\partial_i} \nabla u, \partial_i \rangle\\
&= \sum_i \nabla u \langle \nabla_{\partial_i} \nabla u, \partial_i \rangle. && \text{(metric compatibility, } \nabla \partial_i(p) = 0)
\end{aligned}$$

The first term in the third line is 0 at p, and applying dx_i is the same as applying $\langle \bullet, \partial_i \rangle$ at p. □

The following theorem gives a lower bound for eigenvalues of the Laplacian.

Theorem 7.17 (Lichnerowicz)
If M^n is complete, $\mathrm{Ric} \geq c > 0$, and $\Delta u = -\lambda u$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ with $\int u^2 = 1$, then $\lambda \geq \frac{cn}{n-1}$.

Example 7.18
On a sphere we have $c = n - 1$, so any eigenvalue λ must be at least n. It turns out that coordinate functions u attain this bound and that Lichnerowicz's theorem holds only for the sphere.

Proof. In general,

$$\frac{1}{2} \Delta u^2 = \mathrm{div} \left(\nabla \frac{1}{2} u^2 \right) = \mathrm{div}(u \nabla u) = |\nabla u|^2 + u \Delta u$$

Then in our case,

$$\frac{1}{2} \Delta u^2 = |\nabla u|^2 - \lambda u^2.$$

By the divergence formula, $\int \Delta v = \int \text{Flux} = 0$ because the Flux is across a boundary that does not exist. Integrating the above equation,

$$\int |\nabla u|^2 = \lambda \int u^2 = \lambda.$$

In particular, $\lambda \geq 0$.

By the Bochner formula,

$$\frac{1}{2}\Delta |\nabla u|^2 = |\text{Hess}_u|^2 + \langle \nabla u, \nabla(-\lambda u)\rangle + \text{Ric}(\nabla u, \nabla u),$$

so

$$\begin{aligned} 0 &= \int |\text{Hess}_u|^2 - \lambda \int |\nabla u|^2 + \int \text{Ric}(\nabla u, \nabla u) \\ &\geq \int |\text{Hess}_u|^2 - \lambda^2 + c\lambda \end{aligned}$$

where $\int \text{Ric}(\nabla u, \nabla u) \geq c \int |\nabla u|^2$ by assumption. Then

$$\lambda^2 - c\lambda \geq \int |\text{Hess}_u|^2 .$$

Given a symmetric $n \times n$ matrix S, we have $\operatorname{Tr} S = S \cdot \delta_{ij}$ ($\mathbb{R}^{n^2}$ dot product), so by Cauchy–Schwarz,

$$|\operatorname{Tr} S|^2 = |S \cdot \delta_{ij}|^2 \leq |S|^2 |\delta_{ij}|^2 = n |S|^2 ,$$

so $(\Delta u)^2 \leq n |\text{Hess}_u|^2$. Integrating yields

$$\lambda^2 - c\lambda \geq \int |\text{Hess}_u|^2 \geq \frac{1}{n} \int (\Delta u)^2 = \frac{\lambda^2}{n} \int u^2 = \frac{\lambda^2}{n},$$

which rearranges to what we want: $\lambda \geq \frac{cn}{n-1}$. □

7.3 Isomperimetric and Wirtinger inequalities

Theorem 7.19 (Isoperimetric inequality)
For $\Omega \subset \mathbb{R}^2$, we have

$$\frac{\text{Area}(\Omega)}{L(\partial\Omega)^2} \leq \frac{1}{4\pi}$$

with equality if and only if Ω is a circle.

Proposition 7.20 (Wirtinger inequality)

1. If $f\colon [0, \pi] \to \mathbb{R}$ satisfies $f(0) = f(\pi) = 0$, then

$$\int_0^\pi f^2 \leq \int_0^\pi (f')^2.$$

   Equality holds if and only if $f(x) = \sin x$.

2. If f is a C^1 function on $S^1 = [0, 2\pi]$ and $\int_{S^1} f = 0$, then

$$\int_{S^1} f^2 \leq \int_{S^1} (f')^2.$$

   Equality holds if and only if $f(x) = a\sin x + b\cos x$.

Note that $\frac{\int (f')^2}{\int f^2}$ is not dimensionless. Taking $F(x) = f(\lambda x)$, we have $\frac{\int (F')^2}{\int F^2} = \lambda^2 \frac{\int (f')^2}{\int f^2}$.

Proof. Let $g(x) = f(x)^2 \frac{\cos x}{\sin x}$ which is smooth and equals 0 at the endpoints because $f(x)^2$ vanishes to order 2 while $\sin x$ vanishes to order 1. By the FTC,

$$0 = \int_0^\pi g'\,dx = \int_0^\pi 2ff'\frac{\cos x}{\sin x} - f^2 - f^2\frac{\cos^2 x}{\sin^2 x}\,dx.$$

Then

$$\begin{aligned}\int_0^\pi f^2\,dx &= \int_0^\pi 2ff'\frac{\cos x}{\sin x} - f^2\frac{\cos^2 x}{\sin^2 x}\,dx \\ &= \int_0^\pi (f')^2 - \left(f' - f\frac{\cos x}{\sin x}\right)^2 dx \\ &\le \int_0^\pi (f')^2\,dx.\end{aligned}$$

For equality, we need $\frac{f'}{f} = \frac{(\sin x)'}{\sin x}$ so f is a multiple of $\sin x$. $\square$

Proof of Theorem 7.19. We can assume that Ω is connected because it suffices to prove the inequality on each connected component. Also $\partial\Omega$ is connected, because otherwise filling in any holes of Ω increases the area and decreases the boundary. WLOG suppose $L(\partial\Omega) = 2\pi$, so we wish to show that $\operatorname{Area}(\Omega) \le \pi$.

Let $\partial\Omega$ be given by a map $\gamma\colon S^1 \to \mathbb{R}^2$ where $|\gamma'| = 1$. Writing $\gamma = (\gamma_1, \gamma_2)$, we can translate Ω such that $\int_{S^1}\gamma_1 = \int_{S^1}\gamma_2 = 0$. By the Wirtinger inequality (Proposition 7.20),

$$\int_{S^1}\gamma_1^2 \le \int_{S^1}(\gamma_1')^2$$

and same for γ_2. Adding these inequalities yields

$$\int_{S^1}|\gamma|^2 \le \int_{S^1}|\gamma'|^2 = \int_{S^1} 1 = 2\pi.$$

Since $|\gamma'| = 1$, we can reparameterize $\int_{\partial\Omega}|x|^2 = \int_{S^1}|\gamma|^2 \le 2\pi$ where $x = (x_1, x_2)$. On $\mathbb{R}^2$, $\operatorname{div} x = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} = 2$. Then letting n be the outward unit normal,

$$\begin{aligned}2\operatorname{Area}(\Omega) &= \int_\Omega \operatorname{div} x \\ &= \int_{\partial\Omega} x\cdot n && \text{(divergence theorem)} \\ &\le \left(\int_{\partial\Omega}|x|^2\right)^{\frac12}\left(\int_{\partial\Omega}|n|^2\right)^{\frac12} && \text{(Cauchy–Schwarz)} \\ &\le (2\pi)^{\frac12}(2\pi)^{\frac12} \\ &= 2\pi\end{aligned}$$

where $|n|^2 = 1$ pointwise. For equality in the Cauchy–Schwarz inequality, we need $|x|^2$ to also be constant on $\partial\Omega$. $\square$

7.4 Submanifolds

Let $\Sigma \hookrightarrow M$ be a submanifold, so the induced connection is $\overline{\nabla} u = \nabla^\top u$. Then $\overline{\text{Hess}}_u = \overline{\nabla}\overline{\nabla} u$, and for any tangent vectors V and W,

$$\begin{aligned}
\overline{\text{Hess}}_u(V, W) &= \langle \nabla_V^\top \nabla^\top u, W\rangle \\
&= \langle \nabla_V \nabla^\top u, W\rangle && (W \text{ tangent}) \\
&= \langle \nabla_V (\nabla u - \nabla^\perp u), W\rangle \\
&= \text{Hess}_u(V, W) - \langle \nabla_V \nabla^\perp u, W\rangle \\
&= \text{Hess}_u(V, W) + \langle \nabla^\perp u, \nabla_V W\rangle && (\text{metric compatibility } \nabla^\perp u \perp W) \\
&= \text{Hess}_u(V, W) + \langle A(V, W), \nabla u\rangle.
\end{aligned}$$

Tracing,

$$\begin{aligned}
\overline{\Delta} u &= \sum_{i=1}^n \text{Hess}_u(e_i, e_i) + \langle \nabla u, \mathbf{H}\rangle \\
&= \Delta u + \langle \nabla u, \mathbf{H}\rangle
\end{aligned}$$

where e_i is an orthonormal frame for Σ, and $\mathbf{H}$ is the mean curvature.

Theorem 7.21 (Minkowski)
Let $\Omega \subset \mathbb{R}^{n+1}$ be a compact subset of dimension n with smooth boundary $\partial\Omega$. Let $\vec{n}$ be the outward unit normal on $\partial\Omega$.

1. $\int_{\partial\Omega} \langle x, \vec{n}\rangle = (n+1)\,\text{Vol}(\Omega)$.
2. $\int_{\partial\Omega} \langle x, \vec{n}\rangle \langle \mathbf{H}, \vec{n}\rangle = -\int_{\partial\Omega} n = -n\,\text{Area}(\partial\Omega)$.

A consequence of 2 is that $\partial\Omega$ can never be minimal, because then the integrand and integral are 0.

Proof. 1. This follows from $\operatorname{div} x = n + 1$ on $\mathbb{R}^{n+1}$.

2. We apply the divergence theorem to $x^\top$ on $\partial\Omega$. Letting e_i be an orthonormal frame on $\partial\Omega$,

$$\begin{aligned}
\overline{\text{div}}(x^\top) &= \sum_i \langle \nabla_{e_i} x^\top, e_i\rangle \\
&= \sum_i \langle \nabla_{e_i}(x - x^\perp), e_i\rangle \\
&= \sum_i \langle e_i - \nabla_{e_i} x^\perp, e_i\rangle \\
&= n - \sum_i \langle \nabla_{e_i} x^\perp, e_i\rangle \\
&= n + \sum_i \langle x^\perp, \nabla_{e_i}^\perp e_i\rangle && (\text{metric compatibility}) \\
&= n + \langle x^\perp, \mathbf{H}\rangle.
\end{aligned}$$

Integrating gives the claim.

□

7.5 Spherical harmonics

We specialize to the case of $S^n \hookrightarrow \mathbb{R}^{n+1}$. Let x be the unit normal, so we know that

$$A(V, W) = -\langle V, W\rangle x, \quad \kappa \equiv 1, \quad \mathrm{Ric} = (n-1)g, \quad \mathbf{H} = -nx.$$

Lemma 7.22

The x_i are eigenfunctions with $\overline{\Delta} x_i = -nx_i$.

Note this is the equality case of the Lichnerowicz theorem (Theorem 7.17).

Proof. Since $\mathrm{Hess}_{x_i} \equiv 0$ (the second derivatives of x_i are 0), $\Delta x_i = 0$. Then

$$\overline{\Delta} x_i = \langle \nabla x_i, \mathbf{H}\rangle = \langle \partial_i, -nx\rangle = -nx_i. \qquad \square$$

Definition 7.23 (homogeneous)**.** A function u is *homogeneous* on $\mathbb{R}^{n+1}$ of degree d if $u(sx) = s^d u(x)$ for all $s \in \mathbb{R}_{>0}$.

Taking the s derivative yields

$$\langle \nabla u(sx), x\rangle = ds^{d-1}u(x).$$

Differentiating again yields

$$\mathrm{Hess}_u(x, x)(sx) = d(d-1)s^{d-2}u(x).$$

Plugging in $s = 1$ results in

$$\begin{aligned} \langle \nabla u(x), x\rangle &= du(x) \\ \mathrm{Hess}_u(x, x) &= d(d-1)u(x). \end{aligned}$$

Now consider the sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$. Let $e_1, \ldots, e_n, x$ be an orthonormal frame of $\mathbb{R}^{n+1}$ (x is normal), so

$$\begin{aligned} \overline{\Delta} u &= \sum_{i=1}^{n} \mathrm{Hess}_u(e_i, e_i) - n\langle \nabla u, x\rangle \\ &= \Delta u - \mathrm{Hess}_u(x, x) - n\langle \nabla u, x\rangle \\ &= \Delta u - d(d-1)u - ndu \\ &= \Delta u - d(d+n-1)u. \end{aligned}$$

Note the second equality is because tracing Hess_u over all $n+1$ directions yields Δu.

Theorem 7.24 (Spherical harmonics)

Suppose u is a function on $\mathbb{R}^{n+1}$ that is homogeneous of degree d and $\Delta u \equiv 0$. Then on S^n,

$$\overline{\Delta} u = -d(d+n-1)u,$$

so u is an eigenfunction with eigenvalue $d(d+n-1)$.

- For $d = 1$, we get the coordinate functions x_i, with eigenvalue $\lambda = 1(1+n-1) = n$.
- If $n = 1$, then $\lambda = d^2$, which will yield $r^d \sin d\theta$ and $r^d \cos d\theta$ (we haven't proven this).

8 Minimal submanifolds

8.1 First variation

We show that there exist energy-minimizing curves. Consider the space of curves γ on $[0,L]$ with $\gamma(0)=p$ and $\gamma(L)=q$. We want to show there is a subsequence that converges. To apply Ascoli–Arzelá, we need equicontinuity (we already have uniform boundedness):

$$\begin{aligned}|d(\gamma(x),\gamma(y))| &\le L(\gamma([x,y]))\\ &= \int_x^y |\gamma'(t)|\,dt\\ &\le \left(\int_x^y |\gamma'|^2\right)^{\frac12}\left(\int_x^y 1\right)^{\frac12}\\ &\le c\sqrt{|x-y|}.\end{aligned}$$

This argument can be used in 2 dimensions but fails for 3+ dimensions.

Let $\Sigma^n \hookrightarrow M^N$ be a submanifold. Consider a variation

$$F\colon \Sigma\times(-\epsilon,\epsilon)\to M$$

where $(-\epsilon,\epsilon)$ is parameterized by s, and let $F_s = dF(\partial_s)$ which is a variation vector field. A proper variation means that $F_s\equiv 0$ on the boundary.

Let $\overline{dv}$ denote the induced volume element from M onto $\Sigma\times(-\epsilon,\epsilon)$.

Lemma 8.1 (First variation formula)
$\partial_s\overline{dv} = \left(\overline{\operatorname{div}}(F_s^\top) - \langle F_s^\perp, \mathbf{H}\rangle\right)\overline{dv}$

Proof. Let $x_1,\dots,x_n$ be local coordinates, and let $F_i = dF(\partial_{x_i})$. As before, we have $[F_s,F_i]=0$, and $[F_i,F_j]=0$. The induced metric is $\overline{g}_{ij}=\langle F_i,F_j\rangle$, and the volume element is $\overline{dv}=\sqrt{\det\overline{g}_{ij}}\,dx$. Then

$$\begin{aligned}\partial_s\overline{g}_{ij} &= F_s\langle F_i,F_j\rangle\\ &= \langle\nabla_{F_s}F_i,F_j\rangle + \langle F_i,\nabla_{F_s}F_j\rangle\\ &= \langle\nabla_{F_i}F_s,F_j\rangle + \langle F_s,\nabla_{F_j}F_i\rangle\end{aligned}$$

where

$$\begin{aligned}\langle\nabla_{F_i}F_s,F_j\rangle &= \langle\nabla_{F_i}F_s^\top,F_j\rangle + \langle\nabla_{F_i}F_s^\perp,F_j\rangle\\ &= \langle\nabla_{F_i}F_s^\top,F_j\rangle - \langle F_s^\perp,\nabla_{F_i}F_j\rangle\\ &= \langle\nabla_{F_i}F_s^\top,F_j\rangle - \langle F_s^\perp,\nabla^\perp_{F_i}F_j\rangle\\ &= \langle\nabla_{F_i}F_s^\top,F_j\rangle - \langle F_s^\perp,A(F_i,F_j)\rangle.\end{aligned}$$

Analogously, $\langle F_s,\nabla_{F_j}F_i\rangle = \langle\nabla_{F_j}F_s^\top,F_j\rangle - \langle F_s^\perp,A(F_i,F_j)\rangle$, so

$$\partial_s\overline{g}_{ij} = \langle\nabla_{F_i}F_s^\top,F_j\rangle + \langle\nabla_{F_j}F_s^\top,F_i\rangle - 2\langle F_s^\perp,A(F_i,F_j)\rangle.$$

By Lemma 8.2,

$$\begin{aligned}\partial_s\sqrt{\det\overline{g}_{ij}} &= \frac{1}{2\sqrt{\det\overline{g}_{ij}}}\partial_s(\det\overline{g}_{ij})\\ &= \frac{1}{2\sqrt{\det\overline{g}_{ij}}}\det\overline{g}_{ij}\operatorname{Tr}(\overline{g}^{-1}\partial_s\overline{g}_{ij}).\end{aligned}$$

Next, we compute that

$$\begin{aligned}\frac{1}{2}\operatorname{Tr}(\overline{g}^{-1}\partial_s\overline{g}_{ij}) &= \frac{1}{2}\operatorname{Tr}\left(\overline{g}^{-1}[\langle\nabla_{F_i}F_s^\top, F_j\rangle + \langle\nabla_{F_j}F_s^\perp, F_i\rangle]\right) - \operatorname{Tr}\left(\overline{g}^{-1}\langle F_s^\top, A(F_i,F_j)\rangle\right)\\ &= \overline{\operatorname{div}}(F_s^\top) - \langle F_s^\perp, \mathbf{H}\rangle.\end{aligned}$$

since $\operatorname{Tr}(\overline{g}^{-1}A_{ij}) \equiv \mathbf{H}$. All together,

$$\partial_s\sqrt{\det\overline{g}_{ij}} = \sqrt{\det\overline{g}_{ij}}\left(\overline{\operatorname{div}}(F_s^\top) - \langle F_s^\perp, \mathbf{H}\rangle\right)$$

which implies the desired result. □

Lemma 8.2
Let B and C be $n\times n$ matrices with B invertible. Then

$$\partial_s|_{s=0}\det(B+sC) = (\det B)(\operatorname{Tr}B^{-1}C).$$

Proof. We have

$$\det(B+sC) = \det B\det(I+sB^{-1}C).$$

Differentiating yields

$$\partial_s|_{s=0}\det(B+sc) = (\det B)\left(\partial_s\det(I+sB^{-1}C)|_{s=0}\right) = (\det B)(\operatorname{Tr}B^{-1}C).$$ □

To see the last equality, we do an example:

$$\det\begin{pmatrix}1+sa_{11} & sa_{12}\\ sa_{21} & 1+sa_{22}\end{pmatrix} = 1 + s(a_{11}+a_{22}) + s^2(a_{11}a_{22}-a_{12}a_{21}).$$

The derivative at $s=0$ is the trace $a_{11}+a_{22}$. Then we can prove it inductively on n.

Definition 8.3 (Vol). Given $\Omega\subseteq\Sigma$ compact, let $\operatorname{Vol}_\Omega(s)$ be the volume of the image of Ω at time s:

$$\operatorname{Vol}_\Omega(s) = \int_\Omega \overline{dv}.$$

Corollary 8.4
If Σ is minimal (so $\mathbf{H}\equiv 0$), then $\operatorname{Vol}_\Omega'(0) = 0$ for all compact subdomains $\Omega\subseteq\Sigma$.

Proof. By the first variation formula,

$$\begin{aligned}\partial_s|_{s=0}\operatorname{Vol}(s) &= \int_\Omega\overline{\operatorname{div}}(F_s^\top)\,\overline{dv} - \int_\Omega\langle F_s^\perp, \mathbf{H}\rangle\,\overline{dv}\\ &= -\int_\Omega\langle F_s^\perp, \mathbf{H}\rangle\,\overline{dv}\end{aligned}$$

because the first term is 0 when F_s has compact support (proper variation). The second term is also 0 when $\mathbf{H}\equiv 0$. □

Corollary 8.5

If $\Sigma^n \hookrightarrow \mathbb{R}^N$ minimal, and x_i are the coordinate functions of $\mathbb{R}^N$, then

1. $\overline{\Delta} x_i = 0$.
2. $\overline{\Delta} |x|^2 = 2n$.
3. $\overline{\operatorname{div}}\left(\frac{\nabla^\top |x|^2}{|x|^n}\right) = \frac{2n|x^\perp|^2}{|x|^{n+2}}$.

Proof. 1. We have $\overline{\Delta} x_i = \Delta x_i + \langle \nabla x_i, \mathbf{H}\rangle = 0$ because both terms vanish.

2. Let $e_1, \dots, e_n$ be an orthonormal frame for Σ. On $\mathbb{R}^N$, we have $\nabla |x|^2 = 2x$. Then $\nabla^\top |x|^2 = 2x^\top = 2x - 2x^\perp$. Then

$$\begin{aligned}
\overline{\Delta} |x|^2 &= \overline{\operatorname{div}}(2x - 2x^\perp) \\
&= 2\sum_{i=1}^n \langle \nabla_{e_i}(x - x^\perp), e_i\rangle && (\nabla_{e_i} x = e_i) \\
&= 2n - 2\sum_{i=1}^n \langle \nabla_{e_i} x^\perp e_i\rangle \\
&= 2n + \sum_{i=1}^n \langle x^\perp, \nabla^\perp_{e_i} e_i\rangle && (\text{this is } \mathbf{H}) \\
&= 2n.
\end{aligned}$$

3. By the Leibniz rule, we have

$$\begin{aligned}
\overline{\operatorname{div}}\left(\frac{\nabla^\top |x|^2}{|x|^n}\right) &= \frac{\overline{\nabla} |x|^2}{|x|^n} + \langle \overline{\nabla} |x|^2, \overline{\nabla} |x|^{-n}\rangle && (\text{Leibniz rule}) \\
&= \frac{\overline{\nabla} |x|^2}{|x|^n} + \left\langle \overline{\nabla} |x|^2, \frac{n}{2} |x|^{-(n+2)} \overline{\nabla} |x|^2 \right\rangle \\
&= \frac{2n}{|x|^n} + 2n |x|^{-(n+2)} \left|x^\top\right|^2 \\
&= \frac{2n(|x|^2 - |x^\top|^2)}{|x|^{n+2}} \\
&= \frac{2n\left|x^\perp\right|^2}{|x|^{n+2}}.
\end{aligned}$$

□

Theorem 8.6 (Monotonicity)

If $\Sigma^n \hookrightarrow \mathbb{R}^N$ is minimal, then for any $r_1 < r_2$,

$$\frac{\operatorname{Vol}(B_{r_1} \cap \Sigma)}{r_1^n} \leq \frac{\operatorname{Vol}(B_{r_2} \cap \Sigma)}{r_2^n}.$$

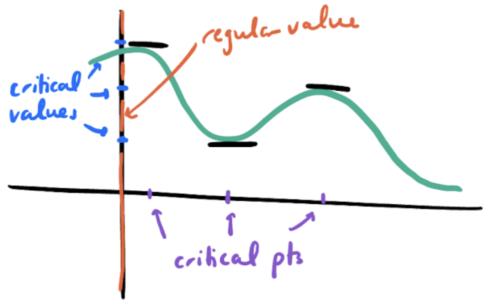
Equality holds if and only if $x^\perp = 0$ between r_1 and r_2, where x is a position vector (same thing as x is tangent to Σ at every point, i.e. being conical).

On $\mathbb{R}^n$, we have $\Delta |x|^2 = 2n$ too. Also,

$$\begin{aligned} 2n \operatorname{Vol}(B_R) &= \int_{B_R} \Delta |x|^2 \\ &= \int_{\partial B_R} \nabla |x|^2 \cdot \frac{x}{|x|} && \text{(divergence theorem)} \\ &= \int_{\partial B_R} 2x \cdot \frac{x}{|x|} \\ &= 2R \operatorname{Area}(\partial B_R) \end{aligned}$$

It's like how integrating $4\pi r^2$ gives $\frac{4}{3}\pi r^3$.

Definition 8.7 (critical, regular value)**.** Recall for a smooth function $f\colon M \to \mathbb{R}$, if $\nabla f(y) = 0$, then y is a *critical point* and $f(y)$ is a *critical value*. The set of *regular values* is $\operatorname{im}(f) \setminus \{\text{critical values}\}$.



- The implicit function theorem says that the inverse image of a regular value is a smooth submanifold.
- Sard's theorem says that the set of critical values has measure 0.

Proof of Theorem 8.6. **Case 1:** r_1 and r_2 are regular values of the function $|x|\colon \Sigma \to \mathbb{R}$.

If r is a regular value, then $\Sigma_r := B_r \cap \Sigma$ is smooth with smooth boundary $\partial\Sigma_r \subset \partial B_r$. We can then compute the volume of Σ_r as

$$\begin{aligned} 2n \operatorname{Vol}(\Sigma_r) &= \int_{\Sigma_r} \overline{\Delta} |x|^2 && \text{(Corollary 8.5)} \\ &= \int_{\partial\Sigma_r} \langle \overline{\nabla} |x|^2, \vec{n} \rangle \\ &= 2 \int_{\partial\Sigma_r} \langle x^\top, \vec{n} \rangle \end{aligned}$$

where $\vec{n} = \frac{x^\top}{|x^\top|}$ is the unit normal vector. Let

$$f(r) := \frac{\operatorname{Vol}(\Sigma_r)}{r^n},$$

so

$$\begin{aligned} 2nf(r_2) - 2nf(r_1) &= 2\int_{\partial\Sigma_{r_2}} \frac{\langle x^\top, \vec{n}\rangle}{|x|^n} - 2\int_{\partial\Sigma_{r_1}} \frac{\langle x^\top, \vec{n}\rangle}{|x|^n} \\ &= 2\int_{\Sigma_{r_2}\setminus\Sigma_{r_1}} \overline{\operatorname{div}}\left(\frac{x^\top}{|x|^n}\right) \\ &= 2\int_{\Sigma_{r_2}\setminus\Sigma_{r_1}} \frac{\left|x^\top\right|^2}{|x|^{n+2}} && \text{(Corollary 8.5)} \\ &> 0. \end{aligned}$$

Case 2: Suppose at least one of r_1, r_2 is not regular. Find a decreasing sequence $\{r_{1,j}\} \to r_1$ of regular values and an increasing sequence $\{r_{2,j}\} \to r_2$ of regular values. We can assume that $r_{1,j} \leq r_{2,j}$ for all j. By the regular case, we have $f(r_{1,j}) \leq f(r_{2,j})$. Also

$$\begin{aligned} f(r_1) &\leq \left(\frac{r_{1,j}}{r_1}\right)^n f(r_{1,j}) \\ &\leq \left(\frac{r_{1,j}}{r_1}\right)^n f(r_{2,j}) \\ &\leq \left(\frac{r_{1,j}}{r_1}\right)^n \left(\frac{r_2}{r_{2,j}}\right)^n f(r_2). \end{aligned}$$

□

Example 8.8

On Euclidean space, the inequality is an equality: we claim that $\frac{\mathrm{Vol}(B_r)}{r^n}$ is constant if and only if $\frac{\mathrm{Vol}^{n-1}(\partial B_r)}{r^{n-1}}$ is constant. We have $\frac{|\partial B_r|}{r^{n-1}} = \int_{\partial B_r} r^{1-n} = \int_{\partial B_r} \frac{\nabla |x|^{2-n}}{2-n} \cdot \vec{n}$ for $\vec{n}$ normal. By the divergence theorem, this equals $\frac{1}{2-n}\int_{B_r \text{ interior}} \operatorname{div}(\nabla |x|^{2-n})$ which is a constant.

8.2 Regularity theory

Recall that $\Sigma^n \hookrightarrow \mathbb{R}^N$ is minimal if $\mathbf{H} = 0$ (mean curvature). For geodesics, it's simple: any energy-minimizing curve between two points is a smooth geodesic. The next case is surfaces in $\mathbb{R}^3$. Take a curve γ and consider the spanning surface Σ of least area. The questions are does Σ exist, and is it smooth?

Example 8.9 (classical plateau problem)

Minimize the area of $F\colon B_1 \to \mathbb{R}^3$ where $B_1 \subset \mathbb{R}^2$ is such that $F(\partial B_1) = \gamma$.

Remark 8.10. A function is a linear functional on C^∞_{comp} (smooth functions with compact support), because a function f determines a map $\phi \mapsto \int f\phi$. Now consider the more general space $\mathcal{D}$ of linear functionals on C^∞_{comp}. An example of another element of $\mathcal{D}$ is $\phi \mapsto \phi(0)$.

If $\Delta f = g$, then we should have for all ϕ that $\int g\phi = \int \phi\Delta f = \int f\Delta\phi$, so $\int f\Delta\phi = \int g\phi$ could be taken as the definition of $\Delta f = g$ in the distributional sense.

In geometric measure theory, we think of submanifolds as functionals on functions f on $\mathbb{R}^3$. There are curves in $\mathbb{R}^3$ that don't have a minimal surface, such as knotted curves, with the simplest being the trefoil.

9 Laplacian comparison

9.1 Laplacian computations in Euclidean space

Question 9.1. What is $\Delta |x|$ on $\mathbb{R}^n \setminus \{0\}$?

We first note that

$$\Delta |x|^2 = \Delta(x_1^2 + \cdots + x_n^2) = 2n.$$

By the product rule,

$$\Delta |x|^2 = \Delta(|x| \cdot |x|) = 2|x| \Delta |x| + 2 |\nabla |x||^2 .$$

By the chain rule,

$$2x = \nabla |x|^2 = 2|x| \nabla |x| \implies \nabla |x| = \frac{x}{|x|},$$

so $|\nabla |x|| = 1$ for $x \neq 0$. All together,

$$2n = \Delta |x|^2 = 2|x| \Delta |x| + 2 \implies \boxed{\Delta |x| = \frac{n-1}{|x|}}$$

on $\mathbb{R}^n \setminus \{0\}$.

In general on M^n with $\operatorname{Ric} \geq 0$, defining $d(x) := d(p,x)$ for a fixed p, we have $\Delta d \leq \frac{n-1}{d}$.

Remark 9.2. We can similarly compute $\operatorname{Hess}_{|x|}$. We know that $\operatorname{Hess}_{|x|^2} = 2\delta_{ij}$, so

$$\begin{aligned} 2\delta_{ij} &= \frac{\partial^2 |x|^2}{\partial x_i \, \partial x_j} \\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} |x|^2\right) \\ &= \frac{\partial}{\partial x_i}\left(2|x| \frac{\partial |x|}{\partial x_j}\right) \\ &= 2|x| \frac{\partial^2 |x|}{\partial x_i \, \partial x_j} + 2\frac{\partial |x|}{\partial x_i}\frac{\partial |x|}{\partial x_j} \\ &= 2|x| \frac{\partial^2 |x|}{\partial x_i \, \partial x_j} + 2\frac{x_i x_j}{|x_i| \, |x_j|}. \end{aligned}$$

In particular,

$$\operatorname{Hess}_{|x|} = \begin{pmatrix} \frac{1}{|x|} \operatorname{Id}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

has eigenvalues of $\frac{1}{|x|}$ (multiplicity $n-1$) and 0 (multiplicity 1). The trace is the Laplacian $\frac{n-1}{|x|}$.

9.2 Distance function

Let $p \in M^n$ be fixed, and define $d(x) := d(p,x)$. Note that d is 1-Lipschitz by the triangle inequality. By *Rademacher's theorem*, d is differentiable almost everywhere.

Proposition 9.3 (Laplacian comparison at smooth points)
If M^n satisfies $\operatorname{Ric} \geq 0$, and q is a smooth point of d, then $\Delta d \leq \frac{n-1}{d}$ at q.

Proposition 9.4

Let $L = d(p, q)$. Suppose $\gamma\colon [0, L] \to M$ is a unit-speed minimizing geodesic with $\gamma(0) = p, \gamma(L) = q$, and that d is smooth in a neighborhood of every point in $\operatorname{im}\gamma \setminus p$. Then

1. $\nabla d(\gamma(t)) = \gamma'(t)$.
2. Hess_d has rank at most $n-1$ along γ and is 0 along γ' itself.

In $\mathbb{R}^n$, think about how along a ray from the origin to a point, the gradient is exactly the tangent to the geodesic (1). The rank of Hess was $n-1$, and the kernel was in the radial direction (2).

Proof. 1. We have $d(\gamma(t)) = t$ because γ is a minimizing unit-speed geodesic. Differentiating with respect to t yields $\langle \nabla d(\gamma(t)), \gamma'(t)\rangle = 1$. Both $\nabla d(\gamma(t))$ and $\gamma'(t)$ are unit vectors, so we need $\nabla d(\gamma(t)) = \gamma'(t)$.

2. Differentiating $\langle \nabla d(\gamma(t)), \gamma'(t)\rangle = 1$ again,

$$\begin{aligned} 0 &= \langle \nabla_{\gamma'} \nabla d(\gamma(t)), \gamma'\rangle + \langle \nabla d, \nabla_{\gamma'}\gamma'\rangle \\ &= \langle \nabla_{\gamma'} \nabla d, \gamma'\rangle \\ &= \langle \mathrm{Hess}_d(\gamma', \gamma'), \gamma'\rangle \end{aligned}$$

which implies γ' is in the kernel of the Hessian. □

Proof of Proposition 9.3. We know $|\nabla d|^2 = 1$ along γ, and in a neighborhood of each point on γ. This means we can take $\Delta |\nabla d|^2$. The Bochner formula (Lemma 7.14) says

$$0 = \frac{1}{2}\Delta |\nabla d|^2 = |\mathrm{Hess}_d|^2 + \langle \nabla d, \nabla \Delta d\rangle + \mathrm{Ric}(\nabla d, \nabla d) \geq |\mathrm{Hess}_d|^2 + 2\langle \nabla d, \nabla \Delta d\rangle.$$

For an $n \times n$ symmetric rank $\leq n-1$ matrix A, we have $(\mathrm{Tr}(A))^2 \leq (n-1)|A|^2$. Letting $A = \mathrm{Hess}_d$, then $(\Delta d)^2 \leq (n-1)|\mathrm{Hess}_d|^2$. Then

$$\begin{aligned} 0 &\geq \frac{1}{n-1}(\Delta d)^2 + \langle \gamma', \nabla \Delta d\rangle \\ &= \frac{1}{n-1}(\Delta d)^2 + \gamma'(\Delta d) \\ &= \frac{1}{n-1}(\Delta d)^2 + (\Delta d)'. \end{aligned}$$

Define $f(t) = (\Delta d)(\gamma(t))$. We have shown that $f' \leq -\frac{f^2}{n-1}$ and will be done by the following lemma. □

Lemma 9.5

If $f\colon [0, L] \to \mathbb{R}$ satisfies $f' \leq -\frac{f^2}{n-1}$, then $f(L) \leq \frac{n-1}{L}$.

Proof. We can assume that $f > 0$, since f is nonincreasing and would then satisfy $f(L) \leq 0 \leq \frac{n-1}{L}$. We can set $u(t) := \frac{1}{f(t)}$, so $u' = -\frac{f'}{f^2} \geq \frac{1}{n-1}$. FTC says that $u(L) - u(\epsilon) \geq \frac{1}{n-1}(L-\epsilon)$. Since $u(\epsilon) > 0$, we have $u(L) \geq \frac{L-\epsilon}{n-1}$ and $f(L) \leq \frac{n-1}{L-\epsilon}$. Since this is true for all $\epsilon > 0$ arbitrarily small, we have $f(L) \leq \frac{n-1}{L}$. □

9.3 Calabi's barriers

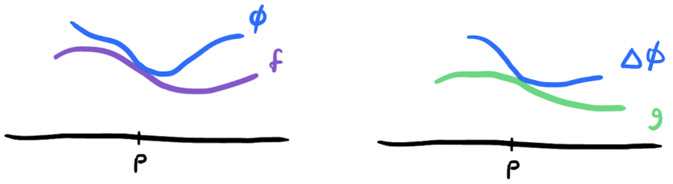
Suppose f, g are continuous functions at p. We give a new definition of $\Delta f \geq g$ even when we can not necessarily take second derivatives of f.

Definition 9.6 (barrier sense)**.** For f, g continuous functions, we say "$\Delta f \geq g$" in the *barrier sense* if for every $\epsilon > 0$, there exists a C^2-function h_ϵ and a neighborhood U_ϵ of p such that

1. $h_\epsilon \leq f$ in U_ϵ with equality at p.
2. $\Delta h_\epsilon \geq g - \epsilon$ at p.

Note that $f - h_\epsilon$ has a local minimum at p. If f is indeed C^2, then at p we have $\nabla f = \nabla h_\epsilon$, and the 2nd derivative test yields $\Delta(f - h_\epsilon) \geq 0$. Then $\Delta f \geq \Delta h_\epsilon \geq g - \epsilon$. So at p, $\Delta f \geq g - \epsilon$ for all $\epsilon > 0$, so $\Delta f \geq g$.

Definition 9.7 (viscosity sense)**.** We say that "$\Delta f \geq g$" at p in the *viscosity sense* if for every open set $U \ni p$ and C^2-function ϕ on U with $f \leq \phi$ and equality at p, then $\Delta\phi(p) \geq g(p)$.



The viscosity sense is weaker than the barrier sense: if $\phi \geq h_\epsilon$ with equality at p, then $\phi - h_\epsilon$ has a local minimum at p. Then $\Delta\phi \geq \Delta h_\epsilon \geq g - \epsilon$ at p.

Theorem 9.8 (Laplacian comparison)

If M^n is complete with $\text{Ric} \geq 0$, then we have $\Delta d \leq \frac{n-1}{d}$ in the barrier sense (and thus in the viscosity sense).

Corollary 9.9

We have $\Delta d^2 \leq 2n$ at every point in the barrier sense.

9.4 Cut points

Let M^n be a complete manifold with no boundary, and fix $p \in M$.

Definition 9.10. For each $V \in S_1^{n-1} \subset T_pM$, there is a unit speed geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = V$. Define

$$T(V) = \sup_{t>0}\{t : d(\gamma(t)) = t\},$$

i.e. for how long the geodesic is minimizing. If $T(V) < \infty$, then $\gamma(T(V))$ is a *cut point*. Let Cut(p) be the union of all cut points as V varies.

Example 9.11

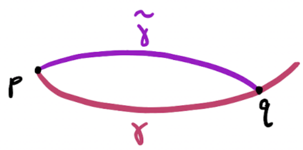
Note that $d(\gamma(t)) = t$ holds at least when t is within the normal neighborhood.

- On S^{n-1}, $T(V) = \pi$ for all V, so $\mathrm{Cut}(p) = \{\text{antipode of } p\}$.
- On $\mathbb{R}^n$, $T(V) = \infty$, so $\mathrm{Cut}(p) = \emptyset$.

Lemma 9.12

If $q \notin \mathrm{Cut}(p)$, then there exists a unique minimizing geodesic from p to q.

Proof. By Hopf–Rinow (Theorem 5.15), there exists a minimizing geodesic γ. Suppose there is a second minimizing geodesic $\widetilde{\gamma}$, which must have the same length from p to q. Extending γ past q, it can no longer minimizing because there is a piecewise geodesic with the same length following $\widetilde{\gamma}$. This implies q is a cut point. ☐



Lemma 9.13

If $q \in \mathrm{Cut}(p)$, then at least one of the following hold.

1. q is the first conjugate point to p along γ.
2. q is the first point where there are distinct minimizers from p.

The sphere satisfies both 1 and 2, while the cylinder satisfies only 2 (it does not have conjugate points).

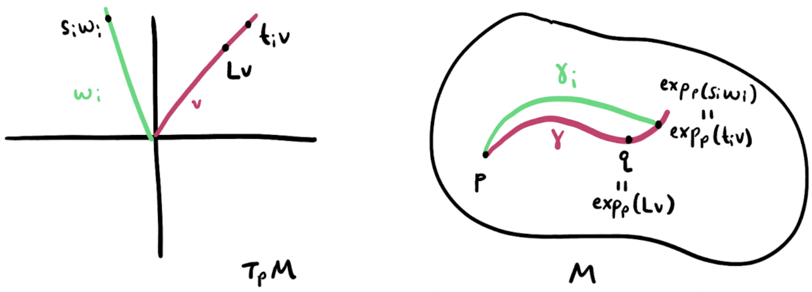
Proof. The reverse direction is straightforward, so consider the forward direction. Let $q \in \mathrm{Cut}(p)$, and γ be a geodesic of unit speed with $L = d(q)$, so $\gamma(L) = q$. Let $V = \gamma'(0)$ be the initial direction. By definition,

- γ is minimizing on $[0, L]$ i.e. $d(\gamma(t)) = t$ for $t \le L$ (it is true for $t < L$, and being minimizing is a closed condition).
- $d(\gamma(t)) < t$ for $t > L$.

Take a decreasing sequence $t_i \to L$. There is some minimizing geodesic $\gamma_i \neq \gamma$ from p to $\gamma(t_i) = \exp_p(t_i V)$ with unit speed, so let $W_i := \gamma_i'(0) \in S^{n-1}$. Note $W_i \neq V$ by uniqueness of ODEs. Define $s_i < t_i$ by $\gamma_i(s_i) = \gamma(t_i)$.

Because S^{n-1} is compact, there is a subsequence $W_i \to W \in S^{n-1}$. Let σ be a unit speed geodesic with $\sigma'(0) = W$, so $\gamma_i \to \sigma$. We have $s_i \to L$ by the squeeze theorem and then $\sigma(L) = q$ by continuity.

If $\sigma \neq \gamma$, then this is 2. If $\sigma = \gamma$, then $W = V$. Since $\exp_p(s_i W_i) = \exp_p(t_i V)$, both sides converge to q as $i \to \infty$. However, notice that $s_i < t_i$, so $s_i W_i$ and $t_i V$ are two distinct points in T_pM. Since they go towards the same point under $\exp_p$, this means $\exp_p\colon T_pM \to M$ is not injective in a neighborhood of LV, so $(d\exp_p)_{LV}$ is not invertible. Let U be the tangent vector such that $(d\exp_p)_{LV}(LU) = 0$. By Lemma 6.5, there is a Jacobi field $J(t) = (d\exp_p)_{\gamma(t)}(tU)$ with $J(0) = J(L) = 0$. Then $\gamma(0) = p$ and $\gamma(L) = q$ are conjugate points. ☐



Corollary 9.14

If γ is a minimizing geodesic from p to q, then there are no cut points before q along γ.

Corollary 9.15

$\text{Cut}(p)$ is closed and has measure 0.

Proof. It has measure 0 because along each ray in T_pM there is at most one point which maps to a cut point.

To show that it is closed, intuitively, the two conditions in Lemma 9.13 are closed. Rigorously, consider $q_i \in \text{Cut}(p)$ with $q_i \to q \in M$, and we must show that $q \in \text{Cut}(p)$. Let $\ell_i = d(q_i)$, γ_i be the minimizing geodesic from p to q_i, and $\ell = d(q)$. By continuity of d, we have $\ell_i \to \ell$. Choose $t_i > \ell_i$ with $t_i \to \ell$. By the definition of a cut point, there exist geodesics σ_i from p to $\gamma_i(t_i)$ of length $s_i < t_i$.

By the compactness of S^{n-1} implies that $\sigma_i \to \sigma$ and $\gamma_i \to \gamma$ for some geodesics σ, γ from p with $\sigma(\ell) = \gamma(\ell) = q$. If $\sigma \neq \gamma$ then 2 in Lemma 9.13 is satisfied. If $\sigma = \gamma$, then like before, $\exp_p$ is not invertible and p and q are conjugate points. ☐

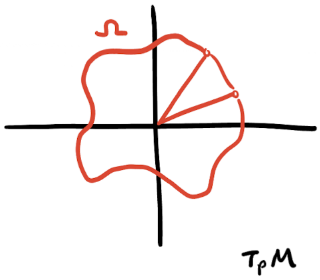
Remark 9.16. Consider a set $A \subset \mathbb{R}^n$, $\delta > 0$, and $p > 0$. We say that the measure $\mathcal{H}^p_\delta(A) \leq K$ if there exists a countable collection of balls $B_{r_i}(x_i)$ with $r_i \leq \delta$ such that $A \subset \bigcup_i B_{r_i}(x_i)$ and $\sum_i r_i^p \leq K$. If $p = n$, then r_i^n is essentially the volume of the balls. The limit $\lim_{\delta \to 0} \mathcal{H}^p_\delta(A)$ exists (may be ∞) and is called the p-dimensional Hausdorff measure $\mathcal{H}^p(A)$.

Proposition 9.17 (smoothness away from cut points)

If $q \notin \text{Cut}(p)$, $q \neq p$, and $d(q) = \ell$, then

1. d is smooth in a neighborhood of q.
2. There exists a unique minimizing geodesic γ from p to q with unit speed such that $\nabla d(\gamma(t)) = \gamma'(t)$ for all $t \leq \ell$.

Proof. 1. Let $\Omega = \exp_p^{-1}(M \setminus (\mathrm{Cut}(p) \cup \{p\}))$. In other words, it is rays of T_pM that go to up until the cut points. Since Ω is starshaped, it is naturally like $\mathbb{R}^n$, and $\exp_p$ is a local diffeomorphism on Ω that maps it to $M \setminus \mathrm{Cut}(p)$. So for any $q \notin \mathrm{Cut}(p) \cup \{p\}$, there exists $x \in \Omega$ such that $\exp_p(x) = q$. Then $d(p,q) = d(p, \exp_p(x)) = |x|$, so d is smooth at q.



2. For any $q \notin \mathrm{Cut}(p) \cup \{p\}$, there exists a unique unit speed geodesic γ with $\gamma(0) = p$, $\gamma(d(p,q)) = q$, and $|\gamma'(0)| = 1$. Fix $V \in T_qM$. Let σ be a unit speed curve with $\sigma(0) = q$ and $\sigma'(0) = V$. For s sufficiently small, $\sigma(s)$ is not a cut point because $\mathrm{Cut}(p)$ is closed and $\sigma(0) = q$ is not a cut point. Then there exists a unique geodesic γ_s from p to $\sigma(s)$. By the first variation formula (Equation (6.2)),

$$\langle V, \nabla d\rangle(q) = \frac{d}{ds} d(\sigma(s))\bigg|_{s=0} = E'(0) = \langle V, \gamma'\rangle(q).$$

Since this holds for any V, we have $\nabla d(q) = \gamma'(q)$. □

Proof of Theorem 9.8. We want to prove that $\Delta d \le \frac{n-1}{d}$ at $q \in \mathrm{Cut}(p)$, which means (c.f. Definition 9.6) for all $\epsilon > 0$, there exists a C^2-function h_ϵ (near q) such that

1. $d \le h_\epsilon$ with equality at q,
2. $\Delta h_\epsilon \le \frac{n-1}{d} + \epsilon$ at q.

Let $\gamma\colon [0,\ell] \to M$ be the unit speed minimizing geodesic from p to q. Given $\delta > 0$, define

$$h(x) = d(\gamma(\delta), x) + \delta.$$

If $\delta = 0$, then $h(x) = d(x)$. Note that $h(q) = \ell = d(q)$. In general, we have by the triangle inequality that

$$d(x) = d(p,x) \le d(p, \gamma(\delta)) + d(\gamma(\delta), x) = \delta + d(\gamma(\delta), x) = h(x),$$

so $d \le h$ with equality at q, giving condition 1.

For condition 2, first note that $q \notin \mathrm{Cut}(\gamma(\delta))$. If $q \in \mathrm{Cut}(\gamma(\delta))$, then $\gamma(\delta) \in \mathrm{Cut}(q)$ because being a cut point is a symmetric condition, but continuing γ past $\gamma(\delta)$ would no longer not be minimizing. Then by smoothness away from cut points (Proposition 9.17) and and Laplacian comparison at smooth points (Proposition 9.3),

$$\Delta h(q) = \Delta d(\gamma(\delta), q) \le \frac{n-1}{d(\gamma(\delta), q)} = \frac{n-1}{d(q) - \delta} \le \frac{n-1}{d(q)} + \epsilon$$

for δ sufficiently small. □

Remark 9.18. We also just showed that $\Delta d^2 \le 2n$, and similarly for other powers d^a. At smooth points,

$$\Delta d^2 = 2d\Delta d + 2\left|\nabla d\right|^2 \le 2d \cdot \frac{n-1}{d} + 2 = 2n.$$

At cut points, use h^2.

9.5 Bishop–Gromov

Theorem 9.19 (Bishop–Gromov)
For M^n with $\operatorname{Ric} \geq 0$, the volume ratio $\frac{\operatorname{Vol}(B_r(p))}{r^n}$ is nonincreasing as r increases.

This is the opposite direction as for minimal submanifolds (Theorem 8.6). In particular, minimal submanifolds can not have nonnegative Ricci curvature.

Corollary 9.20 (Bishop 1964)
$\operatorname{Vol}(B_r(p)) \leq \operatorname{Vol}(B_r \subset \mathbb{R}^n)$.

Corollary 9.21 (volume doubling)

$$\frac{\operatorname{Vol}(B_{2r}(p))}{\operatorname{Vol}(B_r(p))} \leq 2^n.$$

Proof. Rearrange $\frac{\operatorname{Vol}(B_{2r}(p))}{(2r)^n} \leq \frac{\operatorname{Vol}(B_r(p))}{r^n}$. □

Fake proof of Theorem 9.19. A fake proof is that $|\nabla d|^2 = 1$ almost everywhere, and $\Delta d^2 \leq 2n$.
The coarea formula (“slicing”) says that for any $s \in \mathbb{R}$ and functions $f, g \in C(M)$,

$$\int_{f \leq s} g\,|\nabla f| = \int_{\infty}^{s} \int_{\{f=t\}} g\,dt.$$

Define $V(r) \coloneqq \operatorname{Vol}(B_r(p))$ and $A(r) \coloneqq \operatorname{Area}(\partial B_r(p)) = V'(r)$. The fact that $A(r) = V'(r)$ follows from the coarea formula, because

$$\begin{aligned} V(r) &= \int_{d \leq r} |\nabla d| && (|\nabla d| = 1 \text{ a.e.}) \\ &= \int_0^r \int_{\{d=t\}} 1\,dt && (\text{coarea formula}) \\ &= \int_0^r A(t)\,dr, \end{aligned}$$

and then FTC. Then

$$\begin{aligned} V'(r) &= \int_{d=r} |\nabla d| \\ &= \int_{d=r} \frac{|\nabla d^2|}{2d} \\ &= \frac{1}{2r} \int_{d=r} |\nabla d^2| \\ &= \frac{1}{2r} \int_{d=r} \langle \nabla d^2, \text{conormal} \rangle \\ &= \frac{1}{2r} \int_{d<r} \Delta d^2 && (\text{divergence theorem}) \\ &\leq \frac{1}{2r} \int_{d<r} 2n && (\text{Remark 9.18}) \\ &= \frac{n}{r} V(r). \end{aligned}$$

This shows $V' \leq \frac{n}{r}V$, so $(r^{-n}V)' = -\frac{n}{r^{n+1}}V + \frac{V'}{r^n} \leq 0$. ☐

The flaw in this "proof" is that we can't apply the divergence theorem, because it requires the boundary to be smooth.

Real proof of Theorem 9.19. For each $V \in S^{n-1} \subset T_pM$, let

$$T(V) \coloneqq \sup_{t>0}\{t \mid d(\exp_p(tV)) = t\} \in \mathbb{R}^+ \cup \{\infty\},$$

which is the last point where the geodesic from 0 to $\exp_p(tV)$ is smooth. Define $\Omega \subset T_pM$ to be

$$\Omega \coloneqq \bigcup_{V\in S^{n-1}} \bigcup_{t\in[0,T(V))} tV.$$

For example, on $\mathbb{R}^n$, $\Omega = T_pM = \mathbb{R}^n$, while on S^n, $\Omega = B_\pi(0)$. We can show that Ω is open, and that $\exp_p$ is injective and a local diffeomorphism on Ω. Thus $\exp_p$ is a global diffeomorphism onto its image $\operatorname{Im}(\Omega) = M \setminus \operatorname{Cut}(p)$.

Since M is complete, we have $B_r(p) = \exp_p(\Omega \cap B_r(0))$ up to cut points, so

$$V(r) \coloneqq \operatorname{Vol}(B_r(p)) = \operatorname{Vol}(\exp_p(\Omega \cap B_r)).$$

By changing variables, we get

$$V(r) = \int_{\Omega\cap B_r(0)} \det(d\exp_p) = \int_0^r \int_{\partial B_s\cap\Omega} \det(\text{angular part})\, ds,$$

where the last equality is by switching to polar coordinates where there is a radial part and an angular part. Gauss's lemma (Lemma 5.9) says that $d\exp_p$ looks like a 1 on the radial part and an $(n-1)\times(n-1)$ matrix on the angular part.

Let $a(s,\theta)$ be the Jacobian determinant for each $\theta \in \partial B_r \cap \Omega$, so $V(r) = \int_0^r \int_{\theta\in\partial B_s\cap\Omega} a(s,\theta)\, ds$. By the proof of Lemma 8.1, we know how to differentiate the area factor:

$$\begin{aligned} \frac{a'}{a} &= \text{mean curvature of the level set} \\ &= \operatorname{div}(\text{unit normal}) \\ &= \operatorname{div}(\nabla d) = \Delta d \leq \frac{n-1}{d}. \end{aligned}$$

Incomplete proof in class. ☐

9.6 Dirichlet Poincaré inequality

This is a generalization of the Wirtinger inequality. Let M^n be complete with $\operatorname{Ric} \geq 0$ and no boundary.

Theorem 9.22 (Dirichlet Poincaré inequality)
There exists a constant c_n such that if $f \in C(M^n)$ with $f \equiv 0$ on $\partial B_r(p)$, then

$$\int_{B_r(p)} f^2 \leq c_n r^2 \int_{B_r(p)} |\nabla f|^2.$$

Lemma 9.23

Fix a point q, and let $d(x) = d(q, x)$. Then $\Delta d^{-n} \geq 2nd^{-n-2}$.

Proof. Chain rule and Laplacian comparison. □

Proof of Theorem 9.22. Pick any point q such that $d(p, q) = 2r$. Define $w \coloneqq d(q, x)^{-n}$. Lemma 9.23 says

$$\Delta w \geq 2nd(q, x)^{-n-2} \geq 2n(3r)^{-n-2}.$$

On the other hand,

$$|\nabla w| \leq \frac{n}{d(q, x)^{n+1}} \leq \frac{n}{r^{n+1}}$$

on $B_r(p)$.

The function $f^2 \nabla w$ is 0 on $\partial B_r(p)$, so the divergence theorem says

$$\begin{aligned} 0 &= \int_{B_r(p)} \operatorname{div}(f^2 \nabla w) \\ &= \int_{B_r(p)} f^2 \Delta w + 2f\langle \nabla f, \nabla w\rangle \\ &\geq \int_{B_r(p)} 2n(3r)^{-n-2} f^2 + 2f\langle \nabla f, \nabla w\rangle. \end{aligned}$$

Then

$$\begin{aligned} 2n(3r)^{-n-2} \int_{B_r(p)} f^2 &\leq 2 \int_{B_r(p)} |f|\,|\nabla f|\,|\nabla w| \\ &\leq \frac{2n}{r^{n+1}} \int_{B_r(p)} |f|\,|\nabla f| \\ &\leq \frac{2n}{r^{n+1}} \Big(\int_{B_r(p)} f^2 \Big)^{\frac{1}{2}} \Big(\int_{B_r(p)} |\nabla f|^2 \Big)^{\frac{1}{2}} \end{aligned}$$

by Cauchy–Schwarz. Rearrange to get $\frac{3^{-n-2}}{r} \Big(\int_{B_r(p)} f^2 \Big)^{\frac{1}{2}} \leq \Big(\int_{B_r(p)} |\nabla f|^2 \Big)^{\frac{1}{2}}$, so

$$\int_{B_r(p)} f^2 \leq 3^{2n+4} r^2 \int_{B_r(p)} |\nabla f|^2,$$

where $c_n = 3^{2n+4}$. □

10 Gradient estimate and Liouville theorems

10.1 Gradient estimate

Recall that a function u is *harmonic* if $\Delta u = 0$. Liouville's theorem on $\mathbb{R}^n$ says that if $\Delta u = 0$ and $u > 0$ on all of $\mathbb{R}^n$, then u is constant.

In this section, we always assume M^n is complete and connected with $\operatorname{Ric} \geq 0$.

Theorem 10.1 (Cheng–Yau gradient estimate)
If $u \in C(M)$ satisfies $\Delta u = 0$ and $u > 0$ on $B_R(p)$, then $|\nabla \log u| \le \frac{C_n}{R}$ on $B_{R/2}(p)$, for some constant C_n depending only on n.

An estimate like this will blow up near the boundary of $B_R(p)$.

Corollary 10.2 (Yau's Liouville theorem)
If $\Delta u = 0$ and $u > 0$ on all of M, then u is constant.

Proof. Then $|\nabla \log u| \le \frac{C_n}{R}$ holds for every R, and $\frac{C_n}{R} \to 0$. □

Note that $\Delta u = 0$ and $u > L$ for any $L \in \mathbb{R}$ implies u is constant by Liouville's theorem on $u - L$. The same is true if $|u| \le L$.

Corollary 10.3
If $\Delta u = 0$ on $B_R(p)$, then

$$\sup_{x \in B_{R/2}(p)} |\nabla u(x)| \le \frac{3C_n}{R} \sup_{x \in B_R(p)} |u(x)| .$$

In other words, we can control the derivative on a smaller ball by controlling the values on a larger ball.

Proof. Set $L = \sup_{B_R(p)} |u|$ and $w = u + 2L$, so $w > 0$ on $B_R(p)$. By the gradient estimate (Theorem 10.1), $\frac{|\nabla u|}{u+2L} = |\nabla \log w| \le \frac{C_n}{R}$ on $B_{R/2}(p)$, so $|\nabla u| \le \frac{C_n}{R}(u + 2L) \le \frac{C_n}{R}(3L)$. □

Lemma 10.4
For $u \in C(M)$ with $\Delta u = 0$, the function $w := \log u$ satisfies

1. $\Delta w = -|\nabla w|^2$
2. $\frac{1}{2}\Delta |\nabla w|^2 \ge \frac{1}{n} |\nabla w|^4 - \langle \nabla w, \nabla |\nabla w|^2 \rangle$.

Proof.
1. $\Delta w = \operatorname{div}(\nabla w) = \operatorname{div}(\frac{\nabla u}{u}) = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = -|\nabla w|^2$.
2. By the Bochner formula,

$$\frac{1}{2}\Delta |\nabla w|^2 = |\mathrm{Hess}_w|^2 + \langle \nabla w, \nabla \Delta w \rangle + \mathrm{Ric}(\nabla w, \nabla w) \ge |\mathrm{Hess}_w|^2 - \langle \nabla w, \nabla |\nabla w|^2 \rangle.$$

Cauchy–Schwarz $(\operatorname{Tr} A)^2 \le n |A|^2$ on the $n \times n$ symmetric matrix $A = \mathrm{Hess}_w$ implies $(\Delta w)^2 \le n |\mathrm{Hess}_w|^2$. Then $|\mathrm{Hess}_w|^2 \ge \frac{1}{n}(-|\nabla w|^2)^2 = \frac{1}{n} |\nabla w|^4$, and combine with the above inequality.

□

At an interior maximum of F, the first derivative test says that $\nabla F = 0$. The second derivative test says $\mathrm{Hess}_F \le 0$, so tracing yields $\Delta F \le 0$.

At an interior maximum of $\nabla |\nabla w|^2$, by 2 we would have $\nabla |\nabla w|^2 = 0$ and $\Delta |\nabla w|^2 \le 0$. Then by 2, we would have $|\nabla w|^4 = 0 \implies \nabla w = 0$, which is suspicious. The problem is that $|\nabla w|^2$ may not have an interior maximum, but we will use a cutoff function η.

Proof of Theorem 10.1. Define $\eta \geq 0$ with

$$\eta(x) = \begin{cases} R^2 - d(p,x)^2 & \text{if } x \in B_R(p) \\ 0 & \text{else} \end{cases}.$$

Define $F := \eta^2 |\nabla w|^2$ which does have an interior maximum. Choose $y \in B_R(p)$ with $F(y) = \max F$. We will show that $F(y) \leq C_n' R^2$, which will imply that $C_n' R^2 \geq F(x)$ for any x. Then if $x \in B_{R/2}(p)$,

$$C_n' R^2 \geq F(x) = |\nabla w|^2 (x) \eta^2(x) \geq \frac{9}{16} R^4 |\nabla w|^2 (x).$$

Dividing by R^4 yields the gradient estimate.

Assuming that $F(y) \leq C_n' R^2$ for now, at a maximum y for F, the first derivative test says

$$0 = \nabla F = 2\eta |\nabla w|^2 \nabla \eta + 2\eta^2 \nabla |\nabla w|^2 \implies \nabla |\nabla w|^2 = -\frac{|\nabla w|^2 \nabla \eta}{\eta} \tag{10.1}$$

The second derivative test says

$$\begin{aligned} 0 \geq \frac{1}{2}\Delta F &= \frac{1}{2}\Delta(\eta^2 |\nabla w|^2) \\ &= \frac{1}{2} |\nabla w|^2 \Delta \eta^2 + \eta^2 \frac{\Delta |\nabla w|^2}{2} + \langle \nabla \eta^2, \nabla |\nabla w|^2 \rangle \\ &\geq -2nR^2 |\nabla w|^2 + \eta^2 \Big(\frac{1}{n} |\nabla w|^4 - \langle \nabla w, \nabla |\nabla w|^2 \rangle \Big) + 2\eta \langle \nabla \eta, \nabla |\nabla w|^2 \rangle & \text{(Lemma 10.4(2))} \\ &= -2nR^2 |\nabla w|^2 + \frac{\eta^2}{n} |\nabla w|^4 + \eta^2 \Big\langle \nabla w, \frac{|\nabla w|^2 \nabla \eta}{\eta} \Big\rangle - 2\eta \Big\langle \nabla \eta, \frac{|w|^2 \nabla \eta}{\eta} \Big\rangle. & \text{(by (10.1))} \end{aligned}$$

Dividing by $|\nabla w|^2$ and rearranging yields

$$\begin{aligned} 2nR^2 &\geq \eta^2 \frac{|\nabla w|^2}{n} + \eta \langle \nabla w, \nabla \eta \rangle - 2 |\nabla \eta|^2 \\ &\geq \eta^2 \frac{|\nabla w|^2}{n} - \eta |\nabla w| |\nabla \eta| - 2 |\nabla \eta|^2 \\ &= \frac{1}{n} F - \sqrt{F} |\nabla \eta| - 2 |\nabla \eta|^2 \\ &\geq \frac{1}{n} F - \Big(\frac{1}{2n} F + \frac{n}{2} |\nabla \eta|^2 \Big) - 2 |\nabla \eta|^2 & \text{(AM-GM)} \end{aligned}$$

We know $\nabla \eta = -2d + \nabla d$ where $\nabla d \leq 1$, so $|\nabla \eta| \leq 2R$, and $\Delta \eta = -\Delta d^2 \geq -2n$ by Laplacian comparison. Rearranging and using $|\nabla \eta|^2 \leq 4R^2$,

$$F \leq 2n^2 R^2 + n\Big(\frac{n}{2} + 2\Big) R^2 = CR^2$$

where $C = \Theta(n^2)$. □

10.2 Harnack inequality

Theorem 10.5 (Harnack)

If $\Delta u = 0$ and $u > 0$ on $B_R(p)$, then

$$\sup_{B_{R/2}(p)} u \leq C_n \inf_{B_{R/2}(p)} u$$

for some C_n depending only on n.

Proof. Take any $q \in B_{R/2}(p)$ and let γ be a minimizing geodesic from p to q, which is contained in $B_{R/2}(p)$. Let $f(t) := \log u(\gamma(t))$, so

$$\begin{aligned} \left|f'(t)\right| &= \left|\langle \nabla \log u(\gamma(t)), \gamma'(t)\rangle\right| \\ &\le |\nabla \log u(\gamma(t))| && \text{(Cauchy–Schwarz and } |\gamma'| = 1) \\ &\le \frac{C_n'}{R}. && \text{(gradient estimate Theorem 10.1)} \end{aligned}$$

Then FTC implies

$$|\log u(p) - \log u(q)| = \left|\int_0^{d(p,q)} f'(t)\,dt\right| \le \int_0^{d(p,q)} \left|f'(t)\right|\,dt \le \frac{C_n'}{R} d(p,q) \le \frac{C_n'}{2}$$

which implies $u(p) \le u(q)e^{C_n'/2}$, and the inequality holds with $C_n = e^{C_n'}$. ☐

Remark 10.6. By a limiting argument with $u + \epsilon$ and $\epsilon \to 0$, this inequality is also true for $u \ge 0$.

Corollary 10.7

For $\alpha < 1$, we have $\sup_{B_{\alpha R}(p)} u \le C_\alpha \inf_{B_{\alpha R}(p)} u$ where $C_\alpha \to \infty$ as $\alpha \to 1$.

Proof. One way to prove this is with a stronger gradient estimate: for all $q \in B_{\alpha R}(p)$, $|\nabla \log u(q)| \le \frac{C_n}{(1-\alpha)R}$. Another way is to use Harnack's inequality and iterate. ☐

Lemma 10.8 (Kato's inequality)

At any point where $|\nabla u| \ne 0$, $|\nabla |\nabla u|| \le |\mathrm{Hess}_u|$.

Proof. First note

$$2\,|\nabla u|\,\nabla\,|\nabla u| = \nabla\,|\nabla u|^2 = 2\langle \nabla_\bullet \nabla u, \nabla u\rangle = 2\,\mathrm{Hess}_u(\nabla u, \bullet),$$

so dividing by $2\,|\nabla u|$ implies $\nabla\,|\nabla u| = \mathrm{Hess}_u(\frac{\nabla u}{|\nabla u|}, \bullet)$. Choose an orthonormal frame $e_1, \ldots, e_n$ such that $e_1 = \frac{\nabla u}{|\nabla u|}$. Then

$$|\nabla\,|\nabla u||^2 = \sum_{i=1}^n \langle \nabla\,|\nabla u|\,, e_i\rangle^2 = \sum_{i=1}^n u_{1i}^2 \le \sum_{i,j=1}^n u_{ji}^2 = |\mathrm{Hess}_u|^2\,.$$ ☐

Next we give an easier proof of another version of Corollary 10.3, in the case of $\mathbb{R}^n$.

Theorem 10.9

Let M^n be a manifold (not necessarily $\mathrm{Ric} \ge 0$) and $u \in C(M)$ satisfy $\Delta u = 0$. Then there exists a constant $C = C_n$ such that

$$\sup_{x \in B_{R/2}(p)} |\nabla u(x)| \le \frac{C}{R} \sup_{x \in B_R(p)} |u(x)|\,.$$

Proof. WLOG let $p = 0$. Consider the cutoff function $\eta := R^2 = |x|^2$ which vanishes on ∂B_R. We want to bound $\eta^2 |\nabla u|^2$. Recall that

$$\begin{aligned}
\Delta\eta &= -2n \\
\Delta\eta^2 &= 2\eta\Delta\eta + 2|\nabla\eta|^2 \geq -4n\eta \geq -4nR^2 \\
|\nabla\eta| &= 2\eta.
\end{aligned}$$

Kato's inequality says

$$\Delta |\nabla u|^2 = 2|\text{Hess}_u|^2 + 2\langle \nabla u, \nabla\Delta u\rangle \geq 2|\nabla |\nabla u||^2 .$$

Then

$$\begin{aligned}
\Delta(\eta^2 |\nabla u|^2) &= \eta^2\Delta|\nabla u|^2 + |\nabla u|^2 \Delta\eta^2 + 2\langle \nabla|\nabla u|^2, \nabla\eta^2\rangle \\
&\geq 2\eta^2|\nabla|\nabla u||^2 - 4nR^2|\nabla u|^2 + 4\langle\nabla\eta^2, |\nabla|\nabla u|| \, |\nabla u|\rangle \\
&\geq 2\eta^2|\nabla|\nabla u||^2 - 4nR^2|\nabla u|^2 - 8\eta|\nabla\eta| \, |\nabla|\nabla u|| \, |\nabla u| \\
&\geq 2\eta^2|\nabla|\nabla u||^2 - 4nR|\nabla u|^2 - 16R\eta|\nabla u| \, |\nabla|\nabla u|| \qquad (|\nabla\eta| \leq 2R)
\end{aligned}$$

By the absorbing inequality, $16\eta|\nabla u| \, |\nabla|\nabla u|| \leq 2\eta^2|\nabla|\nabla u||^2 + 32|\nabla u|^2 R^2$,so

$$\Delta(\eta^2|\nabla u|^2) \geq -(4n + 32)R^2|\nabla u|^2 .$$

Since $\Delta u^2 = 2u\Delta u + 2|\nabla u|^2 = 2|\nabla u|^2$, we get

$$\Delta(\eta^2|\nabla u|^2 + (2n + 16)R^2u^2) \geq 0.$$

By the maximum principle, $\eta^2|\nabla u|^2 + (2n + 16)R^2u^2$ has its maximum on the boundary. However, $\eta \equiv 0$ on the boundary, so the maximum value is at most $(2n + 16)R^2 \sup_{B_R(p)} |u|^2$. Then

$$\begin{aligned}
(2n + 16)R^2 \sup_{x\in B_R(p)} |u(x)|^2 &\geq \eta(x)^2|\nabla u(x)|^2 + (2n + 16)^2R^2u(x)^2 \\
&\geq \eta(x)^2|\nabla u(x)|^2 \\
&\geq \frac{9R^4}{16}|\nabla u(x)|^2 ,
\end{aligned}$$

which implies the result, where $C = \Theta(n)$. □

10.3 Mean value inequality

Theorem 10.10 (Mean value inequality)

Suppose $\text{Ric} \geq 0$, and let $v \in C(M)$ satisfy $v \geq 0$ and $\Delta v \geq 0$ on $B_{4R}(p)$. Then

$$\sup_{B_R(p)} v^2 \leq C_n \frac{\int_{B_{4R}(p)} v^2}{\text{Vol}(B_{4R}(p))}.$$

In other words, the mean value can be compared to the maximum.

Proof. Let ϕ be a cutoff function such that $\phi \equiv 1$ on $B_{2R}(p)$, $\phi \equiv 0$ on $\partial B_{4R}(p)$, and $|\nabla\phi| \leq \frac{1}{2R}$. Reverse Poincaré (Lemma 7.12) says

$$\int_{B_{2R}(p)} |\nabla v|^2 \leq 4\int_{B_{4R}(p)} v^2|\nabla\phi|^2 \implies \int_{B_{2R}(p)} |\nabla v|^2 \leq \frac{1}{R^2}\int_{B_{4R}(p)} v^2. \qquad (10.2)$$

Now we solve for u such that $\Delta u = 0$ in $B_{2R}(p)$ and $u = v$ on $\partial B_{2R}(p)$. Note that $\Delta(v-u) = \Delta v \geq 0$, so $v - u$ has its maximum on the boundary $\partial B_{2R}(p)$. However, $v - u = 0$ on $\partial B_{2R}(p)$, so $v \leq u$ inside $B_{2R}(p)$. By Lemma 7.11, we have $\int_\Omega |\nabla(u+w)|^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla w|^2$. Then by (10.2),

$$\frac{1}{R^2}\int_{B_{4R}(p)} v^2 \geq \int_{B_{2R}(p)} |\nabla v|^2 = \int_{B_{2R}(p)} |\nabla u|^2 + |\nabla(u-v)|^2 \geq \int_{B_{2R}(p)} |\nabla(u-v)|^2 .$$

By Dirichlet Poincaré (Theorem 9.22), $u - v \equiv 0$ on $\partial B_{2R}(p)$ implies

$$\int_{B_{2R}(p)} (u-v)^2 \leq C_1 R^2 \int_{B_{2R}(p)} |\nabla(u-v)|^2 \leq C_1 \int_{B_{4R}(p)} v^2.$$

Since u is harmonic and $u \geq 0$, the Harnack inequality (Theorem 10.5) says $v^2 \leq u^2 \leq C \inf_{B_R(p)} u^2$. Then

$$\begin{aligned}
\sup_{B_R(p)} v^2 \leq \sup_{B_R(p)} u^2 \leq C \inf_{B_R(p)} u^2 & \\
&\leq \frac{C}{\mathrm{Vol}(B_R(p))}\int_{B_R(p)} u^2 \\
&\leq \frac{C}{\mathrm{Vol}(B_R(p))}\int_{B_{2R}(p)} u^2 \\
&\leq \frac{2C}{\mathrm{Vol}(B_R(p))}\Big(\int_{B_{2R}(p)} v^2 + \int_{B_{2R}(p)} (u-v)^2\Big) \\
&\leq \frac{2C}{\mathrm{Vol}(B_R(p))}\Big(\int_{B_{4R}(p)} v^2 + C_1\int_{B_{4R}(p)} v^2\Big) \\
&\leq \frac{2\cdot 4^n \cdot C(1+C_1)}{\mathrm{Vol}(B_{4R}(p))}\int_{B_{4R}(p)} v^2
\end{aligned}$$

where the last line is by Bishop–Gromov (Theorem 9.19). □

10.4 Harmonic functions of polynomial growth

Recall Liouville's theorem says that any bounded harmonic function is constant. There is also a stronger result: if $\Delta u = 0$ and $\sup_{B_R(p)} |u| \leq C' R^\alpha$ for some $\alpha < 1$, then the easier gradient estimate says that

$$\sup_{B_{R/2}(p)} |\nabla u| \leq \frac{C}{R} \sup_{B_R(p)} u.$$

So if $\Delta u = 0$ and $|u| \leq C'$, then u is constant. On $\mathbb{R}^n$, $u = x_1$ grows like this.

Definition 10.11 (space of harmonic functions)**.** Define $\mathcal{H}^d(M)$ as the space of harmonic functions that grow almost like degree d. In other words, there exist $p \in M$ and $C \in \mathbb{R}$ such that

$$|u(x)| \leq C(1 + d(p,x))^d.$$

In particular, M^n with $\mathrm{Ric} \geq 0$ implies $\mathcal{H}^d(M) = \{0\}$ for $d < 1$.

Example 10.12

On $\mathbb{R}^n$, we can show that $\mathcal{H}^d(\mathbb{R}^n)$ is finite: if $\Delta u = 0$ then $\frac{\partial u}{\partial x_i}$ is harmonic too, and the gradient estimate implies $\frac{\partial u}{\partial x_i} \in \mathcal{H}^{d-1}(\mathbb{R}^n)$. Do this d times to get a constant, which implies u is a polynomial with degree $\leq d$. Then $\dim \mathcal{H}^d(\mathbb{R}^n) = O(n^d)$.

Theorem 10.13 (Colding–Minicozzi 1997)
If $\mathrm{Ric} \geq 0$, then $\dim \mathcal{H}^d(M^n)$ is finite dimensional. It is also true for $(M, \widetilde{g})$ if $\mathrm{Ric}_g \geq 0$ and $g, \widetilde{g}$ are bi-Lipschitz.

In 1998, they furthermore showed that $\dim \mathcal{H}^d(M^n) \leq Cd^{n-1}$.

Theorem 10.14
There exists a constant C_n such that if $u_1, \dots u_N$ are harmonic, $L^2(B_{2r})$-orthonormal, and $\int_{B_r} u_i^2 \geq \alpha > 0$, then $N \leq \frac{C}{\alpha}$.

Theorem 10.15
If $v_1, \dots, v_{2n} \in \mathcal{H}^d(M^n)$ and are linearly independent, then there exist $R > 0$ and $u_1, \dots, u_n$ in the span of the v_i's such that

1. $\int_{B_{2R}} u_i u_j = \delta_{ij}$ ($L^2(B_{2R})$-orthonormal).
2. $\int_{B_R} u_i^2 > 2^{-4(d+n)}$.

Together, these imply $N \leq C2^{4(d+n)}$, so $\dim \mathcal{H}^d(M^n) < \infty$.

Lemma 10.16
Given $x \in B_{2r}$, there exists $y \in S^{N-1}$ satisfying $w = \sum_{i=1}^N y_i u_i$ has $\sum_i u_i(x)^2 = w(x)^2$.

Proof. Let
$$y = \frac{\langle u_1(x), \dots, u_N(x)\rangle}{\sqrt{u_1(x)^2 + \cdots + u_n(x)^2}}.$$
□

Proof of Theorem 10.14. Fix $x \in B_r$, and choose w as in the above lemma, so
$$\sum_{i=1}^n u_i(x)^2 = w(x)^2 \leq \frac{C}{\mathrm{Vol}(B_r(x))} \int_{B_{2r}(x)} w^2 \leq \frac{C}{\mathrm{Vol}(B_r(x))}$$
by the mean value inequality and $\int_{B_{2r}} w^2 = 1$.
Since $B_r(p) \subset B_{2r}(x)$, $\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(B_{2r}(x)) \leq 2^n \,\mathrm{Vol}(B_r(x))$ by Bishop–Gromov. Thus
$$\sum_{i=1}^n u_i^2(x) \leq \frac{C2^n}{\mathrm{Vol}(B_r(p))}$$
where the RHS does not depend on x. Integrating over $B_r(p)$ yields $N\alpha \leq \int_{B_r(p)} \sum_{i=1}^n u_i^2(x) \leq C2^n$. □

Polynomial growth means that if $F(r) = r^d$, then $\frac{F(2r)}{F(r)} = 2^d$.

Lemma 10.17
If $F\colon [1, \infty) \to \mathbb{R}$ satisfies $0 < F(r) \leq Cr^d$ for all r, then for any $\epsilon > 0$, there exist infinitely many $k \in \mathbb{N}$ such that $\frac{F(2^{k+1})}{F(2^k)} \leq 2^{d+\epsilon}$.

Proof. FSOC there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $F(2^{k+1}) > 2^{d+\epsilon} F(2^k)$. This implies

$$F(2^{k_0+m}) > (2^{d+\epsilon})^m F(2^{k_0}).$$

On the other hand, $F(2^{k_0+m}) \leq C(2^{k_0+m})^d$ implies $2^{dm+\epsilon m} F(2^{k_0}) \leq C(2^{k_0+m})^d$. Dividing by 2^{dm} yields

$$2^{\epsilon m} \leq \frac{C}{F(2^{k_0})} 2^{k_0 d}$$

This is a contradiction because as $m \to \infty$, the LHS goes to infinity while the RHS is constant. □

Proof of Theorem 10.15. Let $\Lambda_j = \operatorname{span}\{v_1, \ldots, v_{j-1}\} \subset \mathcal{H}^d(M)$. Fix r, and let $w_{j,r}$ be the $L^2(B_r)$-orthogonal projection of v_j onto Λ_j. Define

$$f_j(r) \coloneqq \int_{B_r} (v_j - w_{j,r})^2 \leq \int_{B_r} (v_j - w)^2$$

for any $w \in \Lambda_j$, $v_j - w_{j,r}$ is perpendicular to Λ_j. Note that

1. The independence of the v_i implies there exists r_j such that $f_j(r_j) > 0$.
2. By letting $w = 0$, we have $f_j(r) \leq \int_{B_r} v_j^2 \leq C_j r^{2d+n}$ for some constant C_j depending only on j.
3. We have $f_j(r_1) \leq f_j(r_2)$ for $r_1 < r_2$ because

$$f_j(r_1) = \int_{B_{r_1}} (v_j - w_{j,r_1})^2 \leq \int_{B_{r_1}} (v_j - w_{j,r_2})^2 \leq \int_{B_{r_2}} (v_j - w_{j,r_2})^2 = f_j(r_2).$$

Now define $F(r) = \prod_{j=1}^{2n} f_j(r)$. By 1, we have $F(r) > 0$ for $r > \max\{r_j\}$. By 2, we have $F(r) \leq C(r^{2d+n})^{2N}$. By, Lemma 10.17, there exists some $R = 2^k$ such that $\frac{F(2R)}{F(R)} \leq 2^{(d+n)2N}$. Every f_i is nondecreasing, so $\frac{\prod_{i=1}^{2N} f_i(2R)}{\prod_{i=1}^{2N} f_i(R)} \leq 2^{(d+n)4N}$ which implies at least N of the f_i's satisfy $\frac{f_i(2R)}{f_i(R)} \leq 2^{4(d+n)}$. For each such i, set

$$u_i = \frac{v_i - w_{i,2R}}{\sqrt{f_i(2R)}}$$

which are orthonormal by Gram–Schmidt. Finally,

$$\begin{aligned}
\int_{B_R} u_i^2 &= \frac{1}{f_i(2R)} \int_{B_{2R}} (v_i - w_{i,2R})^2 \\
&\geq \frac{1}{f_i(2R)} \int_{B_R} (v_i - w_{i,R})^2 \\
&\geq \frac{1}{f_i(2R)} \int_{B_R} (v_i - w_{i,R})^2 \\
&= \frac{f_i(R)}{f_i(2R)} \\
&\geq 2^{-4(d+n)}.
\end{aligned}$$

□